

UNIQUENESS AND STABILITY FOR THE RECOVERY OF A TIME-DEPENDENT SOURCE IN ELASTODYNAMICS

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ABSTRACT. This paper is concerned with inverse source problems for the time-dependent Lamé system in an unbounded domain corresponding to either the exterior of a bounded cavity or the full space \mathbb{R}^3 . If the time and spatial variables of the source term can be separated with compact support, we prove that the vector valued spatial source term can be uniquely determined by boundary Dirichlet data in the exterior of a given cavity. Uniqueness and stability for recovering some class of time-dependent source terms are also obtained by using, respectively, partial and full boundary data.

1. Introduction.

1.1. Statement of the problem. Consider the radiation of an elastic source F outside a cavity D described by the system

$$(1) \quad \rho \partial_t^2 U(x, t) = \mathcal{L}_{\lambda, \mu} U(x, t) + F(x, t), \quad x = (x_1, x_2, x_3) \in \mathbb{R}^3 \setminus \overline{D}, \quad t > 0$$

where ρ denotes the density, μ and λ are the Lamé coefficients, $U = (u_1, u_2, u_3)^\top$ is the displacement vector, $D \subset \mathbb{R}^3$ is the region of the cavity and $\mathcal{L}_{\lambda, \mu} U$ is the Lamé operator defined by

$$(2) \quad \begin{aligned} \mathcal{L}_{\lambda, \mu} U &:= -\mu(x) \nabla \times \nabla \times U + (\lambda(x) + 2\mu(x)) \nabla \nabla \cdot U \\ &\quad + (\nabla \cdot U) \nabla \lambda(x) + ((\nabla U) + (\nabla U)^T) \nabla \mu(x). \end{aligned}$$

Throughout the paper, it is supposed that $\rho > 0$ is a constant and $\mu, \lambda \in \mathcal{C}^3(\mathbb{R}^3)$ satisfy $\mu > 0$, $\lambda \geq 0$. Further, the density function and the Lamé coefficients are supposed to be constants in $|x| > R$ for some sufficiently large $R > 0$ such that $D \subset B_R := \{x \in \mathbb{R}^3 : |x| < R\}$. Together with the governing equation, we impose the initial conditions

$$(3) \quad U(x, 0) = 0, \quad \partial_t U(x, 0) = 0, \quad x \in \mathbb{R}^3 \setminus \overline{D},$$

and the traction-free boundary condition on ∂D :

$$(4) \quad \mathcal{T}U(x, t) = 0, \quad (x, t) \in \partial D \times \mathbb{R}^+,$$

where $\mathcal{T}U$ is the stress boundary condition defined by (11) (see Section 2). In this paper we consider the inverse problem of determining the source term F from

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knowledge of U on the surface $\partial B_R = \{x \in \mathbb{R}^3 : |x| = R\}$ with $R > 0$ sufficiently large. According to [6, Remark 4.5] there is an obstruction for the recovery of general time-dependent source terms F . Facing this obstruction we consider this problem for some specific type of source terms.

1.2. Motivations. We recall that the Lamé system (1)-(2) is frequently used for the study of linear elasticity and imaging problems. In this context our inverse problems can be seen as the recovery of an external force provided by the source term F . Elastodynamics source terms corresponding to the product of a spatial function g and a temporal function f can be regarded as an approximation of the elastic pulse and are commonly used in modeling vibration phenomena in seismology and teleseismic inversion [1, 47]. This type of sources has been also considered in numerous applications in biomedical imaging (see the references [2, 3] and the references therein) where our inverse problem can be seen as the recovery of the information provided by the parameter under consideration.

Let us also observe that the identification of time-dependent sources (see Theorems 1.3 and 1.4) is associated to the recovery of a moving source which can be thought as an approximation of a pulsed signal transmitted by a moving antenna (see [23] and the references therein for more details).

1.3. Known results. Inverse source problems are a class of inverse problems which have received many interest. These problems take different forms and have many applications (environment, imaging, seismology \dots). For an overview of these problems we refer to [28]. Among the different arguments considered for solving these problems we can mention the approach based on applications of Carleman estimates arising from the work of [15] (see also [38, 39]). This approach has been applied successfully to hyperbolic equations by [50] in order to extend his previous work [49] to a wider class of source terms. More precisely, in [50] the author considered the recovery of source terms of the form $f(x)G(x, t)$, where G is known, while in [49] the analysis of the author is restricted to source terms of the form $\sigma(t)f(x)$, with σ known. More recently, the approach of [50] has been extended by [33] to hyperbolic equations with time-dependent second order coefficients and to less regular coefficients by [51]. We mention also the work of [17, 37] using similar approach for inverse source problems stated for parabolic equations and the result of [48] proved by a combination of geometrical arguments and Carleman estimates. Concerning the Lamé system we refer to [25] where a uniqueness result has been stated for the recovery of time-independent source terms by mean of suitable Carleman estimate. We mention also the book [11] for an overview of the approach based on Carleman estimate for solving inverse source problems as well as the works [9, 12, 26, 27, 30] dealing with related problems as well as [10] where an inverse source problem for Biot's equations has been considered. We refer also to the recent work [6] where the recovery of a time-independent source term appearing in the Lamé system in all space has been proved from measurements outside the support of the source under consideration as well as the work of [34] dealing with this inverse source problem for fractional diffusion equations.

In all the above mentioned results the authors considered the recovery of time independent source terms (in other words, the spatial component of the source term). For the recovery of a source depending only on the time variable we refer to [21] where such problems has been considered for fractional diffusion equations and for the recovery of some class of sources depending on both space and time variable

appearing in a parabolic equation on the half space, we refer to [28, Section 6.3]. For hyperbolic equations, we refer to [14, 46] where the recovery of some specific time-dependent source terms have been considered. For Lamé systems, [6, Theorem 4.2] seems to be the only result available in the mathematical literature where such a problem has been addressed for time-dependent source terms. The result of [6, Theorem 4.2] is stated with source terms depending only on the time variable. To the best of our knowledge, except the result of [14], dealing with the recovery of discrete in time sources, and the result of the present paper, there is no result in the mathematical literature treating the recovery of a source term depending on both space and time variables appearing in hyperbolic equations.

1.4. Main results. In the present paper we consider three inverse problems related to the recovery of the source term F . In our first inverse problems we assume that the cavity $D \neq \emptyset$ is a domain with C^3 boundary ∂D , with connected exterior $\mathbb{R}^3 \setminus \overline{D}$, and we consider source terms of the form

$$(5) \quad F(x, t) = f(t)h(x), \quad x \in \mathbb{R}^3 \setminus \overline{D}, \quad t \in (0, +\infty),$$

where f is a real valued function and $h = (h_1, h_2, h_3)^\top : \mathbb{R}^3 \setminus \overline{D} \rightarrow \mathbb{R}^3$ is a vector valued function. Choose $R > 0$ sufficiently large such that B_R also contains the support of h (i.e., $\text{supp}(h) \subset B_R$). We assume that $f \in L^2([0, +\infty))$ is compactly supported and $h \in L^2(\mathbb{R}^3 \setminus \overline{D})^3$. Then, the problem (1)-(4) admits a unique solution

$$U \in \mathcal{C}^1([0, +\infty); L^2(\mathbb{R}^3 \setminus D))^3 \cap \mathcal{C}([0, +\infty); H^1(\mathbb{R}^3 \setminus D))^3.$$

The proof of this result can be carried out by combining the elliptic regularity properties of $\mathcal{L}_{\lambda, \mu}$ (see e.g., [44, Chapters 4 and 10] and [22, Chapter 5]) with [41, Theorem 8.1, Chapter 3] and [41, Theorem 8.2, Chapter 3] (see also the beginning of Sections 4.1 and 4.2 for more details). Our first inverse problem in the exterior of the cavity can be stated as follows.

Inverse Problem 1 (IP1): Assume that f, D are both known in advance. Determine the spatially dependent function h from the radiated field U measured on the surface $\partial B_R \times [0, T_1)$, $T_1 \in (0, +\infty]$.

Below we give a confirmative answer to the uniqueness issue for IP1 in two different cases. For source terms with low regularity we obtain

Theorem 1.1. *Let $f \in \mathcal{C}^1([0, +\infty))$ satisfy $f(0) \neq 0$, $h \in H^1(\mathbb{R}^3 \setminus \overline{D})^3$ and F takes the form (5). Let also $\Omega := B_R \setminus \overline{D}$ and let $d_j, j = 1, 2$, be the Riemannian distance within $\overline{\Omega}$ induced by the metric g_j , where*

$$g_1[x](v, v) = \frac{\rho|v|^2}{\mu(x)}, \quad g_2[x](v, v) = \frac{\rho|v|^2}{2\mu(x) + \lambda(x)}, \quad x \in \overline{\Omega}, \quad v \in \mathbb{R}^3.$$

Then, for

$$T_1 > 2 \left(\max_{j=1,2} \left(\sup_{x \in \overline{\Omega}} d_j(x, \partial B_R) \right) \right),$$

the boundary data $\{U(x, t) : (x, t) \in \partial B_R \times (0, T_1)\}$ uniquely determine h .

By considering measurements for all time ($t \in (0, +\infty)$), we can remove the condition $f(0) \neq 0$ in the following way.

Theorem 1.2. *Let $f \in H_0^1(0, T)$, $h \in H^1(\mathbb{R}^3 \setminus \overline{D})^3$ and F takes the form (5). Then the boundary data $\{U(x, t) : (x, t) \in \partial B_R \times \mathbb{R}^+\}$ uniquely determine h .*

For our last inverse problem, we consider the Lamé system with constant density and Lamé coefficients when the embedded cavity is absent ($D = \emptyset$). We assume here that F takes the form

$$(6) \quad F(\tilde{x}, x_3, t) = g(x_3) f(\tilde{x}, t), \quad \tilde{x} \in \mathbb{R}^2, \quad x_3 \in \mathbb{R}, \quad t \in (0, +\infty),$$

where the vector valued function $f = (f_1, f_2, 0)^\top$ is compactly supported on $\tilde{B}_R \times [0, T)$ and the scalar function g is supported in $(-R, R)$ for some $R > 0$. Here $\tilde{x} = (x_1, x_2) \in \mathbb{R}^2$ for $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and \tilde{B}_R denotes the set $\tilde{B}_R := \{\tilde{x} \in \mathbb{R}^2 : |\tilde{x}| < R\}$. Then our last inverse problem can be stated as follows.

Inverse Problem 2 (IP2): Assume that g is known in advance. Determine the time and space dependent function f from the radiated field U measured on the surface $\Gamma \times (0, T_1)$, with $T_1 > 0$, $R_1 > 0$ sufficiently large and $\Gamma \subset \partial B_{R_1}$ an open set with positive Lebesgue measurement.

In this paper we give a positive answer to (IP2) both in terms of uniqueness and stability. Our uniqueness result can be stated as follows.

Theorem 1.3. *Assume $D = \emptyset$ and ρ, λ, μ are all constants in \mathbb{R}^3 . Assume that F takes the form (6) with $f = (f_1, f_2, 0)^\top \in H^1(0, T; L^2(\mathbb{R}^2))^3$, $g \in L^2(-R, R)$ is non-uniformly vanishing and*

$$f(\tilde{x}, 0) = 0, \quad \tilde{x} \in \mathbb{R}^2.$$

Let $R_1 > \sqrt{2}R$, $T_1 > T + \frac{2R_1\sqrt{\rho}}{\sqrt{\mu}}$ and let $\Gamma \subset \partial B_{R_1}$ be an arbitrary open set with positive Lebesgue measurement. Then the source f can be uniquely determined by the data $U(x, t)$ measured on $\Gamma \times (0, T_1)$.

By assuming that $\Gamma = \partial B_{R_1}$, we can extend this uniqueness result to a log-type stability estimate. For this purpose, we need a priori information on the regularity and upper bound of the source terms f and g .

Theorem 1.4. *Let $R_1 > \sqrt{2}R$, $T_1 > T + \frac{2R_1\sqrt{\rho}}{\sqrt{\mu}}$, ρ, λ, μ be constants and assume that $D = \emptyset$, $f \in H^3(\mathbb{R}^2 \times \mathbb{R})^3 \cap H^4(0, T; L^2(\mathbb{R}^2))^3$ satisfies*

$$f(\tilde{x}, 0) = \partial_t f(\tilde{x}, 0) = \partial_t^2 f(\tilde{x}, 0) = \partial_t^3 f(\tilde{x}, 0) = 0, \quad \tilde{x} \in \mathbb{R}^2.$$

Assume also that g is non-uniformly vanishing with a constant sign ($g \geq 0$ or $g \leq 0$) and that there exists $M > 0$ such that

$$(7) \quad \|f\|_{H^3(\mathbb{R}^2 \times \mathbb{R})^3} + \|f\|_{H^4(0, T; L^2(\mathbb{R}^2))^3} \leq M.$$

Then, there exists $C > 0$ depending on $M, R_1, \rho, \lambda, \mu, T_1$, $\|g\|_{L^1(\mathbb{R})}$ such that

$$(8) \quad \|f\|_{L^2((0, T) \times \tilde{B}_R)} \leq C \left(\|U\|_{H^3(0, T_1; H^{3/2}(\partial B_{R_1}))^3} + \left| \ln \left(\|U\|_{H^3(0, T_1; H^{3/2}(\partial B_{R_1}))^3} \right) \right|^{-1} \right).$$

1.5. Comments about our results. To the best of our knowledge, in Theorems 1.1 and 1.2 we obtain the first results of recovery of source terms stated for the Lamé system with variable coefficients outside a cavity. Indeed, it seems that all other known results are stated on a bounded domain (e.g. [25]) or in the full space \mathbb{R}^3 (e.g. [6]). We emphasize that Theorems 1.1 and 1.2 are valid even if the embedded cavity is unknown. In fact, the unique determination of the embedded cavity can be proved following Isakov's arguments [31, Theorem 5.1] by applying the unique continuation results of [19, 20]; see also the proof of Theorem 1.1. We refer also to

[32] for the determination of other impenetrable scatterers for the wave equation with a single measurement data. The main purpose of this paper is concerned with the identification of elastic sources in an unbounded domain.

Let us observe that in Theorem 1.1, we manage to restrict our measurements to a finite interval of time. However, like in [25, 48, 50, 51], we need to impose the additional condition $f(0) \neq 0$ for a source term F of the form (5). In contrast to Theorem 1.1 and results using Carleman estimates like [9, 12, 25, 48, 50, 51], in Theorem 1.2 we state our result without assuming that the source under consideration is non-vanishing at $t = 0$. For a source term F of the form (5), such assumption will be equivalent to the requirement that $f(0) \neq 0$. From the practical point of view, this means that the results of [25, 48, 50, 51], as well as Theorem 1.2, can only be applied to the determination of a source term associated with a phenomenon which has appeared before the beginning of the measurement. This restriction excludes applications where one wants to determine a phenomenon with measurements that start before its appearance. By removing this restriction in Theorem 1.2 we make our result more suitable for applications in that context. The approach of Theorems 1.1 and 1.2 consists in transforming our problem into the recovery of initial condition. Then, applying some results of unique continuation and global Holmgren theorem borrowed from [19, 20] we complete the proof of Theorem 1.1, 1.2.

Let us remark that, although the main results of [9, 12] correspond to the recovery of coefficients for Lamé system, their analysis can be applied to the stable determination of source terms. Indeed, using an observability inequality obtained by mean of the so called FBI transform, the results of [9] allow to prove the stable recovery of time independent source terms, appearing in a Lamé system on a bounded domain, from measurements on an arbitrary portion of the boundary provided that the source terms are known on a neighborhood of the boundary of the domain. In the same spirit, the result of [12] can be applied to the stable recovery of a source term from internal measurement on a subset corresponding to a neighborhood of the boundary of the domain. These two results can be compared to Corollary 5 where we prove the recovery of the source terms from measurements on a suitable subset. While the results of [9, 12] are stated on a bounded domain, the present work is stated in an unbounded domain with the presence of an obstacle. In addition, in contrast to [9, 12], in Corollary 5 the source terms under consideration are not known and no measurements are made on any neighborhood of the boundary ∂D of the domain $\mathbb{R}^3 \setminus \bar{D}$. Finally, as mentioned above the approach of [9, 12] requires that the source under consideration is non-vanishing at $t = 0$, while this condition is removed in Corollary 5. However, the approach of [9, 12] leads to the stable recovery of a source term stated at finite time while Corollary 5 is restricted to a uniqueness result stated at infinite time.

To the best of our knowledge, even for a bounded domain, Theorems 1.3 and 1.4 seem to be the first results of unique and stable recovery of some general source term depending on both time and space variables appearing in a hyperbolic equation. Indeed, it seems that only results dealing with recovery of source terms depending only on the time variable (see [6, 46]) or space variable (see [6, 25, 48, 50, 51]) are available in the mathematical literature with the exception of [14] where the recovery of discrete in time sources has been considered. Therefore the results of Theorems 1.3 and 1.4 are not only new for the Lamé system but also more general for hyperbolic equations. We mention also that the stability result of Theorem 1.4 requires a result of stability in the unique continuation already considered by [13,

16, 36] for the recovery of time-dependent coefficients. Note also that, in contrast of Theorems 1.2 and 1.1, thanks to the strong Huygens principle we can state Theorems 1.3 and 1.4 at finite time.

In Corollary 1 we prove that the results of Theorem 1.2 can be reformulated in terms of partial recovery of the source term from measurement on a subdomain where the source term or the initial data are known. This situation may for instance occur in several applications where the source under consideration has large support and the data considered in Theorem 1.2 is not accessible. What we prove in Corollary 1 is that even in such context one can expect the recovery of partial information of the source term under consideration by measurements located on some subdomain where the source is known.

Both Theorems 1.1, 1.2 and Corollary 1 remain valid if the cavity D is absent or if D is a rigid elastic body (i.e., U vanishes on ∂D). All the results of this paper can be applied to the wave equation. Actually, the proof for the wave equation will be easier in several aspects and the particular treatment for the Lamé system leads to some difficulties inherent to this type of systems (see for instance the proof of Theorems 1.3 and 1.4).

1.6. Outline. This paper is organized as follows. In Section 2 we study the inverse problem (IP1). More precisely, we prove Theorems 1.1 and 1.2 as well as the Corollary 1. In Section 3 we treat the inverse problem (IP2). We start with the uniqueness result stated in Theorem 1.3. Then, we extend this result by proving the stability estimate stated in Theorem 1.4. We give also some results related to solutions of the problem (1)-(4) in the appendix.

2. Inverse source problem with traction-free boundary condition. This section is devoted to the proofs of Theorems 1.1 and 1.2 for our inverse problem (IP1). More precisely, we consider the radiation of an elastic source in an inhomogeneous medium in the exterior of a cavity D (see Figure 1):

$$(9) \quad \rho \partial_t^2 U(x, t) = \mathcal{L}_{\lambda, \mu} U(x, t) + f(t) g(x), \quad x = (x_1, x_2, x_3) \in \mathbb{R}^3 \setminus \overline{D}, \quad t \in (0, +\infty),$$

where $\mathcal{L}_{\lambda, \mu} U$ stands for the Lamé operator given by (2).

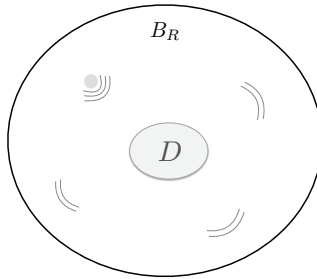


FIGURE 1. Radiation of a source in an inhomogeneous isotropic elastic medium in the exterior of a cavity. Suppose that the cavity D is known. The inverse problem is to determine the source term from the data measured on $\partial B_R = \{x \in \mathbb{R}^3 : |x| = R\}$.

Together with the governing equation (9), we fix the initial conditions at $t = 0$:

$$(10) \quad U(x, 0) = 0, \quad \partial_t U(x, 0) = 0, \quad x \in \mathbb{R}^3 \setminus \overline{D},$$

and the traction-free boundary condition $\mathcal{T}U$ on ∂D given by

$$(11) \quad \mathcal{T}U := \sigma(U)\nu = 0 \quad \text{on} \quad \partial D \times \mathbb{R}^+,$$

where $\nu = (\nu_1, \nu_2, \nu_3)$ stands for the unit normal direction pointing into the exterior of D and the stress tensor $\sigma(U)$ is given by

$$(12) \quad \sigma(U) := \lambda \operatorname{div} U \mathbf{I}_3 + 2\mu E(U), \quad E(U) := \frac{1}{2}((\nabla U) + (\nabla U)^T).$$

Note that \mathbf{I}_3 means the 3-by-3 unit matrix and that the conormal derivative $\sigma(U)\nu$ corresponds to the *stress vector* or *surface traction* on ∂D . With these notations the Lamé operator (2) can be written as $\mathcal{L}_{\lambda,\mu}U = \operatorname{div} \sigma(U)$.

We suppose that $D \subset \mathbb{R}^3$ is a bounded domain with \mathcal{C}^3 -smooth boundary ∂D and with connected exterior $\mathbb{R}^3 \setminus \overline{D}$. If the cavity D is absent (i.e., $D = \emptyset$) and the background medium is homogeneous and isotropic, it was shown in [6] via strong Huygens' principle and Fourier transform that the boundary data of Theorem 1.2 can be used to uniquely determine g . According to [35], in the context of Theorem 1.2, the strong Huygens principle is not valid and we can not even expect integrable local energy decay. For this purpose, we use a different approach based on application of Laplace transform for Theorem 1.2 and unique continuation properties for Theorem 1.1.

Let us first consider the proof of Theorem 1.1 stated with non-vanishing sources at $t = 0$.

Proof of Theorem 1.1. Assuming $U(x, t) = 0$ for $|x| = R$ and $t \in (0, T_1)$, we need to prove that $h \equiv 0$. Since $\operatorname{Supp}(h) \subset B_R$, the wave field U fulfills the homogeneous initial and boundary conditions of the Lamé system in the exterior of B_R :

$$(13) \quad \begin{cases} \rho \partial_t^2 U(x, t) - \mathcal{L}_{\lambda,\mu} U(x, t) = 0 & \text{in } \mathbb{R}^3 \setminus \overline{B}_R \times (0, T_1), \\ U(x, 0) = \partial_t U(x, 0) = 0 & \text{in } \mathbb{R}^3 \setminus \overline{B}_R, \\ U(x, t) = 0 & \text{on } \partial B_R \times \mathbb{R}^+. \end{cases}$$

Applying the elliptic regularity properties of $\mathcal{L}_{\lambda,\mu}$ (see e.g., [44, Chapters 4 and 10] and [22, Chapter 5]) and the results of [41, Theorem 8.1, Chapter 3], [41, Theorem 8.2, Chapter 3] (see also the beginning of Sections 4.1 and 4.2 for more details), one can prove the unique solvability of the initial boundary value problem (13). Consequently, we deduce that $U \equiv 0$ in $(\mathbb{R}^3 \setminus \overline{B}_R) \times [0, T_1)$.

Let us now consider the initial boundary value problem

$$(14) \quad \begin{cases} \rho \partial_t^2 V + \mathcal{L}_{\lambda,\mu} V = 0, & (x, t) \in (\mathbb{R}^3 \setminus \overline{D}) \times (0, +\infty), \\ V(\cdot, 0) = 0, \quad \partial_t V(\cdot, 0) = h, & \text{in } \mathbb{R}^3 \setminus \overline{D}, \\ \mathcal{T}V(x, t) = 0, & (x, t) \in \partial D \times (0, +\infty). \end{cases}$$

Analogous to the boundary value problem (13), one can prove that this exterior problem admits a unique solution $V \in \mathcal{C}([0, +\infty); H^2(\mathbb{R}^3 \setminus D)^3) \cap \mathcal{C}^1([0, +\infty); H^1(\mathbb{R}^3 \setminus D)^3)$. Moreover, one can easily check that the solution U to (9) is connected with V via (which is well-known as Duhamel's principle)

$$(15) \quad U(x, t) = \int_0^t f(t-s)V(x, s)ds, \quad t \in (0, +\infty), \quad x \in \mathbb{R}^3 \setminus \overline{D}.$$

Combining (15) with the fact that $U \equiv 0$ in $(\mathbb{R}^3 \setminus \overline{B}_R) \times [0, T_1)$, we deduce that

$$\int_0^t f(t-s)V(\cdot, s)|_{\mathbb{R}^3 \setminus B_R} ds = 0, \quad t \in [0, T_1).$$

Using the fact that $f \in \mathcal{C}^1([0, T])$, we can differentiate this expression with respect to t in order to get

$$f(0)V(\cdot, t)|_{\mathbb{R}^3 \setminus B_R} + \int_0^t f'(t-s)V(\cdot, s)|_{\mathbb{R}^3 \setminus B_R} ds = 0, \quad t \in [0, T_1].$$

Combining this with the fact that $f(0) \neq 0$, we obtain

$$\|V(\cdot, t)\|_{L^2(\mathbb{R}^3 \setminus B_R)} \leq \frac{\|f\|_{\mathcal{C}^1(0, T_1)}}{|f(0)|} \left(\int_0^t \|V(\cdot, s)\|_{L^2(\mathbb{R}^3 \setminus B_R)} ds \right), \quad t \in [0, T_1].$$

Therefore, applying the Gronwall inequality, we deduce that

$$(16) \quad V(x, t) = 0, \quad t \in [0, T_1], \quad x \in \mathbb{R}^3 \setminus B_R.$$

From this, we deduce that

$$V(x, t) = \mathcal{T}V(x, t) = 0, \quad x \in \partial B_R, \quad t \in [0, T_1].$$

Combining this with the fact that $T_1 > 2 \left(\max_{j=1,2} \left(\sup_{x \in \bar{\Omega}} d_j(x, \partial B_R) \right) \right)$, we deduce that there exist

$$T_2 > \left(\max_{j=1,2} \left(\sup_{x \in \bar{\Omega}} d_j(x, \partial B_R) \right) \right), \quad \epsilon > 0,$$

such that

$$(17) \quad V(x, t) = \mathcal{T}V(x, t) = 0, \quad x \in \partial B_R, \quad t \in [0, 2T_2 + 4\epsilon].$$

Combining the unique continuation result of [19, Theorem 5.5] (see also [8, Theorem 2.1]) with the global Holmgren theorem stated in [20, Theorem 1.2] and repeating arguments similar to [20, Theorem 3.2] (see also the properties of Lamé system recalled at the beginning of Section 3.2 of [20] as well as [20, Remark 3.5]), we deduce that (17) implies

$$V(x, t) = 0, \quad x \in B_R \setminus \bar{D}, \quad t \in (T_2, T_2 + 2\epsilon).$$

Combining this with (16), we get

$$V(x, t) = 0, \quad x \in \mathbb{R}^3 \setminus \bar{D}, \quad t \in (T_2, T_2 + 2\epsilon)$$

and differentiating with respect to t , we get

$$V(x, T_2 + \epsilon) = \partial_t V(x, T_2 + \epsilon) = 0, \quad x \in \mathbb{R}^3 \setminus \bar{D}.$$

Therefore, V restricted to $(\mathbb{R}^3 \setminus \bar{D}) \times (0, T_2 + \epsilon)$ solves the initial boundary value problem

$$\begin{cases} \rho \partial_t^2 V + \mathcal{L}_{\lambda, \mu} V = 0, & (x, t) \in (\mathbb{R}^3 \setminus \bar{D}) \times (0, T_2 + \epsilon), \\ V(\cdot, T_2 + \epsilon) = 0, \quad \partial_t V(\cdot, T_2 + \epsilon) = 0, & \text{in } \mathbb{R}^3 \setminus \bar{D}, \\ \mathcal{T}V(x, t) = 0, & (x, t) \in \partial D \times (0, T_2 + \epsilon). \end{cases}$$

The uniqueness of the solution of this problem implies that

$$V(x, t) = 0, \quad x \in \mathbb{R}^3 \setminus \bar{D}, \quad t \in (0, T_2 + \epsilon)$$

from which we deduce that $h \equiv 0$. □

Now let us consider Theorem 1.2, where we allow $f(0) = 0$ but we make measurements for all time.

Proof of Theorem 1.2. Assuming $U(x, t) = 0$ for $|x| = R$ and $t \in \mathbb{R}^+$, we need to prove that $h \equiv 0$. Repeating the arguments used at the beginning of Theorem 1.1, one can check that

$$U(x, t) = 0, \quad t \in [0, +\infty), \quad x \in \mathbb{R}^3 \setminus B_R.$$

Then, repeating the arguments used in Theorem 1.1 with some minor modifications, we can prove that

$$(18) \quad \int_0^t f(t-s)V(\cdot, s)|_{\mathbb{R}^3 \setminus B_R} ds = 0, \quad t \in [0, +\infty),$$

with V the solution of (14). Since $f(t) \equiv 0$ in $t < 0$, the identity (18) can be rewritten as

$$(19) \quad f(t) * V(\cdot, t)|_{\mathbb{R}^3 \setminus B_R} = \int_0^\infty f(t-s)V(\cdot, s)|_{\mathbb{R}^3 \setminus B_R} ds = 0, \quad t \in [0, +\infty)$$

where the operator $*$ denotes the convolution. For $R_1 > R$, we fix $\Omega_1 := B_{R_1} \setminus \overline{B_R}$. Using standard idea for deriving energy estimates, one can prove that $t \mapsto \|V(\cdot, t)\|_{H^1(\Omega_1)}$ has a long time behavior which is at most of polynomial type (see Proposition 3 in the Appendix). This allows us to define the Laplace transform of $t \mapsto V(\cdot, t)|_{\Omega_1}$ with respect to the time variable as following:

$$\hat{V}(x, \tau) := \int_{\mathbb{R}} V(x, t) e^{-\tau t} dt, \quad \tau > 0, \quad x \in \Omega_1,$$

and $z \mapsto \hat{V}(\cdot, z)|_{\Omega_1}$ is a holomorphic function on $\mathbb{C}_+ := \{z \in \mathbb{C} : \Re z > 0\}$ taking values in $H^1(\Omega_1)^3$. Therefore applying the Laplace transform to both sides of (19), we get

$$(20) \quad \hat{f}(\tau)\hat{V}(x, \tau) = 0, \quad x \in \Omega_1, \tau > 0.$$

Using the fact that $f \in L^1(\mathbb{R}_+)$ is supported in $[0, T]$ and it does not vanish identically, we deduce that the function \hat{f} is holomorphic in \mathbb{C} and not identically zero. Thus, there exists an interval $I \subset (0, +\infty)$ such that $|\hat{f}(\tau)| > 0$ for $\tau \in I$. Combining this with (20), we deduce that

$$\hat{V}(x, \tau) = 0, \quad x \in \Omega_1, \tau \in I$$

and using the fact that $z \mapsto \hat{V}(\cdot, z)|_{\Omega_1}$ is a holomorphic function on $\mathbb{C}_+ := \{z \in \mathbb{C} : \Re z > 0\}$, we deduce that

$$\hat{V}(x, \tau) = 0, \quad x \in \Omega_1, \tau > 0.$$

Then, the injectivity of the Laplace transform, implies

$$V(x, t) = 0, \quad x \in \Omega_1, t > 0.$$

Combining this with the arguments used at the end of Theorem 1.1, we deduce that $h \equiv 0$. \square

We remark that surface data are utilized in the proof of Theorem 1.2. As a corollary, we prove that interior volume observations can also be used to extract partial information of the spatial source term. Below we consider again the problem (9)-(11), with f, g being given as in Theorem 1.2.

Corollary 1. *Suppose that f is given and let Ω be a \mathcal{C}^3 -smooth connected open set of \mathbb{R}^3 satisfying $D \subset \Omega \subset B_R$ for some $R > 0$. Let ω be an open set of \mathbb{R}^3 . Then the wave fields U measured on the volume $\omega \times \mathbb{R}^+$ and on the surface $\partial\Omega \times \mathbb{R}^+$ (see figure 2) uniquely determine $h|_{\Omega}$.*

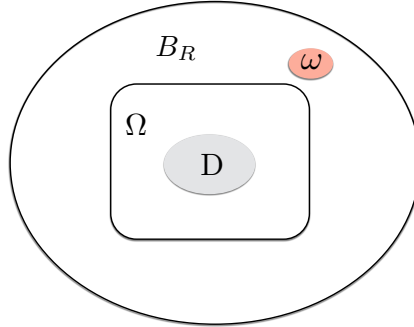


FIGURE 2. Suppose that the data are collected on ω and on $\partial\Omega$. The inverse problem is to determine the value of g on Ω .

Proof. We need to prove that the condition $U(x, t) = 0$ for $(x, t) \in (\omega \times \mathbb{R}^+) \cup (\partial\Omega \times \mathbb{R}^+)$ implies that $h|_{\Omega} \equiv 0$. Repeating the arguments used in Theorem 1.2, for V the solution of (14), we have

$$V(x, t) = 0, \quad x \in \omega \cup \partial\Omega, \quad t > 0.$$

In a similar way to Theorem 1.1, combining the unique continuation result of [19, Theorem 5.5] with the global Holmgren theorem stated in [20, Theorem 1.2], we deduce that the condition

$$V(x, t) = 0, \quad x \in \omega, \quad t > 0$$

implies that there exists $T_3 > 0$ such that

$$V(x, T_3) = \partial_t V(x, T_3) = 0, \quad x \in \Omega.$$

Combining this with the fact that

$$V(x, t) = 0, \quad x \in \partial\Omega, \quad t > 0,$$

we deduce that the restriction of V to $\Omega \times \mathbb{R}^+$ solves the problem

$$(21) \quad \begin{cases} \rho \partial_t^2 V + \mathcal{L}_{\lambda, \mu} V = 0, & (x, t) \in \Omega \times (0, T_3), \\ V(\cdot, T_3) = 0, \quad \partial_t V(\cdot, T_3) = 0, & \text{in } \Omega, \\ V(x, t) = 0, & (x, t) \in \partial\Omega \times (0, T_3) \\ \mathcal{T}V(x, t) = 0, & (x, t) \in \partial D \times (0, T_3). \end{cases}$$

Then, the uniqueness of the solution of this problem implies that

$$V(x, t) = 0, \quad x \in \Omega, \quad t \in [0, T_3].$$

In particular, we obtain $h(x) = \partial_t V(x, t)|_{t=0} = 0$ in Ω . \square

Remark 1. Corollary 1 shows that, the volume observation data on ω and the surface measurements on $\partial\Omega$ determine uniquely the source term h on Ω . This gives partial information of h only. However, in the special case that $\text{Supp}(h) \subset \Omega$ (for instance, $\Omega = B_R$), one may deduce from Corollary 1 that h can be uniquely determined by the data of U on $\omega \times \mathbb{R}^+$.

Remark 2. Assuming that $f(0) \neq 0$, in a similar way to Theorem 1.1, we can restrict the measurements in Corollary 1 to a finite time depending on the coefficients ρ, λ, μ and the domain Ω, ω .

3. Determination of the source term $g(x_3)f(\tilde{x}, t)$. In the previous section, we established uniqueness of recovering a spatial source term in an inhomogeneous background medium with or without embedded obstacles. However, the dependance of the source term on time and spatial variables are completely separated. The counterexamples constructed in [6] show that it is impossible to recover general source terms of the form $F(x, t)$ from the boundary observation on $\partial B_R \times (0, \infty)$. This implies that a priori information on the source term is always necessary in proving uniqueness. In this section we restrict our discussions to the inverse problem (IP2) for alternative source terms of the form $g(x_3)f(\tilde{x}, t)$, where the vectorial function $f = (f_1, f_2, 0)$ is compactly supported on $[0, T] \times \tilde{B}_R$ and the scalar function g is supported in $(-R, R)$ for some $R > 0$. We recall that here $\tilde{x} = (x_1, x_2) \in \mathbb{R}^2$ for $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, and $\tilde{B}_R := \{\tilde{x} : |\tilde{x}| \leq R\}$.

For simplicity, we assume in this section that $D = \emptyset$ and the background medium is homogeneous with constant Lamé coefficients λ, μ and a constant density function ρ . Below we shall consider the initial value problem

$$(22) \quad \begin{cases} \rho \partial_t^2 U(x, t) = \mathcal{L}_{\lambda, \mu} U(x, t) + g(x_3) f(\tilde{x}, t), & x \in \mathbb{R}^3, t > 0, \\ U(x, 0) = \partial_t U(x, 0) = 0, & x \in \mathbb{R}^3. \end{cases}$$

The function $g(x_3) f(\tilde{x}, t)$ can be used to model source terms which mainly radiate over the ox_1x_2 -plane and $g(x_3)$ can be regarded as an approximation of the delta function $\delta(x_3)$ in the x_3 -direction. Suppose that the function g is known in advance. Our inverse problem in this section is concerned with the recovery of f from $U(x, t)$ measured on $\Gamma \times (0, T_1)$ for some $T_1 > 0, R_1 > \sqrt{2}R$ and Γ an open subset of ∂B_{R_1} . The proofs of the uniqueness and stability results (Theorems 1.3 and 1.4) will be presented in the subsequent two subsections.

Recall that by Lemma 4.1 in the appendix, the boundary value problem (22) admits a unique solution in $\mathcal{C}^2([0, +\infty); L^2(\mathbb{R}^3))^3 \cap \mathcal{C}([0, +\infty); H^2(\mathbb{R}^3))^3$ under the assumption $f(\tilde{x}, 0) = 0$. Below we prove the uniqueness with partial boundary data measured over a finite time.

3.1. Proof of Theorem 1.3. By the strong Huygens principle, fixing $\epsilon > 0$, it holds that $U(x, t) = 0$ for all $|x| < R_1 + \epsilon$ and $t > T_1 + 2\epsilon$ (see e.g. [6]). Then, applying the Fourier transform in time to U , with $U|_{\mathbb{R}^3 \times (-\infty, 0]} = 0$, gives

$$(23) \quad \mathcal{L}_{\lambda, \mu} \hat{U}(x, \omega) + \omega^2 \rho \hat{U}(x, \omega) = -g(x_3) \hat{f}(\tilde{x}, \omega), \quad x \in \mathbb{R}^3, \omega \in \mathbb{R},$$

where

$$\hat{U}(x, \omega) := \int_{\mathbb{R}} U(x, t) e^{-i\omega t} dt, \quad \omega \in \mathbb{R}$$

satisfies the Kupradze radiation condition as $|x| \rightarrow \infty$ (see [6, 40]) for any fixed $\omega \in \mathbb{R}$. Here $\hat{f}(\tilde{x}, \omega)$ denotes the Fourier transform of $f(\tilde{x}, t)$ with respect to the

time variable. Evidently, we have the boundary condition $\hat{U}(x, \omega) = 0$, $x \in \Gamma$, $\omega \in \mathbb{R}$. Since, for all $\omega \in \mathbb{R}$, the support of the function $x \mapsto \hat{f}(\tilde{x}, \omega)g(x_3)$ is contained into B_{R_1} , by elliptic interior regularity, we deduce that $x \mapsto \hat{U}(x, \omega)$ is analytic with respect to the spatial variable x in a neighborhood of ∂B_R . By analyticity of both the surface ∂B_{R_1} and the function $\hat{U}(\cdot, \omega)$, we get the vanishing of $\hat{U}(x, \omega)$ on the whole boundary ∂B_{R_1} for any $\omega \in \mathbb{R}$. In view of the uniqueness to the Dirichlet boundary value problem in the unbounded domain $|x| > R_1$ (see e.g., [7]), we get

$$\hat{U}(x, \omega) = 0, \quad |x| > R_1, \quad \omega \in \mathbb{R}.$$

Consequently, we have $T\hat{U}(x, \omega) = 0$ on ∂B_{R_1} . Since the source term $f = (f_1, f_2, 0)^\top$ is compactly supported on \tilde{B}_R , by Hodge decomposition the function \hat{f} can be spatially decomposed into the form

$$(24) \quad \hat{f}(\tilde{x}, \omega) = \begin{pmatrix} \nabla_{\tilde{x}} \hat{f}_p(\tilde{x}, \omega) \\ 0 \end{pmatrix} + \begin{pmatrix} \nabla_{\tilde{x}}^\perp \hat{f}_s(\tilde{x}, \omega) \\ 0 \end{pmatrix},$$

where $\hat{f}_p(\cdot, \omega)$ and $\hat{f}_s(\cdot, \omega)$ are scalar functions compactly supported on \tilde{B}_R as well. Here $\nabla_{\tilde{x}} = (\partial_1, \partial_2)^\top$, $\nabla_{\tilde{x}}^\perp = (-\partial_2, \partial_1)^\top$. For $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ satisfying

$$|\xi| > k_s > k_p, \quad k_p^2 := \frac{\omega^2 \rho}{\lambda + 2\mu}, \quad k_s^2 := \frac{\omega^2 \rho}{\mu},$$

we introduce the test functions

$$V_p(x, \omega) = \begin{pmatrix} -i\xi_1 \\ -i\xi_2 \\ \sqrt{|\xi|^2 - k_p^2} \end{pmatrix} e^{-i\xi \cdot \tilde{x} + \sqrt{|\xi|^2 - k_p^2} x_3},$$

$$V_s(x, \omega) = \begin{pmatrix} i\xi_2 \\ -i\xi_1 \\ 0 \end{pmatrix} e^{-i\xi \cdot \tilde{x} + \sqrt{|\xi|^2 - k_s^2} x_3}.$$

The numbers k_p and k_s denote respectively the compressional and shear wave numbers in the frequency domain. One can easily check that $\nabla_{\tilde{x}}^\perp \cdot V_p \equiv 0$, $\nabla_{\tilde{x}} \cdot V_s \equiv 0$ in \mathbb{R}^3 and, using the fact that

$$\nabla_x \times V_p = 0, \quad (\lambda + 2\mu) \nabla_x \nabla_x \cdot V_p = -\omega^2 \rho V_p,$$

$$\nabla_x \cdot V_s = 0, \quad -\mu \nabla_x \times (\nabla_x \times V_s) = -\omega^2 \rho V_s,$$

we deduce that V_α ($\alpha = p, s$) satisfies the homogeneous Lamé system in the frequency domain

$$\mathcal{L}_{\lambda, \mu} V_\alpha(x, \omega) + \omega^2 \rho V_\alpha(x, \omega) = 0 \quad \text{in } \mathbb{R}^3, \quad x \in \mathbb{R}^3, \quad \alpha = p, s,$$

for any fixed $\omega \in \mathbb{R}$. Now, multiplying V_p to both sides of the equation (23) and applying Betti's formula, we obtain

$$\begin{aligned} & \int_{B_{R_1}} \left(\mathcal{L}_{\lambda, \mu} \hat{U}(x, \omega) + \omega^2 \rho \hat{U}(x, \omega) \right) \cdot V_p(x, \omega) dx \\ &= \int_{\partial B_{R_1}} \mathcal{T} \hat{U}(x, \omega) \cdot V_p(x, \omega) - \mathcal{T} V_p(x, \omega) \cdot \hat{U}(x, \omega) ds(x) \\ &= 0, \end{aligned}$$

where we have used the vanishing of the Cauchy data of \hat{U} on ∂B_{R_1} . On the other hand, making use of (24) together with the relation $\nabla_{\tilde{x}}^\perp \cdot V_p \equiv 0$ yields

$$\begin{aligned} 0 &= \int_{B_{R_1}} V_p(x, \omega) \cdot \hat{f}(x)(\tilde{x}, \omega) g(x_3) dx \\ &= \int_{B_{R_1}} V_p(x, \omega) \cdot \begin{pmatrix} \nabla_{\tilde{x}} \hat{f}_p(\tilde{x}, \omega) \\ 0 \end{pmatrix} g(x_3) dx \\ &= \left(\int_{\tilde{B}_R} \begin{pmatrix} -i\xi_1 \\ -i\xi_2 \\ 0 \end{pmatrix} e^{-i\xi \cdot \tilde{x}} \cdot \begin{pmatrix} \nabla_{\tilde{x}} \hat{f}_p(\tilde{x}, \omega) \\ 0 \end{pmatrix} d\tilde{x} \right) \left(\int_{-R}^R g(x_3) e^{\sqrt{|\xi|^2 - k_p^2} x_3} dx_3 \right) \\ &= |\xi|^2 \left(\int_{\tilde{B}_R} e^{-i\xi \cdot \tilde{x}} \hat{f}_p(\tilde{x}, \omega) d\tilde{x} \right) \left(\int_{-R}^R g(x_3) e^{\sqrt{|\xi|^2 - k_p^2} x_3} dx_3 \right) \end{aligned}$$

for all $\omega \in \mathbb{R}$ and $\xi \in \mathbb{R}^2$ satisfying $|\xi| > k_s$. Since g is compactly supported and lies in the space $L^1((-R, R))$, the function

$$\mathbb{C} \ni z \mapsto \int_{-R}^R g(x_3) e^{z x_3} dx_3$$

is holomorphic in \mathbb{C} . Then, using the fact that g is not uniformly vanishing, for every $\omega \in \mathbb{R}$, we can find an open and not-empty interval $I_\omega \subset (k_s, +\infty)$ such that

$$\int_{-R}^R g(x_3) e^{\sqrt{|\xi|^2 - k_p^2} x_3} dx_3 \neq 0, \quad \xi \in \mathbb{R}^2, |\xi| \in I_\omega.$$

Hence, for every $\omega \in \mathbb{R}$, we have

$$(25) \quad \int_{\tilde{B}_R} e^{-i\xi \cdot \tilde{x}} \hat{f}_p(\tilde{x}, \omega) d\tilde{x} = 0 \quad \text{for all } \xi \in \mathbb{R}^2, |\xi| \in I_\omega.$$

This implies that, for $\omega \in \mathbb{R}$ and for $\hat{f}_p(\cdot, \omega) : \tilde{x} \mapsto \hat{f}_p(\tilde{x}, \omega)$, the Fourier transform $\mathcal{F}_{\tilde{x}}[\hat{f}_p](\xi)$ of $\hat{f}_p(\cdot, \omega)$ with respect to $\tilde{x} \in \mathbb{R}^2$ vanishes for $\xi \in \{\eta \in \mathbb{R}^2 : |\eta| \in I_\omega\}$. On the other hand, since, for all $\omega \in \mathbb{R}$, $\hat{f}_p(\cdot, \omega)$ is supported in \tilde{B}_R , the function

$$\xi \mapsto \int_{\mathbb{R}^3} e^{-i\xi \cdot \tilde{x}} \hat{f}_p(\tilde{x}, \omega) d\tilde{x} = \int_{\tilde{B}_R} e^{-i\xi \cdot \tilde{x}} \hat{f}_p(\tilde{x}, \omega) d\tilde{x}$$

is real analytic with respect to $\xi \in \mathbb{R}^2$. Then, using the fact that the set $\{\xi \in \mathbb{R}^2 : |\xi| \in I_\omega\}$ is an open subset of \mathbb{R}^2 , it follows from (25) that

$$\int_{\tilde{B}_R} e^{-i\xi \cdot \tilde{x}} \hat{f}_p(\tilde{x}, \omega) d\tilde{x} = 0 \quad \text{for all } \xi \in \mathbb{R}^2.$$

Applying the inverse Fourier transform in \tilde{x} , we get $\hat{f}_p(\cdot, \omega) = 0$ for all $\omega \in \mathbb{R}$. Further, applying the inverse Fourier transform in t yields $f_p(\tilde{x}, t) \equiv 0$ for all $\tilde{x} \in \tilde{B}_R$ and $t > 0$. The fact that $f_s \equiv 0$ can be verified analogously by multiplying V_s to both sides of (23). This finishes the proof of the relation $f \equiv 0$ in $\tilde{B}_R \times (0, T)$. \square

To derive a stability estimate of f , we need the dynamic data measured over the whole boundary ∂B_R . In contrast with the proof of Theorem 1.3, we shall carry out the proof of Theorem 1.4 in the time domain without using the Fourier transform in the time variable.

3.2. Proof of Theorem 1.4. As done in (24), we can split f via Hodge decomposition into the form

$$(26) \quad f(\tilde{x}, t) = \begin{pmatrix} \nabla_{\tilde{x}} f_p(\tilde{x}, t) \\ 0 \end{pmatrix} + \begin{pmatrix} \nabla_{\tilde{x}}^{\perp} f_s(\tilde{x}, t) \\ 0 \end{pmatrix},$$

where $f_p(\cdot, t)$ and $f_s(\cdot, t)$ are scalar functions compactly supported on \tilde{B}_R . Fixing $\omega > 0$ and $\xi \in \mathbb{R}^2$ such that

$$(27) \quad |\xi|^2 > k_p^2 := \frac{\omega^2 \rho}{\lambda + 2\mu},$$

we introduce the time-dependent test function

$$V_p(x, t; \xi, \omega) = \begin{pmatrix} -i\xi_1 \\ -i\xi_2 \\ \sqrt{|\xi|^2 - k_p^2} \end{pmatrix} e^{-i\xi \cdot \tilde{x} + \sqrt{|\xi|^2 - k_p^2} x_3} e^{-i\omega t}.$$

In the same way, for

$$(28) \quad |\xi|^2 > k_s^2 := \frac{\omega^2 \rho}{\mu},$$

we introduce the function

$$V_s(x, t; \xi, \omega) = \begin{pmatrix} i\xi_2 \\ -i\xi_1 \\ 0 \end{pmatrix} e^{-i\xi \cdot \tilde{x} + \sqrt{|\xi|^2 - k_s^2} x_3} e^{-i\omega t}.$$

Then, in a similar way to the proof of the uniqueness result, one can check that V_{α} ($\alpha = p, s$) are solutions to the homogeneous elastodynamic equation

$$(29) \quad \rho \frac{\partial^2}{\partial t^2} V_{\alpha}(x, t; \xi, \omega) - \mathcal{L}_{\lambda, \mu} V_{\alpha}(x, t; \xi, \omega) = 0 \quad \text{in } \mathbb{R}^3 \times \mathbb{R}^+$$

for any fixed $\xi \in \mathbb{R}^2$ and $\omega \in \mathbb{R}$ satisfying (27) or (28). Moreover, one can easily check that

$$(30) \quad \nabla_{\tilde{x}}^{\perp} \cdot V_p(x, t) = \nabla_{\tilde{x}} \cdot V_s(x, t) = 0, \quad (x, t) \in \mathbb{R}^3 \times \mathbb{R}.$$

Therefore, multiplying $V_p(x, t; \xi, \omega)$ to the right hand side of the equation (22), using (29) and applying integration by parts yield

$$\begin{aligned} & \int_0^{T_1} \int_{B_{R_1}} \left(\rho \frac{\partial^2}{\partial t^2} U(x, t) - \mathcal{L}_{\lambda, \mu} U(x, t) \right) \cdot V_p(x, t; \xi, \omega) \, dx dt \\ &= - \int_0^{T_1} \int_{\partial B_{R_1}} \mathcal{T}U(x, t) \cdot V_p(x, t; \xi, \omega) - \mathcal{T}V_p(t, x; \xi, \omega) \cdot U(x, t; \xi, \omega) \, ds(x) \, dt \\ & \quad + \int_0^{T_1} \int_{B_{R_1}} \left(\rho \frac{\partial^2}{\partial t^2} U(x, t) \cdot V_p(x, t; \xi, \omega) - \rho \frac{\partial^2}{\partial t^2} V_p(x, t) \cdot U(x, t; \xi, \omega) \right) \, dx dt \\ &= - \int_0^{T_1} \int_{\partial B_{R_1}} \mathcal{T}U(x, t) \cdot V_p(x, t; \xi, \omega) - \mathcal{T}V_p(t, x; \xi, \omega) \cdot U(x, t; \xi, \omega) \, ds(x) \, dt \\ & \quad + \rho \int_{B_{R_1}} \frac{\partial U(x, T_1)}{\partial t} \cdot V_p(x, T_1; \xi, \omega) - \frac{\partial V_p(x, T_1; \xi, \omega)}{\partial t} \cdot U(x, T_1) \, dx. \end{aligned}$$

Again recalling Huygens principle, we know $U(x, T_1) = \partial_t U(x, T_1) = 0$ for all $x \in B_R$ and $T_1 > T + \frac{2R_1\sqrt{\rho}}{\sqrt{\mu}}$. Hence, the integral over B_{R_1} on the right hand side of the previous identity vanishes. Following estimate (42) of Proposition 2 in the

appendix, the traction of U on the boundary ∂B_{R_1} can be bounded by the trace of U itself. Hence, the left hand side can be bounded by

$$\begin{aligned}
 & \left| \int_0^{T_1} \int_{B_{R_1}} \left(\rho \frac{\partial^2}{\partial t^2} U(x, t) - \mathcal{L}_{\lambda, \mu} U(t, x) \right) \cdot V_p(x, t; \xi, \omega) \, dx dt \right| \\
 &= \left| \int_0^{T_1} \int_{\partial B_{R_1}} \mathcal{T}U(x, t) \cdot V_p(x, t; \xi, \omega) - \mathcal{T}V_p(t, x; \xi, \omega) \cdot U(x, t; \xi, \omega) \, ds(x) \, dt \right| \\
 &\leq \| \mathcal{T}U \|_{L^2((0, T_1) \times \partial B_{R_1})^3} \| V_p \|_{L^2((0, T_1) \times \partial B_{R_1})^3} + \\
 &\quad \| U \|_{L^2((0, T_1) \times \partial B_{R_1})^3} \| \mathcal{T}V_p \|_{L^2((0, T_1) \times \partial B_{R_1})^3} \\
 &\leq C \| U \|_{H^3(0, T_1; H^{3/2}(\partial B_{R_1}))^3} \| V_p \|_{L^2((0, T_1) \times \partial B_{R_1})^3} \\
 &\quad + C \| U \|_{L^2((0, T_1) \times \partial B_{R_1})^3} \| V_p \|_{L^2(0, T_1; H^2(B_{R_1}))} \\
 &\leq C \| U \|_{H^3(0, T_1; H^{3/2}(\partial B_{R_1}))^3} \| V_p \|_{L^2(0, T_1; H^2(B_{R_1}))} \\
 (31) \quad &\leq C \| U \|_{H^3(0, T_1; H^{3/2}(\partial B_{R_1}))^3} (1 + |(\xi, \omega)|^3) e^{R\sqrt{|\xi|^2 - k_p^2}}
 \end{aligned}$$

for all $|\xi| > k_p$, where $C > 0$ depends on $M, R_1, T_1, \rho, \lambda$ and μ . On the other hand, using the governing equation (22) together with the relations (26), (30) and using the fact that the sign of g is constant, we obtain a lower bound of the left hand side of (31):

$$\begin{aligned}
 & \left| \int_0^{T_1} \int_{B_{R_1}} \left(\rho \frac{\partial^2}{\partial t^2} U(x, t) - \mathcal{L}_{\lambda, \mu} U(x, t) \right) \cdot V_p(x, t; \xi, \omega) \, dx dt \right| \\
 &= \left| \int_0^{T_1} \int_{B_{R_1}} f(\tilde{x}, t) g(x_3) \cdot V_p(x, t; \xi, \omega) \, dx dt \right| \\
 &= \left| \int_0^{T_1} \int_{B_{R_1}} \begin{pmatrix} \nabla_{\tilde{x}} f_p(\tilde{x}, t) \\ 0 \end{pmatrix} g(x_3) \cdot V_p(x, t; \xi, \omega) \, dx dt \right| \\
 &= |\xi|^2 \left| \int_0^{T_1} \int_{B_{R_1}} f_p(\tilde{x}, t) g(x_3) e^{-i\xi \cdot \tilde{x} + \sqrt{|\xi|^2 - k_p^2} x_3} e^{-i\omega t} \, dx dt \right| \\
 &= |\xi|^2 \left| \left(\int_0^{T_1} \int_{\tilde{B}_R} f_p(\tilde{x}, t) e^{-i\xi \cdot \tilde{x} - i\omega t} \, d\tilde{x} dt \right) \left(\int_{-R}^R g(x_3) e^{\sqrt{|\xi|^2 - k_p^2} x_3} \, dx_3 \right) \right| \\
 &\geq |\xi|^2 \left| \left(\int_0^{T_1} \int_{\tilde{B}_R} f_p(\tilde{x}, t) e^{-i\xi \cdot \tilde{x} - i\omega t} \, d\tilde{x} dt \right) \right| \|g\|_{L^1(\mathbb{R})} e^{-(\sqrt{|\xi|^2 - k_p^2})R},
 \end{aligned}$$

for all $|\xi| > k_p$. Since f_p is supported on $\tilde{B}_R \times (0, T_1)$, the first integral on the right hand of the last identity is the Fourier transform of f_p with respect to (\tilde{x}, t) at the value (ξ, ω) , which we denote by $\hat{f}_p(\xi, \omega)$. Combining the previous two relations we obtain

$$(32) \quad |\hat{f}_p(\xi, \omega)| \leq C \frac{(1 + |(\xi, \omega)|^3) \|U\|_{H^3(0, T; H^{3/2}(\partial B_{R_1}))^3} e^{2R\sqrt{|\xi|^2 - k_p^2}}}{|\xi|^2 \|g\|_{L^1(\mathbb{R})}}$$

for all $|\xi| > k_p(\omega)$. We note that (32) gives the estimate of \hat{f}_p over the cone $\{(\xi, \omega) \in \mathbb{R}^3 : |\xi|^2 > \omega^2 \rho / (\lambda + 2\mu)\}$. In order to derive from (32) a stability estimate of \hat{f}_p on B_r for a large $r > 0$, we will use a result of stability in the analytic continuation, following the arguments presented in [16, 36]. Below we

state a stability estimate for analytic continuation problems; see [5, Theorem 4] (see also [43, 45], where similar results were established).

Proposition 1. *Let $s > 0$ and assume that $g : B_{2s} \subset \mathbb{R}^3 \rightarrow \mathbb{C}$ is a real analytic function satisfying*

$$\|\nabla^\beta g\|_{L^\infty(B_{2s})} \leq \frac{N \beta!}{(s\tau)^{|\beta|}}, \quad \beta = (\beta_1, \beta_2, \beta_3) \in \mathbb{N}^3,$$

for some $N > 0$ and $0 < \tau \leq 1$. Further let $E \subset B_s$ be a measurable set with strictly positive Lebesgue measure. Then,

$$\|g\|_{L^\infty(B_s)} \leq CN^{(1-b)} \|g\|_{L^\infty(E)}^b,$$

where $b \in (0, 1)$, $C > 0$ depend on τ , $|E|$ and s .

Following [36], we introduce the function

$$H_r(\xi, \omega) := \hat{f}_p(r(\xi, \omega)) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} f_p(\tilde{x}, t) e^{-ir(\omega t + \xi \cdot \tilde{x})} d\tilde{x} dt$$

for some $r > 1$ and $|(\xi, \omega)| \leq 2s$. In a similar way to [36], we fix $s = [\max(T_1, 2R)]^{-1} + 1$, choose $N = Ce^{3r}$, with C some constant independent of r , and take $\tau = \frac{[\max(T_1, 2R)]^{-1}}{s} = (s-1)/s$. Then we obtain

$$(33) \quad \left\| \partial_\omega^n \partial_\xi^\beta H_r \right\|_{L^\infty(B_{2s})} \leq C \frac{e^{3r} \beta! n!}{([\max(T, 2R)]^{-1})^{|\beta|+n}} = \frac{N \beta! n!}{(s\tau)^{|\beta|+n}}, \quad n \in \mathbb{N}_+, \beta \in \mathbb{N}_+^2.$$

Moreover, fixing $c := \frac{\rho}{\lambda+2\mu}$, $d := \frac{s}{2\sqrt{1+c}}$ and $a_r \in \left(0, \frac{d}{\sqrt{c}}\right)$, we define

$$E_r := \left\{ (\xi, \omega) \in \tilde{B}_d \times \left[-a_r, \frac{d}{\sqrt{c}}\right] : \max(r^{-2}, \sqrt{c}|\omega|) < |\xi| \right\}.$$

It is easy to check that E_r is a subset of B_s in \mathbb{R}^3 , and it is also a subset of the cone $\{(\xi, \omega) \in \mathbb{R}^3 : |\xi|^2 > \omega^2 \rho / (\lambda + 2\mu)\}$. We remark that $|E_r| = \kappa_r(-a_r)$, where

$$\kappa_r : y \mapsto \int_y^{\frac{d}{\sqrt{c}}} \int_{\max(r^{-2}, \sqrt{c}|\omega|) < |\xi| < d} d\xi d\omega.$$

Note that $\kappa_r\left(-\frac{d}{\sqrt{c}}\right) = 2\kappa_r(0)$ and one can check that

$$\kappa_r(0) = \frac{2\pi d^3}{3\sqrt{c}} + \pi r^{-2} \left(\frac{d^2}{3\sqrt{c}} - \frac{2r^{-4}}{3\sqrt{c}} \right).$$

Thus, there exists $r_0 > 1$ depending only on R, ρ, λ, μ, T , such that

$$\frac{\pi d^3}{2\sqrt{c}} < \kappa_r(0) < \frac{5\pi d^3}{6\sqrt{c}}, \quad r > r_0.$$

Therefore, we have

$$\kappa_r\left(-\frac{d}{\sqrt{c}}\right) = 2\kappa_r(0) > \frac{\pi d^3}{\sqrt{c}} > \frac{5\pi d^3}{6\sqrt{c}} > \kappa_r(0)$$

and, from the continuity of the map κ_r , we deduce that we can choose $a_r \in \left(0, \frac{d}{\sqrt{c}}\right)$ in such way that

$$|E_r| = \kappa_r(-a_r) = \frac{\pi d^3}{\sqrt{c}}, \quad r > r_0.$$

This implies that, with such choice of a_r , the volume $|E_r|$ depends only on R, ρ, λ, μ and T_1 . Consequently, combining (33) with Proposition 1, we deduce that

$$|\hat{f}_p(r(\xi, \omega))| = |H_r(\xi, \omega)| \leq C e^{3(1-b)r} \left(\|H_r\|_{L^\infty(E_r)} \right)^b, \quad |(\xi, \omega)| < s, \quad r > r_0,$$

where $C > 0, b \in (0, 1)$ depend only on R, ρ, λ, μ and T_1 . In addition, applying (32), we get

$$\|H_r\|_{L^\infty(E_r)} \leq C r^4 e^{c_1 r} \|U\|_{H^3(0, T; H^{3/2}(\partial B_R))^3},$$

where C and c_1 depend only on R, ρ, λ, μ and T_1 . Therefore, we can find C, c depending only on R, ρ, λ, μ and T_1 such that

$$|\hat{f}_p(\xi, \omega)| \leq C e^{cr} \|U\|_{H^3(0, T; H^{3/2}(\partial B_R))^3}, \quad |(\xi, \omega)| < r, \quad r > sr_0.$$

It follows that

$$(34) \quad \int_{B_r} |\hat{f}_p(\xi, \omega)|^2 d\xi d\omega \leq C e^{cr} \|U\|_{H^3(0, T; H^{3/2}(\partial B_R))^3}^2, \quad |(\xi, \omega)| < r, \quad r > sr_0,$$

by eventually replacing the constants C and c . On the other hand, using (7) and the fact that $\Delta_{\bar{x}} f_p = \nabla_{\bar{x}} \cdot f$, we deduce that $f_p \in H^2(\mathbb{R}^3)$ and $\|f_p\|_{H^2(\mathbb{R}^3)} \leq CM$, with C depending only on T and R . Thus, we find

$$\begin{aligned} \int_{|(\xi, \omega)| > r} |\hat{f}_p(\xi, \omega)|^2 d\xi d\omega &\leq r^{-4} \int_{|(\xi, \omega)| > r} (1 + |(\xi, \omega)|^4) |\hat{f}_p(\xi, \omega)|^2 d\xi d\omega \\ &\leq r^{-4} \|f_p\|_{H^2(\mathbb{R}^3)}^2 \leq C^2 r^{-4} M^2. \end{aligned}$$

Combining this with (34), we find

$$\begin{aligned} \int_{\mathbb{R}^3} |\hat{f}_p(\xi, \omega)|^2 d\xi d\omega &= \int_{B_r} |\hat{f}_p(\xi, \omega)|^2 d\xi d\omega + \int_{|(\xi, \omega)| > r} |\hat{f}_p(\xi, \omega)|^2 d\xi d\omega \\ &\leq C \left(e^{cr} \|U\|_{H^3(0, T_1; H^{3/2}(\partial B_{R_1}))^3}^2 + r^{-4} \right). \end{aligned}$$

Recalling the Plancherel formula, it holds that

$$\|f_p\|_{L^2((0, T_1) \times \bar{B}_{R_1})} \leq C \left(e^{cr} \|U\|_{H^3(0, T_1; H^{3/2}(\partial B_{R_1}))^3}^2 + r^{-2} \right), \quad r > sr_0.$$

Now, choosing $r = c^{-1} \ln(\|U\|_{H^3(0, T_1; H^{3/2}(\partial B_{R_1}))^3})$, we get for $\|U\|_{H^3(0, T_1; H^{3/2}(\partial B_{R_1}))^3}$ sufficiently small that

$$(35) \quad \|f_p\|_{L^2((0, T_1) \times \bar{B}_{R_1})} \leq C \left(\|U\|_{H^3(0, T_1; H^{3/2}(\partial B_{R_1}))^3}^2 + \left| \ln \left(\|U\|_{H^3(0, T_1; H^{3/2}(\partial B_{R_1}))^3} \right) \right|^{-2} \right),$$

which can be obtained by applying the classical arguments of optimization (see for instance the end of the proof of [36, Theorem 1]). This gives the estimate of f_p by our measurement data taken on ∂B_{R_1} .

Using similar arguments, we can prove

$$(36) \quad \|f_s\|_{L^2((0, T) \times \bar{B}_R)} \leq C \left(\|U\|_{H^3(0, T_1; H^{3/2}(\partial B_R))^3}^2 + \left| \ln \left(\|U\|_{H^3(0, T_1; H^{3/2}(\partial B_R))^3} \right) \right|^{-2} \right).$$

On the other hand, by interpolation and the upper bound (7) we have

$$\begin{aligned}
& \|f\|_{L^2((0,T_1)\times\tilde{B}_R)} \\
& \leq \|\nabla_{\tilde{x}} f_P\|_{L^2((0,T_1)\times\tilde{B}_R)} + \|\nabla_{\tilde{x}}^{\perp} f_S\|_{L^2((0,T_1)\times\tilde{B}_R)} \\
& \leq C(\|f_P\|_{H^1((0,T_1)\times\tilde{B}_R)} + \|f_S\|_{H^1((0,T_1)\times\tilde{B}_R)}) \\
& \leq C(\|f_P\|_{H^2((0,T_1)\times\tilde{B}_R)}^{\frac{1}{2}} \|f_P\|_{L^2((0,T_1)\times\tilde{B}_R)}^{\frac{1}{2}} + \|f_S\|_{H^2((0,T_1)\times\tilde{B}_R)}^{\frac{1}{2}} \|f_S\|_{L^2((0,T_1)\times\tilde{B}_R)}^{\frac{1}{2}}) \\
& \leq C(\|f_P\|_{L^2((0,T_1)\times\tilde{B}_R)}^{\frac{1}{2}} + \|f_S\|_{L^2((0,T_1)\times\tilde{B}_R)}^{\frac{1}{2}}),
\end{aligned}$$

with C depending on M , T_1 and R . Then, combining this with (35)-(36), we obtain (8).

Remark 3. The uniqueness and stability results presented in Theorems 1.3 and 1.4 carry over to the scalar inhomogeneous wave equation of the form

$$\begin{aligned}
\frac{1}{c^2} \partial_t^2 U(x, t) &= \Delta U(x, t) + f(\tilde{x}, t)g(x_3), & (x, t) \in \mathbb{R}^3 \times (0, \infty), \\
U(x, 0) &= U_t(x, 0) = 0, & x \in \mathbb{R}^3,
\end{aligned}$$

where both f and g are compactly supported scalar functions and c is a constant. If the wave speed c and the function $g(x_3)$ are known, one can determine the source term $f(\tilde{x}, t)$ from partial boundary data. In particular, f is allowed to be a moving source with the orbit lying on the ox_1x_2 -plane. In the frequency domain, the above wave equation gives rise to an inverse problem of recovering the wave-number-dependent source term $f(\tilde{x}, k)$ from the multi-frequency boundary observation data of the inhomogeneous Helmholtz equation

$$\Delta u(x, k) + k^2/c^2 u(x, k) = \hat{f}(\tilde{x}, k)g(x_3).$$

Progress along these directions will be reported in our forthcoming publications (see e.g., [24]).

4. Appendix.

4.1. Well-posedness result and estimation of surface traction. In this subsection, we consider the inhomogeneous Lamé system

$$(37) \quad \begin{cases} \rho \partial_t^2 U - \mathcal{L}_{\lambda, \mu} U = F(x, t), & (x, t) \in \mathbb{R}^3 \times (0, +\infty), \\ U(\cdot, 0) = \partial_t U(\cdot, 0) = 0, & x \in \mathbb{R}^3, \end{cases}$$

where the operator $\mathcal{L}_{\lambda, \mu}$ is given by (9). We assume that $\text{supp}(F) \subset [0, T] \times B_R$, with $B_R := \{x \in \mathbb{R}^3 : |x| < R\}$. It is well known that the operator $\mathcal{L}_{\lambda, \mu}$ is an elliptic operator and the standard elliptic regularity holds; see e.g., [44, Chapters 4 and 10] and [22, Chapter 5]. The quadratic form corresponding to $\mathcal{L}_{\lambda, \mu}$ is given by

$$\mathcal{E}(U, V) := \lambda (\text{div } U)(\text{div } V) + 2\mu E(U) : E(V)$$

where the stress tensor E is defined via (12), with the notation $A : B := \sum_{i,j=1}^3 a_{ij} b_{ij}$ for $A = (a_{ij})_{i,j=1}^3$, $B = (b_{ij})_{i,j=1}^3$. Hence, for a bounded Lipschitz domain $D \subset \mathbb{R}^3$ there holds the relation (see e.g., [4, Lemma 3])

$$(38) \quad - \int_D \mathcal{L}_{\lambda, \mu} U \cdot \bar{V} \, dx = \int_D \mathcal{E}(U, \bar{V}) \, dx - \int_{\partial D} \bar{V} \cdot \mathcal{T}U \, ds$$

for all $U, V \in H^2(D)^3$. By the well-known Korn's inequality (see e.g. [44, Theorem 10.2], [18, Chapter 3]), it holds that

$$(39) \quad \int_{\mathbb{R}^3} \mathcal{E}(U, \bar{U}) + c_1 \|U\|_{L^2(\mathbb{R}^3)^3} \geq c_2 \|U\|_{H^1(\mathbb{R}^3)^3}, \quad U \in H^1(\mathbb{R}^3)^3$$

for some constants $c_1, c_2 > 0$. In the particular case of constant Lamé coefficients, we have

$$\mathcal{L}_{\lambda, \mu} U = \mu \Delta U + (\lambda + \mu) \nabla(\nabla \cdot U),$$

and

$$\mathcal{E}(U, V) = 2\mu \sum_{j,k=1}^3 \partial_k U_j \partial_k V_j + \lambda(\operatorname{div} U)(\operatorname{div} V) - \mu \operatorname{curl} U \cdot \operatorname{curl} V.$$

In this case, the surface traction can be simplified to be

$$\mathcal{T}U = 2\mu \partial_\nu U + \lambda(\operatorname{div} U) \nu + \mu \nu \times \operatorname{curl} U \quad \text{on} \quad \partial D.$$

We refer to the monograph [40] for comprehensive studies on the Lamé system. Below we state a well-posedness result to the elastodynamic system in unbounded domains by applying the standard arguments of [41, Chapter 8].

Lemma 4.1. *Let $F \in H^1(\mathbb{R}_+; L^2(\mathbb{R}^3)^3)$. Then problem (37) admits a unique solution*

$$U \in \mathcal{C}^1([0, +\infty); L^2(\mathbb{R}^3)^3) \cap \mathcal{C}([0, +\infty); H^1(\mathbb{R}^3)^3).$$

Under the additional condition $F(\cdot, 0) = x \mapsto F(0, x) = 0$, the unique solution lies in the space

$$U \in \mathcal{C}^2([0, +\infty); L^2(\mathbb{R}^3)^3) \cap \mathcal{C}([0, +\infty); H^2(\mathbb{R}^3)^3).$$

Proof. Without loss of generality, we assume that $\rho = 1$. We define on $L^2(\mathbb{R}^3)^3$ the sesquilinear form a with domain $D(a) := H^1(\mathbb{R}^3)^3$ given by

$$a(U_1, U_2) := \int_{\mathbb{R}^3} \mathcal{E}(U_1, U_2) dx.$$

In view of (38), by density, we find

$$\langle -\mathcal{L}_{\lambda, \mu} U_1, U_2 \rangle_{H^{-1}(\mathbb{R}^3)^3, H^1(\mathbb{R}^3)^3} = a(U_1, U_2), \quad U_1, U_2 \in H^1(\mathbb{R}^3)^3.$$

Therefore, in view of (39), fixing $H = L^2(\mathbb{R}^3)^3$, $V = H^1(\mathbb{R}^3)^3$ and applying [41, Theorem 8.1, Chapter 3] and [41, Theorem 8.2, Chapter 3], we deduce that (37) admits a unique solution $U \in \mathcal{C}^1([0, +\infty); L^2(\mathbb{R}^3)^3) \cap \mathcal{C}([0, +\infty); H^1(\mathbb{R}^3)^3)$. It remains to prove that $U \in \mathcal{C}^2([0, +\infty); L^2(\mathbb{R}^3)^3) \cap \mathcal{C}([0, +\infty); H^2(\mathbb{R}^3)^3)$. For this purpose, we consider $V = \partial_t U$ and, using the fact that $F(\cdot, 0) = 0$, we deduce that V solves

$$(40) \quad \begin{cases} \partial_t^2 V - \mathcal{L}_{\lambda, \mu} V = \partial_t \tilde{F}(x, t), & (x, t) \in \mathbb{R}^3 \times (0, +\infty), \\ V(\cdot, 0) = \partial_t V(\cdot, 0) = 0, & x \in \mathbb{R}^3. \end{cases}$$

Using the fact that $\partial_t F \in L^2((0, +\infty) \times \mathbb{R}^3)$ and applying the above arguments we deduce that $V \in \mathcal{C}^1([0, +\infty); L^2(\mathbb{R}^3)^3)$ and that $U \in \mathcal{C}^2([0, +\infty); L^2(\mathbb{R}^3)^3)$. Therefore, for any $t \in [0, +\infty)$, U is a solution of the boundary value problem

$$(41) \quad -\mathcal{L}_{\lambda, \mu} U(x, t) = -\partial_t^2 U(x, t) + F(x, t), \quad x \in \mathbb{R}^3.$$

Since $\partial_t^2 U(t, \cdot) \in L^2(\mathbb{R}^3)^3$, from the elliptic regularity of the operator $-\mathcal{L}_{\lambda, \mu}$ (see e.g. [22, Theorem 5.8.1]), we deduce that $U(t, \cdot) \in H^2(\mathbb{R}^3)^3$. Moreover, for any $t_1, t_2 \in [0, +\infty)$, we have

$$\begin{aligned} & \|U(\cdot, t_1) - U(\cdot, t_2)\|_{H^2(\mathbb{R}^3)^3} \\ & \leq C(\|\mathcal{L}_{\lambda, \mu}(U(\cdot, t_1) - U(\cdot, t_2))\|_{L^2(\mathbb{R}^3)^3} + \|U(\cdot, t_1) - U(\cdot, t_2)\|_{L^2(\mathbb{R}^3)^3}) \\ & \leq C\left(\|\partial_t^2 U(\cdot, t_1) - \partial_t^2 U(\cdot, t_2)\|_{L^2(\mathbb{R}^3)^3} \right. \\ & \quad \left. + \|U(\cdot, t_1) - U(\cdot, t_2)\|_{L^2(\mathbb{R}^3)^3} + \|F(\cdot, t_1) - F(\cdot, t_2)\|_{L^2(\mathbb{R}^3)^3}\right). \end{aligned}$$

Therefore, using the fact that $U \in \mathcal{C}^2([0, +\infty); L^2(\mathbb{R}^3)^3)$ and the fact that F extended by 0 to $\mathbb{R}^3 \times \mathbb{R}$ is lying in $H^1(\mathbb{R}; L^2(\mathbb{R}^3)^3) \subset \mathcal{C}(\mathbb{R}; L^2(\mathbb{R}^3)^3)$, we deduce that $U \in \mathcal{C}([0, +\infty); H^2(\mathbb{R}^3)^3)$. \square

Using this result for $F \in H^1(0, T; L^2(\mathbb{R}^3)^3)$, we know

$$U(x, t), \mathcal{T}U(x, t) \in \mathcal{C}([0, +\infty); L^2(\partial B_R))^3, \quad (x, t) \in \partial B_R \times [0, \infty).$$

With additional smoothness assumptions we can also estimate $TU|_{\partial B_R \times [0, +\infty)}$ by $U|_{\partial B_R \times [0, +\infty)}$. The main result of this subsection can be stated as follows.

Proposition 2. *Let $T_2 > 0$ and let $F \in H^4(\mathbb{R}_+; L^2(\mathbb{R}^3)^3)$ be such that $F(\cdot, 0) = \partial_t F(\cdot, 0) = \partial_t^2 F(\cdot, 0) = \partial_t^3 F(\cdot, 0) = 0$. Then problem (37) admits a unique solution $U \in \mathcal{C}^4([0, +\infty); L^2(\mathbb{R}^3)^3) \cap \mathcal{C}^3([0, +\infty); H^2(\mathbb{R}^3)^3)$ satisfying the estimate*

$$(42) \quad \|\mathcal{T}U\|_{L^2(\partial B_R \times (0, T_2))^3} \leq C \|U\|_{H^3(0, T_2; H^{\frac{3}{2}}(\partial B_R))^3},$$

with C depending on λ, ρ, μ, T_2 and R .

Proof. Note first that $W = \partial_t^3 U$ solves

$$(43) \quad \begin{cases} \rho \partial_t^2 W - \mathcal{L}_{\lambda, \mu} W = \partial_t^3 F(x, t), & (x, t) \in \mathbb{R}^3 \times (0, +\infty), \\ W(\cdot, 0) = \partial_t W(\cdot, 0) = 0, & x \in \mathbb{R}^3. \end{cases}$$

Using the fact that $\partial_t^3 F \in H^1(\mathbb{R}_+; L^2(\mathbb{R}^3)^3)$ with $\partial_t^3 F(\cdot, 0) = 0$, we can apply Lemma 4.1 to deduce that $W \in \mathcal{C}^2([0, +\infty); L^2(\mathbb{R}^3)^3) \cap \mathcal{C}([0, +\infty); H^2(\mathbb{R}^3)^3)$. Thus, we have $U \in \mathcal{C}^4([0, +\infty); L^2(\mathbb{R}^3)^3) \cap \mathcal{C}^3([0, +\infty); H^2(\mathbb{R}^3)^3)$. This implies that $g := U|_{\partial B_R \times [0, +\infty)} \in \mathcal{C}^3([0, +\infty); H^{\frac{3}{2}}(\partial B_R))^3$. Hence, the restriction of U to $(\mathbb{R}^3 \setminus B_R) \times (0, T_2)$ solves the initial boundary value problem

$$(44) \quad \begin{cases} \rho \partial_t^2 U - \mathcal{L}_{\lambda, \mu} U = 0, & (x, t) \in (\mathbb{R}^3 \setminus B_R) \times (0, T_2), \\ U(\cdot, 0) = \partial_t U(\cdot, 0) = 0, & x \in \mathbb{R}^3 \setminus B_R, \\ U = g, & (x, t) \in \partial B_R \times (0, T_2). \end{cases}$$

Using the fact that $g(\cdot, 0) = \partial_t g(\cdot, 0) = \partial_t^2 g(\cdot, 0) = 0$, from a classical lifting result, one can find $G \in \mathcal{C}^3([0, +\infty); H^2(\mathbb{R}^3 \setminus B_R))^3$ such that $G|_{\partial B_R \times (0, T_2)} = g$, $G(\cdot, 0) = \partial_t G(\cdot, 0) = \partial_t^2 G(\cdot, 0) = 0$ and

$$(45) \quad \|G\|_{H^3(0, T_2; H^2(\mathbb{R}^3)^3)} \leq C \|g\|_{H^3(0, T_2; H^{\frac{3}{2}}(\partial B_R))^3},$$

with C depending only on T_2 and R . Therefore, we can split U to $U = V + G$ on $(0, T_2) \times (\mathbb{R}^3 \setminus B_R)$, with V the solution of

$$\begin{cases} \rho \partial_t^2 V - \mathcal{L}_{\lambda, \mu} V = -(\rho \partial_t^2 G - \mathcal{L}_{\lambda, \mu} G) := H, & (x, t) \in (\mathbb{R}^3 \setminus B_R) \times (0, T_2), \\ V(\cdot, 0) = \partial_t V(\cdot, 0) = 0, & x \in \mathbb{R}^3 \setminus B_R, \\ V = 0, & (x, t) \in \partial B_R \times (0, T_2). \end{cases}$$

Using the fact that $H \in H^1(0, T_2; L^2(\mathbb{R}^3 \setminus B_R))$ and $H(\cdot, 0) = 0$, in a similar way to Lemma 4.1 we can prove that $V \in C^2([0, T_2]; L^2(\mathbb{R}^3 \setminus B_R))^3 \cap C([0, T_2]; H^2(\mathbb{R}^3 \setminus B_R))^3$, with

$$\|V\|_{L^2(0, T_2; H^2(\mathbb{R}^3 \setminus B_R))^3} \leq C \|H\|_{H^1(0, T_2; L^2(\mathbb{R}^3 \setminus B_R))^3} \leq C \|G\|_{H^3(0, T_2; H^2(\mathbb{R}^3))^3},$$

with C depending on λ, ρ, μ, T_2 and R . Combining this with (45), we deduce that

$$\|U\|_{L^2(0, T_2; H^2(\mathbb{R}^3 \setminus B_R))^3} \leq C \|g\|_{H^3(0, T_2; H^{\frac{3}{2}}(\partial B_R))^3}$$

and using the continuity of the trace map, we obtain

$$\|\mathcal{T}U\|_{L^2((0, T_2) \times \partial B_R)^3} \leq C \|U\|_{L^2(0, T_2; H^2(\mathbb{R}^3 \setminus B_R))^3}.$$

Combining the last two estimates we finally obtain (42). \square

4.2. Long time asymptotic behavior of the solution on a bounded domain.

In this subsection we fix Ω_1 a bounded C^2 domain of \mathbb{R}^3 . We consider the bilinear form a with domain $D(a) = H^1(\Omega_1)^3$ given by

$$\begin{aligned} a(U_1, U_2) &= \int_{\Omega_1} \mathcal{E}(U, \bar{V}) \, dx \\ &= \int_{\Omega_1} \left[(\lambda(x) + 2\mu(x))(\nabla \cdot U_1(x)) \overline{(\nabla \cdot U_2(x))} - \mu(x) \nabla \times U_1(x) \cdot \overline{\nabla \times U_2(x)} \right] dx. \end{aligned}$$

Then, for $U_1 \in H^1(\Omega_1)^3$ such that $\mathcal{L}_{\lambda, \mu} U_1 \in L^2(\Omega_1)^3$ and $\mathcal{T}U_1 = 0$ on $\partial\Omega_1$, we have

$$(46) \quad a(U_1, U_2) = - \int_{\Omega_1} \mathcal{L}_{\lambda, \mu} U_1(x) \cdot \overline{U_2(x)} dx.$$

Fixing $H = L^2(\Omega_1)^3$, $V = H^1(\Omega_1)^3$ and applying [22, Chapter 5], [41, Theorem 8.1, Chapter 3] and [41, Theorem 8.2, Chapter 3], we deduce that, for $F \in L^2((0, +\infty) \times \Omega_1)^3$, $V_0 \in V$ and $V_1 \in H$, the problem

$$(47) \quad \begin{cases} \rho \partial_t^2 U - \mathcal{L}_{\lambda, \mu} U = F(x, t), & (x, t) \in \Omega_1 \times (0, +\infty), \\ U(\cdot, 0) = V_0, \quad \partial_t U(\cdot, 0) = V_1, & x \in \Omega_1 \\ \mathcal{T}U(x, t) = 0, & (x, t) \in \partial\Omega_1 \times (0, +\infty), \end{cases}$$

admits a unique solution in $C^1([0, +\infty); L^2(\Omega_1))^3 \cap C([0, +\infty); H^1(\Omega_1))^3$. Below we show the long time behavior of the solution of (47).

Proposition 3. *Let $F \in L^2((0, +\infty) \times \Omega_1)^3$ be such that $\text{supp}(F) \subset \overline{\Omega_1} \times [0, T)$ and let $V_0 \in H^1(\Omega_1)^3$, $V_1 \in L^2(\Omega)^3$. Then problem (47) admits a unique solution $U \in C^1([0, +\infty); L^2(\Omega_1))^3 \cap C([0, +\infty); H^1(\Omega_1))^3$ satisfying*

$$(48) \quad \|U(t, \cdot)\|_{H^1(\Omega_1)^3} \leq \tilde{C} (\|F\|_{L^2((0, T) \times \Omega_1)} + \|V_0\|_{H^1(\Omega_1)^3} + \|V_1\|_{L^2(\Omega_1)^3})(t + 1), \quad t > 0,$$

with \tilde{C} independent of t .

Proof. Without lost of generality we may assume that $V_0 = V_1 = 0$. Indeed the result with non-vanishing initial conditions can be carried out in a similar way. Let us first assume that $F \in H_0^1(0, T; L^2(\Omega_1))^3$. Repeating the arguments in the proof of Lemma 4.1, we can prove that the regularity of U can be improved to be

$$U \in C^2([0, +\infty); L^2(\Omega_1))^3 \cap C([0, +\infty); H^2(\Omega_1))^3.$$

Now let us consider the energy

$$J(t) := \int_{\Omega_1} \rho |\partial_t U(x, t)|^2 + \mathcal{E}(U(x, t), U(x, t)) \, dx.$$

For simplicity, we assume that F takes values in \mathbb{R}^3 such that U takes also values in \mathbb{R}^3 , otherwise our arguments may be extended without any difficulty to function F taking values in \mathbb{C}^3 . It is clear that $J \in \mathcal{C}^1([0, +\infty))$ and

$$J'(t) = 2 \int_{\Omega_1} \rho \partial_t^2 U(x, t) \cdot \partial_t U(x, t) + \mathcal{E}(U(x, t), \partial_t U(x, t)) \, dx.$$

Using the fact that $\mathcal{T}U = 0$ on $[0, +\infty) \times \partial\Omega_1$, we can integrate by parts to obtain (see (46))

$$J'(t) = 2 \int_{\Omega_1} [\rho \partial_t^2 U(x, t) - \mathcal{L}_{\lambda, \mu} U(x, t)] \cdot \partial_t U(x, t) \, dx = 2 \int_{\Omega_1} F(x, t) \cdot \partial_t U(x, t).$$

Thus, using the fact that $\rho > 0$ we get

$$\begin{aligned} & \int_{\Omega_1} |\partial_t U(x, t)|^2 \, dx \leq \rho^{-1} J(t) \\ & \leq 2\rho^{-1} \int_0^t \left| \int_{\Omega_1} F(x, s) \cdot \partial_t U(x, s) \, dx \right| \, ds \\ & \leq 2\rho^{-1} \int_0^T \int_{\Omega_1} |F(x, s)| |\partial_t U(x, s)| \, dx \, ds \\ (49) \quad & \leq 2\rho^{-1} T^{\frac{1}{2}} \|F\|_{L^2((0, T) \times \Omega_1)^3} \|\partial_t U\|_{L^\infty(0, T; L^2(\Omega_1))^3}. \end{aligned}$$

Combining this with a classical estimate of $\|\partial_t U\|_{L^\infty(0, T; L^2(\Omega_1))^3}$ (e.g. [41, Formula (8.15), Chapter 3]), we deduce that

$$\|\partial_t U\|_{L^\infty(0, T; L^2(\Omega_1))^3} \leq \tilde{C} \|F\|_{L^2((0, T) \times \Omega_1)^3}$$

with \tilde{C} depending only on T , Ω_1 , ρ and $\mathcal{L}_{\lambda, \mu}$. It then follows from (49) that

$$\int_{\Omega_1} |\partial_t U(x, t)|^2 \, dx \leq \tilde{C} \|F\|_{L^2((0, T) \times \Omega_1)^3}^2.$$

Combining this with the fact that

$$U(t, \cdot) = \int_0^t \partial_t U(\cdot, s) \, ds,$$

we obtain that

$$\begin{aligned} \|U(\cdot, t)\|_{L^2(\Omega_1)^3} &= \left\| \int_0^t \partial_t U(\cdot, s) \, ds \right\|_{L^2(\Omega_1)^3} \leq \int_0^t \|\partial_t U(\cdot, s)\|_{L^2(\Omega_1)^3} \, ds \\ (50) \quad &\leq \tilde{C} \|F\|_{L^2((0, T) \times \Omega_1)^3} t. \end{aligned}$$

By standard density argument, we can extend this estimate to $F \in L^2((0, T_1) \times \Omega_1)^3$.

Applying Korn's inequality gives the estimate

$$\begin{aligned} \|U(\cdot, t)\|_{H^1(\Omega_1)^3}^2 &\leq \tilde{C} \left(\|U(\cdot, t)\|_{L^2(\Omega_1)^3}^2 + \int_{\Omega_1} \mathcal{E}(U(x, t), U(x, t)) \, dx \right) \\ (51) \quad &\leq \tilde{C} (\|U(\cdot, t)\|_{L^2(\Omega_1)^3}^2 + J(t)). \end{aligned}$$

Multiplying U to both sides of (47) and integrating by part with respect to x over Ω_1 , we can estimate $J(t)$ by

$$(52) \quad J(t) = \int_{\Omega_1} F(x, t) \cdot U(x, t) \, dx \leq \|F(\cdot, t)\|_{L^2(\Omega_1)^3} \|U(\cdot, t)\|_{L^2(\Omega_1)^3}.$$

Now, inserting (52) into (51) and making use of (50), we finally obtain

$$\|U(\cdot, t)\|_{H^1(\Omega_1)^3}^2 \leq \tilde{C} \|F\|_{L^2((0,T)\times\Omega_1)^3}^2 (1 + t^2),$$

which proves (48). \square

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