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Uniqueness in determining rectangular grating profiles with a single incoming wave (part II): TM polarization case

Jianli Xiang[1] **and Guanghui Hu**[2,*]

[1] Three Gorges Mathematical Research Center, College of Science, China Three Gorges University, Yichang 443002, People's Republic of China
[2] School of Mathematical Sciences and LPMC, Nankai University, Tianjin 300071, People's Republic of China

E-mail: ghhu@nankai.edu.cn

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Abstract

This paper is concerned with an inverse transmission problem for recovering the shape of a penetrable rectangular grating sitting on a perfectly conducting plate. We consider a general transmission problem with the coefficient $\lambda \neq 1$ which covers the transverse magnetic (TM) polarization case. It is proved that a rectangular grating profile can be uniquely determined by the near-field observation data incited by a single plane wave and measured on a line segment above the grating. In comparison with the transverse electric (TE) case ($\lambda = 1$), the wave field cannot lie in H^2 around each corner point, bringing essential difficulties in proving uniqueness with one plane wave. Our approach relies on singularity analysis for Helmholtz transmission problems in a right-corner domain and also provides an alternative idea for treating the TE transmission conditions which were considered in the authors' previous work (Xiang and Hu 2023 *Inverse Problems* **39** 055004).

Keywords: inverse scattering, penetrable rectangular grating, uniqueness, transmission conditions, TM polarization case.

(Some figures may appear in colour only in the online journal)

* Author to whom any correspondence should be addressed.

1. Introduction and main result

Consider the time-harmonic electromagnetic scattering of a plane wave from a penetrable rectangular grating which remains invariant along one surface direction x_3. The diffractive grating is supposed to sit on the perfectly conducting substrate $x_2 < 0$. In transverse electric (TE) and transverse magnetic (TM) polarization cases, the wave scattering can be modeled by a transmission problem for the Helmholtz equation over the ox_1x_2-plane with a boundary condition on $x_2 = 0$ and an appropriate radiation condition as $x_2 \to \infty$. In this paper the medium above the grating profile is supposed to be isotropic and homogeneous. For rectangular gratings, the cross-section Λ of the grating surface in the ox_1x_2-plane consists of line segments that are perpendicular to either the x_1- or x_2-axis. More precisely, we define a set $\mathcal{A}$ of the so-called rectangular grating profiles by (see figure 1)

$$\mathcal{A} = \{\Lambda \mid \Lambda \text{ is a non-self-intersecting curve in } \mathbb{R}^2_+ \text{ which is } 2\pi\text{-periodic in } x_1, \\ \Lambda \text{ is piecewise linear and any linear part is parallel to the } x_1\text{- or } x_2\text{-axis}\}.$$

Note that $\Lambda \in \mathcal{A}$ cannot contain any crack, for instance, a line segment intersecting the other part of Λ at one ending point. The rectangular gratings defined above include the class of binary gratings, whose grooves have the same height. Denote by Ω^+_Λ the unbounded periodic domain above Λ, that is, the component of $\mathbb{R}^2_+$ separated by Λ which is connected to $x_2 = +\infty$. Let Ω^-_Λ be the periodic domain below Λ but above the substrate $x_2 = 0$. Let $\nu = (\nu_1, \nu_2) \in \mathbb{S} := \{x \in \mathbb{R}^2 : |x| = 1\}$ be the normal direction at Λ pointing into Ω^+_Λ. Suppose that a plane wave in the (x_1, x_2)-plane given by

$$u^i(x_1, x_2) = e^{i\alpha x_1 - i\beta x_2}, \quad \alpha = k_1 \sin\theta, \quad \beta = k_1 \cos\theta$$

with some incident angle $\theta \in (-\pi/2, \pi/2)$ and wave number $k_1 > 0$, is incident upon the grating Λ from the top. Consider a general transmission problem for finding the total field $u = u(x_1, x_2)$ such that

$$\begin{cases} \Delta u + k_1^2 u = 0, & \text{in} \quad \Omega^+_\Lambda, \\ \Delta u + k_2^2 u = 0, & \text{in} \quad \Omega^-_\Lambda, \\ u^+ = u^-, \quad \partial^+_\nu u = \lambda \partial^-_\nu u, & \text{on} \quad \Lambda, \\ u = u^i + u^s, & \text{in} \quad \Omega^+_\Lambda, \\ \partial_\nu u = 0, & \text{on} \quad \Gamma_0, \end{cases} \tag{1.1}$$

with the following radiation condition as $x_2 \to +\infty$:

$$u^s(x) := u - u^i = \sum_{n \in \mathbb{Z}} A_n \, e^{i\alpha_n x_1 + i\beta_n x_2} \qquad \text{in } x_2 > \Lambda^+ := \max_{(x_1, x_2) \in \Lambda} x_2. \tag{1.2}$$

In (1.1), we have $k_j > 0$ for $j = 1, 2$, $k_1 \neq k_2$, $\lambda > 0, \lambda \neq 1$, $\alpha_n := n + \alpha$ and

$$\beta_n := \begin{cases} \sqrt{k_1^2 - \alpha_n^2} & \text{if } |\alpha_n| \leqslant k_1, \\ i\sqrt{\alpha_n^2 - k_1^2} & \text{if } |\alpha_n| > k_1. \end{cases}$$

The notations $(\cdot)^\pm$ stand for the limits of u and $\partial_\nu u$ on Λ obtained from above $(+)$ or below $(-)$ and $\Gamma_h = \{(x_1, h) : 0 < x_1 < 2\pi\}$ for $h \in \mathbb{R}$. Note that the TM polarization case corresponds to the special case that $\lambda = (k_1/k_2)^2$. The expansion in (1.2) is the well-known Rayleigh expansion (see e.g. [10, 17, 19]), $A_n \in \mathbb{C}$ are called Rayleigh coefficients. The series (1.2) together with their derivatives are uniformly convergent in any compact set in $x_2 > \Lambda^+$,

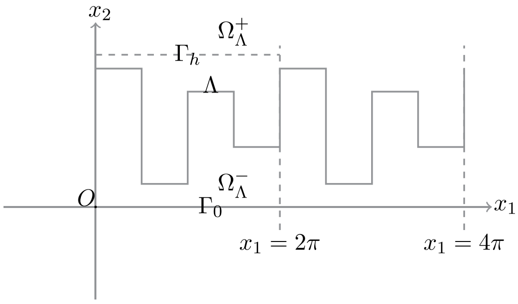

Figure 1. Rectangular periodic structures.

because $u \in H^1_\alpha(S_h)$ (see below for the definition) and the scattered fields consist of infinitely many surface waves which exponentially decay as $x_2 \to +\infty$. We will look for weak solutions to (1.1) and (1.2) in the α-quasiperiodic Sobolev space

$$H^1_\alpha(S_h) := \left\{u \in H^1_{\mathrm{loc}}(S_h),\ \mathrm{e}^{-\mathrm{i}\alpha x_1}u \text{ is } 2\pi\text{-periodic in } x_1\right\},$$

with $S_h := \{x \in \mathbb{R}^2 : 0 < x_2 < h\}$ for any $h > \Lambda^+$. Note that, since we are interested in quasiperiodic solutions, the notations $\Omega_\Lambda^\pm, \Lambda, S_h$ and Γ_h always denote the corresponding sets in one periodicity cell $0 < x_1 < 2\pi$. Uniqueness, existence and regularity results on solutions to the forward scattering problem will be summarized as follows.

Proposition 1.1. *(i) There exists at least one solution $u \in H^1_\alpha(S_h)$ to the forward scattering problem (1.1) and (1.2), where $h > \Lambda^+$ is arbitrary. Moreover, uniqueness holds true if $k_1^2 \geqslant \lambda k_2^2$.*

(ii) Let $u \in H^1_\alpha(S_h)$ be a solution to the forward scattering problem (1.1) and (1.2) corresponding to some rectangular grating $\Lambda \in \mathcal{A}$. Then we have $u \in H^{1+s}_\alpha(S_h) \cap H^2_\alpha(S_h^\pm)$ for any $s \in [0, 1/2)$, where $S_h^\pm := S_h \cap \Omega_\Lambda^\pm$. Moreover, u is real-analytic on $\overline{S_h^+}$ and $\overline{S_h^-}$ except at the finite number of corner points of Λ.

Uniqueness and existence of the above transmission problem have been sufficiently investigated in the literature by applying the Dirichlet-to-Neumann map; see e.g. [1, 2, 4, 9] in periodic structures. In particular, the uniqueness proof for rectangular gratings with the condition $k_1^2 \geqslant \lambda k_2^2$ follows directly from the authors' previous paper [24, appendix]. If $k_1^2 \geqslant \lambda k_2^2$ does not hold, guided Bloch waves might exist and additional constraint should be imposed on the total field to ensure uniqueness; see the recent publication [12] for a sharp radiation condition derived from the limiting absorption principle under the Dirichlet boundary condition. The second assertion, which states smoothness of the solution around a corner point and up to a flat interface, follows from standard elliptic regularity result for interface problems in a right-corner domain; see e.g. in [9, 14, 15, 18, 20]. We refer to the appendix of this paper for the proof of proposition 1.1.

Now we formulate the inverse problem with a single measurement data above the grating.

(IP): Let $h > \Lambda^+$ be a fixed constant and suppose $u = u(x_1, x_2)$ is a solution to the direct problem (1.1) and (1.2). Given the transmission coefficient $\lambda > 0\ (\neq 1)$ and the wavenumbers k_1 and k_2, determine the periodic interface $\Lambda \in \mathcal{A}$ from knowledge of the near-field data $u(x_1, h)$ for all $0 < x_1 < 2\pi$.

The main uniqueness result of this paper is stated as follows.

Theorem 1.2. *Let u_1 and u_2 be solutions to the direct diffraction problem (1.1) and (1.2) corresponding to $(\Lambda_1, k_1, k_2, \lambda)$ and $(\Lambda_2, k_1, k_2, \lambda)$, respectively. If*

$$u_1(x_1, h) = u_2(x_1, h) \quad \text{for all } x_1 \in (0, 2\pi), \tag{1.3}$$

where $h > \max\{\Lambda_1^+, \Lambda_2^+\}$ is a fixed constant, then $\Lambda_1 = \Lambda_2$.

It is well known that a general grating profile cannot be uniquely determined by one plane wave in a lossless media. In the literature there are uniqueness results using many incoming waves of different kinds, for instance, quasiperiodic waves with the same phase-shift [13], fixed-direction multifrequency plane waves [10] and fixed-frequency multi-direction plane waves [25]. Binary gratings have very important applications in industry, because they can be easily fabricated [22, 23]. The inverse problem of identifying parameters of binary gratings plays a major role in quality control and optimal design of diffractive elements with prescribed far field patterns [1, 5, 9]. In the authors' previous work [24], a global uniqueness result in the TE polarization case (i.e. $\lambda = 1$) was verified. The approach of [24] was based on the singularity analysis of an overdetermined Cauchy problem for an inhomogeneous Laplacian equation in a corner domain. The singular behavior of the wave field encodes partial information on the unknown grating, including the position of singular points lying on the interface and also the physical parameters around them (see e.g. [11, 25]). In the TE case the singularity of the wave field near corners also yields knowledge of the wave number beneath the grating profile (that is, k_2). If $\lambda \neq 1$, the wave field cannot lie in H^2 around each corner point. This weaker smoothness gives rise to difficulties in carrying out approach of [24] to the transmission conditions with $\lambda \neq 1$. It seems non-trivial to recover the parameters λ and k_2 from the corner singularities in the TM case, and hence we can only get weaker uniqueness results than [24]. The aim of this paper is to develop a different approach only for identifying the shape of a rectangular grating profile stated as in theorem 1.2. Numerically, optimization-based iterative schemes are usually utilized for solving the inverse problem. One may conclude from theorem 1.2 that the global minimizer of the object functional within the class of rectangular gratings is unique. The proof of theorem 1.2 also implies that wave fields must be singular (that is, non-analytic) at the corner point.

2. Preliminary lemmas

The singularity analysis seems natural for justifying uniqueness to inverse scattering from penetrable scatterers whose boundary contains corner points; see e.g. [6, 7, 24] where the TE transmission conditions (i.e. $\lambda = 1$) were considered. In this section, we prepare several lemmas for the proof of theorem 1.2. They are mostly motivated by the papers [6, 7, 24], but are interesting on their own right. Throughout the whole paper, we let (r, θ) be the polar coordinates of $x = (x_1, x_2)$ in $\mathbb{R}^2$, and let B_R denote the disk centered at origin with radius $R > 0$. The corner domains Ω_ℓ and the line segments Π_ℓ $(\ell = 1, 2)$ are defined as (see figure 2):

$$\Omega_1 := \{(r, \theta) : 0 < r < R,\ 0 < \theta < 3\pi/2\}, \quad \Pi_1 := \{(r, 0) : 0 \leqslant r \leqslant R\},$$
$$\Omega_2 := \{(r, \theta) : 0 < r < R,\ -\pi/2 < \theta < 0\}, \quad \Pi_2 := \{(r, 3\pi/2) : 0 \leqslant r \leqslant R\}.$$

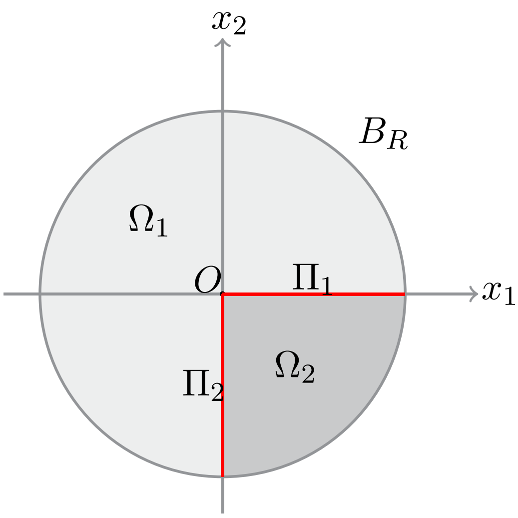

Figure 2. Illustration of two domains Ω_ℓ and two line segments Π_ℓ ($\ell = 1,2$).

Lemma 2.1. *Let q_1 and q_2 be two constants in B_R and let λ be a positive constant. Suppose that u_1 and u_2 satisfy the Helmholtz equations*

$$\Delta u_\ell + q_\ell u_\ell = 0 \quad \text{in } B_R, \quad \ell = 1,2$$

subject to the transmission conditions

$$u_1 = u_2, \quad \frac{\partial u_1}{\partial \nu} = \lambda \frac{\partial u_2}{\partial \nu} \quad \text{on } \Pi_1 \cup \Pi_2. \tag{2.1}$$

If $q_1 \neq q_2$ and $\lambda \neq 1$, then $u_1 = u_2 \equiv 0$ in B_R.

The proof of lemma 2.1 will be postponed to the appendix. When $\lambda = 1$, the proof was given in [7, proposition 2.1] in a general corner domain. The assumption $\lambda \neq 1$ brings additional complexities even in a right corner domain and the results does hold true for general planer angles. In our uniqueness proof, we need a weak version of lemma 2.1, which is stated below.

Lemma 2.2. *Suppose $\rho_1(r,\theta) \equiv 0$ in Ω_1 and $\rho_1(r,\theta) \equiv \rho \in \mathbb{C}, \rho \neq 0$ in Ω_2. Let v_1, v_2 be solutions to*

$$\Delta v_1 + k^2(1+\rho_1)v_1 = 0, \quad \Delta v_2 + k^2 v_2 = 0 \quad \text{in } B_R,$$

subject to the transmission conditions (2.1). Then $v_1 = v_2 \equiv 0$ in B_R.

Proof. Set $q_1 := k^2(1+\rho_1)$ in Ω_2. Since the Cauchy data of v_2 are analytic on $\Pi_1 \cup \Pi_2$, the Cauchy data of v_1 are also analytic thereby the transmission boundary conditions. Since v_1 is analytic in Ω_2, by the Cauchy–Kowalewski theorem in a piecewise analytic domain (see [16, lemma 2.1]), the function v_1 can be analytically extended from Ω_2 to a full neighboring area of the corner as a solution of the Helmholtz equation $\Delta w_1 + q_1 w_1 = 0$, where w_1 denotes the extended solution. Now applying lemma 2.1 to w_1 and v_2 gives $w_1 = v_2 \equiv 0$ near the origin. This together with the unique continuation leads to $v_1 = v_2 \equiv 0$ in B_R. □

To investigate the regularity of solutions to the Helmholtz equation in a corner domain, we consider the transmission problem

$$\begin{cases} \Delta u_\ell + k_\ell^2 u_\ell = 0, & \text{in} \quad \Omega_\ell, \\ u_1 = u_2, \quad \partial_\nu u_1 = \lambda \partial_\nu u_2, & \text{on} \quad \Pi_\ell, \end{cases} \tag{2.2}$$

where k_ℓ $(\ell = 1,2)$ are constants satisfying $k_1 \neq k_2$ and the unit normal vector ν at Π_ℓ is supposed to point into Ω_1. To rewrite the system (2.2) into a divergence form, we define

$$\hat{a}(\theta) := \begin{cases} 1, & \text{in } \Omega_1, \\ \lambda, & \text{in } \Omega_2, \end{cases} \qquad \hat{\kappa}(\theta) := \begin{cases} k_1^2, & \text{in } \Omega_1, \\ \lambda k_2^2, & \text{in } \Omega_2, \end{cases} \qquad \hat{u}(r,\theta) := \begin{cases} u_1, & \text{in } \Omega_1, \\ u_2, & \text{in } \Omega_2. \end{cases}$$

Then the transmission problem (2.2) can be equivalently written as

$$\nabla \cdot (\hat{a}(\theta) \nabla \hat{u}) + \hat{\kappa}(\theta) \hat{u} = 0 \quad \text{in } B_R.$$

By a decomposition theorem (see e.g. [9, 20, 21]), one obtains

$$\hat{u} = \hat{w} + \sum_{j=1}^{m} c_j r^{\eta_j} \varphi_j(\theta) (\ln r)^{p_j} \quad \text{in} \quad B_R, \quad p_j \in \{0,1,\ldots\},$$

where $\hat{w} \in H^2(\Omega_\ell)$ $(\ell = 1,2)$ and $\eta_j \in (0,1)$ are eigenvalues of the following positive definite Sturm–Liouville eigenvalue problem:

$$\begin{cases} \varphi_j''(\theta) + \eta_j^2 \varphi_j(\theta) = 0, & \theta \in (0, 3\pi/2) \cup (-\pi/2, 0), \\ \varphi_{j,+}(0) = \varphi_{j,-}(0), & \varphi_{j,+}'(0) = \lambda \varphi_{j,-}'(0), \\ \varphi_j(3\pi/2) = \varphi_j(-\pi/2), & \varphi_j'(3\pi/2) = \lambda \varphi_j'(-\pi/2). \end{cases} \tag{2.3}$$

In (2.3), the subscripts '+' and '−' denote the limits from Ω_1 and Ω_2, respectively. It is obvious that $\eta_0 = 0$ is an eigenvalue with the eigenfunction $\varphi_{j,\pm} \equiv C \in \mathbb{C}$. A general solution to (2.3) takes the form

$$\varphi_j(\theta) = \begin{cases} A_j^+ \cos(\eta_j \theta) + B_j^+ \sin(\eta_j \theta), & \theta \in (0, 3\pi/2), \\ A_j^- \cos(\eta_j \theta) + B_j^- \sin(\eta_j \theta), & \theta \in (-\pi/2, 0), \end{cases} \tag{2.4}$$

where the non-vanishing coefficients $A_j^\pm$, $B_j^\pm$ are uniquely determined by the transmission conditions through a homogeneous 4-by-4 algebraic system. Lengthy calculations give the first positive eigenvalue (see appendix)

$$\eta_1 = \frac{1}{\pi} \arccos\left(-\frac{\lambda^2 + 6\lambda + 1}{2(\lambda+1)^2} \right) > \frac{2}{3}, \tag{2.5}$$

which yields the leading singularity of $\hat{u}$ around the origin.

Lemma 2.3. *For $\theta \in [0,\pi]$, we have $\varphi_j(\theta) = \varphi_j(\theta + \pi/2)$ if and only if $\eta_j = 4N$; $\varphi_j(\theta) + \varphi_j(\theta + \pi/2) = 0$ if and only if $\eta_j = 4N + 2$. Here $N \in \mathbb{N}$.*

Proof. Recalling the expression of $\varphi_j(\theta)$ in (2.4), we have

$$\varphi_j(\theta + \pi/2) = A_j^+ \cos(\eta_j (\theta + \pi/2)) + B_j^+ \sin(\eta_j (\theta + \pi/2)), \quad \theta \in [0, \pi].$$

For $\eta_j = 4N$, we obtain

$$\varphi_j(\theta+\pi/2) = A_j^+ \cos(4N\theta) + B_j^+ \sin(4N\theta) = \varphi_j(\theta).$$

If $\eta_j = 4N+2$, then

$$\varphi_j(\theta+\pi/2) = -A_j^+ \cos((4N+2)\theta) - B_j^+ \sin((4N+2)\theta) = -\varphi_j(\theta).$$

Conversely, if $\varphi_j(\theta) = \varphi_j(\theta+\pi/2)$ for $\theta \in [0,\pi]$, then $\eta_j \neq 4N+2$. In the following, we only need to show that the eigenvalue η_j cannot be a fractional number which implies $\eta_j = 4N$. Setting $\theta = 0$ and $\theta = \pi$ in the equality $\varphi_j(\theta) = \varphi_j(\theta+\pi/2)$ yields

$$\begin{pmatrix} 1-\cos(\pi\eta_j/2) & -\sin(\pi\eta_j/2) \\ \cos(\pi\eta_j) - \cos(3\pi\eta_j/2) & \sin(\pi\eta_j) - \sin(3\pi\eta_j/2) \end{pmatrix} \begin{pmatrix} A_j^+ \\ B_j^+ \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

By simple calculation,

$$\begin{vmatrix} 1-\cos(\pi\eta_j/2) & -\sin(\pi\eta_j/2) \\ \cos(\pi\eta_j) - \cos(3\pi\eta_j/2) & \sin(\pi\eta_j) - \sin(3\pi\eta_j/2) \end{vmatrix} = 2\sin(\pi\eta_j)\left[1-\cos(\pi\eta_j/2)\right],$$

which cannot vanish when η_j is a fractional number. Hence, $A_j^+ = B_j^+ = 0$, which is impossible.

Similarly, if $\varphi_j(\theta) + \varphi_j(\theta+\pi/2) = 0$ for $\theta \in [0,\pi]$, then $\eta_j \neq 4N$. To show that the eigenvalue η_j cannot be a fractional number, we take $\theta = 0$ and $\theta = \pi$ in the equality $\varphi_j(\theta) + \varphi_j(\theta+\pi/2) = 0$. It then follows the linear system

$$\begin{pmatrix} 1+\cos(\pi\eta_j/2) & \sin(\pi\eta_j/2) \\ \cos(\pi\eta_j) + \cos(3\pi\eta_j/2) & \sin(\pi\eta_j) + \sin(3\pi\eta_j/2) \end{pmatrix} \begin{pmatrix} A_j^+ \\ B_j^+ \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

In this case the determinant of coefficient matrix is given by $2\sin(\pi\eta_j)\left[1+\cos(\pi\eta_j/2)\right]$, which does not vanish when η_j is a fractional number. Hence, $\eta_j = 4N+2$ for some $N \in \mathbb{N}$.

□

In the subsequent sections, we normalize the eigenfunctions in $L^2(-\pi/2, 3\pi/2)$, that is, $\varphi_0(\theta) = 1/\sqrt{2\pi}$ and

$$\int_{-\pi/2}^{3\pi/2} |\varphi_j(\theta)|^2 \mathrm{d}_\theta = 1, \quad \int_{-\pi/2}^{3\pi/2} \varphi_j(\theta)\overline{\varphi_l(\theta)}\mathrm{d}_\theta = \delta_{jl} := \begin{cases} 1, & \text{if } j = l, \\ 0, & \text{if } j \neq l. \end{cases}$$

Then, we make an ansatz on the solution $\hat{u}$ to (2.2) of the form (refer to (2.33) in [3])

$$\hat{u}(r,\theta) = \sum_{j\geqslant 0} \alpha_j r^{\eta_j} \varphi_j(\theta) + \sum_{j\geqslant 0} e_j(r)\varphi_j(\theta), \quad \alpha_j \in \mathbb{C}. \tag{2.6}$$

Note that the second part is required to satisfy the inhomogeneous equation

$$\sum_{j\geqslant 0} \nabla\cdot\left[\hat{a}(\theta)\nabla\left(e_j(r)\varphi_j(\theta)\right)\right] = f(r,\theta),$$

with $f(r,\theta) := -\hat{\kappa}(\theta)\hat{u}(r,\theta)$ in B_R. The first part on the right hand side of (2.6) satisfies the homogeneous equation with $f \equiv 0$. Since $\hat{a}(\theta)$ is a piecewise constant function, it holds that

$$\sum_{j\geqslant 0}\left[\frac{1}{r}\left(re_j'\right)' - \frac{\eta_j^2}{r^2}e_j\right]\varphi_j(\theta) = \frac{f(r,\theta)}{\hat{a}(\theta)}.$$

Multiplying $\overline{\varphi_l(\theta)}$ to both sides of the above equation and integrating over $(-\pi/2, 3\pi/2)$ with respect to θ yields

$$\frac{1}{r}\left(re_j'\right)' - \frac{\eta_j^2}{r^2}e_j = f_j(r),$$

where

$$f_j(r) = -\int_{-\pi/2}^{0} k_2^2 u_2(r,\theta)\overline{\varphi_j(\theta)}\mathrm{d}_\theta - \int_0^{3\pi/2} k_1^2 u_1(r,\theta)\overline{\varphi_j(\theta)}\mathrm{d}_\theta. \tag{2.7}$$

An explicit expression of e_j is given by (see e.g. [3])

$$e_j(r) = \frac{r^{\eta_j}}{2\eta_j}\int_{r_0/2}^{r} f_j(s)\, s^{1-\eta_j}\mathrm{d}s - \frac{r^{-\eta_j}}{2\eta_j}\int_0^r f_j(s)\, s^{1+\eta_j}\mathrm{d}s \quad \text{for } j>0,\ 0<r_0<r.$$

In the special case $j = 0$, one has

$$\frac{1}{r}\left(re_0'(r)\right)' = f_0(r) := -\frac{1}{\sqrt{2\pi}}\int_{-\pi/2}^{0} k_2^2 u_2(r,\theta)\,\mathrm{d}_\theta - \frac{1}{\sqrt{2\pi}}\int_0^{3\pi/2} k_1^2 u_1(r,\theta)\,\mathrm{d}_\theta. \tag{2.8}$$

Straight forward calculations yield the leading terms of f_0 and e_0.

Lemma 2.4. *Let $u_0 = u_1(O) = u_2(O)$. we have*

$$f_0(r) = -\frac{\pi}{2}\left(k_2^2 + 3k_1^2\right)\frac{u_0}{\sqrt{2\pi}} + o(1), \quad e_0(r) = -\frac{\pi}{8}\left(k_2^2+3k_1^2\right)\frac{u_0}{\sqrt{2\pi}}r^2 + o\left(r^2\right), \quad \text{as } r\to 0.$$

3. Proof of theorem 1.2

From the coincidence of u_1 and u_2 on Γ_h, we obtain $u_1 = u_2$ in $x_2 > h$. The unique continuation of solutions to the Helmholtz equation leads to

$$u_1(x_1,x_2) = u_2(x_1,x_2) \quad \text{for all } x\in\Omega_{\Lambda_1}^+\cap\Omega_{\Lambda_2}^+. \tag{3.1}$$

Assume on the contrary that $\Lambda_1 \neq \Lambda_2$. Switching the notations for Λ_1 and Λ_2 if necessary, we only need to consider the following cases:

- Case one: there exists a corner point O of Λ_1 such that $O \in \Omega_{\Lambda_2}^+$ (see figure 3);
- Case two: all corners of Λ_1 and Λ_2 coincide but $\Lambda_1 \neq \Lambda_2$ (see figure 4);
- Case three: there exists a corner point O of Λ_2 lying on Λ_1, but O is not a corner of Λ_1 (see figure 5).

Obviously, the corners of Λ_1 and Λ_2 do not coincide completely in the first and last cases. Using coordinate translation, we suppose that the corner O is located at the origin. Below we shall prove that neither of previous three cases occurs. This contraction yields $\Lambda_1 = \Lambda_2$.

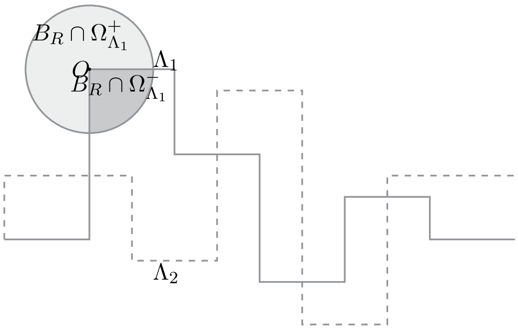

Figure 3. Case one: there exists a corner point O of Λ_1 such that $O \in \Omega^+_{\Lambda_2}$.

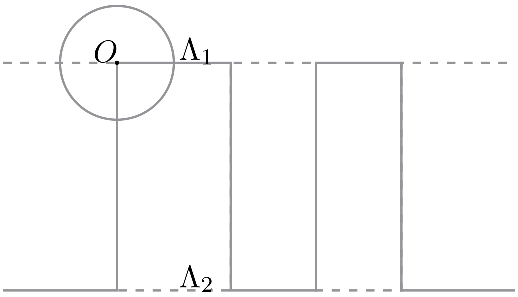

Figure 4. Case two: corners of Λ_1 and Λ_2 are identical but $\Lambda_1 \neq \Lambda_2$.

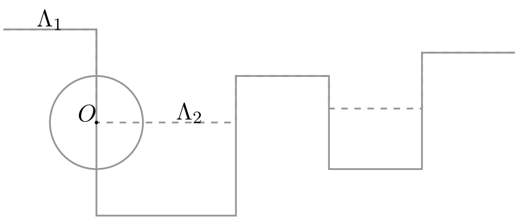

Figure 5. Case three: $O \in \Lambda_1 \cap \Lambda_2$ is a corner of Λ_2 but not a corner of Λ_1.

3.1. Case one

Choose $R > 0$ such that $B_R \subseteq \Omega^+_{\Lambda_2}$. Since the corner point $O \in \Omega^+_{\Lambda_2}$ stays away from Λ_2, the function u_2 satisfies the Helmholtz equation with the wave number k_1 in B_R, while u_1 fulfills the Helmholtz equation with the variable potential $k_1^2(1+\rho_1)$. Here, $\rho_1(x)$ is a piecewise constant function defined by

$$\rho_1(x) := \begin{cases} 0, & \text{in} \quad B_R \cap \Omega^+_{\Lambda_1}, \\ \left(\frac{k_2}{k_1}\right)^2 - 1, & \text{in} \quad B_R \cap \Omega^-_{\Lambda_1}. \end{cases}$$

Recalling the transmission conditions in (1.1), we find that the pair (u_1, u_2) is a solution to

$$\begin{cases} \Delta u_1 + k_1^2 (1 + \rho_1(x)) u_1 = 0, & \text{in} \quad B_R, \\ \Delta u_2 + k_1^2 u_2 = 0, & \text{in} \quad B_R, \\ u_1 = u_2, \quad \lambda \frac{\partial u_1^-}{\partial \nu} = \frac{\partial u_2}{\partial \nu}, & \text{on} \quad B_R \cap \Lambda_1. \end{cases}$$

Here, the symbol $(\cdot)^-$ denotes the limit from $\Omega_{\Lambda_1}^-$. Applying lemma 2.2, we obtain $u_1 = 0$ in B_R and thus $u_1 = 0$ in $\mathbb{R}^2$, which is impossible (see [24]).

3.2. Case two

The corners of Λ_1 and Λ_2 coincide (see figure 4), implying that Λ_1 and Λ_2 have the same height and also the same grooves but with different opening directions. This section relies on ingenious analysis on the regularity of solutions to the Helmholtz equation in a corner domain. We refer to [20] for an overview of the interface problem of the Laplacian equation.

Choose a corner point $O \in \Lambda_1 \cap \Lambda_2$ and $R > 0$ sufficiently small such that the disk $B_R := \{x \in \mathbb{R}^2 : |x| < R\}$ does not contain other corners. We can conclude from proposition 1.1 that $u_1, u_2 \in H^{1+s}(B_R)$ $(0 \leqslant s < 1/2)$ fulfill the system

$$\begin{cases} \nabla \cdot (a(\theta) \nabla u_1) + \kappa(\theta) u_1 = 0, & \text{in} \quad B_R, \\ \nabla \cdot (a(\theta + \pi/2) \nabla u_2) + \kappa(\theta + \pi/2) u_2 = 0, & \text{in} \quad B_R, \end{cases} \tag{3.2}$$

where

$$a(\theta) := \begin{cases} 1, & \text{if} \quad \theta \in (0, 3\pi/2), \\ \lambda, & \text{if} \quad \theta \in (-\pi/2, 0), \end{cases} \qquad \kappa(\theta) := \begin{cases} k_1^2, & \text{if} \quad \theta \in (0, 3\pi/2), \\ \lambda k_2^2, & \text{if} \quad \theta \in (-\pi/2, 0), \end{cases}$$

and $a(\theta \pm 2\pi) = a(\theta)$, $\kappa(\theta \pm 2\pi) = \kappa(\theta)$. It is obvious that u_2 coincides with u_1 after a rotation about the angle $\pi/2$, that is, $u_2(r, \theta) = u_1(r, \theta + \pi/2)$. In lemma 3.1 below, we shall derive a more explicit expression of u_ℓ $(\ell = 1, 2)$ under the condition (3.1).

Lemma 3.1. *Let $u_1, u_2 \in H^{1+s}(B_R)$ $(0 \leqslant s < 1/2)$ be solutions to (3.2). If*

$$u_1(r, \theta) = u_2(r, \theta) \quad \text{for all} \quad \theta \in (0, \pi), r \in [0, R),$$

then

$$u_\ell(r, \theta) = \sum_{n, m \in \mathbb{N}: n + m \geqslant 0} a_{n,m}^{(\ell)} r^{2(n+m)} \psi_{2n}^{(\ell)}(\theta), \quad \ell = 1, 2 \tag{3.3}$$

where $\psi_{2n}^{(1)}(\theta)$ is the normalized eigenfunction of (2.3) corresponding to the eigenvalue $\eta = 2n$ and $\psi_{2n}^{(2)}(\theta) = \psi_{2n}^{(1)}(\theta + \pi/2)$.

Proof. To prove (3.3), it suffices to verify for all $l \in \mathbb{N}$ that

$$u_\ell(r, \theta) = \sum_{0 \leqslant n + m \leqslant l} a_{n,m}^{(\ell)} r^{2(n+m)} \psi_{2n}^{(\ell)}(\theta) + o\left(r^{2l}\right), \quad \text{as} \quad r \to 0. \tag{3.4}$$

Recalling (2.6) and lemma 2.4, we have

$$u_\ell(r, \theta) = u_0 + \sum_{j \geqslant 1} \alpha_j^{(\ell)} r^{\eta_j} \varphi_j^{(\ell)}(\theta) + e_{0,0}^{(\ell)}(r) \varphi_0^{(\ell)}(\theta) + \sum_{j \geqslant 1} e_{j,0}^{(\ell)}(r) \varphi_j^{(\ell)}(\theta), \qquad \ell = 1, 2, \tag{3.5}$$

where $\varphi_j^{(1)}(\theta) := \varphi_j(\theta)$ are normalized eigenfunctions, $\varphi_j^{(2)}(\theta) := \varphi_j^{(1)}(\theta+\pi/2)$ and

$$e_{j,0}^{(\ell)}(r) = \frac{r^{\eta_j}}{2\eta_j}\int_{r_0/2}^{r} f_{j,0}^{(\ell)}(s)\, s^{1-\eta_j}\mathrm{d}s - \frac{r^{-\eta_j}}{2\eta_j}\int_0^r f_{j,0}^{(\ell)}(s)\, s^{1+\eta_j}\mathrm{d}s, \quad \text{for } j>0,\ \ell=1,2. \tag{3.6}$$

Here the functions $f_{j,0}^{(\ell)}$ with $\ell = 1,2$ are defined analogously to (2.7) and $0 < r_0 < r$. By (2.5), we know that $\eta_j > 2/3$ for $j \geqslant 1$, which together with $e_{j,0}^{(\ell)}(r) = o(r)$ $(\ell = 1,2)$ implies that (3.4) holds with $l = n+m = 0$ and $a_{0,0}^{(1)} = a_{0,0}^{(2)} = \sqrt{2\pi}u_0$.

Step 1: prove that (3.4) holds for $l = 1$. It is obvious that if $l = n+m = 1$ for some $n,m \in \mathbb{N}$, then $n = 0$, $m = 1$ or $n = 1$, $m = 0$. Hence, it suffices to prove

$$u_\ell(r,\theta) = a_{0,0}^{(\ell)}\psi_0^{(\ell)}(\theta) + \left[a_{0,1}^{(\ell)}\psi_0^{(\ell)}(\theta) + a_{1,0}^{(\ell)}\psi_2^{(\ell)}(\theta)\right] r^2 + o\left(r^2\right), \quad \text{as} \quad r\to 0,\ \ell = 1,2,$$

with some $a_{0,1}^{(\ell)}$, $a_{1,0}^{(\ell)} \in \mathbb{C}$ for $\ell = 1,2$. Recalling from the definition of $e_{j,0}^{(\ell)}$ $(j \geqslant 0,\ \ell = 1,2)$ in (3.6), we obtain

$$e_{j,0}^{(\ell)}(r) = \begin{cases} \frac{\sqrt{2\pi}}{4-\eta_j^2}\, d_{j,0}\, u_0\, r^2 + o\left(r^3\right), & \text{if} \quad \eta_j \neq 2, \\ \frac{\sqrt{2\pi}}{4}\, d_{j,0}\, u_0\, r^2 \ln r + o\left(r^3\right), & \text{if} \quad \eta_j = 2, \end{cases} \qquad \text{as } r \to 0, \tag{3.7}$$

where $d_{j,0} \in \mathbb{C}$ are given by

$$d_{j,0} := -\left[k_2^2 \int_{-\pi/2}^{0} \psi_0^{(1)}(\theta)\,\overline{\varphi_j^{(1)}(\theta)}\,\mathrm{d}_\theta + k_1^2 \int_0^{3\pi/2} \psi_0^{(1)}(\theta)\,\overline{\varphi_j^{(1)}(\theta)}\,\mathrm{d}_\theta\right], \quad \eta_j \geqslant 0. \tag{3.8}$$

Hence, it follows from (3.5) that

$$u_\ell(r,\theta) = u_0 + \sum_{0<\eta_j<2} \alpha_j^{(\ell)} r^{\eta_j}\varphi_j^{(\ell)}(\theta) + o\left(r^{l_0}\right),$$

where $l_0 = \max\{\eta_j : 0 < \eta_j < 2\}$. Recalling $u_1(r,\theta) = u_2(r,\theta)$, $\partial_\theta u_1(r,\theta) = \partial_\theta u_2(r,\theta)$ $(\theta \in [0,\pi])$, we obtain

$$\alpha_j^{(1)}\varphi_j^{(1)}(\theta) = \alpha_j^{(2)}\varphi_j^{(2)}(\theta), \quad \alpha_j^{(1)}\left[\varphi_j^{(1)}(\theta)\right]' = \alpha_j^{(2)}\left[\varphi_j^{(2)}(\theta)\right]', \quad \forall\,\theta \in (0,\pi),\ \eta_j \in (0,2),$$

which can be rewritten as the linear system

$$\begin{pmatrix} A_j^+\cos(\eta_j\theta) + B_j^+\sin(\eta_j\theta) & -A_j^+\cos\left(\eta_j\left(\theta+\frac{\pi}{2}\right)\right) - B_j^+\sin\left(\eta_j\left(\theta+\frac{\pi}{2}\right)\right) \\ B_j^+\cos(\eta_j\theta) - A_j^+\sin(\eta_j\theta) & A_j^+\sin\left(\eta_j\left(\theta+\frac{\pi}{2}\right)\right) - B_j^+\cos\left(\eta_j\left(\theta+\frac{\pi}{2}\right)\right) \end{pmatrix}$$
$$\times \begin{pmatrix} \alpha_j^{(1)} \\ \alpha_j^{(2)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since the determinant of coefficient matrix is $\left[(A_j^+)^2 + (B_j^+)^2\right]\sin\left(\frac{\pi}{2}\eta_j\right) > 0$, we obtain $\alpha_j^{(1)} = \alpha_j^{(2)} = 0$ for $0 < \eta_j < 2$. It then follows from (3.5) that

$$u_\ell(r,\theta) = u_0 + \sum_{2\leqslant\eta_j<4} \alpha_j^{(\ell)} r^{\eta_j}\varphi_j^{(\ell)}(\theta) + \sum_{\eta_j\geqslant 0} e_{j,0}^{(\ell)}(r)\,\varphi_j^{(\ell)}(\theta) + o\left(r^{l_1}\right), \qquad \text{as } r\to 0,$$

where $l_1 = \max\{\eta_j : 2 < \eta_j < 4\}$. Hence,

$$u_\ell(r,\theta) = u_0 + a_{1,0}^{(\ell)} r^2 \psi_2^{(\ell)}(\theta) + \sum_{2<\eta_j<4} \alpha_j^{(\ell)} r^{\eta_j} \varphi_j^{(\ell)}(\theta) + \frac{\sqrt{2\pi}}{4} D_{2,0} u_0 r^2 \ln r\, \psi_2^{(\ell)}(\theta)$$
$$+ u_0 r^2 \sum_{\eta_j \neq 2} \frac{\sqrt{2\pi}\, d_{j,0}}{4-\eta_j^2} \varphi_j^{(\ell)}(\theta) + o\left(r^{l_1}\right), \qquad \text{as } r \to 0, \tag{3.9}$$

where $a_{1,0}^{(\ell)} = \alpha_j^{(\ell)}$, $D_{2,0} = d_{j,0}$ for $\eta_j = 2$. Equating the coefficients of the terms r^2 and $r^2 \ln r$ yields

$$D_{2,0} u_0 \left[\psi_2^{(1)}(\theta) - \psi_2^{(2)}(\theta)\right] = 0,$$
$$\left[a_{1,0}^{(1)} \psi_2^{(1)}(\theta) - a_{1,0}^{(2)} \psi_2^{(2)}(\theta)\right] + \sum_{\eta_j \neq 2} \frac{\sqrt{2\pi} d_{j,0} u_0}{4-\eta_j^2} \left[\varphi_j^{(1)}(\theta) - \varphi_j^{(2)}(\theta)\right] = 0,$$

for all $\theta \in (0,\pi)$. Since $\psi_2^{(2)}(\theta) = -\psi_2^{(1)}(\theta)$, by linear independence of trigonometric functions, we conclude that

$$D_{2,0} u_0 = 0, \quad a_{1,0}^{(1)} + a_{1,0}^{(2)} = 0 \qquad \text{and} \qquad d_{j,0} u_0 = 0 \text{ if } \varphi_j^{(1)}(\theta) \neq \varphi_j^{(2)}(\theta),\ \eta_j \neq 2.$$

If $\varphi_j^{(1)}(\theta) = \varphi_j^{(2)}(\theta)$, we have $\eta_j = 4N$ by lemma 2.3 and

$$d_{j,0} = \begin{cases} 0, & \text{if } \eta_j = 4N,\ N \neq 0, \\ -\frac{1}{4}\left(3k_1^2 + k_2^2\right), & \text{if } \eta_j = 0 \ (\text{i.e. } j = 0). \end{cases}$$

This implies that the terms with $j \neq 0$ in the following summation all vanish, i.e.

$$r^2 u_0 \sum_{\eta_j \neq 2} \frac{\sqrt{2\pi}\, d_{j,0}}{4-\eta_j^2} \varphi_j^{(\ell)}(\theta) = r^2 u_0 \frac{\sqrt{2\pi}\, d_{0,0}}{4} \varphi_0^{(\ell)}(\theta).$$

Inserting these results into (3.9) yields as $r \to 0$ that

$$u_\ell(r,\theta) = \sum_{0 \leqslant n+m \leqslant 1} a_{n,m}^{(\ell)} r^{2(n+m)} \psi_{2n}^{(\ell)}(\theta) + \sum_{2<\eta_j<4} \alpha_j^{(\ell)} r^{\eta_j} \varphi_j^{(\ell)}(\theta) + \sum_{\eta_j \geqslant 0} e_{j,1}^{(\ell)}(r) \varphi_j^{(\ell)}(\theta) + o\left(r^{l_1}\right)$$

where $a_{0,0}^{(\ell)} = \sqrt{2\pi} u_0$, $a_{0,1}^{(\ell)} = \sqrt{2\pi}\, d_{0,0} u_0/4$, $a_{1,0}^{(1)} = -a_{1,0}^{(2)}$. Further, we have $a_{0,1}^{(\ell)} = a_{0,0}^{(\ell)} d_{0,0}/4$ and

$$a_{0,0}^{(\ell)} d_{j,0} = 0 \quad \text{for } \eta_j \neq 0; \quad a_{n,m}^{(1)} \psi_{2n}^{(1)}(\theta) = a_{n,m}^{(2)} \psi_{2n}^{(2)}(\theta) \quad \text{for all } 0 \leqslant n+m \leqslant 1,$$
$$e_{j,1}^{(\ell)}(r) = e_{j,0}^{(\ell)}(r) - \begin{cases} \frac{\sqrt{2\pi}}{4-\eta_j^2} d_{j,0} u_0 r^2, & \text{if } \eta_j \neq 2, \\ \frac{\sqrt{2\pi}}{4} d_{j,0} u_0 r^2 \ln r, & \text{if } \eta_j = 2. \end{cases}$$

It is seen from (3.7) that $e_{j,1}^{(\ell)}(r) = o(r^3)$. This finishes the step 1.

Step 2: induction arguments. We make an induction hypothesis that for some $N \geqslant 1$,

$$\begin{cases} u_\ell(r,\theta) = \sum\limits_{0\leqslant n+m\leqslant N} a^{(\ell)}_{n,m} r^{2(n+m)} \psi^{(\ell)}_{2n}(\theta) + \sum\limits_{2N<\eta_j<2N+2} \alpha^{(\ell)}_j r^{\eta_j} \varphi^{(\ell)}_j(\theta) \\ \qquad + \sum\limits_{\eta_j\geqslant 0} e^{(\ell)}_{j,N}(r) \varphi^{(\ell)}_j(\theta) + o\left(r^{l_N}\right); \\ a^{(\ell)}_{n,m} = \frac{a^{(\ell)}_{n,m-1} D_{2n,2n}}{(2N)^2-(2n)^2}, \ \forall n+m=N, 0\leqslant n\leqslant N-1; \\ a^{(\ell)}_{n,m} d_{j,2n} = 0, \quad \text{for } \eta_j \neq 2n, \ \forall 0\leqslant n+m\leqslant N-1; \\ a^{(1)}_{n,m}\psi^{(1)}_{2n}(\theta) = a^{(2)}_{n,m}\psi^{(2)}_{2n}(\theta), \quad \forall 0\leqslant n+m\leqslant N, \end{cases} \tag{3.10}$$

where $e^{(\ell)}_{j,N}(r)$ $(\ell=1,2)$ is defined as (2.8), (3.6) with $f^{(\ell)}_{j,0}$ replaced by $f^{(\ell)}_{j,N}$:

$$\begin{aligned} f^{(\ell)}_{j,N}(r) = &-\int_0^{3\pi/2} k_1^2 \left[u_\ell(r,\theta) - \sum_{0\leqslant n+m\leqslant N-1} a^{(\ell)}_{n,m} r^{2(n+m)} \psi^{(\ell)}_{2n}(\theta)\right] \overline{\varphi^{(\ell)}_j(\theta)} \mathrm{d}_\theta \\ &-\int_{-\pi/2}^0 k_2^2 \left[u_\ell(r,\theta) - \sum_{0\leqslant n+m\leqslant N-1} a^{(\ell)}_{n,m} r^{2(n+m)} \psi^{(\ell)}_{2n}(\theta)\right] \overline{\varphi^{(\ell)}_j(\theta)} \mathrm{d}_\theta; \end{aligned}$$

$l_N := \max\{\eta_j : 2N < \eta_j < 2N+2\}$;

$$\begin{aligned} d_{j,2n} &= -\left[k_2^2 \int_{-\pi/2}^0 \psi^{(1)}_{2n}(\theta) \overline{\varphi^{(1)}_j(\theta)} \mathrm{d}_\theta + k_1^2 \int_0^{3\pi/2} \psi^{(1)}_{2n}(\theta) \overline{\varphi^{(1)}_j(\theta)} \mathrm{d}_\theta\right] \\ &= \begin{cases} -k_1^2 + \left(k_1^2 - k_2^2\right) \int_{-\pi/2}^0 |\psi^{(1)}_{2n}(\theta)|^2 \mathrm{d}_\theta, & \text{if} \quad \eta_j = 2n, \\ \left(k_1^2 - k_2^2\right) \int_{-\pi/2}^0 \psi^{(1)}_{2n}(\theta) \overline{\varphi^{(1)}_j(\theta)} \mathrm{d}_\theta, & \text{if} \quad \eta_j \neq 2n, \end{cases} \end{aligned} \tag{3.11}$$

for $0 \leqslant n \leqslant N-1$; $D_{2n,2n} := d_{j,2n}$ when $\eta_j = 2n$.

Note that the above induction hypothesis with $N=1$ has been proved in step one. Now we want to prove that (3.10) holds for $N+1$. By the definition of $e^{(\ell)}_{j,N}$, straightforward calculations show that

$$e^{(\ell)}_{j,N}(r) = \begin{cases} \frac{r^{2N+2}}{(2N+2)^2-\eta_j^2} \sum\limits_{n+m=N} a^{(\ell)}_{n,m} d_{j,2n} + o\left(r^{2N+3}\right), & \text{if} \quad \eta_j \neq 2N+2, \\ \frac{a^{(\ell)}_{N,0} D_{2N+2,2N}}{4N+2} r^{2N+2} \ln r + o\left(r^{2N+3}\right), & \text{if} \quad \eta_j = 2N+2. \end{cases} \tag{3.12}$$

Here $D_{2N+2,2N} := d_{j,2N}$ with $\eta_j = 2N+2$ and $d_{j,2N}$ is defined analogously by (3.11).

Using the relations $u_1(r,\theta) = u_2(r,\theta)$, $\partial_\theta u_1(r,\theta) = \partial_\theta u_2(r,\theta)$ $(\theta \in [0,\pi])$, we deduce from the expressions of u_l in (3.10) that

$$\begin{gathered} \alpha^{(1)}_j \varphi^{(1)}_j(\theta) = \alpha^{(2)}_j \varphi^{(2)}_j(\theta), \quad \alpha^{(1)}_j \left[\varphi^{(1)}_j(\theta)\right]' = \alpha^{(2)}_j \left[\varphi^{(2)}_j(\theta)\right]', \\ \forall \theta \in (0,\pi), \ \eta_j \in (2N, 2N+2). \end{gathered}$$

Similarly, we can obtain an equation system about the unknowns $\alpha^{(1)}_j$ and $\alpha^{(2)}_j$, where the determinant of coefficient matrix is still not equal to zero for $2N < \eta_j < 2N+2$.

Consequently, we achieve that $\alpha_j^{(1)} = \alpha_j^{(2)} = 0$ for $2N < \eta_j < 2N+2$. Inserting this into (3.10) gives

$$u_\ell(r,\theta) = \sum_{0\leqslant n+m\leqslant N} a_{n,m}^{(\ell)} r^{2(n+m)} \psi_{2n}^{(\ell)}(\theta) + \sum_{2N+2\leqslant \eta_j<2N+4} \alpha_j^{(\ell)} r^{\eta_j} \varphi_j^{(\ell)}(\theta) + \sum_{\eta_j\geqslant 0} e_{j,N}^{(\ell)}(r)\varphi_j^{(\ell)}(\theta) + o\left(r^{l_N}\right), \quad \ell = 1,2.$$

Using the relations in (3.12), we can obtain

$$\begin{aligned} u_\ell(r,\theta) = &\sum_{0\leqslant n+m\leqslant N} a_{n,m}^{(\ell)} r^{2(n+m)} \psi_{2n}^{(\ell)}(\theta) + r^{2N+2} \sum_{0\leqslant n\leqslant N-1}^{n+m=N+1} a_{n,m}^{(\ell)} \psi_{2n}^{(\ell)}(\theta) + a_{N+1,0}^{(\ell)} r^{2N+2}\psi_{2N+2}^{(\ell)}(\theta) \\ &+ \frac{a_{N,0}^{(\ell)} D_{2N+2,2N}}{4N+2} r^{2N+2}\ln r\, \psi_{2N+2}^{(\ell)}(\theta) + \sum_{\eta_j\neq 2N+2} \frac{a_{N,0}^{(\ell)} d_{j,2N}}{(2N+2)^2-\eta_j^2} r^{2N+2}\varphi_j^{(\ell)}(\theta) \\ &+ \sum_{2N+2<\eta_j<2N+4} \alpha_j^{(\ell)} r^{\eta_j}\varphi_j^{(\ell)}(\theta) + o\left(r^{l_{N+1}}\right), \quad \ell = 1,2. \end{aligned}$$

Here, $a_{N+1,0}^{(\ell)} := \alpha_j^{(\ell)}$ for $\eta_j = 2N+2$, $l_{N+1} := \max\{\eta_j : 2N+2 < \eta_j < 2N+4\}$ and

$$a_{n,m}^{(\ell)} = \frac{a_{n,m-1}^{(\ell)} D_{2n,2n}}{(2N+2)^2-(2n)^2}, \qquad \forall 0\leqslant n\leqslant N-1, n+m = N+1. \tag{3.13}$$

Applying the induction hypothesis $a_{n,m}^{(1)}\psi_{2n}^{(1)}(\theta) = a_{n,m}^{(2)}\psi_{2n}^{(2)}(\theta)$ for all $0\leqslant n+m\leqslant N$ into (3.13), we have

$$a_{n,m}^{(1)}\psi_{2n}^{(1)}(\theta) = a_{n,m}^{(2)}\psi_{2n}^{(2)}(\theta), \qquad \forall 0\leqslant n\leqslant N-1, n+m = N+1. \tag{3.14}$$

Comparing the expressions of u_1 and u_2 and using the fact that $u_1 = u_2$ for all $\theta\in(0,\pi)$ yields

$$a_{N,0}^{(1)} D_{2N+2,2N}\psi_{2N+2}^{(1)}(\theta) = a_{N,0}^{(2)} D_{2N+2,2N}\psi_{2N+2}^{(2)}(\theta),$$

and

$$\begin{aligned} &a_{N+1,0}^{(1)}\psi_{2N+2}^{(1)}(\theta) + \sum_{\eta_j\neq 2N+2} \frac{a_{N,0}^{(1)} d_{j,2N}}{(2N+2)^2-\eta_j^2}\varphi_j^{(1)}(\theta) \\ &\quad = a_{N+1,0}^{(2)}\psi_{2N+2}^{(2)}(\theta) + \sum_{\eta_j\neq 2N+2} \frac{a_{N,0}^{(2)} d_{j,2N}}{(2N+2)^2-\eta_j^2}\varphi_j^{(2)}(\theta). \end{aligned}$$

Since $a_{N,0}^{(1)} = (-1)^N a_{N,0}^{(2)}$, $\psi_{2N+2}^{(2)}(\theta) = (-1)^{N+1}\psi_{2N+2}^{(1)}(\theta)$, we conclude that

$$a_{N,0}^{(\ell)} D_{2N+2,2N} \psi_{2N+2}^{(\ell)}(\theta) = 0,$$

and

$$\left[a_{N+1,0}^{(1)} - (-1)^{N+1} a_{N+1,0}^{(2)}\right] \psi_{2N+2}^{(1)}(\theta) + \sum_{\eta_j \neq 2N+2} \frac{a_{N,0}^{(1)} d_{j,2N}}{(2N+2)^2 - \eta_j^2} \times \left[\varphi_j^{(1)}(\theta) - (-1)^N \varphi_j^{(2)}(\theta)\right] = 0.$$

Using lemma 2.3 and the linear independence of trigonometric functions, we conclude that

$$a_{N+1,0}^{(1)} \psi_{2N+2}^{(1)}(\theta) = a_{N+1,0}^{(2)} \psi_{2N+2}^{(2)}(\theta), \tag{3.15}$$

and

$$a_{N,0}^{(1)} d_{j,2N} = \begin{cases} 0, & \text{if} \quad \varphi_j^{(1)}(\theta) \neq \varphi_j^{(2)}(\theta),\ N \text{ is an even number}, \\ 0, & \text{if} \quad \varphi_j^{(1)}(\theta) + \varphi_j^{(2)}(\theta) \neq 0,\ N \text{ is an odd number}. \end{cases}$$

Recalling lemma 2.3 and the definition of $d_{j,2N}$, we find that

$$d_{j,2N} = \begin{cases} 0, & \text{if} \quad \eta_j = 4l \text{ and } N \text{ is an even number},\ l \neq N/2, \\ 0, & \text{if} \quad \eta_j = 4l+2 \text{ and } N \text{ is an odd number},\ l \neq (N-1)/2. \end{cases}$$

Based on the above results, we conclude that

$$a_{N,0}^{(\ell)} d_{j,2N} = 0, \quad \text{for } \eta_j \neq 2N,\ \ell = 1, 2.$$

Combining the previous equalities with the following two induction hypothesis

$$\begin{cases} a_{n,m}^{(\ell)} = \frac{a_{n,m-1}^{(\ell)} D_{2n,2n}}{(2N)^2 - (2n)^2}, & \forall n+m = N,\ 0 \leqslant n \leqslant N-1, \\ a_{n,m}^{(\ell)} d_{j,2n} = 0, & \text{for } \eta_j \neq 2n,\ \forall 0 \leqslant n+m \leqslant N-1, \end{cases}$$

we find that

$$a_{n,m}^{(\ell)} d_{j,2n} = 0, \quad \text{for } \eta_j \neq 2n,\ \forall 0 \leqslant n \leqslant N,\ n+m = N. \tag{3.16}$$

Hence,

$$u_\ell(r,\theta) = \sum_{0 \leqslant n+m \leqslant N+1} a_{n,m}^{(\ell)} r^{2(n+m)} \psi_{2n}^{(\ell)}(\theta) + \sum_{2N+2 < \eta_j < 2N+4} \alpha_j^{(\ell)} r^{\eta_j} \varphi_j^{(\ell)}(\theta) + \sum_{\eta_j \geqslant 0} e_{j,N+1}^{(\ell)}(r) \varphi_j^{(\ell)}(\theta) + o\left(r^{J_{N+1}}\right), \quad \ell = 1, 2, \tag{3.17}$$

where $e_{j,N+1}^{(\ell)}$ is defined in the same way as $e_{j,N}^{(\ell)}$, $D_{2N,2N}$ equals to $d_{j,2N}$ when $\eta_j = 2N$ and

$$a_{N,1}^{(\ell)} = \frac{a_{N,0}^{(\ell)} D_{2N,2N}}{(2N+2)^2 - (2N)^2}, \quad \ell = 1, 2. \tag{3.18}$$

Then, the relation $a_{N,0}^{(1)}\psi_{2N}^{(1)}(\theta)=a_{N,0}^{(2)}\psi_{2N}^{(2)}(\theta)$ gives that

$$a_{N,1}^{(1)}\psi_{2N}^{(1)}(\theta)=a_{N,1}^{(2)}\psi_{2N}^{(2)}(\theta). \tag{3.19}$$

Therefore, relations (3.13)–(3.19) imply that (3.10) still holds for $N+1$.

Step 3: by the induction argument, we know that (3.10) holds for any $N\in\mathbb{N}$ which implies (3.4) for all $l\in\mathbb{N}$. Hence, the proof of (3.3) is complete. □

By lemma 3.1, we have

$$u_1(r,\theta)=\begin{cases}\sum\limits_{n+m\geqslant 0}a_{n,m}^{(1)}r^{2(n+m)}\left[A_n^-\cos(2n\theta)+B_n^-\sin(2n\theta)\right], & \theta\in(-\pi/2,0),\\ \sum\limits_{n+m\geqslant 0}a_{n,m}^{(1)}r^{2(n+m)}\left[A_n^+\cos(2n\theta)+B_n^+\sin(2n\theta)\right], & \theta\in(0,3\pi/2).\end{cases}$$

Now, using the transmission condition of u_1 on Π_ℓ one can repeat the proof in the proof of lemma 2.1 to obtain $u_1\equiv 0$ around O, which is impossible. This excludes the case two.

3.3. Case three

Assume there exists a corner O of Λ_2 such that $O\in\Lambda_1$, but O is not a corner point of Λ_1. Without loss of generality, we suppose that O is located on a vertical line segment of Λ_1 (see figure 5). Choose $R>0$ sufficiently small such that the disk B_R does not contain any other corners. We can see that $u_1,u_2\in H^{1+s}(B_R)$ $(0\leqslant s<1/2)$ are solutions to the systems

$$\begin{cases}\Delta u_1+k_1^2u_1=0, & \text{in}\quad \theta\in[0,\pi/2)\cup(3\pi/2,2\pi],\\ \Delta u_1+k_2^2u_1=0, & \text{in}\quad \theta\in(\pi/2,3\pi/2),\\ u_1^+=u_1^-,\quad \partial_\nu^+u_1=\lambda\partial_\nu^-u_1, & \text{on}\quad \theta=\pi/2,3\pi/2,\end{cases} \tag{3.20}$$

$$\begin{cases}\Delta u_2+k_1^2u_2=0, & \text{in}\quad \theta\in(0,\pi/2),\\ \Delta u_2+k_2^2u_2=0, & \text{in}\quad \theta\in(\pi/2,2\pi),\\ u_2^+=u_2^-,\quad \partial_\nu^+u_2=\lambda\partial_\nu^-u_2, & \text{on}\quad \theta=0,\pi/2.\end{cases} \tag{3.21}$$

By proposition 1.1 (ii), the Cauchy data $(u_1^+,\partial_\nu u_1^+)$ are analytic on $B_R\cap\Lambda_2$. Then, the coincidence $u_1(r,\theta)=u_2(r,\theta)$ for all $\theta\in[0,\pi/2]$ implies that u_2^+ and $\partial_\nu u_2^+$ are both analytic on $B_R\cap\Lambda_2$. By the Cauchy–Kowalewski theorem in a piecewise analytic domain (refer to lemma 2.1 in [16]), we conclude that there exists $R_1\in(0,R)$ such that u_2 can be extended analytically from $B_{R_1}\cap\Omega_{\Lambda_2}^+$ to B_{R_1} and the extended function w_2 satisfies that

$$\begin{cases}\Delta w_2+k_1^2w_2=0, & \text{in}\quad B_{R_1},\\ w_2=u_2^+,\quad \partial_\nu w_2=\partial_\nu u_2^+, & \text{on}\quad B_{R_1}\cap\Lambda_2.\end{cases}$$

Recalling the transmission boundary in (3.21) and the fact that λ is a constant, we also find that u_2^- and $\partial_\nu u_2^-$ are both analytic on $B_R\cap\Lambda_2$. Similarly, the solution u_2 can be extended analytically from $B_{R_2}\cap\Omega_{\Lambda_2}^-$ to B_{R_2} $(R_2\in(0,R_1))$ by the Cauchy–Kowalewski theorem. Denote by v_2 the extended function in B_{R_2}, which satisfies

$$\begin{cases}\Delta v_2+k_2^2v_2=0, & \text{in}\quad B_{R_2},\\ v_2=u_2^-,\quad \partial_\nu v_2=\partial_\nu u_2^-, & \text{on}\quad B_{R_2}\cap\Lambda_2.\end{cases}$$

Again using the transmission conditions in (3.21) yields

$$\begin{cases} \Delta w_2 + k_1^2 w_2 = 0, & \text{in} \quad B_{R_2}, \\ \Delta v_2 + k_2^2 v_2 = 0, & \text{in} \quad B_{R_2}, \\ w_2 = v_2, \quad \partial_\nu w_2 = \lambda \partial_\nu v_2, & \text{on} \quad B_{R_2} \cap \Lambda_2. \end{cases}$$

Since $k_1 \neq k_2$, we obtain $w_2 = v_2 \equiv 0$ in B_{R_2} by lemma 2.1, that is, $u_2 \equiv 0$ in B_{R_2}. This together with the unique continuation leads to $u_2 \equiv 0$ in B_R, which is impossible.

Remark 3.2. We remark that the uniqueness proof for treating case three is also valid in the TE polarization case, providing a different method to the approach present in [24, section 4.3].

Data availability statement

No new data were created or analyzed in this study.

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Appendix

This section is devoted to the regularity problem around a corner point and up to the flat interface. For the readers' convenience, we also justify the well-posedness of solutions to the forward scattering (1.1) and (1.2). The proof of lemma 2.1 will be given in section A.4.

A.1. Regularity around a corner

Firstly, we investigate the regularity of a solution to the transmission problem of the Helmholtz equation in a right angle domain (see the figure 6).

Theorem A.1. *The solution $\hat{u}$ to (2.2) has the regularity $\hat{u} \in H^{1+s}(B_R) \cap H^{1+2/3}(\Omega_\ell)$ for any $0 \leqslant s < 1/2$ ($\ell = 1, 2$).*

Proof. For the sake of notational simplicity, we write $\varphi(\theta) := \varphi_j(\theta)$, $\eta := \eta_j$ for some fixed j. A general solution to (2.3) takes the form

$$\varphi(\theta) = \begin{cases} A^+ \cos(\eta\theta) + B^+ \sin(\eta\theta), & \theta \in (0, 3\pi/2), \\ A^- \cos(\eta\theta) + B^- \sin(\eta\theta), & \theta \in (-\pi/2, 0). \end{cases} \tag{A.1}$$

Using the transmission boundary conditions in (2.3) yields

$$A^+ = A^-, \quad A^+ \cos(3\pi\eta/2) + B^+ \sin(3\pi\eta/2) = A^- \cos(\pi\eta/2) - B^- \sin(\pi\eta/2).$$

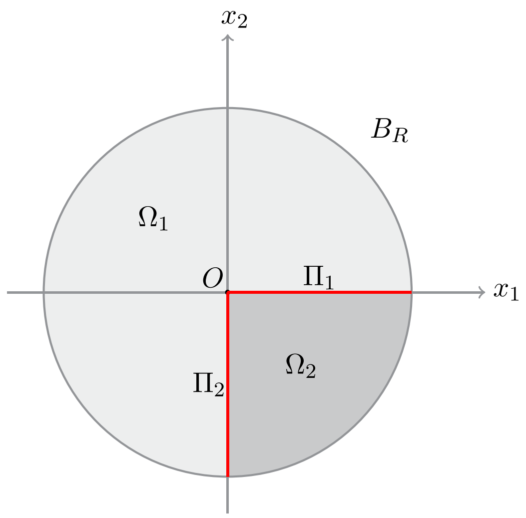

Figure 6. Sketch map of Ω_ℓ and Π_ℓ ($\ell = 1,2$).

Since

$$\varphi'(\theta) = \begin{cases} -\eta A^+ \sin(\eta\theta) + \eta B^+ \cos(\eta\theta), & \theta \in (0, 3\pi/2), \\ -\eta A^- \sin(\eta\theta) + \eta B^- \cos(\eta\theta), & \theta \in (-\pi/2, 0), \end{cases}$$

we have

$$B^+ = \lambda B^-, \quad -A^+ \sin(3\pi\eta/2) + B^+ \cos(3\pi\eta/2) = \lambda\left[A^- \sin(\pi\eta/2) + B^- \cos(\pi\eta/2)\right].$$

That is, (A^+, A^-, B^+, B^-) satisfies the following 4-by-4 algebraic system:

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ \cos(3\pi\eta/2) & -\cos(\pi\eta/2) & \sin(3\pi\eta/2) & \sin(\pi\eta/2) \\ 0 & 0 & 1 & -\lambda \\ \sin(3\pi\eta/2) & \lambda\sin(\pi\eta/2) & -\cos(3\pi\eta/2) & \lambda\cos(\pi\eta/2) \end{pmatrix} \begin{pmatrix} A^+ \\ A^- \\ B^+ \\ B^- \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

We denote the fourth order matrix on the left by M. Then simple calculation shows that

$$\begin{aligned} |M| &= \begin{vmatrix} 1 & -1 & 0 & 0 \\ \cos(3\pi\eta/2) & -\cos(\pi\eta/2) & \sin(3\pi\eta/2) & \sin(\pi\eta/2) \\ 0 & 0 & 1 & -\lambda \\ \sin(3\pi\eta/2) & \lambda\sin(\pi\eta/2) & -\cos(3\pi\eta/2) & \lambda\cos(\pi\eta/2) \end{vmatrix} \\ &= \begin{vmatrix} \cos(3\pi\eta/2) - \cos(\pi\eta/2) & \sin(3\pi\eta/2) & \sin(\pi\eta/2) \\ 0 & 1 & -\lambda \\ \lambda\sin(\pi\eta/2) + \sin(3\pi\eta/2) & -\cos(3\pi\eta/2) & \lambda\cos(\pi\eta/2) \end{vmatrix} \\ &= \begin{vmatrix} \cos(3\pi\eta/2) - \cos(\pi\eta/2) & 0 & \sin(\pi\eta/2) + \lambda\sin(3\pi\eta/2) \\ 0 & 1 & -\lambda \\ \lambda\sin(\pi\eta/2) + \sin(3\pi\eta/2) & 0 & \lambda\cos(\pi\eta/2) - \lambda\cos(3\pi\eta/2) \end{vmatrix} \\ &= \begin{vmatrix} \cos(3\pi\eta/2) - \cos(\pi\eta/2) & \sin(\pi\eta/2) + \lambda\sin(3\pi\eta/2) \\ \lambda\sin(\pi\eta/2) + \sin(3\pi\eta/2) & \lambda\cos(\pi\eta/2) - \lambda\cos(3\pi\eta/2) \end{vmatrix}. \end{aligned}$$

That is,

$$\begin{aligned}|M| &= -\lambda\left[\cos(3\pi\eta/2)-\cos(\pi\eta/2)\right]^2\\&\quad-\left[\lambda\sin(\pi\eta/2)+\sin(3\pi\eta/2)\right]\left[\sin(\pi\eta/2)+\lambda\sin(3\pi\eta/2)\right]\\&=2\lambda\cos(3\pi\eta/2)\cos(\pi\eta/2)-\left(\lambda^2+1\right)\sin(3\pi\eta/2)\sin(\pi\eta/2)-2\lambda\\&=(\lambda+1)^2\cos^2(\pi\eta)-\frac{(\lambda-1)^2}{2}\cos(\pi\eta)-\frac{\lambda^2+6\lambda+1}{2}=0,\end{aligned}$$

which implies that

$$\cos(\pi\eta)=-\frac{\lambda^2+6\lambda+1}{2(\lambda+1)^2}\quad\text{or}\quad\cos(\pi\eta)=1.$$

Hence,

$$\eta=\frac{1}{\pi}\arccos\left(-\frac{\lambda^2+6\lambda+1}{2(\lambda+1)^2}\right)\quad\text{or}\quad\eta=2l,\quad l\in\mathbb{N}.$$

Note that, $\eta\in(0,1)$ and

$$\frac{\lambda^2+6\lambda+1}{2(\lambda+1)^2}=\frac{(\lambda+1)^2+4\lambda}{2(\lambda+1)^2}=\frac{1}{2}+\frac{2\lambda}{(\lambda+1)^2}\in(1/2,1),\quad\text{i.e.}\ -1<\cos(\pi\eta)<-\frac{1}{2}.$$

Therefore,

$$\eta=\frac{1}{\pi}\arccos\left(-\frac{\lambda^2+6\lambda+1}{2(\lambda+1)^2}\right)>\frac{2}{3}.$$

The proof is complete. □

A.2. Regularity up to flat interface

In this subsection we suppose that the angle is π and consider the transmission problem

$$\begin{cases}\Delta v_\ell+k_\ell^2v_\ell=0, & \text{in}\quad\widetilde{\Omega}_\ell,\\ v_1=v_2,\quad\partial_\nu v_1=\lambda\partial_\nu v_2, & \text{on}\quad\widetilde{\Pi}_\ell,\end{cases}\tag{A.2}$$

where k_ℓ are constants and $k_1\neq k_2$, the unit normal vector ν at $\widetilde{\Pi}_\ell$ is pointing into $\widetilde{\Omega}_1$. The two semi-circles $\widetilde{\Omega}_\ell$ and their boundaries $\widetilde{\Pi}_\ell$ ($\ell=1,2$) are defined as (see the figure 7):

$$\widetilde{\Omega}_1:=\{(r,\theta):0<r<R,\ 0\leqslant\theta<\pi/2\text{ or }3\pi/2<\theta\leqslant2\pi\},\quad\widetilde{\Pi}_1:=\{(r,\pi/2):0\leqslant r\leqslant R\},$$
$$\widetilde{\Omega}_2:=\{(r,\theta):0<r<R,\ \pi/2<\theta<3\pi/2\},\quad\widetilde{\Pi}_2:=\{(r,3\pi/2):0\leqslant r\leqslant R\}.$$

In order to rewrite the equation (A.2) into the divergence form, we define

$$\tilde{a}(\theta):=\begin{cases}1, & \text{in }\widetilde{\Omega}_1,\\ \lambda, & \text{in }\widetilde{\Omega}_2,\end{cases}\qquad\tilde{\kappa}(\theta):=\begin{cases}k_1^2, & \text{in }\widetilde{\Omega}_1,\\ \lambda k_2^2, & \text{in }\widetilde{\Omega}_2,\end{cases}\qquad\tilde{v}(r,\theta):=\begin{cases}v_1, & \text{in }\widetilde{\Omega}_1,\\ v_2, & \text{in }\widetilde{\Omega}_2.\end{cases}$$

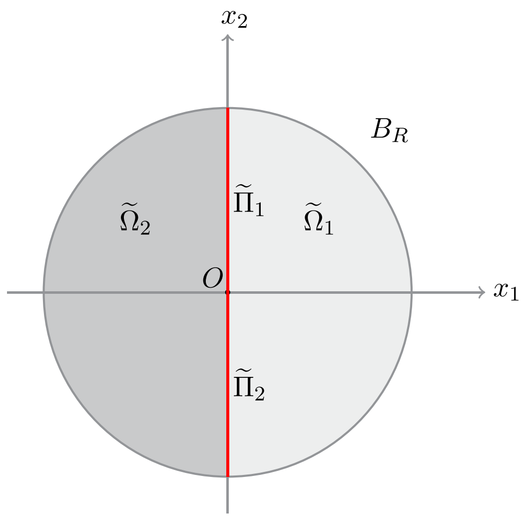

Figure 7. Sketch map of $\widetilde{\Omega}_\ell$ and $\widetilde{\Pi}_\ell$ ($\ell = 1, 2$).

Then (A.2) is equivalent to

$$\nabla \cdot (\tilde{a}(\theta) \nabla \tilde{v}) + \tilde{\kappa}(\theta) \tilde{v} = 0 \quad \text{in } B_R.$$

By the decomposition theorem, $\tilde{v} = \tilde{w} + \sum_{j=1}^{m} \tilde{c}_j r^{\delta_j} \phi_j(\theta)(\ln r)^{\tilde{p}_j}$ in B_R with $\tilde{p}_j \in \{0, 1, \ldots\}$. Here, $\tilde{w} \in H^2(\widetilde{\Omega}_\ell)$ ($\ell = 1, 2$), and $\delta_j \in (0, 1)$ are eigenvalues of the following positive definite Sturm–Liouville:

$$\begin{cases} \phi_j''(\theta) + \delta_j^2 \phi_j(\theta) = 0, & \text{in} \quad \theta \in [0, \pi/2) \cup (\pi/2, 3\pi/2) \cup (3\pi/2, 2\pi], \\ \phi_{j,+}(\pi/2) = \phi_{j,-}(\pi/2), & \phi'_{j,+}(\pi/2) = \lambda \phi'_{j,-}(\pi/2), \\ \phi_{j,+}(3\pi/2) = \phi_{j,-}(3\pi/2), & \phi'_{j,+}(3\pi/2) = \lambda \phi'_{j,-}(3\pi/2). \end{cases} \tag{A.3}$$

Here, $\phi_{j,+}$, $\phi'_{j,+}$ denote the limits from $\widetilde{\Omega}_1$ and $\phi_{j,-}$, $\phi'_{j,-}$ the limits from $\widetilde{\Omega}_2$.

Theorem A.2. *The solution $\tilde{v}$ to (A.2) has the regularity $\tilde{v} \in H^{1+s}(B_R) \cap H^2(\widetilde{\Omega}_\ell)$ for any $0 \leqslant s < 1/2$, and $\tilde{v}$ is analytic on the closure of $\widetilde{\Omega}_\ell$ ($\ell = 1, 2$).*

Proof. Write $\phi(\theta) := \phi_j(\theta)$, $\delta_j := \delta$ for some fixed j. A general solution to (A.3) takes the form

$$\phi(\theta) = \begin{cases} \tilde{A}^+ \cos(\delta\theta) + \tilde{B}^+ \sin(\delta\theta), & \theta \in [0, \pi/2) \cup (3\pi/2, 2\pi], \\ \tilde{A}^- \cos(\delta\theta) + \tilde{B}^- \sin(\delta\theta), & \theta \in (\pi/2, 3\pi/2). \end{cases}$$

Using the transmission boundary conditions in (A.3) yields

$$\begin{cases} \tilde{A}^+ \cos(\pi\delta/2) + \tilde{B}^+ \sin(\pi\delta/2) = \tilde{A}^- \cos(\pi\delta/2) + \tilde{B}^- \sin(\pi\delta/2), \\ \tilde{A}^+ \cos(3\pi\delta/2) + \tilde{B}^+ \sin(3\pi\delta/2) = \tilde{A}^- \cos(3\pi\delta/2) + \tilde{B}^- \sin(3\pi\delta/2). \end{cases}$$

Since

$$\phi'(\theta) = \begin{cases} -\delta\tilde{A}^+ \sin(\delta\theta) + \delta\tilde{B}^+ \cos(\delta\theta), & \theta \in [0, \pi/2) \cup (3\pi/2, 2\pi], \\ -\delta\tilde{A}^- \sin(\delta\theta) + \delta\tilde{B}^- \cos(\delta\theta), & \theta \in (\pi/2, 3\pi/2), \end{cases}$$

then we obtain that

$$\begin{cases} -\tilde{A}^+ \sin(\pi\delta/2) + \tilde{B}^+ \cos(\pi\delta/2) = \lambda\left[-\tilde{A}^- \sin(\pi\delta/2) + \tilde{B}^- \cos(\pi\delta/2)\right], \\ -\tilde{A}^+ \sin(3\pi\delta/2) + \tilde{B}^+ \cos(3\pi\delta/2) = \lambda\left[-\tilde{A}^- \sin(3\pi\delta/2) + \tilde{B}^- \cos(3\pi\delta/2)\right]. \end{cases}$$

That is, $(\tilde{A}^-, \tilde{B}^-, \tilde{A}^+, \tilde{B}^+)$ satisfies the following equation system:

$$\begin{pmatrix} \cos(\pi\delta/2) & \sin(\pi\delta/2) & -\cos(\pi\delta/2) & -\sin(\pi\delta/2) \\ \cos(3\pi\delta/2) & \sin(3\pi\delta/2) & -\cos(3\pi\delta/2) & -\sin(3\pi\delta/2) \\ -\lambda\sin(\pi\delta/2) & \lambda\cos(\pi\delta/2) & \sin(\pi\delta/2) & -\cos(\pi\delta/2) \\ -\lambda\sin(3\pi\delta/2) & \lambda\cos(3\pi\delta/2) & \sin(3\pi\delta/2) & -\cos(3\pi\delta/2) \end{pmatrix} \begin{pmatrix} \tilde{A}^- \\ \tilde{B}^- \\ \tilde{A}^+ \\ \tilde{B}^+ \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

We denote the fourth order matrix on the left by $\tilde{M}$. Then simple calculation shows that

$$\begin{aligned} |\tilde{M}| &= \begin{vmatrix} \cos(\pi\delta/2) & \sin(\pi\delta/2) & 0 & 0 \\ \cos(3\pi\delta/2) & \sin(3\pi\delta/2) & 0 & 0 \\ -\lambda\sin(\pi\delta/2) & \lambda\cos(\pi\delta/2) & (1-\lambda)\sin(\pi\delta/2) & (\lambda-1)\cos(\pi\delta/2) \\ -\lambda\sin(3\pi\delta/2) & \lambda\cos(3\pi\delta/2) & (1-\lambda)\sin(3\pi\delta/2) & (\lambda-1)\cos(3\pi\delta/2) \end{vmatrix} \\ &= -(\lambda-1)^2 \begin{vmatrix} \cos(\pi\delta/2) & \sin(\pi\delta/2) & 0 & 0 \\ \cos(3\pi\delta/2) & \sin(3\pi\delta/2) & 0 & 0 \\ 0 & 0 & \sin(\pi\delta/2) & \cos(\pi\delta/2) \\ 0 & 0 & \sin(3\pi\delta/2) & \cos(3\pi\delta/2) \end{vmatrix} \\ &= (\lambda-1)^2 \sin^2(\pi\delta) = 0. \end{aligned}$$

That is, $\sin(\pi\delta) = 0$ and then $\delta \in \mathbb{N}$, which implies that $\tilde{v} \in H^1(B_R) \cap H^2(\widetilde{\Omega}_\ell)$ and $\tilde{v}$ is analytic up to the boundary of $\widetilde{\Pi}_1 \cup \widetilde{\Pi}_2$. The proof is complete. □

A.3. Uniqueness and existence of forward scattering problem

Define the DtN mapping $T : H_\alpha^{1/2}(\Gamma_h) \to H_\alpha^{-1/2}(\Gamma_h)$ by

$$(Tf)(x_1) := \sum_{n\in\mathbb{Z}} i\beta_n f_n \mathrm{e}^{\mathrm{i}\alpha_n x_1}, \qquad \text{where } f(x_1) = \sum_{n\in\mathbb{Z}} f_n \mathrm{e}^{\mathrm{i}\alpha_n x_1} \in H_\alpha^{1/2}(\Gamma_h).$$

Introduce the piecewise analytic functions

$$a(x) := \begin{cases} 1 & \text{in} \quad S_h^+, \\ \lambda & \text{in} \quad S_h^-, \end{cases} \qquad \kappa(x) := \begin{cases} k_1^2 & \text{in} \quad S_h^+, \\ \lambda k_2^2 & \text{in} \quad S_h^-. \end{cases}$$

The scattering problem (1.1) and (1.2) can be equivalently formulated as the following divergence form in the truncated domain S_h:

$$\begin{cases} \nabla\cdot(a(x)\nabla u) + \kappa(x)u = 0, & \text{in} \quad S_h, \\ \partial_2 u = Tu + \left(\partial_2 u^i - Tu^i\right), & \text{on} \quad \Gamma_h, \\ u = 0, & \text{on} \quad \Gamma_0. \end{cases} \tag{A.4}$$

Theorem A.3. *The boundary value problem* (A.4) *has at least one solution* $u \in H_\alpha^1(S_h)$ *for any fixed* $h > \Lambda^+$. *Moreover, uniqueness remains true for any* $k_1, k_2 > 0$ *under the following monotonicity conditions on the medium:*

$$k_1^2 > \lambda k_2^2. \tag{A.5}$$

Proof. From the definition of T, it follows that for $f \in H_\alpha^{1/2}(\Gamma_h)$,

$$\mathrm{Re}\,\langle Tf,f\rangle = -\sum_{|\alpha_n|>k_1} |\beta_n|\,|f_n|^2 \leqslant 0, \quad \mathrm{Im}\,\langle Tf,f\rangle = \sum_{|\alpha_n|\leqslant k_1} |\beta_n\,|f_n|^2 \geqslant 0, \quad \text{(A.6)}$$

where the pair $\langle\cdot,\cdot\rangle$ denotes the duality between $H_\alpha^{-1/2}$ and $H_\alpha^{1/2}$ on Γ_h. The variational formulation for (A.4) can be written as: find $u \in H_\alpha^1(S_h)$ such that for all $v \in H_\alpha^1(S_h)$,

$$L(u,v) := \int_{S_h} [a(x)\nabla u \cdot \nabla \overline{v} - a(x)\kappa(x)u\overline{v}]\,\mathrm{d}x - \int_{\Gamma_h} Tu\overline{v}\,\mathrm{d}s = \int_{\Gamma_h}\left(Tu^i - \frac{\partial u^i}{\partial x_2}\right)\overline{v}\,\mathrm{d}s. \tag{A.7}$$

Using (A.6), one can conclude that the above sesquilinear form gives rise to a strongly elliptic operator $\mathcal{L}$ such that $L(u,v) = \langle \mathcal{L}u, v\rangle$ for all $u,v \in H_\alpha^{1/2}(S_h)$ (see also e.g. [5, 9]), where $\langle\cdot,\cdot\rangle$ denotes the inner product over the Hilbert space $H_\alpha^1(S_h)$. On the other hand, the adjoint of $\mathcal{L}$: $H_\alpha^1(S_h) \to H_\alpha^1(S_h)$ takes the explicit form

$$\langle \mathcal{L}^* u, v\rangle = \overline{L(v,u)} = \int_{S_h} [a(x)\nabla u \cdot \nabla \overline{v} - a(x)\kappa(x)u\overline{v}]\,\mathrm{d}x + 2\pi \sum_{n\in\mathbb{Z}} i\overline{\beta_n} u_n \overline{v}_n, \quad u,v \in H_\alpha^1(S_h).$$

Here, u_n and v_n denote the Fourier coefficients of $\mathrm{e}^{-\mathrm{i}\alpha x_1} u|_{\Gamma_h}$ and $\mathrm{e}^{-\mathrm{i}\alpha x_1} v|_{\Gamma_h}$, respectively. Taking the imaginary part on both sides of the previous identity with $v = u$ and using (A.6), we get $\sum_{|\alpha_n|\leqslant k_1} |\beta_n|\,|u_n|^2 = 0$ for $u \in \mathrm{Ker}(\mathcal{L}^*)$. This implies that

$$\int_{\Gamma_h}\left(Tu^i - \frac{\partial u^i}{\partial x_2}\right)\overline{v}\,\mathrm{d}s = 0 \quad \text{for all} \quad v \in \mathrm{Ker}(\mathcal{L}^*).$$

By Fredholm alternative, there always exists a solution $u \in H_\alpha^1(S_h)$ to (A.4).

To prove uniqueness, we suppose that $u^i \equiv 0$. Then u satisfies the upward Rayleigh expansion radiation condition. Taking the real part on both sides of (A.7) with $v = u$ and $u^i = 0$ and using (A.6), we obtain

$$I_1 := \int_{S_h} \left[a(x)|\nabla u|^2 - a(x)\kappa(x)|u|^2\right]\mathrm{d}x = -\sum_{|\alpha_n|>k_1} |\beta_n|\,|u_n|^2\,\mathrm{e}^{-2|\beta_n|h} \leqslant 0.$$

Multiplying the Helmholtz equation by $x_2\,\partial_2\overline{u}$ and integrating by part over $S_h^\pm$ yield the Rellich's identities:

$$\begin{aligned}
I^+ &= \left(\int_{\Gamma_h} - \int_{\Lambda}\right) x_2\left[-\nu_2|\nabla u|^2 + \nu_2 k_1^2|u|^2 + 2\mathrm{Re}\left(\partial_2\overline{u^+}\,\partial_\nu u^+\right)\right]\mathrm{d}s \\
&\quad + \int_{S_h^+} |\nabla u|^2 - k_1^2|u|^2 - 2|\partial_2 u|^2\,\mathrm{d}x = 0, \\
I^- &= \int_{\Lambda} x_2\left[-\nu_2|\nabla u|^2 + \nu_2 k_2^2|u|^2 + 2\mathrm{Re}\left(\partial_2\overline{u^-}\,\partial_\nu u^-\right)\right]\mathrm{d}s \\
&\quad + \int_{S_h^-} |\nabla u|^2 - k_1^2|u|^2 - 2|\partial_2 u|^2\,\mathrm{d}x = 0.
\end{aligned}$$

The integrand over Λ is well-defined because, for rectangular gratings it holds that $u \in H_\alpha^{3/2+\epsilon}(S_h^\pm)$ for some $\epsilon > 0$ depending on λ (see e.g. [20, chapter 2.4.3] and [9, section 3.3]). Straightforward calculations show that

$$\int_{\Gamma_h} x_2 \left[-\nu_2|\nabla u|^2 + \nu_2 k_1^2|u|^2 + 2\mathrm{Re}\left(\partial_2 \bar{u}\, \partial_\nu u\right)\right] \mathrm{d}s = h \sum_{|\alpha_n|\leqslant k_1} |\beta_n|\,|u_n|^2 = 0,$$

and

$$\begin{aligned} 0 &= I^+ + \lambda I^- \\ &= -\int_\Lambda \left[\lambda(\lambda-1)\,|\partial_\nu u^-|^2 + (\lambda-1)\,|\partial_\tau u^-|^2 + \left(k_1^2 - \lambda k_2^2\right)|u|^2\right]\nu_2 x_2\,\mathrm{d}s \\ &\quad 2\int_{S_h} a(x)\,|\partial_2 u|^2\,\mathrm{d}x + I_1, \end{aligned}$$

where ∂_τ denotes the tangential derivative on Λ with $\tau := (-\nu_2, \nu_1)$. By the assumptions (A.5) on k_1, k_2 and recalling the fact that $\nu_2 \geqslant 0$ on Λ, we conclude that the integral over Λ is non-positive, so that each term in the above expression vanishes. Consequently, we get $\partial_2 u \equiv 0$ in S_h and $I_1 = 0$, implying that $u_n = 0$ for all $|\alpha_n| > k_1$. Therefore,

$$u = A_n \mathrm{e}^{\mathrm{i}k_1 x_1} + A_m\,\mathrm{e}^{-\mathrm{i}k_1 x_1} \quad \text{in} \quad \Omega_\Lambda^+, \qquad A_n, A_m \in \mathbb{C},$$

if $\alpha_n = k_1$ or $\alpha_m = -k_1$ for some $n, m \in \mathbb{Z}$ (that is, Rayleigh frequencies occurs). Note that the above expression of u is well-defined in $\mathbb{R}^2$. Since $\nu_2 = 1$ on the line segment of Λ parallel to the x_1-axis and $k_1^2 > \lambda k_2^2$, one can also deduce from (A.8) that $u \equiv 0$ on this segment, which gives $A_n = A_m = 0$ and thus $u \equiv 0$.

□

A.4. Proof of lemma 2.1

Proof. Recalling the Taylor expansion of analytic solutions of the Helmholtz equation (see [7, 8]), we have

$$u_\ell(r,\theta) = \sum_{n,m\in\mathbb{N}:n+2m\geqslant 0} r^{n+2m}\left(a_{n,m}^{(\ell)}\cos(n\theta) + b_{n,m}^{(\ell)}\sin(n\theta)\right), \quad \text{for } 0 \leqslant r < R,$$

where the coefficients $a_{n,m}^{(\ell)}$ and $b_{n,m}^{(\ell)}$ fulfill the recurrence relations

$$a_{n,m+1}^{(\ell)} = \frac{-q_\ell}{4(m+1)(n+m+1)} a_{n,m}^{(\ell)},\ b_{n,m+1}^{(\ell)} = \frac{-q_\ell}{4(m+1)(n+m+1)} b_{n,m}^{(\ell)},\ \forall\, n,m \in \mathbb{N}. \tag{A.8}$$

The transmission conditions in (2.1) are equivalent to the four relations:

$$\sum_{n,m\in\mathbb{N}}^{n+2m=l} a_{n,m}^{(1)} = \sum_{n,m\in\mathbb{N}}^{n+2m=l} a_{n,m}^{(2)}, \qquad \sum_{n,m\in\mathbb{N}}^{n+2m=l} n b_{n,m}^{(1)} = \lambda \sum_{n,m\in\mathbb{N}}^{n+2m=l} n b_{n,m}^{(2)},$$

$$\sum_{n,m\in\mathbb{N}}^{n+2m=l}\left[a_{n,m}^{(1)}\cos(n\pi/2) - b_{n,m}^{(1)}\sin(n\pi/2)\right] = \sum_{n,m\in\mathbb{N}}^{n+2m=l}\left[a_{n,m}^{(2)}\cos(n\pi/2) - b_{n,m}^{(2)}\sin(n\pi/2)\right],$$

$$\sum_{n,m\in\mathbb{N}}^{n+2m=l} n\left[a_{n,m}^{(1)}\sin(n\pi/2) + b_{n,m}^{(1)}\cos(n\pi/2)\right] = \lambda \sum_{n,m\in\mathbb{N}}^{n+2m=l} n\left[a_{n,m}^{(2)}\sin(n\pi/2) + b_{n,m}^{(2)}\cos(n\pi/2)\right].$$

Case one: $n = 2k+1$ for some $k \in \mathbb{N}$. In this case the transmission conditions can be simplified to be

$$\begin{cases} \sum_{2k+1+2m=l} a^{(1)}_{2k+1,m} = \sum_{2k+1+2m=l} a^{(2)}_{2k+1,m}, \\ \sum_{2k+1+2m=l} (2k+1)(-1)^k a^{(1)}_{2k+1,m} = \lambda \sum_{2k+1+2m=l} (2k+1)(-1)^k a^{(2)}_{2k+1,m}, \end{cases} \tag{A.9}$$

$$\begin{cases} \sum_{2k+1+2m=l} (2k+1)\, b^{(1)}_{2k+1,m} = \lambda \sum_{2k+1+2m=l} (2k+1)\, b^{(2)}_{2k+1,m}, \\ \sum_{2k+1+2m=l} (-1)^k b^{(1)}_{2k+1,m} = \sum_{2k+1+2m=l} (-1)^k b^{(2)}_{2k+1,m}. \end{cases} \tag{A.10}$$

It suffices to show $a^{(\ell)}_{2k+1,m} = b^{(\ell)}_{2k+1,m} = 0$ for all $k, m \in \mathbb{N}$, $\ell = 1, 2$.

We first consider the case: $l = 2k+1+2m = 1$, that is $k = 0$, $m = 0$. From (A.9) and (A.10) we deduce that

$$a^{(1)}_{1,0} = a^{(2)}_{1,0}, \quad a^{(1)}_{1,0} = \lambda a^{(2)}_{1,0}; \qquad b^{(1)}_{1,0} = \lambda b^{(2)}_{1,0}, \quad b^{(1)}_{1,0} = b^{(2)}_{1,0}.$$

Since $\lambda \neq 1$, we obtain $a^{(1)}_{1,0} = a^{(2)}_{1,0} = b^{(1)}_{1,0} = b^{(2)}_{1,0} = 0$. By the recurrence relation (A.8), we have $a^{(\ell)}_{1,m} = b^{(\ell)}_{1,m} = 0$ for all $m \in \mathbb{N}$, $\ell = 1, 2$.

We carry out the proof by induction. Supposing for some $M \in \mathbb{N}$ that

$$a^{(1)}_{2k+1,m} = a^{(2)}_{2k+1,m} = 0, \quad b^{(1)}_{2k+1,m} = b^{(2)}_{2k+1,m} = 0, \quad \text{for } k \leqslant M, \quad k, m \in \mathbb{N}. \tag{A.11}$$

We need to prove the above relations in (A.11) with M replaced by $M+1$. For this purpose, it is sufficient to verify

$$a^{(1)}_{2M+3,0} = a^{(2)}_{2M+3,0} = 0, \quad b^{(1)}_{2M+3,0} = b^{(2)}_{2M+3,0} = 0.$$

Setting $l = 2k+1+2m = 2M+3$ in (A.9) and (A.10) and using the relations in (A.11), we obtain

$$a^{(1)}_{2M+3,0} = a^{(2)}_{2M+3,0}, \quad a^{(1)}_{2M+3,0} = \lambda a^{(2)}_{2M+3,0}; \qquad b^{(1)}_{2M+3,0} = \lambda b^{(2)}_{2M+3,0}, \quad b^{(1)}_{2M+3,0} = b^{(2)}_{2M+3,0}.$$

Again using $\lambda \neq 1$ yields $a^{(1)}_{2M+3,0} = a^{(2)}_{2M+3,0} = b^{(1)}_{2M+3,0} = b^{(2)}_{2M+3,0} = 0$. Consequently, we achieve that $a^{(\ell)}_{2k+1,m} = b^{(\ell)}_{2k+1,m} = 0$ for all $k, m \in \mathbb{N}$, $\ell = 1, 2$.

Case two: $n = 2k$ for $k \in \mathbb{N}$. It then follows from the transmission conditions that

$$\sum_{2k+2m=l} a^{(1)}_{2k,m} = \sum_{2k+2m=l} a^{(2)}_{2k,m}, \qquad \sum_{2k+2m=l} (-1)^k a^{(1)}_{2k,m} = \sum_{2k+2m=l} (-1)^k a^{(2)}_{2k,m}, \tag{A.12}$$

$$\sum_{2k+2m=l} k b^{(1)}_{2k,m} = \lambda \sum_{2k+2m=l} k b^{(2)}_{2k,m}, \qquad \sum_{2k+2m=l} (-1)^k k b^{(1)}_{2k,m} = \lambda \sum_{2k+2m=l} (-1)^k k b^{(2)}_{2k,m}. \tag{A.13}$$

Suppose $\tilde{l} := k+m = 0$, that is $k=0$, $m=0$. From the relation (A.12), we obtain $a_{0,0}^{(1)} = a_{0,0}^{(2)}$. Then we set $\tilde{l} = k+m = 1$ in (A.12) and (A.13), that is $k=1$, $m=0$ or $k=0$, $m=1$. This gives the relations $b_{2,0}^{(1)} = \lambda b_{2,0}^{(2)}$ and

$$a_{2,0}^{(1)} + a_{0,1}^{(1)} = a_{2,0}^{(2)} + a_{0,1}^{(2)}, \quad -a_{2,0}^{(1)} + a_{0,1}^{(1)} = -a_{2,0}^{(2)} + a_{0,1}^{(2)},$$

which imply that $a_{0,1}^{(1)} = a_{0,1}^{(2)}$ and $a_{2,0}^{(1)} = a_{2,0}^{(2)}$. Since $a_{0,0}^{(1)} = a_{0,0}^{(2)}$, $a_{0,1}^{(1)} = a_{0,1}^{(2)}$, $a_{0,1}^{(\ell)} = -\frac{q_\ell}{4} a_{0,0}^{(\ell)}$ and $q_1 \neq q_2$, we obtain that

$$a_{0,m}^{(1)} = a_{0,m}^{(2)} = 0, \quad \forall\, m \in \mathbb{N}.$$

Set $\tilde{l} = k+m = 2$ in (A.12) and (A.13), that is $k=2$, $m=0$ or $k=1$, $m=1$ or $k=0$, $m=2$, we have

$$\begin{cases} a_{4,0}^{(1)} + a_{2,1}^{(1)} = a_{4,0}^{(2)} + a_{2,1}^{(2)}, \\ a_{4,0}^{(1)} - a_{2,1}^{(1)} = a_{4,0}^{(2)} - a_{2,1}^{(2)}, \end{cases} \qquad \begin{cases} 2b_{4,0}^{(1)} + b_{2,1}^{(1)} = \lambda \left(2b_{4,0}^{(2)} + b_{2,1}^{(2)}\right), \\ 2b_{4,0}^{(1)} - b_{2,1}^{(1)} = \lambda \left(2b_{4,0}^{(2)} - b_{2,1}^{(2)}\right), \end{cases}$$

which lead to that

$$a_{4,0}^{(1)} = a_{4,0}^{(2)}, \quad a_{2,1}^{(1)} = a_{2,1}^{(2)}; \qquad b_{4,0}^{(1)} = \lambda b_{4,0}^{(2)}, \quad b_{2,1}^{(1)} = \lambda b_{2,1}^{(2)}.$$

Since $a_{2,0}^{(1)} = a_{2,0}^{(2)}$, $a_{2,1}^{(1)} = a_{2,1}^{(2)}$, $a_{2,1}^{(\ell)} = -\frac{q_\ell}{12} a_{2,0}^{(\ell)}$ and $q_2 \neq q_1$, we conclude that

$$a_{2,m}^{(1)} = a_{2,m}^{(2)} = 0, \quad \forall\, m \in \mathbb{N}.$$

Since $b_{2,0}^{(1)} = \lambda b_{2,0}^{(2)}$, $b_{2,1}^{(1)} = \lambda b_{2,1}^{(2)}$ and $b_{2,1}^{(\ell)} = -\frac{q_\ell}{12} b_{2,0}^{(\ell)}$, we arrive at

$$0 = b_{2,1}^{(1)} - \lambda b_{2,1}^{(2)} = -\frac{q_1}{12} b_{2,0}^{(1)} + \lambda \frac{q_2}{12} b_{2,0}^{(2)} = \lambda \frac{q_2 - q_1}{12} b_{2,0}^{(2)}.$$

That is $b_{2,0}^{(2)} = 0$ for $q_2 \neq q_1$, $\lambda \neq 0$. By the recurrence relation (A.8), we conclude

$$b_{2,m}^{(1)} = b_{2,m}^{(2)} = 0, \quad \forall\, m \in \mathbb{N}.$$

We shall finish the proof by induction. Supposing for some $M \in \mathbb{N}$ that

$$a_{2k-2,m}^{(1)} = a_{2k-2,m}^{(2)} = 0, \quad a_{2M,0}^{(1)} = a_{2M,0}^{(2)}, \quad \text{for } 1 \leqslant k \leqslant M, \quad m \in \mathbb{N}; \quad \text{(A.14)}$$

$$b_{2k-2,m}^{(1)} = b_{2k-2,m}^{(2)} = 0, \quad b_{2M,0}^{(1)} = \lambda b_{2M,0}^{(2)}, \quad \text{for } 1 \leqslant k \leqslant M, \quad m \in \mathbb{N}. \quad \text{(A.15)}$$

We need to prove all relations in (A.14) and (A.15) with M replaced by $M+1$. For this purpose, it is sufficient to verify

$$a_{2M,0}^{(1)} = a_{2M,0}^{(2)} = 0, \quad a_{2(M+1),0}^{(1)} = a_{2(M+1),0}^{(2)}; \qquad b_{2M,0}^{(1)} = b_{2M,0}^{(2)} = 0, \quad b_{2M+2,0}^{(1)} = \lambda b_{2M+2,0}^{(2)}.$$

Setting $\tilde{l} = k+m = M+1$ in (A.12) and using (A.14), we obtain

$$a_{2(M+1),0}^{(1)} + a_{2M,1}^{(1)} = a_{2(M+1),0}^{(2)} + a_{2M,1}^{(2)}, \qquad a_{2(M+1),0}^{(1)} - a_{2M,1}^{(1)} = a_{2(M+1),0}^{(2)} - a_{2M,1}^{(2)}.$$

That is, $a^{(1)}_{2(M+1),0} = a^{(2)}_{2(M+1),0}$ and $a^{(1)}_{2M,1} = a^{(2)}_{2M,1}$. Since $a^{(1)}_{2M,1} = a^{(2)}_{2M,1}$, $a^{(1)}_{2M,0} = a^{(2)}_{2M,0}$, $a^{(\ell)}_{2M,1} = \frac{-q_\ell}{4(2M+1)} a^{(\ell)}_{2M,0}$ and $q_1 \neq q_2$, it follows that $a^{(1)}_{2M,0} = a^{(2)}_{2M,0} = 0$. Similarly, setting $\tilde{l} = k + m = M + 1$ in (A.13) and using (A.15) will lead to $b^{(1)}_{2(M+1),0} = \lambda b^{(2)}_{2(M+1),0}$ and $b^{(1)}_{2M,0} = b^{(2)}_{2M,0} = 0$.

□

ORCID iDs

Jianli Xiang https://orcid.org/0000-0002-8585-6793
Guanghui Hu https://orcid.org/0000-0002-8485-9896

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