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ScienceDirect

J. Differential Equations 257 (2014) 469–489

Journal of Differential Equations

www.elsevier.com/locate/jde

Recovering complex elastic scatterers by a single far-field pattern

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Received 5 December 2013

Available online 29 April 2014

Abstract

We consider the inverse scattering problem of reconstructing multiple impenetrable bodies embedded in an unbounded, homogeneous and isotropic elastic medium. The inverse problem is nonlinear and ill-posed. Our study is conducted in an extremely general and practical setting: the number of scatterers is unknown in advance; and each scatterer could be either a rigid body or a cavity which is not required to be known in advance; and moreover there might be components of multiscale sizes presented simultaneously. We develop several locating schemes by making use of only a single far-field pattern, which is widely known to be challenging in the literature. The inverse scattering schemes are of a totally "direct" nature without any inversion involved. For the recovery of multiple small scatterers, the nonlinear inverse problem is linearized and to that end, we derive sharp asymptotic expansion of the elastic far-field pattern in terms of the relative size of the cavities. The asymptotic expansion is based on the boundary-layer-potential technique and the result obtained is of significant mathematical interest for its own sake. The recovery of regular-size/extended scatterers is based on projecting the measured far-field pattern into an admissible solution space. With a local tuning technique, we can further recover multiple multiscale elastic scatterers.

MSC: primary 74J20, 74J25; secondary 35Q74, 35R30

Keywords: Inverse elastic scattering; Multiscale scatterers; Asymptotic estimate; Indicator functions; Locating

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http://dx.doi.org/10.1016/j.jde.2014.04.007
0022-0396/

1. Introduction

This work concerns the time-harmonic elastic scattering from cavities (e.g., empty or fluid-filled cracks and inclusions) and rigid bodies, which has its origin in industrial and engineering applications; see, e.g., [30,9,21,22] and the references therein. In seismology and geophysics, it is important to understand how anomalies diffract the detecting elastic waves and to characterize them from the surface measurement data. This leads to the inverse problem of determining the position and shape of an elastic scatterer; see, e.g., [1,12,13]. The inverse elastic scattering problem also plays a key role in many other science and technology such as petroleum and mine exploration, nondestructive testing of concrete structures etc. The inverse problem is nonlinear and ill-posed and far from well understood. In this work, we shall develop several qualitative inverse elastic scattering schemes in an extremely general and practical scenario. In what follows, we first present the mathematical formulations of the forward and inverse elastic scattering problems for the present study, and then we briefly discuss the results obtained.

Consider a time-harmonic elastic plane wave $u^{in}(x)$, $x \in \mathbb{R}^3$ (with the time variation of the form $e^{-i\omega t}$ being factorized out, where $\omega \in \mathbb{R}_+$ denotes the frequency) impinging on a scatterer $D \subset \mathbb{R}^3$ embedded in an infinite isotropic and homogeneous elastic medium in $\mathbb{R}^3$. The incident elastic plane wave is of the following general form

$$u^{in}(x) = u^{in}\big(x; d, d^\perp, \alpha, \beta, \omega\big) = \alpha d e^{ik_p x\cdot d} + \beta d^\perp e^{ik_s x\cdot d}, \quad \alpha, \beta \in \mathbb{C}, \tag{1.1}$$

where $d \in \mathbb{S}^2 := \{x \in \mathbb{R}^3 : |x| = 1\}$, is the impinging direction, $d^\perp \in \mathbb{S}^2$ satisfying $d^\perp \cdot d = 0$ denotes the polarization direction; and $k_s := \omega/\sqrt{\mu}$, $k_p := \omega/\sqrt{\lambda + 2\mu}$ denote the shear and compressional wave numbers, respectively. If $\alpha = 1, \beta = 0$ for u^{in} in (1.1), then $u^{in} = u_p^{in} := de^{ik_p x\cdot d}$ is the (normalized) plane pressure wave; and if $\alpha = 0, \beta = 1$ for u^{in} in (1.1), then $u^{in} = u_s^{in} := d^\perp e^{ik_s x\cdot d}$ is the (normalized) plane shear wave. Let $u(x) \in \mathbb{C}^3$, $x \in \mathbb{R}^3\backslash\overline{D}$ denote the total displacement field, and define the linearized strain tensor by

$$\varepsilon(u) := \frac{1}{2}\big(\nabla u + \nabla u^\top\big) \in \mathbb{C}^{3\times 3}, \tag{1.2}$$

where ∇u and $\nabla u^\top$ stand for the Jacobian matrix of u and its adjoint, respectively. By Hooke's law the strain tensor is related to the stress tensor via the identity

$$\sigma(u) = \lambda(\operatorname{div} u)\mathbf{I} + 2\mu\varepsilon(u) \in \mathbb{C}^{3\times 3} \tag{1.3}$$

with the Lamé constants λ, μ satisfying $\mu > 0$ and $3\lambda + 2\mu > 0$. Here and in what follows, $\mathbf{I}$ denotes the 3×3 identity matrix. The surface traction (or the stress operator) on ∂D is defined as

$$Tu = T_\nu u := \nu \cdot \sigma(u) = (2\mu\nu \cdot \operatorname{grad} + \lambda\nu \operatorname{div} + \mu\nu \times \operatorname{curl})u, \tag{1.4}$$

where ν denotes the unit normal vector to ∂D pointing into $\mathbb{R}^3\backslash\overline{D}$. We suppose that $D \subset \mathbb{R}^3$ is a bounded C^2 domain such that $\mathbb{R}^3\backslash\overline{D}$ is connected. For the subsequent use, we also introduce $Ru := u$. In the present study, the elastic body D is supposed to be either a cavity or a rigid body for which u satisfies the following boundary condition

$$\mathcal{B}u = 0 \quad \text{on } \partial D,\ \mathcal{B} = T \text{ or } R. \tag{1.5}$$

If D is a cavity, then $\mathcal{B} = T$ in (1.5), and if D is a *rigid* body, then $\mathcal{B} = R$.

In the exterior of D, the propagation of elastic waves is governed by the following reduced Navier equation (or Lamé system)

$$\left(\Delta^* + \omega^2\right)u = 0 \quad \text{in } \mathbb{R}^3\backslash\overline{D},\ \Delta^* := \mu\Delta + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div} \tag{1.6}$$

where we note that the density of the background elastic medium has been normalized to be one.

Define $u^{sc} := u - u^{in}$ to be the *scattered wave*, which can be easily verified to satisfy the Navier equation (1.6) as well. u^{sc} can be decomposed into the sum

$$u^{sc} := u_p^{sc} + u_s^{sc}, \quad u_p^{sc} := -\frac{1}{k_p^2}\,\mathrm{grad}\,\mathrm{div}\,u^{sc},\ u_s^{sc} := \frac{1}{k_s^2}\,\mathrm{curl}\,\mathrm{curl}\,u^{sc},$$

where the vector functions u_p^{sc} and u_s^{sc} are referred to as the pressure (longitudinal) and shear (transversal) parts of u^{sc}, respectively, satisfying

$$\begin{aligned} &\left(\Delta + k_p^2\right)u_p^{sc} = 0, \qquad \mathrm{curl}\,u_p^{sc} = 0, \quad \text{in } \mathbb{R}^3\backslash\overline{D},\\ &\left(\Delta + k_s^2\right)u_s^{sc} = 0, \qquad \mathrm{div}\,u_s^{sc} = 0, \quad \text{in } \mathbb{R}^3\backslash\overline{D}. \end{aligned}$$

Moreover, the scattered field u^{sc} is required to satisfy Kupradze's radiation conditions (see, e.g. [3])

$$\lim_{r\to\infty}\left(\frac{\partial u_p^{sc}}{\partial r} - ik_p u_p^{sc}\right) = 0, \qquad \lim_{r\to\infty}\left(\frac{\partial u_s^{sc}}{\partial r} - ik_s u_s^{sc}\right) = 0, \quad r = |x|, \tag{1.7}$$

uniformly in all directions $\hat{x} = x/|x| \in \mathbb{S}^2$. The radiation conditions in (1.7) lead to the P-part (longitudinal part) u_p^∞ and the S-part (transversal part) u_s^∞ of the far-field pattern of u^{sc}, which can be read off from the large $|x|$ asymptotics (after some normalization)

$$u^{sc}(x) = \frac{\exp(ik_p|x|)}{4\pi(\lambda+\mu)|x|}u_p^\infty(\hat{x}) + \frac{\exp(ik_s|x|)}{4\pi\mu|x|}u_s^\infty(\hat{x}) + \mathcal{O}\left(\frac{1}{|x|^2}\right), \quad |x| \to +\infty. \tag{1.8}$$

$u_p^\infty(\hat{x})$ and $u_s^\infty(\hat{x})$ are also known as the far-field patterns of u_p^{sc} and u_s^{sc}, respectively. In this work, we define the *full far-field pattern* u^∞ of the scattered field u^{sc} as the sum of u_p^∞ and u_s^∞; that is,

$$u^\infty(\hat{x}) := u_p^\infty(\hat{x}) + u_s^\infty(\hat{x}). \tag{1.9}$$

Since $u_p^\infty(\hat{x})$ is normal to $\mathbb{S}^2$ and $u_s^\infty(\hat{x})$ is tangential to $\mathbb{S}^2$, there holds

$$u_p^\infty(\hat{x}) = \left(u^\infty(\hat{x})\cdot\hat{x}\right)\hat{x}, \qquad u_s^\infty(\hat{x}) = \hat{x}\times u^\infty(\hat{x})\times\hat{x}.$$

The direct elastic scattering problem (1.1)–(1.8) is well understood and particularly there admits a unique solution $u \in C^2(\mathbb{R}^3\backslash\overline{D})^3 \cap C^1(\mathbb{R}^3\backslash D)^3$ (cf. [21]).

Throughout the rest of the paper, $u_\tau^\infty(\hat{x})$ with $\tau = \emptyset$ signifies the full far-field pattern defined in (1.9). We shall also write $u_\tau^\infty(\hat{x}; D, d, d^\perp, \alpha, \beta, \omega)$ ($\tau = p, s$ or $\emptyset$) to signify the dependence of the far-field pattern on the scatterer D and the detecting incident plane wave u^{in}. The inverse elastic scattering problem concerns the recovery of D from knowledge (i.e., the measurement) of the far-field pattern $u_\tau^\infty(\hat{x}; d, d^\perp, \alpha, \beta, \omega)$ ($\tau = p, s$ or $\emptyset$). If one introduces an abstract operator $\mathcal{F}$ (defined by the elastic scattering system described earlier) which sends the scatterer D to the corresponding far-field pattern u_τ^∞, then the inverse problem can be formulated as the following operator equation

$$\mathcal{F}(D) = u_\tau^\infty\big(\hat{x}; d, d^\perp, \alpha, \beta, \omega\big). \tag{1.10}$$

It is easily verified that (1.10) is *nonlinear*, and moreover it is widely known to be *ill-posed* in the Hadamard sense. For the measurement data $u_\tau^\infty(\hat{x}; d, d^\perp, \alpha, \beta, \omega)$ in (1.10), we always assume that they are collected for all $\hat{x} \in \mathbb{S}^2$. On the other hand, it is remarked that u_τ is a real-analytic function on $\mathbb{S}^2$, and hence if it is known on any open portion of $\mathbb{S}^2$, then it is known on the whole sphere by analytic continuation. Moreover, if the data set is given for a single quintuplet of $(d, d^\perp, \alpha, \beta, \omega)$, then it is called *a single far-field pattern*, otherwise it is called *multiple far-field patterns*. Physically, a single far-field pattern can be obtained by sending a single incident plane wave and then measuring the scattered wave field far away in every possible observation direction.

Due to its practical importance, the inverse elastic scattering problem has been extensively studied in the literature. We refer to the theoretical uniqueness results proved in [16,25–29], and numerical reconstruction schemes developed in [2,7,5,4,6,14,18]. However, one usually needs multiple or even infinitely many far-field patterns. Based on the reflection principle for the Navier system under the third or fourth kind boundary conditions, a global uniqueness with a single far-field pattern was shown in [11] for bounded impenetrable elastic bodies of polyhedral type. However, the uniqueness proof there does not apply to the more practical case of rigid or traction-free bodies.

In this work, we shall consider the inverse problem (1.10) with a single measurement of the P-part far-field pattern u_p^∞, or the S-part far-field pattern u_s^∞, or the full far-field pattern u^∞. The inverse problem is formally posed with a single far-field pattern. Moreover, we shall consider our study in an extremely general and practical setting. The number of scatterers is unknown in advance, and each scatterer could be rigid or traction-free which is not required to be known in advance either. Furthermore, there might be multiscale components presented simultaneously. Here, the size of an elastic scatterer is interpreted in terms of the detecting wavelength. We develop several qualitative locating schemes by making use of only a single far-field pattern. The inverse scattering schemes are of a totally "direct" nature without any inversion involved. The present work significantly extends our recent study in [19], where the locating of only rigid bodies was considered. For the recovery of multiple small scatterers, the nonlinear inverse problem (1.10) would be linearized and to that end, we derive sharp asymptotic expansion of the corresponding scattered wave filed from multiple small traction-free cavities. The asymptotic expansion is based on the boundary-layer-potential technique and the result obtained is of significant mathematical interest for its own sake. Indeed, the asymptotic scattering estimates from small acoustic and electromagnetic bodies are of critical importance in the corresponding study of regularized approximate invisibility cloaking of acoustic and electromagnetic waves; see [23, 24,8]. Our result on the scattering estimate from traction-free bodies shall find important application in regularized approximate cloaking of elastic waves. The recovery of regular-size/extended scatterers is based on projecting the measured far-field pattern into an admissible solution space. With a local tuning technique, we can further recover multiple multiscale elastic bodies.

The rest of the paper is organized as follows. In Section 2, we derive the asymptotic expansion of the scattered wave field from multiple small elastic scatterers. In Section 3, we develop the inverse scattering schemes of locating multiple small, extended and multiscale elastic scatterers with the corresponding theoretical justifications. The paper is concluded in Section 4 with some remarks.

2. Elastic scattering from multiscale scatterers

In this section, we consider the elastic scattering from multiple multiscale scatterers. To that end, we first recall the fundamental solution (Green's tensor) to the Navier equation (1.6) given by

$$\begin{aligned}\Pi(x,y)&=\Pi^{(\omega)}(x,y)\\&=\frac{k_s^2}{4\pi\omega^2}\frac{e^{ik_s|x-y|}}{|x-y|}\mathbf{I}+\frac{1}{4\pi\omega^2}\operatorname{grad}_x\operatorname{grad}_x^\top\left[\frac{e^{ik_s|x-y|}}{|x-y|}-\frac{e^{ik_p|x-y|}}{|x-y|}\right],\end{aligned}\tag{2.1}$$

for $x,y\in\mathbb{R}^3$, $x\neq y$. Let D_j, $j\in\mathbb{N}$ be a bounded simply connected domain in $\mathbb{R}^3$ with C^2 boundary ∂D_j. Define the single and double layer potential operators, respectively, by

$$(S_j\varphi)(x)=(S_{D_j}\varphi)(x):=2\int_{\partial D_j}\Pi(x,y)\varphi(y)\,ds(y),\quad \varphi\in C(\partial D_j),\quad x\in\partial D_j,\tag{2.2}$$

$$(K_j\varphi)(x)=(K_{D_j}\varphi)(x):=2\int_{\partial D_j}\frac{\partial\Pi(x,y)}{\partial\nu(y)}\varphi(y)\,ds(y),\quad \varphi\in C(\partial D_j),\quad x\in\partial D_j,\tag{2.3}$$

where $\partial_{\nu(y)}\Pi(x,y)$ is a matrix function whose l-th column vector is given by

$$\left[\frac{\partial\Pi(x,y)}{\partial\nu(y)}\right]^\top\mathbf{e}_l=T_{\nu(y)}\big[\Pi(x,y)\mathbf{e}_l\big]=\nu(y)\cdot\big[\sigma\big(\Pi(x,y)\mathbf{e}_l\big)\big]\quad\text{on }\partial D_j,$$

for $x\neq y$, $l=1,2,3$. Here, $\mathbf{e}_l$, $1\le l\le 3$ are the standard Euclidean base vectors in $\mathbb{R}^3$, and $T_{\nu(y)}$ is the stress operator defined in (1.4). The adjoint operator K_j' of K_j is given by

$$\big(K_j'\varphi\big)(x)=\big(K_{D_j}'\varphi\big)(x):=2\int_{\partial D_j}\frac{\partial\Pi(x,y)}{\partial\nu(x)}\varphi(y)\,ds(y),\quad \varphi\in C(\partial D_j),\ x\in\partial D_j.\tag{2.4}$$

As seen by interchanging the order of integration, K_j' and K_j are adjoint with respect to the dual system $(C(\partial D_j),C(\partial D_j))$ defined by

$$(f,g):=\int_{\partial D_j}fg\,ds,\quad f,g\in C(\partial D_j).$$

Using Taylor series expansion for exponential functions, one can rewrite the matrix $\Pi^{(\omega)}(x, y)$ as the series (see, e.g., [6])

$$\Pi^{(\omega)}(x,y) = \frac{1}{4\pi}\sum_{n=0}^{\infty}\frac{(n+1)(\lambda+2\mu)+\mu}{\mu(\lambda+2\mu)}\frac{(i\omega)^n}{(n+2)n!}|x-y|^{n-1}\mathbf{I} - \frac{1}{4\pi}\sum_{n=0}^{\infty}\frac{\lambda+\mu}{\mu(\lambda+2\mu)}\frac{(i\omega)^n(n-1)}{(n+2)n!}|x-y|^{n-3}(x-y)\otimes(x-y), \tag{2.5}$$

from which it follows that

$$\Pi^{(\omega)}(x,y) = \frac{\lambda+3\mu}{8\pi\mu(\lambda+2\mu)}\frac{1}{|x-y|}\mathbf{I} + i\omega\frac{2\lambda+5\mu}{12\pi\mu(\lambda+2\mu)}\mathbf{I} + \frac{\lambda+\mu}{8\pi\mu(\lambda+2\mu)}\frac{1}{|x-y|^3}(x-y)\otimes(x-y) + o(1)\omega^2 \tag{2.6}$$

as $x \to y$. Taking $\omega \to +0$ in (2.6), we obtain the fundamental tensor of the Lamé system with $\omega = 0$

$$\widetilde{\Pi}(x,y) = \Pi^{(0)}(x,y) := \frac{\lambda+3\mu}{8\pi\mu(\lambda+2\mu)}\frac{1}{|x-y|}\mathbf{I} + \frac{\lambda+3\mu}{8\pi\mu(\lambda+2\mu)}\frac{1}{|x-y|^3}(x-y)\otimes(x-y). \tag{2.7}$$

Similar to the definitions of S_j, K_j, K'_j, we define the operators $\widetilde{S}_j$, $\widetilde{K}_j$, $\widetilde{K}'_j$ in the same way as (2.2), (2.3) and (2.4), but with the tensor $\Pi^{(\omega)}(x, y)$ replaced by $\Pi^{(0)}(x, y)$. By comparing (2.5), (2.6) and (2.7), we obtain

$$\Pi^{(\omega)}(x,y) - \Pi^{(0)}(x,y) = i\omega\frac{2\lambda+5\mu}{12\pi\mu(\lambda+2\mu)}\mathbf{I} + \omega^2\mathcal{O}\left(|x-y|\mathbf{I} - \frac{(x-y)\otimes(x-y)}{|x-y|}\right),$$

which together with (1.3) yields

$$\left|\sigma\left(\left[\Pi^{(\omega)}(x,y) - \Pi^{(0)}(x,y)\right]\mathbf{e}_j\right)\right| \le C(\lambda,\mu)\omega^2, \quad C > 0, \tag{2.8}$$

uniformly in $j = 1, 2, 3$ as $x \to y$. The operators K_j, K'_j, $\widetilde{K}_j$ and $\widetilde{K}'_j$ all have weakly singular kernels and therefore are compact on $C(\partial D_j)$; see, e.g., [17,21].

Throughout the rest of the paper, in order to simplify the exposition, we shall assume that $\omega \sim 1$. Hence, the size of a scatterer can be interpreted in terms of its Euclidean diameter. Next, we first consider the scattering from multiple sparsely distributed scatterers. Let $l \in \mathbb{N}$ and D_j, $1 \le j \le l$ be bounded simply-connected domains in $\mathbb{R}^3$ with C^2-smooth boundaries. Set

$$D = \bigcup_{j=1}^{l} D_j \quad \text{and} \quad L = \min_{j\neq j', 1\le j,j'\le l} \operatorname{dist}(\overline{D}_j, \overline{D}_{j'}). \tag{2.9}$$

Lemma 2.1. *Consider an elastic scatterer with multiple components given in (2.9), where each component D_j, $1 \le j \le l$ is either traction-free or rigid. For L sufficiently large, we have*

$$u^\infty(\hat{x}; D) = \sum_{j=1}^{l} u^\infty(\hat{x}; D_j) + \mathcal{O}(L^{-1}). \tag{2.10}$$

Proof. The case that all the components of D are rigid was considered in [19]. In what follows, for simplicity we first assume that $l = 2$, and moreover we assume that both D_1 and D_2 are traction-free and ω^2 is not an eigenvalue for $-\Delta^*$ in D_j associated with the homogeneous traction-free boundary condition on ∂D_j, $j = 1, 2$.

The scattered field $u^{sc}(x; D_j)$ corresponding to D_j can be represented as the single layer potential

$$u^{sc}(x; D_j) = \int_{\partial D_j} \Pi(x, y)\varphi_j(y)\, ds(y), \quad x \in \mathbb{R}^3 \backslash \overline{D}_j,$$

where the density function $\varphi_j \in C(\partial D_j)$ is uniquely determined from the traction-free boundary condition on ∂D_j, and is implied in the boundary integral equation

$$\varphi_j = 2\big(I - K_j'\big)^{-1}\big(Tu^{in}|_{\partial D_j}\big), \quad j = 1, 2.$$

The uniqueness and existence of φ_j follow from the Fredholm alternative applied to the operator $I - K_j'$. To prove the lemma for the scatterer $D = D_1 \cup D_2$, we make the ansatz

$$u^{sc}(x; D) = \sum_{j=1,2} \left\{ \int_{\partial D_j} \Pi(x, y)\phi_j(y)\, ds(y) \right\}, \quad x \in \mathbb{R}^3 \backslash \overline{D},$$

with $\phi_j \in C(\partial D_j)$. By using the boundary condition $T(u^{sc} + u^{in}) = 0$ on each ∂D_j, we obtain the system of integral equations

$$\begin{pmatrix} I - K_1' & J_2 \\ J_1 & I - K_2' \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = 2 \begin{pmatrix} Tu^{in}|_{\partial D_1} \\ Tu^{in}|_{\partial D_2} \end{pmatrix}, \tag{2.11}$$

where the operators $J_1 : C(\partial D_1) \to C(\partial D_2)$, $J_2 : C(\partial D_2) \to C(\partial D_1)$ are defined respectively by

$$(J_1\phi_1)(x) := -2 \int_{\partial D_1} \big[T_{\nu(x)}\Pi(x, y)\big]\phi_1(y)\, ds(y), \quad x \in \partial D_2,$$

$$(J_2\phi_2)(x) := -2 \int_{\partial D_2} \big[T_{\nu(x)}\Pi(x, y)\big]\phi_2(y)\, ds(y), \quad x \in \partial D_1.$$

Since $L \gg 1$, using the fundamental solution (2.1) one readily estimates

$$\|J_1\phi_1\|_{C(\partial D_2)} \le C_1 L^{-1}\|\phi_1\|_{C(\partial D_1)}, \qquad \|J_2\phi_2\|_{C(\partial D_1)} \le C_2 L^{-1}\|\phi_2\|_{C(\partial D_2)}, \quad C_1, C_2 > 0.$$

Hence, it follows from (2.11) and the invertibility of $I - K'_j : C(\partial D_j) \to C(\partial D_j)$ that

$$\begin{aligned}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} &= \begin{pmatrix}(I-K'_1)^{-1} & 0\\ 0 & (I-K'_2)^{-1}\end{pmatrix}\begin{pmatrix}2Tu^{in}|_{\partial D_1}\\ 2Tu^{in}|_{\partial D_2}\end{pmatrix} + \mathcal{O}\big(L^{-1}\big)\\ &= \begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} + \mathcal{O}\big(L^{-1}\big).\end{aligned}$$

This implies that

$$u^{sc}(x; D) = u^{sc}(x; D_1) + u^{sc}(x; D_2) + \mathcal{O}\big(L^{-1}\big),$$

which further leads to

$$u^{\infty}(\hat{x}; D) = u^{\infty}(\hat{x}; D_1) + u^{\infty}(\hat{x}; D_2) + \mathcal{O}\big(L^{-1}\big).$$

The case that D has more than two components or D has mixed type components can be proved in a similar manner by making use of the integral equation method. In the argument above, there is a technical assumption that ω^2 is not an eigenvalue for $-\Delta^*$ in D_j with the traction-free boundary condition. If the eigenvalue problem happens, one can make use of the combined layer potentials (cf. [17,21]) and then by a completely similar argument as above, one can show (2.10).

The proof is completed. $\square$

Next, we consider the scattering from multiple small scatterers. Let $l_s \in \mathbb{N}$ and let M_j, $1 \le j \le l_s$, be bounded simply-connected domains in $\mathbb{R}^3$ with C^2-smooth boundaries. It is supposed that M_j, $1 \le j \le l_s$, contains the origin and its diameter is comparable with the S-wavelength or P-wavelength, i.e., $\mathrm{diam}(M_j) \sim \mathcal{O}(1)$. For $\rho \in \mathbb{R}_+$, we introduce a scaling/dilation operator Λ_ρ by

$$\Lambda_\rho M_j := \{\rho x : x \in M_j\} \tag{2.12}$$

and set

$$D_j^\rho := z_j + \Lambda_\rho M_j, \quad z_j \in \mathbb{R}^3,\ 1 \le j \le l_s. \tag{2.13}$$

Let

$$D^\rho := \bigcup_{j=1}^{l_s} D_j^\rho. \tag{2.14}$$

Theorem 2.1. *Consider an elastic scatterer D^ρ given in (2.14). Assume that $\rho \ll 1$, $\omega \sim 1$ and*

$$L_s = \min_{j \ne j', 1 \le j, j' \le l_s} \mathrm{dist}(z_j, z_{j'}) \gg 1. \tag{2.15}$$

Moreover, we assume that D_j^ρ, $j=1,2,\ldots,l_s$, are all traction-free cavities. Then we have

$$u_p^\infty(\hat{x}; D^\rho) = -\rho^3\left[(\hat{x}\otimes\hat{x})\sum_{j=1}^{l_s} e^{-ik_p\hat{x}\cdot z_j} U_j(\hat{x};\alpha,\beta,d,d^\perp) + \mathcal{O}(\rho + L_s^{-1})\right],$$

$$u_s^\infty(\hat{x}; D^\rho) = -\rho^3\left[(\mathbf{I}-\hat{x}\otimes\hat{x})\sum_{j=1}^{l_s} e^{-ik_s\hat{x}\cdot z_j} U_j(\hat{x};\alpha,\beta,d,d^\perp) + \mathcal{O}(\rho + L_s^{-1})\right] \tag{2.16}$$

where

$$U_j(\hat{x};\alpha,\beta,d,d^\perp) := \sum_{n=0}^{1}\sum_{m=-n}^{n}\left(\beta e^{ik_s z_j\cdot d} C_{n,j}^{m,s} + \alpha e^{ik_p z_j\cdot d} C_{n,j}^{m,p}\right) Y_n^m(\hat{x}).$$

Here, α, β are the coefficients attached to the incident plane wave (1.1), $Y_n^m(\hat{x})$ are the spherical harmonics and $C_{n,j}^{m,p}, C_{n,j}^{m,s} \in \mathbb{C}^3$ are complex-valued vectors independent of ρ, l_s, L_s and z_j.

Proof. By Lemma 2.1, it suffices to analyze the asymptotics of the far-field patterns for only one single scatterer component. For notational convenience, we employ $\Omega = z + \Lambda_\rho M$ to denote $D_j^\rho = z + \Lambda_\rho M_j$ with any fixed $j \in \{1,2,\cdots,l_s\}$. For $f \in C(\partial\Omega)$ and $g \in C(\partial M)$, we introduce the transforms

$$\hat{f}(\xi) = f^\wedge = f(\rho\xi + z), \quad \xi \in \partial M, \qquad \check{g}(x) = g^\vee := g\big((x-z)/\rho\big), \quad x \in \partial\Omega.$$

Using change of variables it is not difficult to verify that (see, e.g., [15,10])

$$K_\Omega\varphi = (K_M\hat{\varphi})^\vee, \qquad (I-K_\Omega)\varphi = \big((I-K_M)\hat{\varphi}\big)^\vee, \qquad (I-K_\Omega)^{-1}\psi = \big((I-K_M)^{-1}\hat{\psi}\big)^\vee,$$

and similarly

$$K'_\Omega\varphi = (K'_M\hat{\varphi})^\vee, \qquad (I-K'_\Omega)\varphi = \big((I-K'_M)\hat{\varphi}\big)^\vee, \qquad (I-K'_\Omega)^{-1}\psi = \big((I-K'_M)^{-1}\hat{\psi}\big)^\vee.$$

These identities also hold for $\widetilde{K}$ and $\widetilde{K}'$ defined via the tensor $\widetilde{\Pi}(x,y) = \Pi^{(0)}(x,y)$. Hence, using (2.8), there hold

$$\begin{aligned}(I-K'_\Omega)\varphi - \big((I-\widetilde{K}'_M)\hat{\varphi}\big)^\vee &= (I-K'_\Omega)\varphi - (I-\widetilde{K}'_\Omega)\varphi \\ &= (\widetilde{K}'_\Omega - K'_\Omega)\varphi \\ &= 2\int_\Omega \nu(x)\cdot\big[\sigma\big(\Pi^{(0)}(x,y) - \Pi^{(\omega)}(x,y)\big)\big]\varphi(y)\,ds(y) \\ &\le C(\lambda,\mu)\rho^2\|\varphi\|_{C(\partial\Omega)},\end{aligned}$$

as $\rho \to +0$. Since $\rho \ll 1$, by the Neumann series we have

$$(I-K'_\Omega)^{-1}\varphi = \big((I-\widetilde{K}'_M)^{-1}\hat{\varphi}\big)^\vee + \mathcal{O}(\rho^2) \quad \text{as } \rho \to 0. \tag{2.17}$$

It is worth pointing out that the operator $I - \widetilde{K}'_M$ is bijective over the space (see, e.g., [17])

$$\left\{\psi \in C(\partial M) : \int_{\partial M} \psi(x) \cdot (\mathbf{a} + \mathbf{b} \times x)\, ds(x) = 0 \text{ for all } \mathbf{a}, \mathbf{b} \in \mathbb{C}^3\right\}.$$

To proceed with the proof, we represent the scattered field $u^{sc}(x, \Omega)$ as the single layer potential

$$u^{sc}(x, \Omega) = \int_{\partial \Omega} \Pi(x, y)\varphi(y)\, ds(y), \quad x \in \mathbb{R}^3 \backslash \overline{\Omega},$$

with the density function $\varphi \in C(\partial \Omega)$ given by

$$\varphi = 2\big(I - K'_\Omega\big)^{-1}\big(Tu^{in}|_{\partial \Omega}\big).$$

Then, the P-part and S-part far-field patterns of u^∞ are given respectively by (after the normalization used in (1.8))

$$\begin{aligned} u_p^\infty(\hat{x}, \Omega) &= 2(\hat{x} \otimes \hat{x}) \int_{\partial \Omega} e^{-ik_p \hat{x} \cdot y}\big[\big(I - K'_\Omega\big)^{-1}\varphi\big](y)\, ds(y), \\ u_s^\infty(\hat{x}, \Omega) &= 2(\mathbf{I} - \hat{x} \otimes \hat{x}) \int_{\partial \Omega} e^{-ik_s \hat{x} \cdot y}\big[\big(I - K'_\Omega\big)^{-1}\varphi\big](y)\, ds(y), \end{aligned} \tag{2.18}$$

where $\varphi := Tu^{in}|_{\partial \Omega} = \nu \cdot \sigma(u^{in})|_{\partial \Omega}$. In the rest of the proof, we only justify the asymptotic behavior of $u_p^\infty(\hat{x}, \Omega)$ as $\rho \to +0$. Our argument can be readily adapted to the case of the S-part far-field pattern.

Changing the variable $y = z + \rho\xi$ with $\xi \in \partial M$ in (2.18) and making use of the estimate (2.17), we find

$$u_p^\infty(\hat{x}, \Omega) = 2(\hat{x} \otimes \hat{x})\rho^2 \int_{\partial M} e^{-ik_p \hat{x} \cdot (z + \rho\xi)}\big[\big(I - \widetilde{K}'_M\big)^{-1}\hat{\varphi}(\xi) + \mathcal{O}\big(\rho^2\big)\big]\, ds(\xi). \tag{2.19}$$

Expanding the exponential function $\xi \to \exp(-ik_p \hat{x} \cdot (z + \rho\xi))$ around z yields

$$\exp\big(-ik_p \hat{x} \cdot (z + \rho\xi)\big) = \exp(-ik_p \hat{x} \cdot z) - ik_p\rho(\hat{x} \cdot \xi)\exp(-ik_p \hat{x} \cdot z) + \mathcal{O}\big(k_p^2\rho^2\big) \tag{2.20}$$

as $\rho \to +0$. Inserting (2.20) into (2.19) gives

$$\begin{aligned} u_p^\infty(\hat{x}, \Omega) = {}& 2(\hat{x} \otimes \hat{x})\rho^2 e^{-ik_p \hat{x} \cdot z}\left(\int_{\partial M} \big(I - \widetilde{K}'_M\big)^{-1}\hat{\varphi}(\xi)\, ds(\xi)\right) \\ & - 2i(\hat{x} \otimes \hat{x})k_p\rho^3 e^{-ik_p \hat{x} \cdot z}\left(\int_{\partial M} (\hat{x} \cdot \xi)\big(I - \widetilde{K}'_M\big)^{-1}\hat{\varphi}(\xi)\, ds(\xi)\right) \\ & + \mathcal{O}\big(\rho^4\big). \end{aligned} \tag{2.21}$$

To estimate the integrals on the right hand side of (2.21), we will investigate the incident plane pressure and shear waves, respectively. The asymptotics for general plane waves of the form (1.1) can be derived by linear superposition.

Case (i): $\beta = 1$, $\alpha = 0$, i.e., $u^{in} = u_s^{in} = d^\perp e^{ik_s x\cdot d}$ is an incident plane shear wave.

Since $\operatorname{div} u_s^{in} = 0$, by (1.3) we have $\sigma(u_s^{in}) = \mu(\nabla u_s^{in} + \nabla u_s^{in\top})$. Expanding the function $\xi \to (\nabla u_s^{in\wedge})(\xi) = \nabla u_s^{in}(\rho\xi + z)$ around z,

$$\left(\nabla u_s^{in\wedge}\right)(\xi) = ik_s\left(d^\perp \otimes d\right)\left[e^{ik_s z\cdot d} + ik_s\rho e^{ik_s z\cdot d}(d\cdot\xi) + \mathcal{O}\left(k_s^2\rho^2\right)\right] \quad \text{as } \rho \to +0.$$

Hence,

$$\left(\sigma\left(u_s^{in}\right)^\wedge\right)(\xi) = i\mu k_s e^{ik_s z\cdot d}\mathbb{H}(d)\left[1 + ik_s\rho(d\cdot\xi) + \mathcal{O}\left(k_s^2\rho^2\right)\right], \tag{2.22}$$

where $\mathbb{H}(d) := (d^\perp \otimes d) + (d^\perp \otimes d)^\top$. Recalling that (which can actually be proved by using the jump relations for the double layer potential with constant density, see, e.g., [20, Example 6.14])

$$\widetilde{K}_M 1 = 2\int_{\partial M} \frac{\widetilde{\Pi}(x,y)}{\partial\nu(y)}\, ds(y) = -1,$$

we see $(I - \widetilde{K}_M)^{-1}1 = 1/2$, and thus

$$\begin{aligned}\int_{\partial M} \left(I - \widetilde{K}'_M\right)^{-1}\hat{\varphi}(\xi)\, ds(\xi) &= \int_{\partial M} \hat{\varphi}(\xi)(I - \widetilde{K}_M)^{-1}1\, ds(\xi) \\ &= \frac{1}{2}\int_{\partial M} \nu(\xi)\cdot\left(\sigma\left(u_s^{in}\right)^\wedge\right)(\xi)\, ds(\xi).\end{aligned} \tag{2.23}$$

Inserting (2.22) into (2.23) and applying Gauss's theorem yield

$$\begin{aligned}&\int_{\partial M} \left(I - \widetilde{K}'_M\right)^{-1}\hat{\varphi}(\xi)\, ds(\xi) \\ &\quad = i\mu k_s e^{ik_s z\cdot d}/2\left\{\int_M \operatorname{div}_\xi\left[\mathbb{H}(d)(1 + ik_s\rho d\cdot\xi)\right] ds(\xi)\right\} + \mathcal{O}\left(\rho^2\right) \\ &\quad = -\rho d^\perp e^{ik_s z\cdot d}|M|/2 + \mathcal{O}\left(\rho^2\right).\end{aligned} \tag{2.24}$$

Note that $|M|$ denotes the volume of M and that the last equality follows from the relation

$$\operatorname{div}_\xi\left[\mathbb{H}(d)(1 + ik_s\rho d\cdot\xi)\right] = ik_s\rho d\cdot\mathbb{H}(d) = ik_s\rho d^\perp.$$

Again using (2.22) we can evaluate the second integral over ∂M on the right hand of (2.21) as follows:

$$\begin{aligned}
&\int\limits_{\partial M} (\hat{x}\cdot\xi)\big(I-\widetilde{K}'_M\big)^{-1}\hat{\varphi}(\xi)\,ds(\xi) \\
&\quad = i\mu k_s e^{ik_s z\cdot d}\bigg\{\int\limits_{\partial M}(\hat{x}\cdot\xi)\big(I-\widetilde{K}'_M\big)^{-1}\big(\nu(\xi)\cdot\mathbb{H}(d)\big)\,ds(\xi)\bigg\}+\mathcal{O}(\rho) \\
&\quad = -i\mu k_s e^{ik_s z\cdot d}(\hat{x}\cdot\mathbb{M})\cdot\mathbb{H}(d)+\mathcal{O}(\rho),
\end{aligned} \tag{2.25}$$

where the polarization tensor $\mathbb{M}$ depending only on M is defined as

$$\mathbb{M} = -\int\limits_{\partial M}\xi\otimes\big(I-\widetilde{K}'_M\big)^{-1}\nu(\xi)\,ds(\xi). \tag{2.26}$$

Now, combining (2.25), (2.24) and (2.21) gives the asymptotics

$$u_p^\infty(\hat{x},\Omega) = -(\hat{x}\otimes\hat{x})\rho^3 e^{iz\cdot(k_s d-k_p\hat{x})}\big[d^\perp|M|+2(\hat{x}\cdot\mathbb{M})\cdot\mathbb{H}(d)\big]+\mathcal{O}\big(\rho^4\big), \tag{2.27}$$

as $\rho\to+0$. Recalling that the spherical harmonics $Y_0^0=\sqrt{1/4\pi}$ and that each Cartesian component of the vector $\hat{x}\in\mathbb{R}^3$ can be expressed in terms of Y_1^{-1}, Y_1^0, Y_1^1, we may reformulate the previous identity as

$$u_p^\infty(\hat{x},\Omega) = -(\hat{x}\otimes\hat{x})\rho^3 e^{iz\cdot(k_s d-k_p\hat{x})}\Bigg[\sum_{n=0}^{1}\sum_{m=-n}^{n}C_n^m Y_n^m(\hat{x})\Bigg]+\mathcal{O}\big(\rho^4\big), \tag{2.28}$$

with $C_0^0=d^\perp|M|2\sqrt{\pi}$ and $C_1^m\in\mathbb{C}^3$ for $m=-1,0,1$. This proves (2.16) when $\alpha=0$, $\beta=1$ and $l_s=1$. In the case $\alpha=0$, $\beta\in\mathbb{C}$ and $l_s>1$, there holds

$$u_p^\infty(\hat{x},\Omega) = -(\hat{x}\otimes\hat{x})\rho^3\sum_{j=1}^{l_s}e^{-ik_p z_j\cdot\hat{x}}\Bigg[\sum_{n=0}^{1}\sum_{m=-n}^{n}\beta e^{ik_s z_j\cdot d}C_{n,j}^{m,s}Y_n^m(\hat{x})\Bigg]+\mathcal{O}\big(\rho^4+L_s^{-1}\big),$$

with $C_{0,j}^{0,s}=d^\perp|M_j|2\sqrt{\pi}$ and $C_{1,j}^{m,s}\in\mathbb{C}^3$ for $m=-1,0,1$, and $j=1,2,\cdots,l_s$. Note that the constant $C_{n,j}^{m,s}$ are independent of z_j, ρ, l_s and L_s.

Case (ii): $\beta=0,\alpha=1$, i.e., $u^{in}=u_p^{in}=de^{ik_p x\cdot d}$ is an incident plane pressure wave.

We sketch the proof, since it can be carried out analogously to Case (i). The corresponding expansion of $(\sigma(u_p^{in}))^\wedge$ to (2.22) reads as follows:

$$\big(\sigma\big(u_p^{in}\big)^\wedge\big)(\xi) = i(\lambda+2\mu)k_p e^{ik_p z\cdot d}\mathbb{L}(\lambda,\mu,d)\big[1+ik_p\rho(d\cdot\xi)+\mathcal{O}\big(k_p^2\rho^2\big)\big] \tag{2.29}$$

when $\rho\to+0$, where $\mathbb{L}(\lambda,\mu,d):=(\lambda\mathbf{I}+2\mu(d\otimes d))/(\lambda+2\mu)$. As a consequence of (2.23), we have for $\varphi=\nu\cdot\sigma(u_p^{in})|_{\partial\Omega}$,

$$
\begin{aligned}
&\int_{\partial M} \left(I - \widetilde{K}'_M\right)^{-1} \hat{\varphi}(\xi)\, ds(\xi) \\
&\quad = i(\lambda + 2\mu) k_p e^{i k_p z \cdot d}/2 \left\{ \int_M \operatorname{div}_\xi \left[\mathbb{L}(\lambda, \mu, d)(1 + i k_p \rho d \cdot \xi) \right] ds(\xi) \right\} + \mathcal{O}\left(\rho^2\right) \\
&\quad = -\rho d e^{i k_p z \cdot d} |M|/2 + \mathcal{O}\left(\rho^2\right).
\end{aligned} \tag{2.30}
$$

Similar to (2.25), one has

$$
\begin{aligned}
&\int_{\partial M} (\hat{x} \cdot \xi)\left(I - \widetilde{K}'_M\right)^{-1} \hat{\varphi}(\xi)\, ds(\xi) \\
&\qquad = -i(\lambda + 2\mu) k_p e^{i k_p z \cdot d} (\hat{x} \cdot \mathbb{M}) \cdot \mathbb{L}(\lambda, \mu, d) + \mathcal{O}(\rho),
\end{aligned} \tag{2.31}
$$

where the polarization tensor $\mathbb{M}$ is given as the same in (2.26). Therefore, the insertion of (2.30) and (2.31) into (2.21) yields

$$
u_p^\infty(\hat{x}, \Omega) = -(\hat{x} \otimes \hat{x}) \rho^3 e^{i k_p \hat{x} \cdot (d - z)} \left[d|M| + 2(\hat{x} \cdot \mathbb{M}) \cdot \mathbb{L}(\lambda, \mu, d) \right] + \mathcal{O}\left(\rho^4\right), \tag{2.32}
$$

as $\rho \to +0$, and it further leads to the asymptotic behavior in (2.16) when $\alpha = 1$, $\beta = 0$ and $l_s = 1$.

The proof is completed. $\square$

The asymptotic expansions of the far-field patterns corresponding to small rigid bodies was considered in [10,19], and for completeness and also the subsequent use, we include it in the following theorem.

Theorem 2.2. *Consider an elastic scatterer D^ρ given in (2.14). Assume that $\rho \ll 1$ and $L_s = \min_{j \neq j', 1 \le j, j' \le l_s} \operatorname{dist}(z_j, z_{j'}) \gg 1$. Moreover, we assume that D_j^ρ, $j = 1, 2, \ldots, l_s$, are all traction-free cavities. Then we have*

$$
\begin{aligned}
u_p^\infty\left(\hat{x}; D^\rho\right) &= \frac{\rho}{4\pi(\lambda + 2\mu)} (\hat{x} \otimes \hat{x}) \left[\sum_{j=1}^{l_s} e^{-i k_p \hat{x} \cdot z_j} \left(C_{p,j} \alpha e^{i k_p z_j \cdot d} + C_{s,j} \beta e^{i k_s z_j \cdot d} \right) \right] \\
&\quad + \mathcal{O}\left(\rho^2 l_s \left(1 + L_s^{-1}\right)\right), \\
u_s^\infty\left(\hat{x}; D^\rho\right) &= \frac{\rho}{4\pi\mu} (\mathbf{I} - \hat{x} \otimes \hat{x}) \left[\sum_{j=1}^{l_s} e^{-i k_s \hat{x} \cdot z_j} \left(C_{p,j} \alpha e^{i k_p z_j \cdot d} + C_{s,j} \beta e^{i k_s z_j \cdot d} \right) \right] \\
&\quad + \mathcal{O}\left(\rho^2 l_s \left(1 + L_s^{-1}\right)\right),
\end{aligned}
$$

where $C_{p,j}, C_{s,j} \in \mathbb{C}^3$ are constant vectors independent of ρ, l_s, L_s and z_j.

Finally, for our subsequent study on the inverse scattering problem, we shall also need some results on the scattering from extended elastic bodies. Let Σ be a bounded simply-connected set that contains the origin. Denote by $\mathscr{R} := \mathscr{R}(\theta, \phi, \psi) \in SO(3)$ the 3D rotation matrix around the origin whose Euler angles are $\theta \in [0, 2\pi]$, $\phi \in [0, 2\pi]$ and $\psi \in [0, \pi]$; and define $\mathscr{R}\Sigma := \{\mathscr{R}x : x \in \Sigma\}$. We introduce

$$\mathscr{A} := \{\Sigma_j\}_{j=1}^{l'}, \quad l' \in \mathbb{N} \tag{2.33}$$

where each $\Sigma_j \subset \mathbb{R}^3$ is a bounded simply-connected C^2 domain containing the origin. $\mathscr{A}$ is called a *base scatterer class*, and each base scatterer Σ_j, $1 \le j \le l'$, could be either rigid or traction-free. Next, we introduce the multiple extended scatterers for our study via the base class $\mathscr{A}$ in (2.33). Let $l_e \in \mathbb{N}$ and for $j = 1, 2, \cdots, l_e$, set $r_j \in \mathbb{R}_+$ such that

$$r_j \in [R_0, R_1], \quad 0 < R_0 < R_1 < +\infty, \ R_0 \sim \mathcal{O}(1),$$

and moreover, let $(\theta_j, \phi_j, \psi_j) \in [0, 2\pi]^2 \times [0, \pi]$, $j = 1, 2 \ldots, l_e$, be l_e Euler angles. For $z_j \in \mathbb{R}^3$, we let

$$D = \bigcup_{j=1}^{l_e} D_j, \quad D_j := z_j + \mathscr{R}_j \Lambda_{r_j} E_j, \ E_j \in \mathscr{A}, \ \mathscr{R}_j := \mathscr{R}(\theta_j, \phi_j, \psi_j). \tag{2.34}$$

The physical property of D_j is inherited from that of the base scatterer E_j; namely, if E_j is traction-free (resp. rigid), then D_j is also traction-free (resp. rigid). For technical purpose, we impose the following sparsity assumption on the extended scatterer D introduced in (2.34),

$$L_e = \min_{j \ne j', 1 \le j, j' \le l_e} \operatorname{dist}(D_j, D_{j'}) \gg 1. \tag{2.35}$$

Theorem 2.3. *Consider an elastic scatterer D given in (2.34). Assume that the sparsity condition (2.35) is satisfied. If $\alpha = 1$ and $\beta = 0$, then*

$$\begin{aligned} u_\tau(\hat{x}; D, d, d, \omega) = {} & \sum_{j=1}^{l_e} \kappa(z_j) r_j \mathscr{R}_j u_\tau^\infty\big(\mathscr{R}_j^\perp \hat{x}; E_j, \mathscr{R}_j^\perp d, \mathscr{R}_j^\perp, r_j \omega\big) \\ & + \mathcal{O}\big(L_e^{-1}\big), \quad \tau = p, s \end{aligned} \tag{2.36}$$

where

$$\kappa(z_j) = e^{ik_p(d-\hat{x})\cdot z_j} \quad \text{if } \tau = p; \ e^{i(k_p d - k_s \hat{x})\cdot z_j} \ \text{if } \tau = s. \tag{2.37}$$

If $\alpha = 0$ and $\beta = 1$, then one has a similar expansion as that in (2.36) but with

$$\kappa(z_j) = e^{i(k_s d - k_p \hat{x})\cdot z_j} \quad \text{if } \tau = p; \ e^{ik_s(d-\hat{x})\cdot z_j} \ \text{if } \tau = s. \tag{2.38}$$

Proof. We only consider the first case with $\alpha = 1$ and $\beta = 0$, and the second case with $\alpha = 0$ and $\beta = 1$ can be proved in a similar manner. If E_j is a rigid elastic body, the following identities were proved in [19]:

$$u_\tau^\infty(\hat{x}; z_j + E_j) = \kappa(z_j) u_\tau^\infty(\hat{x}; E_j) \quad \text{where } \kappa(z_j) \text{ is given in (2.37)}, \tag{2.39}$$

and

$$\mathscr{R}u_\tau^\infty(\hat{x}; E_j, d, d^\perp) = u_\tau^\infty(\mathscr{R}\hat{x}; \mathscr{R}E_j, \mathscr{R}d, \mathscr{R}d^\perp), \tag{2.40}$$

and

$$u_\tau^\infty(\hat{x}; \Lambda_{r_j} E_j, \omega) = r_j u_\tau^\infty(\hat{x}; E_j, r_j\omega). \tag{2.41}$$

By following a completely similar argument, one can show that the above identities also hold when E_j is a traction-free cavity. Finally, by using Lemma 2.1, and (2.39)–(2.41), one can show (2.37), which completes the proof. □

3. Locating multiple multiscale elastic scatterers

In this section, we consider the inverse scattering problem of recovering multiple elastic scatterers. We first consider the locating of multiple small scatterers introduced in (2.14), and then we consider the locating of multiscale scatterers with both small components of those in (2.14) and extended components of those described in (2.34). The key ingredients of the developed inverse scattering schemes are some indicator functions, whose local maximum behaviors can be used to identify the multiple elastic bodies in an effective and efficient manner.

3.1. Locating small scatterers

Let D^ρ be a small elastic scatterer consisting of multiple components as introduced in (2.14). In order to present the scheme of locating the multiple components of D^ρ, we introduce the following three indicator functions

$$I_1(z) = \frac{1}{\|u_p^\infty(\hat{x}; D^\rho)\|_{L^2}^2} \sum_{n=0}^{1}\sum_{m=-n}^{n}\sum_{l=1}^{3} \left|\left\langle u_p^\infty(\hat{x}; D^\rho), (\hat{x}\otimes\hat{x})Y_n^m(\hat{x})\mathbf{e}_l e^{-ik_p\hat{x}\cdot z}\right\rangle\right|^2, \tag{3.1}$$

$$I_2(z) = \frac{1}{\|u_s^\infty(\hat{x}; D^\rho)\|_{L^2}^2} \sum_{n=0}^{1}\sum_{m=-n}^{n}\sum_{l=1}^{3} \left|\left\langle u_s^\infty(\hat{x}; D^\rho), (\mathbf{I}-\hat{x}\otimes\hat{x})Y_n^m(\hat{x})\mathbf{e}_l e^{-ik_s\hat{x}\cdot z}\right\rangle\right|^2, \tag{3.2}$$

$$I_3(z) = \frac{1}{\|u^\infty(\hat{x}; D^\rho)\|_{L^2}^2} \sum_{n=0}^{1}\sum_{m=-n}^{n}\sum_{l=1}^{3} \left|f_{n,m,l}(z)\right|^2, \tag{3.3}$$

where

$$f_{n,m,l}(z) := \left\langle u^\infty(\hat{x}; D^\rho), \left[(\hat{x}\otimes\hat{x})e^{-ik_p\hat{x}\cdot z} + (\mathbf{I}-\hat{x}\otimes\hat{x})e^{-ik_s\hat{x}\cdot z}\right]Y_n^m(\hat{x})\mathbf{e}_l\right\rangle.$$

Here and in what follows, the notation $\langle\cdot,\cdot\rangle$ denotes the inner product in $L^2 := L^2(\mathbb{S}^2)^3$ with respect to the variable $\hat{x}\in\mathbb{S}^2$, defined as $\langle\mathbf{u},\mathbf{v}\rangle := \int_{\mathbb{S}^2}\mathbf{u}(\hat{x})\cdot\overline{\mathbf{v}(\hat{x})}\,ds(\hat{x})$. Clearly, I_m $(m = 1, 2, 3)$ are all nonnegative functions and they can be obtained, respectively, by using a single P-part far-field pattern $(m = 1)$, S-part far-field pattern $(m = 2)$, or the full far-field pattern $(m = 3)$. The functions introduced above possess certain indicating behaviors, which lies in the essence of our inverse scattering schemes. Before stating the theorem of the indicating behaviors for those functions, we introduce the following real numbers

$$K_1^j := \frac{\|u_p^\infty(\hat{x}; D_j^\rho)\|_{L^2}^2}{\|u_p^\infty(\hat{x}; D^\rho)\|_{L^2}^2}, \qquad K_2^j := \frac{\|u_s^\infty(\hat{x}; D_j^\rho)\|_{L^2}^2}{\|u_s^\infty(\hat{x}; D^\rho)\|_{L^2}^2}, \qquad K_3^j := \frac{\|u^\infty(\hat{x}; D_j^\rho)\|_{L^2}^2}{\|u^\infty(\hat{x}; D^\rho)\|_{L^2}^2}, \tag{3.4}$$

for $1 \le j \le l_s$.

Theorem 3.1. *Consider the elastic scatterer D^ρ described in (2.14), and assume that D^ρ is traction-free. For K_m^j, $m = 1, 2, 3$, defined in (3.4), we have*

$$K_m^j = \widetilde{K}^j + \mathcal{O}\big(L_s^{-1} + \rho\big), \quad 1 \le j \le l_s, \ m = 1, 2, 3, \tag{3.5}$$

where $\widetilde{K}^j$'s are positive numbers independent of L_s, ρ and m. Moreover, there exists an open neighborhood of z_j, neigh(z_j), *such that*

$$I_m(z) \le \widetilde{K}^j + \mathcal{O}\big(L_s^{-1} + \rho\big) \quad \textit{for all } z \in \mathrm{neigh}(z_j), \tag{3.6}$$

and $I_m(z)$ achieves its maximum value at z_j in neigh(z_j), *i.e.,*

$$I_m(z_j) = \widetilde{K}^j + \mathcal{O}\big(L_s^{-1} + \rho\big). \tag{3.7}$$

Proof. For notational convenience we write

$$A_j =: \sum_{n=0}^{1} \sum_{m=-n}^{n} |B_{n,m,j}|^2, \quad B_{n,m,j} = \beta e^{ik_s z_j \cdot d} C_{n,j}^{m,s} + \alpha e^{ik_p z_j \cdot d} C_{n,j}^{m,p}, \tag{3.8}$$

where the constants $C_{n,j}^{m,p}$, $C_{n,j}^{m,s}$ are those given in (2.16). Then, it is seen from Theorem 2.1 and the orthogonality of Y_n^m that

$$\big\|u_p^\infty\big(\hat{x}; D_j^\rho\big)\big\|_{L^2}^2 = \rho^6 A_j + \mathcal{O}\big(\rho^7\big) \quad \text{as } \rho \to +0.$$

Under the sparsity assumption (2.15), by using the Riemann–Lebesgue lemma about oscillating integrals, we can obtain

$$\big\|u_p^\infty\big(\hat{x}; D^\rho\big)\big\|_{L^2}^2 = \rho^6 \sum_{j=1}^{l_s} A_j + \mathcal{O}\big(\rho^7\big) + \mathcal{O}\big(L_s^{-1}\big).$$

Hence,

$$K_1^j = \frac{\|u_p^\infty(\hat{x}; D_j^\rho)\|_{L^2}^2}{\|u_p^\infty(\hat{x}; D^\rho)\|_{L^2}^2} = \widetilde{K}^j + \mathcal{O}\big(\rho + L_s^{-1}\big), \quad \widetilde{K}^j := \frac{A_j}{\sum_{j=1}^{l_s} A_j}. \tag{3.9}$$

This proves (3.5) for $m = 1$. The case of using the S-part far-field pattern (i.e., $m = 2$) can be treated analogously. To treat the case $m = 3$, we shall use the orthogonality of u_p^∞ and u_s^∞. Since $\langle \mathbf{I} - \hat{x} \otimes \hat{x}, \hat{x} \otimes \hat{x} \rangle = 0$, again applying Theorem 2.1 to D^ρ and D_j^ρ yields

$$\left\| u^\infty(\hat{x}; D^\rho) \right\|_{L^2}^2 = 2\rho^6 \sum_{j=1}^{l_s} A_j + \mathcal{O}(\rho^7) + \mathcal{O}(L_s^{-1}),$$

$$\left\| u^\infty(\hat{x}; D_j^\rho) \right\|_{L^2}^2 = 2\rho^6 A_j + \mathcal{O}(\rho^7).$$

Hence, (3.5) is proved with $\widetilde{K}^j$ defined as in (3.9).

To verify (3.6) and (3.7), without loss of generality we only consider the indicating behavior of $I_1(z)$ in a small neighborhood of z_j for some fixed $1 \le j \le l_s$, i.e., $z \in \mathrm{neigh}(z_j)$. We assume further that $|z - z_j| < \rho$. Clearly, under the assumption (2.15),

$$\omega |z_{j'} - z| \sim \omega L_s \gg 1, \quad \text{for all } z \in \mathrm{neigh}(z_j),\ j' \ne j.$$

By using the Riemann–Lebesgue lemma and Theorem 2.1, one can obtain

$$\begin{aligned}
&\sum_{l'=1}^{3} \left| \left\langle u_p^\infty(\hat{x}; D^\rho), (\hat{x} \otimes \hat{x}) Y_{n'}^{m'}(\hat{x}) \mathbf{e}_{l'} e^{-ik_p \hat{x}\cdot z} \right\rangle \right|^2 \\
&= \rho^6 \sum_{l'=1}^{3} \left| \left\langle e^{-ik_p \hat{x}\cdot z_j} \sum_{n=0}^{1} \sum_{m=-n}^{n} B_{n,m,j} Y_n^m(\hat{x}), Y_{n'}^{m'}(\hat{x}) \mathbf{e}_{l'} e^{-ik_p \hat{x}\cdot z} \right\rangle \right|^2 + \mathcal{O}(\rho^7 + L_s^{-1}) \\
&\le \rho^6 \sum_{l'=1}^{3} |B_{n',m',j} \cdot e_{l'}|^2 + \mathcal{O}(\rho^7 + L_s^{-1}) \\
&= \rho^6 |B_{n',m',j}|^2 + \mathcal{O}(\rho^7 + L_s^{-1}),
\end{aligned} \tag{3.10}$$

where inequality (3.10) follows from the Cauchy–Schwarz inequality and $B_{n',m',j} \in \mathbb{C}^3$ are given in (3.8). Moreover, the strict inequality in (3.10) holds if $z \ne z_j$ and the equal sign holds only when $z = z_j$. Therefore, by the definitions of I_1, A_j and $\widetilde{K}^j$,

$$I_1(z) \le \frac{\rho^6 \sum_{n'=0}^{1} \sum_{m'=-n'}^{n'} |B_{n',m',j}|^2 + \mathcal{O}(\rho^7 + L_s^{-1})}{\rho^6 \sum_{j=1}^{l_s} A_j + \mathcal{O}(\rho^7 + L_s^{-1})} = \widetilde{K}^j + \mathcal{O}(\rho + L_s^{-1}),$$

where the equality holds only when $z = z_j$. This proves (3.6) and (3.7). The indicating behavior of I_2 and I_3 can be verified in the same way. □

Remark 3.1. The local maximum behavior of $I_m(z)$ can be used to locate the scatterer components of D^ρ, namely z_j, $1 \le j \le l_s$. Such indicating behavior is much evident if one considers the case that D^ρ has only one component, i.e., $l_s = 1$. In the one-component case, one has that

$$\widetilde{K}^j = 1, \qquad I_m(z) < 1 + \mathcal{O}(\rho) \quad \text{for all } m = 1, 2, 3,\ z \ne z_1,$$

and

$$I_m(z_1) = 1 + \mathcal{O}(\rho), \quad m = 1, 2, 3.$$

That is, z_1 is a global maximizer for $I_m(z)$.

Remark 3.2. In Theorem 3.1, we only consider that D^ρ is a traction-free scatterer. If D^ρ is a rigid scatterer, by using Theorem 2.2 and following a similar argument, one can show that Theorem 3.1 remains valid. Moreover, by Theorem 2.2, it is easily seen that in the rigid case the terms with the index $n = 1$ in $I_m(z)$ are high-order terms and hence can be eliminated. Eliminating the terms with the index $n = 1$ in (3.1), (3.2) and (3.3) actually gives the indicator functions proposed in [19]. However, it is clear that the indicator functions proposed in [19] work only for locating rigid bodies. The indicator functions proposed in (3.1)–(3.3) works for locating both rigid and traction-free cavities. Furthermore, we can consider an even more general case by assuming $D_s = D^{\rho_1} \cup D^{\rho_2}$ with D^{ρ_j}, $j = 1, 2$, both of the form (2.14). D^{ρ_1} and D^{ρ_2}, respectively, contain the rigid bodies and traction-free cavities. It is assumed that $\rho_1 \sim \rho_2^3 \ll 1$. This means that both D^{ρ_1} and D^{ρ_2} are small scatterers, and by Theorems 2.1 and 2.2, the scattering strengths from the components of D^{ρ_1} and D^{ρ_2} are comparable. Then it is straightforward to show that Theorem 3.1 remains valid for the scatterer D_s described above.

Based on Theorem 3.1, it is ready to formulate a reconstruction scheme of locating the multiple scatterers of D^ρ in (2.14) as follows.

Scheme I	Locating small scatterers of D^ρ in (2.14).
Step 1	For an unknown scatterer D^ρ with multiple components in (2.14), collect the P-part ($m = 1$), S-part ($m = 2$) or the full far-field data ($m = 3$) by sending a single detecting plane wave (1.1).
Step 2	Select a sampling region with a mesh $\mathcal{T}_h$ containing D^ρ.
Step 3	For each sampling point $z \in \mathcal{T}_h$, calculate $I_m(z)$ ($m = 1, 2, 3$) according to the measurement data.
Step 4	Locate all the local maximizers of $I_m(z)$ on $\mathcal{T}_h$, which represent locations of the scatterer components of D^ρ.

3.2. *Locating multiscale scatterers*

Consider an elastic scatterer with multiscale components of the following form

$$D_m := D^\rho \cup D, \tag{3.11}$$

where D^ρ given in (2.14) and D given in (2.33)–(2.35) represent, respectively, the collections of small-size and extended-size scatterers. For D_m introduced above, we assume that $\operatorname{dist}(\overline{D^\rho}, \overline{D}) \gg 1$. Next, we consider the recovery of the multiple multiscale scatterer components of D_m, under the a priori knowledge that the base scatterer class $\mathscr{A}$ in (2.33) is known in advance. In the present section, $\mathscr{A}$ is also referred to as an *admissible class*. If D_m consists of only rigid bodies, the recovery was considered in [19]. By using Scheme I developed for locating small scatterers, together with the help of Lemma 2.1 and Theorem 2.3, and some slight necessary modifications, the inverse scattering scheme developed in [19] for locating multiscale rigid bodies can be readily extended to the locating of the more general multiscale scatterers contained in D_m. In what follows, for completeness and self-containedness, we sketch the reconstruction procedure.

First, for the admissible class $\mathscr{A}$ and a sufficiently small $\epsilon \in \mathbb{R}_+$, we introduce an ϵ-net $\mathscr{A}_\epsilon := \{\widetilde{\Sigma}_j\}_{j=1}^{l''}$ of $\mathscr{A}$, such that for any $\Sigma \in \mathscr{A}$, there exists $\widetilde{\Sigma} \in \mathscr{A}_\epsilon$ satisfying $d_H(\Sigma, \widetilde{\Sigma}) \le \epsilon$, where d_H denotes the Hausdorff distance. It is assumed that

(a) $u_\tau^\infty(\hat{x}; \widetilde{\Sigma}_j) \neq u_\tau^\infty(\hat{x}; \widetilde{\Sigma}_{j'})$ for $\tau = s, p$ or $\emptyset$, and $j \neq j'$, $1 \le j, j' \le l''$.
(b) $\|u_\tau^\infty(\hat{x}; \widetilde{\Sigma}_j)\|_{L^2} \ge \|u_\tau^\infty(\hat{x}; \widetilde{\Sigma}_{j'})\|_{L^2}$ for $\tau = s, p$ or $\emptyset$, and $j < j'$, $1 \le j, j' \le l''$.

Assumption (a) is the generic uniqueness for the inverse elastic scattering problem, whereas assumption (b) can be achieved by reordering if necessary. Next, for simplicity, we only consider the case by making use of the P-part far-field pattern. However, all the results presented below still hold when the S-part or full far-field patterns are employed, if one replaces the locating functional by the corresponding functionals developed in [19]. Let either α or β be taken to be zero in the detecting plane wave (1.1) and define

$$J_j(z) = \frac{1}{\|u_p^\infty(\hat{x}; \widetilde{\Sigma}_j)\|_{L^2}^2} \left|\left\langle u_p^\infty(\hat{x}; D_m), e^{-ik_p\hat{x}\cdot z} u_p^\infty(\hat{x}; \widetilde{\Sigma}_j)\right\rangle\right|^2, \quad z \in \mathbb{R}^3. \tag{3.12}$$

Since $\widetilde{\Sigma}_j \in \mathscr{A}$ is known in advance, $J_j(z)$ is actually obtained by projecting the scattering measurement data into a space generated by the scattering data from the admissible base scatterers. Then, one starts with the indicator function $J_1(z)$ to locate all the local maximum points on a sampling mesh $\mathcal{T}$ containing the target scatterer. We denote the obtained local maximum points by $z_1^1, z_1^2, \ldots, z_l^{l_1}$, which represent the approximate locations of scatterer components of the form $z_1^j + \widetilde{\Sigma}_1$, $j = 1, 2, \ldots, l_1$. With the located scatterer components $z_1^j + \widetilde{\Sigma}_1$, one updates the P-part of the far-field pattern according to the following formula,

$$u_p^\infty(\hat{x}) := u_p^\infty(\hat{x}; D_m) - \sum_{j=1}^{l_1} \kappa\left(z_1^j\right) u_p^\infty(\hat{x}; \widetilde{\Sigma}_1),$$

where $\kappa(z_j)$ is given in (2.37)–(2.38). Using the updated far-field pattern as the measurement data, one continues the locating procedure with the indicator function $J_2(z)$ and finds the corresponding local maximum points on $\mathcal{T}$, say, $z_2^1, z_2^2, \ldots, z_2^{l_2}$, which represent the approximate locations of scatterer components of the form $z_2^j + \widetilde{\Sigma}_2$, $j = 1, 2, \ldots, l_2$. By continuing the above procedure, one can find $z_j^1, z_j^2, \ldots, z_j^{l_j}$, $j = 3, \ldots, l''$, which represent the approximate locations of the scatterer components of the form $z_j^m + \widetilde{\Sigma}_j$, $m = 1, 2, \ldots, l_j$. It is emphasized that it may happen that $l_j = 0$ for some $1 \le j \le l''$, which means that the scatterer components obtained from the base scatterer $\widetilde{\Sigma}_j$ does not appear in the target elastic scatterer D_m.

In the above step, one finds $\bigcup_{j=1}^{l''} \bigcup_{m=1}^{l_j} \{z_j^m\}$, and from which one recovers the extended scatterer components of D in (3.11) in an approximate manner. Next, one proceeds to the recovery of the small scatterer components of D^ρ. To that end, we let $U(z_j^m)$ denote an open neighborhood of z_j^m and V_j^m be an h-net of $U(z_j^m)$ with $h \ll 1$. Each set

$$\bigcup_{j=1}^{l''} \bigcup_{m=1}^{l_j} \{\widetilde{z_j^m}\}, \quad z_j^m \in V_j^m, \tag{3.13}$$

is a called a *local tuneup* of $\bigcup_{j=1}^{l''} \bigcup_{m=1}^{l_j} \{z_j^m\}$ relative to $\bigcup_{j=1}^{l''} \bigcup_{m=1}^{l_j} V_j^m$. For a local tuneup in (3.13), let

$$u_p^\infty(\hat{x}; D^\rho) := u_p^\infty(\hat{x}; D_m) - \sum_{j=1}^{l''} \sum_{m=1}^{l_j} \kappa(z_j^m) u_p^\infty(\hat{x}; \widetilde{\Sigma}_j), \tag{3.14}$$

where $\kappa(z_j)$ is given in (2.37)–(2.38). Applying $u_p(\hat{x}; D^\rho)$ as the measurement data to Scheme I developed at the end of Section 3.1, and then locate all the local maximum points of the corresponding indicator function. By running through all the possible local tuneups and repeating the above procedure, one can locate the clustered local maximum points, which represent the locations of the small scatterer components of D^ρ in (3.11).

4. Concluding remarks

In this paper, we consider the inverse scattering problem of reconstructing cavities and rigid bodies embedded in an unbounded, homogeneous and isotropic elastic medium. The present study is conducted in an extremely general and practical setting. There might be multiple components with the number unknown in advance. Each scatterer components could be either rigid or traction-free, which is not required to be known in advance. Moreover, there might be multiscale components (in terms of the detecting wavelength) presented simultaneously. We develop several schemes that can recover the multiple multiscale scatterers in an effective and efficient manner. The key ingredients of our methods are some indicator functions, which are directly obtained by using a single P-part, or S-part, or full elastic far-field pattern. The local maximum behaviors of the proposed indicator functions can be used to locate the multiple multiscale scatterers. Rigorous mathematical justifications are presented. To that end, we derived sharp asymptotic expansion of the far-field pattern corresponding to multiple small elastic scatterers. The proof is based on the boundary-layer-potential technique. The results obtained are of significant mathematical interests for their own sake, particularly for the regularized approximate cloaking of elastic waves, and we shall explore this aspect in our future study. Clearly, the reconstruction methods developed in this paper can be readily adapted into certain numerical recovery schemes, and we shall present such a numerical study, together with some other interesting issues, in a forthcoming paper.

Acknowledgment

This work was supported by the Deutsche Forschungsgemeinschaft (DFG), Grant No. HU 2111/1-1; the National Natural Science Foundation of China (Nos. 11371115, 11201453) and the NSF grant, DMS 1207784.

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