Research Article

Mathematical Methods in the Applied Sciences

Received 16 March 2010 Published online 22 September 2010 in Wiley Online Library

(wileyonlinelibrary.com) DOI: 10.1002/mma.1375
MOS subject classification: 35 R 30; 78 A 46; 35 Q 61; 35 B 27

The linear sampling method for the inverse electromagnetic scattering by a partially coated bi-periodic structure

Guanghui Hu[a] and Bo Zhang[b*†]

Communicated by A. Kirsch

In this paper, we consider the inverse problem of recovering a doubly periodic Lipschitz structure through the measurement of the scattered field above the structure produced by point sources lying above the structure. The medium above the structure is assumed to be homogeneous and lossless with a positive dielectric coefficient. Below the structure is a perfect conductor partially coated with a dielectric. A periodic version of the linear sampling method is developed to reconstruct the doubly periodic structure using the near field data. In this case, the far field equation defined on the unit ball of $\mathbb{R}^3$ is replaced by the near field equation which is a linear integral equation of the first kind defined on a plane above the periodic surface. Copyright © 2010 John Wiley & Sons, Ltd.

Keywords: inverse problems; linear sampling method; doubly periodic structure; partially coated dielectric

1. Introduction

Consider the problem of scattering of electromagnetic waves by a doubly periodic structure of period $\Lambda=(2\pi,2\pi)$ defined by

$$\tilde{\Gamma}=\{x_3=f(x_1,x_2)\,|\,f(x_1+2n_1\pi,x_2+2n_2\pi)=f(x_1,x_2)>0\quad \forall n=(n_1,n_2)\in\mathbb{Z}^2\},$$

where the function f is assumed to be Lipschitz continuous so that the periodic structure $\tilde{\Gamma}$ is a Lipschitz surface. The medium above the structure is assumed to be homogeneous with a constant dielectric coefficient $\varepsilon_0>0$, and below the structure is a perfect conductor with a partially coated dielectric boundary. The magnetic permeability is assumed to be a positive constant μ_0 throughout $\mathbb{R}^3$. Given the structure and a time-harmonic electromagnetic wave incident on the structure, the direct scattering problem is to compute the electric and magnetic distributions away from the structure. In this paper, we are interested in the inverse problem of reconstructing the shape of the bi-periodic structure from a knowledge of the incident and scattered fields. The purpose of this paper is to develop a periodic version of the Linear Sampling Method for such an inverse problem. We refer to [1] for historical remarks and details of the applications of the scattering theory in periodic structures and [2] for a recent overview of the linear sampling method.

Physically, the propagation of time-harmonic electromagnetic waves (with the time variation of the form $e^{-i\omega t}$, $\omega>0$) in a homogeneous isotropic medium in $\mathbb{R}^3$ is modeled by the time-harmonic Maxwell equations:

$$\operatorname{curl} E-ikH=0,\quad \operatorname{curl} H+ikE=0. \tag{1}$$

Here, we assume that the medium above the structure is lossless, that is, k is a positive wave number given by $k=\sqrt{\varepsilon_0\mu_0}\,\omega$ in terms of the frequency ω, the electric permittivity ε_0, and the magnetic permeability μ_0. Consider the time-harmonic plane wave

$$E^i=pe^{ikx\cdot d},\quad H^i=qe^{ikx\cdot d},$$

incident on $\tilde{\Gamma}$ from the top region $\tilde{\Omega}:=\{x\in\mathbb{R}^3\,|\,x_3>f(x_1,x_2)\}$, where $d=(\alpha_1,\alpha_2,-\beta)=(\cos\theta_1\cos\theta_2,\cos\theta_1\sin\theta_2,-\sin\theta_1)$ is the incident direction specified by θ_1 and θ_2 with $0<\theta_1<\pi$, $0<\theta_2\leqslant 2\pi$ and the vectors p and q are polarization directions satisfying that $p=\sqrt{\mu_0/\varepsilon_0}(q\times d)$ and $q\perp d$.

[a] *Weierstrass Institute for Applied Analysis and Stochastics, Mohrenstr. 39, 10117 Berlin, Germany*
[b] *LSEC and Institute of Applied Mathematics, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, 100190 Beijing, People's Republic of China*
* *Correspondence to: Bo Zhang, LSEC and Institute of Applied Mathematics, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, 100190 Beijing, People's Republic of China.*
† *E-mail: b.zhang@amt.ac.cn*

In this paper, we assume that the boundary $\tilde{\Gamma}$ has a Lipschitz dissection $\tilde{\Gamma}=\tilde{\Gamma}_D\cup\Sigma\cup\tilde{\Gamma}_I$, where $\tilde{\Gamma}_D$ and $\tilde{\Gamma}_I$ are disjoint, relatively open subsets of $\tilde{\Gamma}$ having Σ as their common boundary. Suppose that below $\tilde{\Gamma}$ is a perfect conductor partially coated by a dielectric on $\tilde{\Gamma}_I$. The problem of scattering of time-harmonic electromagnetic waves is modeled by the following exterior mixed boundary value problem:

$$\operatorname{curl}\operatorname{curl} E-k^2E=0 \quad \text{in } \tilde{\Omega}, \tag{2}$$

$$\nu\times E=0 \quad \text{on } \tilde{\Gamma}_D, \tag{3}$$

$$\nu\times\operatorname{curl} E-i\lambda(\nu\times E)\times\nu=0 \quad \text{on } \tilde{\Gamma}_I, \tag{4}$$

$$E=E^i+E^s \quad \text{in } \tilde{\Omega}, \tag{5}$$

where ν is the unit normal of $\tilde{\Gamma}$ pointing into $\tilde{\Omega}$. We assume throughout this paper that λ is a positive constant and the mixed boundary conditions on $\tilde{\Gamma}$ are 2π-periodic with respect to x_1 and x_2.

Set $\alpha=(\alpha_1,\alpha_2,0)\in\mathbb{R}^3$, $n=(n_1,n_2)\in\mathbb{Z}^2$. Motivated by the periodicity of the medium we look for α-quasi-periodic solutions in the sense that $E(x_1,x_2,x_3)e^{-i\alpha\cdot x}$ is 2π periodic with respect to x_1 and x_2, respectively. As the domain is unbounded in the x_3-direction, a radiation condition must be imposed. Physically it is required that the scattered fields remain bounded as x_3 tends to $+\infty$, which leads to the so-called outgoing wave condition of the form:

$$E^s(x)=\sum_{n\in\mathbb{Z}^2} E_n e^{i(\alpha_n\cdot x+\beta_n x_3)}, \quad x_3>\max_{x_1,x_2} f(x_1,x_2), \tag{6}$$

where $E_n=(E_n^{(1)},E_n^{(2)},E_n^{(3)})\in\mathbb{C}^3$ are constant vectors and

$$\alpha_n=(\alpha_1+n_1,\alpha_2+n_2,0)\in\mathbb{R}^3, \quad \beta_n=\begin{cases}(k^2-|\alpha_n|^2)^{\frac{1}{2}} & \text{if } |\alpha_n|<k,\\ i(|\alpha_n|^2-k^2)^{\frac{1}{2}} & \text{if } |\alpha_n|>k,\end{cases} \tag{7}$$

with $i^2=-1$. Furthermore, we assume that $\beta_n\neq 0$ for all $n\in\mathbb{Z}^2$. The series expansion in (6) will be considered as the Rayleigh series of the scattered field, and the condition is called the Rayleigh expansion radiation condition. The coefficients E_n in (6) are also called the Rayleigh sequence. From the fact that $\operatorname{div} E^s(x)=0$ in $\tilde{\Omega}$ it is clear that

$$\alpha_n\cdot E_n+\beta_n E_n^{(3)}=0. \tag{8}$$

The inverse problem considered in this paper is only concerned with determining the profile $\tilde{\Gamma}$ from a knowledge of the incident wave E^i and the tangential component of the total electric field, $\nu\times E$, on a plane $\tilde{\Gamma}_b=\{x\in\mathbb{R}^3\,|\,x_3=b\}$ above the structure. We refer to [3] for reconstructing the impedance coefficient for cracks from far field measurements. The uniqueness of this inverse problem was proved in [4] for the case when the incident waves are electric dipoles. Precisely, it was shown in [4] that, if the tangential components on $\tilde{\Gamma}_b$ of two scattered electric fields are identical for all incident electric dipoles $E^{in}(x;y)=\operatorname{curl}_x\operatorname{curl}_x\{PG(x,y)\}$ with all $y\in\tilde{\Gamma}_b$ and three linear independent vectors $P\in\mathbb{R}^3$, then their corresponding scattered periodic structures $\tilde{\Gamma}_j (j=1,2)$ and the impedance coefficients $\lambda_j (j=1,2)$ on $\tilde{\Gamma}_I$ must coincide, where $G(x,y)$ is the free-space quasi-periodic Green function (see Section 2). In this paper, we are interested in numerically reconstructing the shape of the periodic structure $\tilde{\Gamma}$ by using the idea of the linear sampling method. The linear sampling method was proposed in [5] for numerically reconstructing the shape and location of the obstacle in the inverse acoustic obstacle scattering problems. This method has attracted extensive attention in recent years since it does not need to know the physical property of the scattering obstacles in advance. The application of the linear sampling method to the inverse electromagnetic scattering problems can be found in [6–8]. Recently in [9], a periodic version of the linear sampling method was proposed and implemented for the two-dimensional TE polarization case of the inverse problem considered in this paper, where the Maxwell equations are replaced by the scalar Helmholtz equation and the boundary conditions on $\tilde{\Gamma}_D$ and $\tilde{\Gamma}_I$ are replaced with the Dirichlet and impedance conditions, respectively. In [10], Kirsch proposed a mathematically justified version of the linear sampling method, the so-called factorization method. However, it is still an open question to characterize a bounded conducting obstacle for the Maxwell equations using the factorization method (see [11]). We refer to [12–14] for the application of the factorization method to the 2D inverse problems by diffraction gratings with the Dirichlet, impedance, and transmission conditions and to [15] for a recent convergence result of the linear sampling method as well as a connection between the linear sampling and factorization methods.

The inverse scattering problem by a smooth doubly periodic structure has been studied in [16, 17] for the case when $\tilde{\Gamma}_I=\emptyset$. With a lossy medium (i.e. $\operatorname{Im}(k)>0$) above the conductor, Ammari [16] proved a global uniqueness result for the inverse problem with one incident plane wave. For the case of lossless medium (i.e. $\operatorname{Im}(k)=0$) above the conductor, a local uniqueness result was obtained by Bao and Zhou in [17] for the inverse problem with one incident plane wave by establishing a lower bound of the first eigenvalue of the $\operatorname{curl}\operatorname{curl}$ operator with the boundary condition (3) in a bounded, smooth convex domain in $\mathbb{R}^3$. The stability of the inverse problem was also studied in [17]. Recently in [18] it was proved that one incident plane wave is enough to uniquely determine a bi-periodic polyhedral structure except for several extremely exceptional cases.

Note that the inverse problem considered in this paper involves in the near field measurements since only a finite number of terms in (6) are upward propagating plane waves and the rest are evanescent modes that decay exponentially with distance away from the grating. Thus we use near field data rather than far field data to reconstruct the grating structure, which implies that the far field equation defined on the unit ball of $\mathbb{R}^3$ for the non-periodic case must be replaced by a near field equation defined on a plane above the structure. On the other hand, instead of using electromagnetic Herglotz pairs in the case of bounded obstacle scattering problems, we consider another kind of incident electric fields (see Section 3 and Remark 4.2) which lead to a denseness range result on the grating structure since scattering occurs in a half space and the solution is α-quasi-periodic depending on the incident angle of the incident direction. This differs from the original version of the linear sampling method which makes use of incident plane waves of all incident directions with three linearly independent polarization directions (cf. [7]).

The remainder of the paper is organized as follows. Section 2 is devoted to the basic quasi-periodic function spaces used in the study of electromagnetic scattering problems by periodic structures. Section 3 gives several important lemmas which are necessary for establishing the main result. The main result on the periodic Linear Sampling Method and the numerical strategies on the implementation of the linear sampling method are presented in Section 4.

2. Basic function spaces

In this section we introduce some quasi-periodic Sobolev spaces which are well-suited for our problems. Owing to the periodicity of the problem, the original problem can be reduced to a problem in a single periodic cell of the grating profile. To this end and for the subsequent analysis, we reformulate the following notations:

$$\Gamma_I=\{x\in\tilde{\Gamma}_I|0<x_1,x_2<2\pi\},\quad \Gamma_D=\{x\in\tilde{\Gamma}_D|0<x_1,x_2<2\pi\},$$
$$\Gamma=\{x_3=f(x_1,x_2)|0<x_1,x_2<2\pi\},\quad \Gamma_b=\{x_3=b|0<x_1,x_2<2\pi\},$$
$$\Omega=\{x\in\mathbb{R}^3|x_3>f(x_1,x_2),0<x_1,x_2<2\pi\},\quad \Omega_b=\{x\in\Omega|x_3<b\},$$
$$\mathbb{R}^3_\pi=\{x\in\mathbb{R}^3|0<x_1,x_2<2\pi\}$$

for any $b>\max\{f(x_1,x_2)\}$. We now introduce the scalar quasi-periodic Sobolev space:

$$H^1(\Omega_b)=\{u(x)=\sum_{n\in\mathbb{Z}^2}u_n\exp[i(\alpha_n\cdot x+\beta_n x_3)]|u\in L^2(\Omega_b),\nabla u\in(L^2(\Omega_b))^3,u_n\in\mathbb{C},\ \alpha_n \text{ and } \beta_n \text{ are defined by (7)}\}.$$

Denote by $H^{1/2}(\Gamma_b)$ the trace space of $H^1(\Omega_b)$ on Γ_b with the norm

$$||F||^2_{H^{\frac{1}{2}}(\Gamma_b)}=\sum_{n\in\mathbb{Z}^2}|F_n|^2(1+|\alpha_n|^2)^{\frac{1}{2}},\quad F\in H^{\frac{1}{2}}(\Gamma_b),$$

where $F_n=(F,\exp(i\alpha_n\cdot x))_{L^2(\Gamma_b)}$. Write $H^{-\frac{1}{2}}(\Gamma_b)=(H^{\frac{1}{2}}(\Gamma_b))'$, the dual space to $H^{\frac{1}{2}}(\Gamma_b)$. We also need some vector spaces. Let

$$H(\mathrm{curl},\Omega_b)=\left\{E(x)=\sum_{n\in\mathbb{Z}^2}E_n\exp[i(\alpha_n\cdot x+\beta_n x_3)]|E_n\in\mathbb{C}^3,E\in(L^2(\Omega_b))^3,\mathrm{curl}\,E\in(L^2(\Omega_b))^3\right\}$$

with the norm

$$||E||^2_{H(\mathrm{curl},\Omega_b)}=||E||^2_{L^2(\Omega_b)}+||\mathrm{curl}\,E||^2_{L^2(\Omega_b)}$$

and let

$$H_0(\mathrm{curl},\Omega_b)=\{E\in H(\mathrm{curl},\Omega_b),\nu\times E=0\quad\text{on }\Gamma_b\}.$$

Define

$$X:=X(\Omega_b,\Gamma_I)=\{E\in H(\mathrm{curl},\Omega_b),\nu\times E|_{\Gamma_I}\in L^2_t(\Gamma_I)\}$$

with the norm

$$||E||^2_X=||E||^2_{H(\mathrm{curl},\Omega_b)}+||\nu\times E||^2_{L^2_t(\Gamma_I)},$$

where $L^2_t(\Gamma)=\{E\in(L^2(\Gamma))^3,\nu\cdot E=0\text{ on }\Gamma\}$. For $x'=(x_1,x_2,b)\in\Gamma_b$, $s\in\mathbb{R}$ define

$$H^s_t(\Gamma_b)=\left\{E(x')=\sum_{n\in\mathbb{Z}^2}E_n\exp(i\alpha_n\cdot x')|E_n\in\mathbb{C}^3,e_3\cdot E=0,\|E\|^2_{H^s(\Gamma_b)}=\sum_{n\in\mathbb{Z}^2}(1+|\alpha_n|^2)^s|E_n|^2<+\infty\right\}$$

$$H^s_t(\mathrm{div},\Gamma_b)=\left\{E(x')=\sum_{n\in\mathbb{Z}^2}E_n\exp(i\alpha_n\cdot x')|E_n\in\mathbb{C}^3,e_3\cdot E=0,||E||^2_{H^s(\mathrm{div},\Gamma_b)}=\sum_{n\in\mathbb{Z}^2}(1+|\alpha_n|^2)^s(|E_n|^2+|E_n\cdot\alpha_n|^2)<+\infty\right\}$$

$$H^s_t(\mathrm{curl},\Gamma_b)=\left\{E(x')=\sum_{n\in\mathbb{Z}^2}E_n\exp(i\alpha_n\cdot x')|E_n\in\mathbb{C}^3,e_3\cdot E=0,||E||^2_{H^s(\mathrm{curl},\Gamma_b)}=\sum_{n\in\mathbb{Z}^2}(1+|\alpha_n|^2)^s(|E_n|^2+|E_n\times\alpha_n|^2)<+\infty\right\}$$

and write $L_t^2(\Gamma_b)=H_t^0(\Gamma_b)$. Recall that

$$H_t^{-1/2}(\mathrm{div},\Gamma_b)=\{e_3\times E|_{\Gamma_b}|E\in H(\mathrm{curl},\Omega_b)\}$$

and that the trace mapping from $H(\mathrm{curl},\Omega_b)$ to $H_t^{-1/2}(\mathrm{div},\Gamma_b)$ is continuous and surjective (see [19] and the references there). The trace space on the complementary part Γ_D of $X(\Omega_b,\Gamma_I)$ is

$$Y(\Gamma_D)=\{F\in(H^{-1/2}(\Gamma_D))^3|\exists E\in H_0(\mathrm{curl},\Omega_b)\ \text{such that}\ v\times E|_{\Gamma_I}\in L_t^2(\Gamma_I), v\times E|_{\Gamma_D}=F\}$$

which is a Banach space with the norm

$$||F||^2_{Y(\Gamma_D)}=\inf\{||E||^2_{H(\mathrm{curl},\Omega_b)}+||v\times E||^2_{L_t^2(\Gamma_I)}|E\in H_0(\mathrm{curl},\Omega_b), v\times E|_{\Gamma_I}\in L_t^2(\Gamma_I), v\times E|_{\Gamma_D}=F\}.$$

An equivalent norm to $||\cdot||_{Y_{\Gamma_D}}$ is given by (see [7, 20, 21])

$$|||F|||_1=\sup_{V\in X(\Omega_b,\Gamma_I)}\frac{|<F,V>_1|}{||V||_{X(\Omega_b,\Gamma_I)}},$$

where, for $E\in H_0(\mathrm{curl},\Omega_b)$ satisfying that $v\times E|_{\Gamma_I}\in L_t^2(\Gamma_I)$ and $v\times E|_{\Gamma_D}=F$, we have

$$<F,V>_1=\int_{\Omega_b}(\mathrm{curl}\,E\cdot V-E\cdot\mathrm{curl}\,V)dx-\int_{\Gamma_I}v\times E\cdot V ds\quad\forall V\in X(\Omega_b,\Gamma_I). \tag{9}$$

In particular, $Y(\Gamma_D)$ is a Hilbert space, and (9) can be considered as a duality between $Y(\Gamma_D)$ and its dual space $Y(\Gamma_D)'$. From (9) it can be seen that $\varphi\in Y(\Gamma_D)'$ can be extended as a function $\tilde{\varphi}\in H_{\mathrm{curl}}^{-1/2}(\Gamma)$ defined on the whole boundary Γ such that $\tilde{\varphi}|_{\Gamma_I}\in L_t^2(\Gamma_I)$.

We conclude this section with introducing the following free space Green function $\Phi(x,y)$ for the Helmholtz equation $(\Delta+k^2)u=0$ in $\mathbb{R}^3$:

$$\Phi(x,y)=\frac{e^{ik|x-y|}}{4\pi|x-y|}$$

and the following free space α-quasi-periodic Green function $G(x,y)$ for the Helmholtz equation:

$$G(x,y)=\frac{1}{8\pi^2}\sum_{n\in\mathbb{N}^2}\frac{1}{i\beta_n}\exp(i\alpha_n\cdot(x-y)+i\beta_n|x_3-y_3|) \tag{10}$$

with α_n,β_n defined in (7) in the introduction.

3. Several lemmas

In this section we prove several important lemmas that are necessary for the proof of the main theorem. We first define the incident electric field

$$E^{\mathrm{in}}(x;g):=\mathrm{curl}_x\mathrm{curl}_x\int_{\Gamma_b}\overline{g(y)G(y,x)}\mathrm{d}s(y) \tag{11}$$

for $g\in L_t^2(\Gamma_b)$. From the definition of $G(x,y)$ it is seen that $E^{\mathrm{in}}(x;g)$ satisfies the radiation condition (6) in the region above Γ_b. This means that physically the above incident field propagates upward and does not appear to be meaningful as incident waves. Thus the total electric field corresponding to $E^{\mathrm{in}}(x;g)$ cannot be generated directly. We will discuss how to solve the direct scattering problem for such incident waves in the final section. For any $g\in L_t^2(\Gamma_b)$ we next define a function $(Hg)\in B:=Y(\Gamma_D)\times L_t^2(\Gamma_I)$ by

$$(Hg)(x)=\begin{cases}v(x)\times E^{in}(x;g) & \text{on }\Gamma_D,\\ v(x)\times\mathrm{curl}\,E^{\mathrm{in}}(x;g)-i\lambda E^{\mathrm{in}}(x;g)_T & \text{on }\Gamma_I,\end{cases}$$

where, for any vector field V, $V_T:=(v\times V)\times v$ denotes its tangential component on a surface.

Lemma 3.1
The range of H is dense in B.

Proof

For $F\times h\in B^*:=Y(\Gamma_D)^{'}\times L_t^2(\Gamma_I)$, we are going to prove that $F=0, h=0$ under the assumption that $<Hg, F\times h>_{B,B^*}=0$ for any $g\in L_t^2(\Gamma_b)$. Recalling that the duality between $Y(\Gamma_D)$ and $Y(\Gamma_D)^{'}$ is defined by (9) and the duality between $L_t^2(\Gamma_I)$ and $L_t^2(\Gamma_I)$ is the L^2 scalar product, we have

$$0=\int_{\Gamma_D}\nu(x)\times\left[\mathrm{curl}_x\mathrm{curl}_x\int_{\Gamma_b}g(y)G(y,x)\mathrm{d}s(y)\right]\cdot F(x)\mathrm{d}s(x)+\int_{\Gamma_I}\left\{\nu(x)\times\left[\mathrm{curl}_x\mathrm{curl}_x\mathrm{curl}_x\int_{\Gamma_b}g(y)G(y,x)\mathrm{d}s(y)\right]\right.$$
$$\left.+i\lambda\left[\mathrm{curl}_x\mathrm{curl}_x\int_{\Gamma_b}g(y)G(y,x)\mathrm{d}s(y)\right]_T\right\}\cdot h(x)\mathrm{d}s(x)$$

As $F\in Y(\Gamma_D)^{'}$, there is an extension $\widetilde{F}\in H_{\mathrm{curl}}^{-1/2}(\Gamma)$ of F, defined on Γ, satisfying that $\widetilde{F}|_{\Gamma_I}\in L_t^2(\Gamma_I)$. Thus the above equation can be rewritten as

$$0=\int_{\Gamma}\nu(x)\times\left[\mathrm{curl}_x\mathrm{curl}_x\int_{\Gamma_b}g(y)G(y,x)\mathrm{d}s(y)\right]\cdot\widetilde{F}(x)\mathrm{d}s(x)-\int_{\Gamma_I}\nu(x)\times\left[\mathrm{curl}_x\mathrm{curl}_x\int_{\Gamma_b}g(y)G(y,x)\mathrm{d}s(y)\right]\cdot\widetilde{F}(x)\mathrm{d}s(x)$$
$$+\int_{\Gamma_I}\left\{\nu(x)\times\left[\mathrm{curl}_x\mathrm{curl}_x\mathrm{curl}_x\int_{\Gamma_b}g(y)G(y,x)\mathrm{d}s(y)\right]+i\lambda\left[\mathrm{curl}_x\mathrm{curl}_x\int_{\Gamma_b}g(y)G(y,x)\mathrm{d}s(y)\right]_T\right\}\cdot h(x)\mathrm{d}s(x).$$

Making use of the vector identity:

$$\{\mathrm{curl}_x\mathrm{curl}_x[g(y)G(y,x)]\}\cdot h(x)=\{\mathrm{curl}_y\mathrm{curl}_y[h(x)G(y,x)]\}\cdot g(y),$$

we obtain by a direct computation that for any $g\in L_t^2(\Gamma_b)$,

$$k^2\int_{\Gamma_b}E(y)\cdot g(y)\mathrm{d}s(y)=0,$$

where, for $y\in\mathbb{R}_\pi^3/\Gamma$,

$$E(y)=\frac{1}{k^2}\{\mathrm{curl}_y\mathrm{curl}_y\int_{\Gamma}G(y,x)\widetilde{F}(x)\times\nu(x)\mathrm{d}s(x)-\mathrm{curl}_y\mathrm{curl}_y\int_{\Gamma_I}G(y,x)\widetilde{F}(x)\times\nu(x)\mathrm{d}s(x)$$
$$+k^2\mathrm{curl}_y\int_{\Gamma_I}G(y,x)h(x)\times\nu(x)\mathrm{d}s(x)+i\lambda\mathrm{curl}_y\mathrm{curl}_y\int_{\Gamma_I}G(y,x)h(x)\mathrm{d}s(x)\}.$$

Note that the above definition of E is well-defined since Γ and Γ_I are bounded surfaces. Furthermore, since the integrands in the definition of E are periodic in x_1 and x_2, then the jump relations and other properties of the classical single and double potentials on closed surfaces (see [22, 23]) can be transferred to the present quasi-periodic case; see e.g. [24] for the treatment of boundary integral operators for diffraction problems by coated gratings in 2D. We also refer to [25, 26] for quasi-periodic potential theories with C^2-smooth periodic surfaces for solutions to the Maxwell equations. Noting that $\widetilde{F}\in H^{-1/2}(\mathrm{curl},\Gamma)$, one can interpret the corresponding jump relations in the sense of the L^2-limit (see [27, pp. 172]).

It is clear that

$$\mathrm{curl}\,\mathrm{curl}\,E-k^2E=0\qquad y\in\mathbb{R}_\pi^3\backslash\Gamma,$$
$$\nu\times E=0\qquad y\in\Gamma_b,$$

and that $E(y)$ propagates upward above Γ satisfying the Rayleigh expansion radiation condition (6) and propagates downward below Γ satisfying the Rayleigh expansion radiation condition (6) with α replaced by $-\alpha$. By the uniqueness of the radiating solution to the exterior problem of the Maxwell equations with the perfectly conducting condition and the analytic continuation of the solution of the Maxwell equations, it follows that $E(y)\equiv 0$ for $y_3>f(y_1,y_2)$. When $y\to\Gamma$, the following jump relations hold on Γ:

$$\nu\times E^+-\nu\times E^-=0\quad\text{on }\Gamma_D,\tag{12}$$
$$i\lambda E^+-i\lambda E^-=-i\lambda h\quad\text{on }\Gamma_I,\tag{13}$$
$$\nu\times\mathrm{curl}\,E^+-\nu\times\mathrm{curl}\,E^-=i\lambda h\quad\text{on }\Gamma_I,\tag{14}$$

where the superscripts + and − indicate the limit obtained from Ω and $\mathbb{R}_\pi^3\backslash\overline{\Omega}$, respectively. Combining these jump relations and using the fact that $\nu\times E^+=\nu\times\mathrm{curl}\,E^+=0$ lead to

$$\mathrm{curl}\,\mathrm{curl}\,E-k^2E=0\qquad y_3<f(y_1,y_2),$$
$$\nu\times E^-=0\qquad\text{on }\Gamma_D,$$
$$\nu\times\mathrm{curl}\,E^-+i\lambda E_T^-=0\qquad\text{on }\Gamma_I.$$

A similar argument as in [4] can be applied to the above problem to show that $E(y)\equiv 0$ for $y_3<f(x_1,x_2)$. Thus, we have

$$F=[\mathrm{curl}\,E]|_{\Gamma_D}=0,\quad h=-[\nu\times E]|_{\Gamma_I}=0,$$

where $[\cdot]|_{\Gamma_A}$ stands for the jump across Γ_A of a function with $A=D,I$. The proof of Lemma 3.1 is thus completed. □

The near field operator N is defined by a bounded operator from B into $H_t^{-1/2}(\,\mathrm{div}\,,\Gamma_b)$ which maps the boundary data $(h_1,h_2)\in B$ to the tangential component $e_3\times E^s(x)|_{\Gamma_b}$ of the near electric field. Here, E^s stands for the unique Rayleigh expansion radiating solution to the Maxwell equations with the following boundary conditions:

$$\nu\times E^s=h_1\quad\text{on}\ \Gamma_D,\quad \nu\times\mathrm{curl}\,E^s-i\lambda(E^s)_T=h_2\quad\text{on}\ \Gamma_I.$$

By the well-posedness of the direct problem (see [4]) it is known that N is injective and bounded. Furthermore, N is a compact operator. To see this, we need the following periodic representation formula.

Lemma 3.2
Assume that E satisfies the Rayleigh expansion radiation condition (6) and the Maxwell equations in Ω. Then for any $x\in\Omega$ we have

$$E(x)=\mathrm{curl}_x\int_\Gamma \nu(y)\times E(y)G(x,y)\mathrm{d}s(y)+\frac{1}{k^2}\mathrm{curl}_x\mathrm{curl}_x\int_\Gamma \nu(y)\times\mathrm{curl}\,E(y)G(x,y)\,\mathrm{d}s(y),$$

where $G(x,y)$ is the quasi-periodic Green function defined by (10).

Proof
For arbitrarily fixed $x\in\Omega$ and an arbitrary constant vector $P\in\mathbb{R}^3$ let $F(x,y)=PG(x,y)$ with $y\in\Omega$. Assume that $x\in\Omega_b$ for some $b>0$. Denote by $B_\delta(x)$ the small ball centered at x with radius δ such that $B_\delta(x)\subset\Omega_b$. It is clear that both E and $F(x,\cdot)$ satisfy the vector Helmholtz equation in $\Omega/B_\delta(x)$. Using Green's second vector theorem and the quasi-periodicity of E and F we have

$$\begin{aligned}0&=\int_{\Omega_b\backslash B_\delta(x)}E(y)\cdot\triangle F(x,y)-\triangle E(y)\cdot F(x,y)\mathrm{d}y\\&=\left(-\int_\Gamma+\int_{\Gamma_b}+\int_{|y-x|=\delta}\right)\left\{\nu\times E\cdot\mathrm{curl}_yF+\nu\cdot E\mathrm{div}_yF-\nu\times F\cdot\mathrm{curl}\,E\right\}\mathrm{d}s(y)\\&:=-I_1+I_2+I_3.\end{aligned}\tag{15}$$

By a direct computation, we have

$$\begin{aligned}I_1&:=\int_\Gamma\{\nu\times E\cdot\mathrm{curl}_y[PG(x,y)]+\nu\cdot E\mathrm{div}_y[PG(x,y)]-\nu\times[PG(x,y)]\cdot\mathrm{curl}\,E\}\,\mathrm{d}s(y)\\&=\int_\Gamma\{(\nu\times E)\times\nabla_yG(x,y)+(\nu\cdot E)\nabla_yG(x,y)+(\nu\times\mathrm{curl}\,E)G(x,y)\}\,\mathrm{d}s(y)\cdot P\\&:=\int_\Gamma T_G(E)(y)\,\mathrm{d}s(y)\cdot P\end{aligned}\tag{16}$$

with

$$\begin{aligned}T_G(E)(y)&=(\nu\times E(y))\times\nabla_yG(x,y)+(\nu(y)\cdot E(y))\nabla_yG(x,y)+(\nu\times\mathrm{curl}\,E(y))G(x,y)\\&=\mathrm{curl}_x[\nu(y)\times E(y)G(x,y)]-\nabla_x[\nu(y)\cdot E(y)G(x,y)]+(\nu\times\mathrm{curl}\,E(y))G(x,y),\end{aligned}$$

where we have used the fact that $\nabla_yG(x,y)=-\nabla_xG(x,y)$ to get the second equality,

$$\begin{aligned}I_2&:=\int_{\Gamma_b}T_G(E)(y)\,\mathrm{d}s(y)\cdot P\\&=\int_{\Gamma_b}\{[(\nabla_yG(x,y)\times P)\times e_3]\cdot E(y)+(\nabla_yG(x,y)\cdot P)(e_3\cdot E(y))-PG(x,y)\cdot(\mathrm{curl}\,E(y)\times e_3)\}\,\mathrm{d}s(y)=0,\end{aligned}\tag{17}$$

where the last equality follows from the Rayleigh expansion condition (6) for $E(y)$, the definition of $G(x,y)$ and the fact that $\mathrm{div}E\equiv 0$ in Ω,

$$I_3:=\int_{|y-x|=\delta}T_G(E)(y)\,\mathrm{d}s(y)\cdot P=\left(\int_{|y-x|=\delta}T_{G-\Phi}(E)(y)\,\mathrm{d}s(y)+\int_{|y-x|=\delta}T_\Phi(E)(y)\,\mathrm{d}s(y)\right)\cdot P.\tag{18}$$

As $G(x,y)-\Phi(x,y)$ is a C^∞-function with respect to y in $B_\delta(x)$ (see [26]), we have $\int_{|y-x|=\delta} T_{G-\Phi}(E)(y)\,ds(y)\to 0$ as $\delta\to 0$. The application of the mean value theorem yields that $\int_{|y-x|=\delta} T_\Phi(E)(y)\,ds(y))\to E(x)$ as $\delta\to 0$ (cf. [27]). Thus it follows from (15)–(18) that

$$E(x)=\operatorname{curl}_x\int_\Gamma \nu(y)\times E(y)G(x,y)\,ds(y)+\int_\Gamma \nu(y)\times\operatorname{curl}E(y)G(x,y)\,ds(y)-\nabla_x\int_\Gamma \nu(y)\cdot E(y)G(x,y)\,ds(y)$$

which is analogous to the well-known non-periodic Stratton-Chu representation theorem ([27, Theorem 6.1]). Finally, the application of the Stokes theorem together with the vector identity $\operatorname{curl}\operatorname{curl}=-\Delta+\nabla(\nabla\cdot)$ gives the desired result. □

It is seen from Lemma 3.2 and the well-posedness of the direct scattering problem that N is a composition of a bounded operator mapping the boundary data into the scattered field with a compact operator taking the scattered field to its tangential component of E^s on Γ_b. Thus N is compact. We now prove, with the help of Lemma 3.2, that N has a dense range in $H_t^{-1/2}(\operatorname{div},\Gamma_b)$.

Lemma 3.3
The set $\{N(\varphi,\psi)|\varphi\in Y(\Gamma_D),\psi\in L_t^2(\Gamma_I)\}$ is dense in $H_t^{-1/2}(\operatorname{div},\Gamma_b)$.

Proof
Let $h\in H_t^{-1/2}(\operatorname{curl},\Gamma_b)=H_t^{-1/2}(\operatorname{div},\Gamma_b)'$ satisfy

$$<N(\varphi,\psi),h>=0\quad \forall\varphi\in Y(\Gamma_D),\psi\in L_t^2(\Gamma_I), \tag{19}$$

where $<\cdot,\cdot>$ denotes the duality between $H_t^{-1/2}(\operatorname{curl},\Gamma_b)$ and $H_t^{-1/2}(\operatorname{div},\Gamma_b)$. Then it is sufficient to prove that $h=0$. By the definition of N and the well-posedness of the direct scattering problem there exists a unique $E\in H_{loc}(\operatorname{curl},\Omega)$ satisfying the Rayleigh expansion radiation condition (6) such that $N(\varphi,\psi)=e_3\times E$ on Γ_b. From Lemma 3.2 it follows that

$$\begin{aligned}<N(\varphi,\psi),h> &= \int_{\Gamma_b} e_3\times E(x)\cdot\overline{h(x)}\,ds(x)\\ &=\int_{\Gamma_b}(\overline{h(x)}\times e_3)\cdot\left\{\operatorname{curl}_x\int_\Gamma \nu(y)\times E(y)G(x,y)\,ds(y)\right\}ds(x)+\frac{1}{k^2}\int_{\Gamma_b}(\overline{h(x)}\times e_3)\cdot\left\{\operatorname{curl}_x\operatorname{curl}_x\int_\Gamma \nu(y)\times\operatorname{curl}E(y)G(x,y)\,ds(y)\right\}ds(x)\\ &:=I_1+I_2.\end{aligned}$$

Interchanging the order of integration gives

$$I_1=-\int_\Gamma \nu(y)\times E(y)\cdot\left\{\operatorname{curl}_y\int_{\Gamma_b}G(x,y)\overline{h(x)}\times e_3\,ds(x)\right\}ds(y),$$

$$I_2=\frac{1}{k^2}\int_\Gamma \nu(y)\times\operatorname{curl}E(y)\cdot\left\{\operatorname{curl}_y\operatorname{curl}_y\int_{\Gamma_b}G(x,y)\overline{h(x)}\times e_3\,ds(x)\right\}ds(y).$$

Let

$$F(y):=\frac{1}{k^2}\operatorname{curl}_y\operatorname{curl}_y\int_{\Gamma_b}G(x,y)\overline{h(x)}\times e_3\,ds(x),\quad y\in\mathbb{R}^3_\pi/\Gamma_b.$$

Then, since $\operatorname{curl}\operatorname{curl}=-\Delta+\nabla(\nabla\cdot)$, we have

$$\operatorname{curl}F(y)=-\operatorname{curl}_y\int_{\Gamma_b}\overline{h(x)}\times e_3G(x,y)\,ds(x).$$

Thus

$$<N(\varphi\times\psi),h>=\int_\Gamma \nu(y)\times E(y)\cdot\operatorname{curl}F(y)-\nu(y)\times F(y)\cdot\operatorname{curl}E(y)\,ds(y). \tag{20}$$

Let $\tilde E$ be the $-\alpha$-quasi-periodic Rayleigh expansion radiating solution to the problem:

$$\operatorname{curl}\operatorname{curl}\tilde E-k^2\tilde E=0\quad\text{in }\Omega,$$

$$\nu\times\tilde E=\nu\times F\quad\text{on }\Gamma_D, \tag{21}$$

$$\nu\times\operatorname{curl}\tilde E-i\lambda\tilde E_T=\nu\times\operatorname{curl}F-i\lambda F_T\quad\text{on }\Gamma_I. \tag{22}$$

From Green's second vector theorem and the Rayleigh expansion of $\tilde E$ and E it follows that

$$\int_\Gamma \nu(y)\times E(y)\cdot\operatorname{curl}\tilde E(y)-\nu(y)\times\tilde E(y)\cdot\operatorname{curl}E(y)\,ds(y)=\int_{\Gamma_b}\nu(y)\times E(y)\cdot\operatorname{curl}\tilde E(y)-\nu(y)\times\tilde E(y)\cdot\operatorname{curl}E(y)\,ds(y)=0,$$

which, in conjunction with the boundary conditions (21) and (22), $\nu\times E=\varphi$ on Γ_D and $\nu\times \mathrm{curl}\,E-i\lambda E_T=\psi$ on Γ_I, implies that

$$\int_{\Gamma_D}\nu\times F\cdot \mathrm{curl}\,E+\int_{\Gamma_I}(\nu\times \mathrm{curl}\,F-i\lambda F_T)\cdot E=\int_{\Gamma_D}\varphi\cdot \mathrm{curl}\,\tilde{E}+\int_{\Gamma_I}\psi\cdot\tilde{E}.$$

This together with (20) yields

$$\begin{aligned}<N(\varphi,\psi),h> &=\int_{\Gamma_D}\varphi\cdot \mathrm{curl}\,F-\nu\times F\cdot \mathrm{curl}\,E+\int_{\Gamma_I}\psi\cdot F-(\nu\times \mathrm{curl}\,F-i\lambda F_T)\cdot E\\ &=\int_{\Gamma_D}\varphi\cdot[\mathrm{curl}\,F-\mathrm{curl}\,\tilde{E}]+\int_{\Gamma_I}\psi\cdot(F-\tilde{E}).\end{aligned}$$

It is seen from the above identity that the conjugate operator of N is given by

$$N^*h=((\overline{\mathrm{curl}\,F-\mathrm{curl}\,\tilde{E}})_T,(\overline{F-\tilde{E}})_T)\in B.$$

Combining (19) and the boundary conditions (21) and (22) gives $\nu\times F=\nu\times\tilde{E}$ and $\nu\times \mathrm{curl}\,F=\nu\times \mathrm{curl}\,\tilde{E}$ on Γ. By Holmgren's uniqueness theorem, $F\equiv\tilde{E}$ in Ω_b. As $\nu\times F=\nu\times\tilde{E}$ on Γ_b and both F and $\tilde{E}$ satisfy the $-\alpha$-quasi-periodic Rayleigh expansion radiation condition for $x_3>b$, it follows from the uniqueness result for the exterior Dirichlet problem that $F\equiv\tilde{E}$ for $x_3>b$. Now, in view of the fact that $\tilde{E}$ is analytic in Ω, we have by the jump relation of $\mathrm{curl}\,F(y)$ as $y\to\Gamma_b$ that

$$\overline{h}=[\mathrm{curl}\,F^+-\mathrm{curl}\,F^-]|_{\Gamma_b}=[\mathrm{curl}\,E^+-\mathrm{curl}\,E^-]|_{\Gamma_b}=0,$$

which completes the proof of the lemma. □

4. The linear sampling method

For $g\in L_t^2(\Gamma_b)$ consider $E^{\mathrm{in}}(x;g)$ defined by (11) as incident waves. Denote by $E^s(x;g)$ the scattered solution of the problem (2)-(5) corresponding to $E^{\mathrm{in}}(x;g)$. To derive a periodic version of the linear sampling method consider the following near field equation:

$$\mathscr{F}(g_z):=\int_{\Gamma_b}e_3\times E^s(x,g_z)\,\mathrm{d}s(x)=e_3\times \mathrm{curl}_x\mathrm{curl}_x\{PG(x,z)\}\quad \text{on }\Gamma_b, \tag{23}$$

where $z\in\{z\in\mathbb{R}^3|0<z_3<b\}$ and $P\in\mathbb{R}^3$ is a polarization vector. It is clear that

$$NH(g)=-\mathscr{F}(g). \tag{24}$$

Theorem 4.1
Assume that Γ is Lipschitz continuous with the dissection $\Gamma=\Gamma_D\cup\Sigma\cup\Gamma_I$ and $\Gamma_I\neq\emptyset$.

(1) If $z\in\mathbb{R}^3_\pi\backslash\overline{\Omega}$, then for any small $\varepsilon>0$, there exists a $g^\varepsilon_{z,P}\in L_t^2(\Gamma_b)$ such that

$$||(\mathscr{F}g^\varepsilon_{z,P})-\nu\times \mathrm{curl}\,\mathrm{curl}\,\{PG(\cdot,z)\}||_{L_t^2(\Gamma_b)}<\varepsilon$$

and

$$||g^\varepsilon_{z,P}||_{L_t^2(\Gamma_b)}\to\infty\quad \text{as } z\to\Gamma^-.$$

(2) If $z\in\Omega$, then for any small $\varepsilon>0$ and $\delta>0$ there exists a $g^{\varepsilon,\delta}_{z,P}\in L_t^2(\Gamma_b)$ such that

$$||(\mathscr{F}g^{\varepsilon,\delta}_{z,P})-\nu\times \mathrm{curl}\,\mathrm{curl}\,\{PG(\cdot,z)\}||_{L_t^2(\Gamma_b)}<\varepsilon+\delta$$

and

$$||g^{\varepsilon,\delta}_{z,P}||_{L_t^2(\Gamma_b)}\to\infty\quad \text{as } \delta\to 0.$$

Proof
(1) Let $z\in\mathbb{R}^3_\pi/\overline{\Omega}$. In this case, $e_3\times \mathrm{curl}_x\mathrm{curl}_x\{PG(x,z)\}|_{\Gamma_b}$ is in the range of N since it is the tangential component of the electric field $E^s_{P,z}:=\mathrm{curl}_x\mathrm{curl}_x\{PG(x,z)\}$ which is a solution of the exterior mixed boundary value problem with boundary data $h_1=\nu\times E^s_{P,z}$ on Γ_D and $h_2=\nu\times \mathrm{curl}\,E^s_{P,z}-i\lambda(E^s_{P,z})_T$ on Γ_I, that is,

$$e_3\times \mathrm{curl}\,\mathrm{curl}\,\{PG(x,z)\}|_{\Gamma_b}=N(h_1,h_2). \tag{25}$$

It can then be seen from the denseness of the range of H that, for every $\varepsilon>0$ there is a $g^\varepsilon_{P,z}:=g(\cdot;\varepsilon,P,z)\in L_t^2(\Gamma_b)$ such that

$$||H(g^\varepsilon_{P,z})+(h_1,h_2)||_{Y(\Gamma_D)\times L_t^2(\Gamma_I)}<\varepsilon. \tag{26}$$

The boundedness of N implies that

$$||NHg^{\varepsilon}_{P,z}+N(h_1,h_2)||_{L^2_t(\Gamma_b)}<C\varepsilon$$

for some positive constant C. From this, (24) and (25) it follows that

$$||(\mathscr{F}g^{\varepsilon}_{z,P})-\nu\times \mathrm{curl}\,\mathrm{curl}\,\{PG(\cdot,z)\}||_{L^2_t(\Gamma_b)}<C\varepsilon.$$

Furthermore, if $z\to\Gamma^-$, then we have

$$||(h_1,h_2)||_{Y(\Gamma_D)\times L^2_t(\Gamma_I)}\to\infty$$

due to the singularity of h_1 and h_2 as $z\to\Gamma^-$. This, together with (26), gives rise to

$$\lim_{z\to\Gamma^-}||Hg^{\varepsilon}_{P,z}||_{Y(\Gamma_D)\times L^2_t(\Gamma_I)}=\infty,$$

which together with the boundedness of H implies that

$$\lim_{z\to\Gamma^-}||g^{\varepsilon}_{P,z}||_{L^2_t(\Gamma_b)}=\infty.$$

(2) Let $z\in\Omega$. In this case, $e_3\times\mathrm{curl}\,\mathrm{curl}\,\{PG(x,z)\}|_{\Gamma_b}$ is not in the range of N since, otherwise, $\mathrm{curl}\,\mathrm{curl}\,\{PG(x,z)\}$ will be a solution to the Maxwell equations in $\mathbb{R}^3_\pi/\overline{\Omega}$ which is impossible due to its singularity at z. However, using the Tikhonov regularization, we can construct a regularized solution to the near field equation (23) since, by Lemmas 3.1 and 3.3, $\mathscr{F}$ is compact and has a dense range. Specifically, for an arbitrary $\delta>0$ there exist functions $(h^{\delta}_{1,P,z},h^{\delta}_{2,P,z})\in Y(\Gamma_D)\times L^2_t(\Gamma_I)$ corresponding to some parameter $\alpha=\alpha(\delta)$ chosen by a regularization strategy (e.g. the Morozov discrepancy principle) such that

$$||N(h^{\delta}_{1,P,z},h^{\delta}_{2,P,z})-\nu\times\mathrm{curl}\,\mathrm{curl}\,\{PG(\cdot,z)\}||_{L^2_t(\Gamma_b)}<\delta. \tag{27}$$

Furthermore, using the regularization strategy and the Picard theorem (see [27]) we get

$$\lim_{\delta\to 0}\alpha(\delta)=0,\qquad \lim_{\delta\to 0}||(h^{\delta}_{1,P,z},h^{\delta}_{2,P,z})||_{Y(\Gamma_D)\times L^2_t(\Gamma_I)}=\infty. \tag{28}$$

Then by Lemma 3.1 and the boundedness of N, for any $\varepsilon>0$ it is possible to find a $g^{\varepsilon,\delta}_{P,z}\in L^2_t(\Gamma_b)$ such that

$$||N(Hg^{\varepsilon,\delta}_{P,z})-N(h^{\delta}_{1,P,z},h^{\delta}_{2,P,z})||_{L^2_t(\Gamma_b)}<\varepsilon. \tag{29}$$

Thus we have from (27) and (29) that

$$||(\mathscr{F}g^{\varepsilon,\delta}_{P,z})-\nu\times\mathrm{curl}\,\mathrm{curl}\,\{PG(\cdot,z)\}||_{L^2_t(\Gamma_b)}<\varepsilon+\delta.$$

Finally, by (28) and (29) in conjunction with the boundedness of H and N, we have

$$||g^{\varepsilon,\delta}_{z,P}||_{L^2_t(\Gamma_b)}\to\infty\quad\text{as}\quad\delta\to 0.$$

The proof is thus completed. □

We now discuss some numerical strategies on the implementation of the above linear sampling method.

As stated in Section 3, the incident waves $E^{\mathrm{in}}(x;g)$ defined by (11) are not of physical relevance since they propagate away from the surface. Thus $E^s(x;g(y))$, the scattered field corresponding to $E^{\mathrm{in}}_g(x)$, can not be generated directly. In what follows, we make use of the method of Arens and Kirsch [13] to generate $E^s(x;g)$. We first examine that

$$\begin{aligned}G(x,y)-\overline{G(y,x)}&=\frac{1}{8\pi^2}\left\{\sum_{\alpha_n\leqslant k}\frac{1}{i\beta_n}e^{i(\alpha_n\cdot(x-y)-\beta_n(y_3-x_3))}+\sum_{\alpha_n\leqslant k}\frac{1}{i\beta_n}e^{i(\alpha_n\cdot(x-y)+\beta_n(y_3-x_3))}\right\}\\&:=\Delta^{(U)}(x;y)+\Delta^{(D)}(x;y)\end{aligned} \tag{30}$$

for $y\in\Gamma_b$ and $x\in\Omega_b$. Note that $\Delta^{(U)}(x;y)$ and $\Delta^{(D)}(x;y)$ are upward and downward propagating modes, respectively. Set

$$E^{(U)}(x;g):=\mathrm{curl}_x\mathrm{curl}_x\int_{\Gamma_b}\overline{g(y)}\Delta^{(U)}(x;y)\,\mathrm{d}s(y),$$

$$E^{(D)}(x;g):=\mathrm{curl}_x\mathrm{curl}_x\int_{\Gamma_b}\overline{g(y)}\Delta^{(D)}(x;y)\,\mathrm{d}s(y),$$

$$\widetilde{E}^{in}(x;g):=\mathrm{curl}_x\mathrm{curl}_x\int_{\Gamma_b}\overline{g(y)}\{G(x;y)-\Delta^{(D)}(x,y)\}\,\mathrm{d}s(y).$$

Clearly, $\widetilde{E}^{\text{in}}(x;g)$ is propagating towards the scattering surface, hence the corresponding unique scattered field $\widetilde{E}^s(x;g)$ can be computed directly. It is seen from (30) and the boundary value of $\widetilde{E}^s(x,g)$ that

$$\begin{aligned}\nu\times\{\widetilde{E}^s(x;g)|_{\Gamma_D}+E^{(U)}(x;g)|_{\Gamma_D}\} &= \nu\times\{\widetilde{E}^s(x;g)|_{\Gamma_D}+\widetilde{E}^{\text{in}}(x;g)|_{\Gamma_D}-E^{\text{in}}(x;g)|_{\Gamma_D}\}\\ &= -\nu\times E^{\text{in}}(x;g)|_{\Gamma_D}.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\nu\times\text{curl}\,[\widetilde{E}^s(x;g)+E^{(U)}(x;g)]-i\lambda[\widetilde{E}^s(x;g)+E^{(U)}(x;g)]_T &= \nu\times\text{curl}\,[\widetilde{E}^s(x;g)+\widetilde{E}^{\text{in}}(x;g)-E^{\text{in}}(x;g)]-i\lambda[\widetilde{E}^s(x;g)+\widetilde{E}^{\text{in}}(x;g)-E^{\text{in}}(x;g)]_T\\ &= -\{\nu\times\text{curl}\,E^{\text{in}}(x;g)-i\lambda E^{\text{in}}(x;g)_T\}.\end{aligned}$$

It follows from the uniqueness of the direct scattering problem that $E^s(x;g)=\widetilde{E}^s(x;g)+E^{(U)}(x;g)$. Thus we can exactly generate $E^s(x,g)$ using the incident field $\widetilde{E}^{\text{in}}(x;g)$.

Remark 4.2
The reason why we use $E^{\text{in}}(x;g)$ defined by (11) as incident waves to reconstruct the periodic structure is that, by Lemma 3.1, such kinds of incident waves lead to a dense range in B of the operator H. Note that the conjugation in the definition of E_d^{in} is needed to ensure the trivialness of the solution to the problem (12)–(13). It should be remarked that, if $\Gamma_I=\emptyset$, that is, the total electric field $E(x)$ satisfies the perfectly conducting boundary condition $\nu\times E=0$ on Γ, then we are allowed to choose the following field as incident waves:

$$E_d^{\text{in}}(x;g):=\text{curl}_x\text{curl}_x\int_{\Gamma_b}g(y)G(x,y)\,\text{d}s(y),\quad x_3<b$$

with $g\in L_t^2(\Gamma_b)$. This kind of an incident field leads to a dense range in $H^{-1/2}(\text{ div},\Gamma)$ of the operator mapping g into the tangential component on Γ of $E_d^{\text{in}}(x;g)$. As $E_d^{\text{in}}(x;g)$ propagates downward in Ω_b, the corresponding scattered field can be produced directly. Thus the above strategy of generating the scattered field corresponding to $E^{\text{in}}(x;g)$ can be avoided.

Our reconstruction algorithm consists of the following three steps:

Step 1. Select a mesh of sampling points in a computing region $\Sigma_b=\{x\in\mathbb{R}^3_\pi|0<x_3<b,0<x_1,x_2<2\pi\}$ which contains the grating surface.
Step 2. Making use of the Tikhonov regularization and the Morozov discrepancy principle to compute an approximate solution $g^{\varepsilon}_{z,P}$ to the near field equation (23).
Step 3. Consider $||g^{\varepsilon}_{z,P}||_{L_t^2(\Gamma_b)}$ as an indicator function of the sampling points z and get the contour plot of $||g^{\varepsilon}_{z,P}||_{L_t^2(\Gamma_b)}$ as a function of z.

Remark 4.3
The above algorithm has been implemented in [9] for the two-dimensional TE polarization case, where the Maxwell equations are replaced by the scalar Helmholtz equation and the boundary conditions on Γ_D and Γ_I are replaced with the Dirichlet and impedance conditions, respectively. The numerical reconstruction results presented in [9] have shown the efficiency of the algorithm. The implementation of the above algorithm for the three-dimensional case of the full Maxwell equations is still in progress.

Remark 4.4
It follows from Theorem 4.1 that the indicator function can be used to characterize the different regions below and above the grating surface. The numerical implementation of the linear sampling method can be found in [7] for inverse electromagnetic scattering problems by general bounded obstacles, which has been proven to be very successful and effective once the necessary direct scattering data are available. In order to get a better reconstruction result the mesh in Step 1 must be fine so that the characterization of the grating surface could be clear. But this also increases the computational cost since the near field equation (23) must be solved at each sampling point. To avoid this, a multilevel linear sampling method was proposed in [28] for the inverse acoustic obstacle scattering problems. The multilevel linear sampling method has been shown to be effective and to possess asymptotically optimal computational complexity and thus provides a fast numerical technique to implement the linear sampling method.

Acknowledgements

The authors greatly acknowledge the referees' suggestions and comments in improving the original manuscript.
This work was supported by the NNSF of China under grant No. 11071244 and the German Research Foundation (DFG) under Grant No. 584/1-2.

References

1. Petit R (ed.). *Electromagnetic Theory of Gratings*. Springer: Berlin, 1980.
2. Cakoni F, Colton D. *Qualitative Methods in Inverse Scattering Theory*: *An Introduction*. Springer: Berlin, 2006.
3. Liu J, Sini M. Reconstruction of cracks of different type from far field measurements. *Mathematical Methods in the Applied Sciences* 2010; **33**:950–973.
4. Hu G, Qu F, Zhang B. The direct and inverse problems for electromagnetic diffraction by a doubly periodic structure with a partially coated dielectric. *Mathematical Methods in the Applied Sciences* 2010; **33**:147–156.
5. Colton D, Kirsch A. A simple method for solving inverse scattering problems in the resonance region. *Inverse Problems* 1996; **12**:383–393.
6. Colton D, Haddar H, Monk P. The linear sampling method for solving the electromagnetic inverse scattering problem. *SIAM Journal on Scientific Computing* 2002; **24**:719–731.
7. Cakoni F, Colton D, Monk P. The electromagnetic inverse scattering problem for partially coated Lipschitz domains. *Proceedings of the Royal Society of Edinburgh* 2004; **134A**:661–682.
8. Cakoni F, M'Barek F, Haddar H. Analysis of two linear sampling methods applied to electromagnetic imaging of buried objects. *Inverse Problems* 2006; **22**:845–867.
9. Hu G, Qu F, Zhang B. A linear sampling method for inverse problems of diffraction gratings of mixed type. 2009.
10. Kirsch A. Characterization of the shape of a scattering obstacle using the spectral data of the far field operator. *Inverse Problems* 1998; **14**:1489–1512.
11. Kirsch A. The factorization method for Maxwell's equations. *Inverse Problems* 2004; **20**:S117–S134.
12. Arens T, Grinberg N. A complete factorization method for scattering by periodic surface. *Computing* 2005; **75**:111–132.
13. Arens T, Kirsch A. The factorization method in inverse scattering from periodic structures. *Inverse Problems* 2003; **19**:1195–1211.
14. Lechleiter A. Imaging of periodic dielectrics. *BIT Numerical Mathematics* 2010; **50**:59–83.
15. Arens T, Lechleiter A. The linear sampling method revisited. *Journal of Integral Equations and Applications* 2009; **21**:179–202.
16. Ammari H. Uniqueness theorems for an inverse probelms in a doubly periodic structure. *Inverse Problems* 1995; **11**:823–833.
17. Bao G, Zhou Z. An inverse problem for scattering by a doubly periodic structure. *Transactions of the American Mathematical Society* 1998; **350**:4089–4103.
18. Bao G, Zhang H, Zou J. Unique determination of periodic polyhedral structures by scattered electromagnetic fields. *Transactions of the American Mathematical Society* 2010.
19. Buffa A, Costabel M, Sheen D. On traces for $H(\mathrm{curl}, \Omega)$ in Lipschitz domains. *Journal of Mathematical Analysis and Applications* 2002; **276**:847–867.
20. Chen Z, Du Q, Zou J. Finite element methods with matching and nonmatching meshes for Maxwell equations with discontinuous coefficients. *SIAM Journal on Numerical Analysis* 2000; **37**:1542–1570.
21. Monk P. *Finite Element Method for Maxwell's Equations*. Oxford Univ Press: Oxford, 2003.
22. Colton D, Kress R. *Integral Equation Method in Scattering Theory*. Wiley: New York, 1983.
23. Mclean W. *Strongly Elliptic Systems and Boundary Integral Equations*. Cambridge University Press: Cambridge, 2000.
24. Schmidt G. Integral equations for conical diffraction by coated gratings. *WIAS Preprint No. 1296*, 2008.
25. Chen X, Frideman A. Maxwell's equations in a periodic structure. *Transactions of the American Mathematical Society* 1991; **323**:465–507.
26. Nédélec J-C, Starling F. Integral equation method in a quasi-periodic diffraction problem for the time-harmonic Maxwell equations. *SIAM Journal on Mathematical Analysis* 1991; **22**:1679–1701.
27. Colton D, Kress R. *Inverse Acoustic and Electromagnetic Scattering Theory* (2nd edn). Springer: Berlin, 1998.
28. Li J, Liu H, Zou J. Multilevel linear sampling method for inverse scattering problems. *SIAM Journal on Scientific Computing* 2008; **30**:1228–1250.