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An inverse electromagnetic scattering problem for a bi-periodic inhomogeneous layer on a perfectly conducting plate

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This article is concerned with uniqueness for reconstructing a periodic inhomogeneous medium sitting on a perfectly conducting plate. We deal with the problem in the framework of time-harmonic Maxwell systems without *TE* or *TM* polarization. An orthogonal relation is obtained for two refractive indices and then used to prove that the refractive index can be uniquely identified from a knowledge of the incident fields and the total tangential electric field on a plane above the inhomogeneous medium, utilizing the eigenvalues and eigenfunctions of a quasi-periodic Sturm–Liouville eigenvalue problem.

Keywords: inverse electromagnetic scattering; uniqueness; periodic inhomogeneous layer; Maxwell's equations

AMS Subject Classifications: 35R30; 35P25; 35B27; 35Q60

1. Introduction

Scattering theory in periodic structures has many applications in micro-optics, radar imaging and non-destructive testing. We refer to [1] for historical remarks and details of these applications. Consider a time-harmonic electromagnetic plane wave incident on a bi-periodic layer sitting on a perfectly conducting plate in $\mathbb{R}^3$. We assume that the medium inside the layer consists of some inhomogeneous isotropic conducting or dielectric material, whereas the medium above the layer consists of some homogeneous dielectric material. Suppose the magnetic permeability is a fixed positive constant throughout the whole space. The material properties of the media are then characterized completely by an index of refraction in the layer and a positive constant above the layer. The direct scattering problem is, given the incident field and the bi-periodic refractive index, to study the electromagnetic distributions, whereas the inverse scattering problem is to determine the refractive index from the knowledge of the incident waves and their corresponding measured scattered fields.

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Adopting the Cartesian axis $ox_1x_2x_3$ with the x_3-axis vertically upwards, perpendicular to the plate. If the refractive index is invariant in the x_2-direction, the direct and inverse problems (DP and IP) as indicated above can be dealt with in the TE polarization case where the electric field $E(x)$ is transversal to the (x_1, x_3)-plane by assuming $E = (0, u(x_1, x_3), 0)$, or in the TM polarization case where the magnetic field $H(x)$ is transversal to the (x_1, x_3)-plane by assuming $H = (0, u(x_1, x_3), 0)$. In the case of TE polarization, Kirsch [2] has studied the direct scattering problem via the variational method, and for the IP, instead of constructing the complex geometrical optical solutions as in the Calderóns problem (see [3,4]), he considered a class of eigenfunctions to a special kind of quasi-periodic Sturm–Liouville eigenvalue problem. Relying on the asymptotic behaviour of those eigenvalues, the uniqueness result for the IP can be proved once the orthogonal relation for two different refractive indices has obtained. See also [5,6] for the direct and inverse acoustic scattering by periodic, inhomogeneous, penetrable medium in the whole $\mathbb{R}^2$. Other uniqueness results for reconstructing the profile of a bi-periodic perfectly conducting grating can be seen in [7–9].

In this article, we are mainly concerned with the uniqueness issue for reconstructing the refractive index in the framework of time-harmonic Maxwell equations without TE or TM polarization. The uniqueness result for the IP in this article is most closely related in terms of result and method of argument to Kirsch on the determination of the refractive index in the TE polarization. Inspired by [10] and [11], we obtain an orthogonality relation for two different refractive indices by using a D-to-N map on an artificial boundary on which the tangential electric fields are identical for an integral type of incident electric field. It should be remarked that the method for constructing geometry optical solutions in [3,10,11] for non-periodic inverse conductivity problems does not work since the solutions are required to be quasi-periodic in the periodic case. To reconstruct the refractive index, we follow Kirsch's idea [2] (see also [6]) by considering a kind of Sturm–Liouville eigenvalue problems. We shall prove the uniqueness result when the index depends only on one direction (x_1 or x_2). However, we expect the result to hold in a more general case by constructing special solutions with suitable asymptotic behaviours for the Maxwell equations.

Scattering by bi-periodic structures have been studied by many authors using both integral equation methods and variational methods (see, e.g. [12–19]). It is known that, for all but possibly a discrete set of frequencies, the direct scattering problem has a unique weak solution in the case of bi-periodic inhomogeneous medium in the whole $\mathbb{R}^3$, of which an absorbing medium always leads to a uniqueness result for any frequency. When the refractive index is non-absorbing, uniqueness can be guaranteed in the TE mode if the refractive index satisfies an increasing criterion in the x_3-direction [5,20]. See also [21] and [22] for the uniqueness results of more general rough surface scattering by an inhomogeneous medium in a half space in the TE or TM mode. In this article, we assume that the medium inside the layer is absorbing so that the uniqueness result for the direct problem holds, implying that the D-to-N map T (at the end of Section 3), which depends on the refractive index, is well-defined.

The rest of the article is organized as follows. In Section 2, we set up the precise mathematical framework and introduce some quasi-periodic function spaces needed. In Section 3, we consider a quasi-periodic boundary value problem (QPBVP) in a periodic cell via the variational approach which is used for the study of the IP. Uniqueness and existence of solutions to the QPBVP are justified by the classic Hodge decomposition and the Fredholm alternative. This leads to the definition of a D-to-N map on an artificial boundary which is continuous and depends on the refractive index. Section 4 is devoted to the well-posedness of the scattering problem. In Section 5, we establish a uniqueness result for the inverse scattering problem.

2. Time-harmonic Maxwell equations and quasi-periodic function spaces

2.1. *Time-harmonic Maxwell equations*

Let $\mathbb{R}^3_+ = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 > 0\}$ and assume that $\mathbb{R}^3_+$ is filled with an inhomogeneous, isotropic, conducting or dielectric medium of electric permittivity $\epsilon > 0$, magnetic permeability $\mu > 0$ and electric conductivity $\sigma \geq 0$. Suppose the medium is non-magnetic, that is, the magnetic permeability μ is a fixed constant in $\mathbb{R}^3_+$ and the field is source free. Then the electromagnetic wave propagation is governed by the time-harmonic Maxwell equations (with the time variation of the form $e^{-i\omega t}, \omega > 0$)

$$\operatorname{curl} E - i\omega\mu H = 0, \quad \operatorname{curl} H + i\omega\left(\epsilon + i\frac{\sigma}{\omega}\right)E = 0, \tag{2.1}$$

where E and H are the electric field and magnetic field, respectively. Suppose the inhomogeneous medium is 2π-periodic with respect to x_1- and x_2-directions, that is, for all $n = (n_1, n_2) \in \mathbb{Z}^2$,

$$\epsilon(x_1 + 2\pi n_1, x_2 + 2\pi n_2, x_3) = \epsilon(x_1, x_2, x_3), \quad \sigma(x_1 + 2\pi n_1, x_2 + 2\pi n_2, x_3) = \sigma(x_1, x_2, x_3).$$

Further, assume that $\epsilon(x) = \epsilon_0$, $\sigma = 0$ for $x_3 > b$ (which means that the medium above the layer is lossless) and that the inhomogeneous medium has a perfectly conducting boundary $\Gamma_0 := \{x \in \mathbb{R}^3 \mid x_3 = 0\}$.

Consider a time-harmonic electromagnetic wave (E^i, H^i) incident on the periodic inhomogeneous layer from the top region $\Omega := \{x \in \mathbb{R}^3 \mid x_3 > b\}$. The incident wave (E^i, H^i) will be assumed to be a solution of the time-harmonic Maxwell equations (2.1) in the region Ω where $\sigma = 0$. We will further assume that E^i and H^i are α-*quasi-periodic* with respect to x_1 and x_2, that is, that for some $\alpha = (\alpha_1, \alpha_2, 0) \in \mathbb{R}^3$, $E^i(x_1, x_2, x_3)e^{-i\alpha\cdot x}$ and $H^i(x_1, x_2, x_3)e^{-i\alpha\cdot x}$ are 2π periodic with respect to x_1 and x_2, respectively. The problem of scattering of time-harmonic electromagnetic waves in this model leads to the following problem (the magnetic field H is eliminated):

$$\operatorname{curl}\operatorname{curl} E - k^2 E = 0 \quad \text{in } x_3 > b, \tag{2.2}$$

$$\operatorname{curl}\operatorname{curl} E - k^2 q E = 0 \quad \text{in } \Omega_b, \tag{2.3}$$

$$\nu \times E = 0 \quad \text{on } \Gamma_0, \tag{2.4}$$

$$E = E^i + E^s \quad \text{in } \mathbb{R}^3_+, \tag{2.5}$$

where $k = \sqrt{\epsilon_0\mu}\,\omega$ is the wave number, $q(x) = (\epsilon(x) + i\sigma(x)/\omega)/\epsilon_0$ is the refractive index, ν is the unit normal at the boundary and E^s is the scattered electric field.

The periodicity of the medium motivates us to look for α-quasi-periodic solutions. Since the domain is unbounded in the x_3-direction, a radiation condition must be imposed. It is required physically that the scattered field remains bounded as x_3 tends to $+\infty$, which leads to the so-called outgoing wave condition in the form of

$$E^s(x) = \sum_{n\in\mathbb{Z}^2} E_n e^{i(\alpha_n\cdot x+\beta_n x_3)}, \quad x_3 > b, \tag{2.6}$$

where $\alpha_n = (\alpha_1 + n_1, \alpha_2 + n_2, 0) \in \mathbb{R}^3$, $E_n = (E_n^{(1)}, E_n^{(2)}, E_n^{(3)}) \in \mathbb{C}^3$ are constant vectors and

$$\beta_n = \begin{cases} (k^2 - |\alpha_n|^2)^{\frac{1}{2}} & \text{if } |\alpha_n| < k, \\ i(|\alpha_n|^2 - k^2)^{\frac{1}{2}} & \text{if } |\alpha_n| > k, \end{cases}$$

with $i^2 = -1$. Furthermore, we assume that $\beta_n \neq 0$ for all $n \in \mathbb{Z}^2$. The series expansion in (2.6) is considered as the Rayleigh series of the scattered field and the condition is called the Rayleigh expansion radiation condition. The coefficients E_n in (2.6) are also called the Rayleigh sequence. From the fact that $\operatorname{div} E^s(x) = 0$ it is clear that

$$\alpha_n \cdot E_n + \beta_n E_n^{(3)} = 0.$$

The DP is to compute the total field E in $\mathbb{R}^3_+$, given the incident wave E^i, the refractive index $q(x)$ and the boundary condition on Γ_0. Since only a finite number of terms in (2.6) are upward propagating plane waves and the rest is evanescent modes that decay exponentially with distance away from the periodic medium, we use the near field data rather than the far field data to reconstruct the refractive index $q(x)$. Thus, our IP is to determine the periodic medium $q(x)$ from a knowledge of the incident wave E^i and the total tangential electric field $\nu \times E$ on a plane $\Gamma_a = \{x \in \mathbb{R}^3 \mid x_3 = a\}$ above the layer ($a > b$).

Remark 2.1 Two frequently used incident waves are the plane waves

$$E^i = pe^{ikx\cdot d}, \quad H^i = se^{ikx\cdot d},$$

where $d = (\alpha_1, \alpha_2, -\beta) = (\cos\theta_1 \cos\theta_2, \cos\theta_1 \sin\theta_2, -\sin\theta_1)$ is the incident wave vector whose direction is specified by θ_1 and θ_2 with $0 < \theta_1 < \pi$, $0 < \theta_2 \leq 2\pi$ and the vectors p and s are polarization directions satisfying that $p = \sqrt{\mu/\varepsilon}(s \times d)$ and $s \perp d$, and the electric dipole of the form

$$E^i(x, g) = \operatorname{curl}_x \operatorname{curl}_x \int_{\Gamma_a} G(x, y)\, g(y) \mathrm{d}s(y), \quad x_3 < a \ (a > b) \tag{2.7}$$

for some function g, where $G(x, y)$ is the free space α-quasi-periodic Green function for the Helmholtz equation $(\Delta + k^2)u = 0$ in $\mathbb{R}^3$ given by [18]

$$G(x, y) = \frac{1}{8\pi^2} \sum_{n\in\mathbb{Z}^2} \frac{1}{i\beta_n} \exp(i\alpha_n \cdot (x - y) + i\beta_n |x_3 - y_3|). \tag{2.8}$$

For our IP we will use the above electric dipole as incident waves.

2.2. *Quasi-periodic function spaces*

In this section we introduce some function spaces needed for the scattering problem (2.2)–(2.5). These spaces will play a crucial role not only in the study of the DP but also in the IP. In [13,14,19], the authors always seek the H^1-variational solution for the magnetic field H, based on the facts that the magnetic permeability $\mu > 0$ is a constant and that any vector field $H \in L^2(D)^3$ satisfying that $\nabla \times H \in L^2(D)^3$ and $\nabla \cdot H \in L^2(D)^3$ belongs to $H^1_{\text{loc}}(D)^3$ for any bounded domain $D \subset \mathbb{R}^3$. In this article, based on the classic Hodge decomposition, we are interested in weak solutions in $H_{\text{loc}}(\text{curl}, \mathbb{R}^3)$ of the problem (2.2)–(2.5), that is, both E and $\nabla \times E$ belong to $L^2_{\text{loc}}(\mathbb{R}^3_+)^3$. This allows us to solve the scattering problem in a general case when μ is a periodic variable function other than a constant.

The scattering problem can be reduced to a single periodic cell. To this end, we reformulate the following notations:

$$\Gamma_b = \{x_3 = b \mid 0 < x_1, x_2 < 2\pi\}, \quad \Omega_b = \{x \in \mathbb{R}^3_+ \mid x_3 < b,\ 0 < x_1, x_2 < 2\pi\}.$$

We also need the following scalar quasi-periodic Sobolev space:

$$H^1(\Omega_b) = \left\{ u(x) = \sum_{n \in \mathbb{Z}^2} u_n(x_3) \exp(i\alpha_n \cdot x) \mid u \in L^2(\Omega_b), \nabla u \in (L^2(\Omega_b))^3, u_n \in \mathbb{C} \right\}.$$

Denote by $H^{\frac{1}{2}}(\Gamma_b)$ the trace space of $H^1(\Omega_b)$ on Γ_b with the norm

$$\|f\|^2_{H^{\frac{1}{2}}(\Gamma_b)} = \sum_{n \in \mathbb{Z}^2} |f_n|^2 (1 + |\alpha_n|^2)^{\frac{1}{2}}, \quad f \in H^{\frac{1}{2}}(\Gamma_b),$$

where $f_n = (f, \exp(i\alpha_n \cdot x))_{L^2(\Gamma b)}$ and write $H^{-\frac{1}{2}}(\Gamma_b) = (H^{\frac{1}{2}}(\Gamma_b))'$, the dual space to $H^{\frac{1}{2}}(\Gamma_b)$.

We now introduce some vector spaces. Let

$$H(\text{curl}, \Omega_b) = \left\{ E(x) = \sum_{n \in \mathbb{Z}^2} E_n(x_3) \exp(i\alpha_n \cdot x) \mid E_n \in \mathbb{C}^3, \right.$$
$$\left. E \in (L^2(\Omega_b))^3,\ \text{curl}\, E \in (L^2(\Omega_b))^3 \right\}$$

with the norm

$$\|E\|^2_{H(\text{curl},\, \Omega_b)} = \|E\|^2_{L^2(\Omega_b)} + \|\text{curl}\, E\|^2_{L^2(\Omega_b)}.$$

Note that the α-quasi-periodic space $H(\text{curl}, \Omega_b)$ is a subset of the classical vector space $\mathbb{H}(\text{curl}, \Omega_b)$ defined by

$$\mathbb{H}(\text{curl}, \Omega_b) = \{E \in (L^2(\Omega_b))^3 \mid \text{curl}\, E \in (L^2(\Omega_b))^3\}$$

with the norm $\|E\|^2_{\mathbb{H}(\text{curl},\, \Omega_b)} = \|E\|^2_{L^2(\Omega_b)} + \|\text{curl}\, E\|^2_{L^2(\Omega_b)}$. Further, it was shown in [23] that $H(\text{curl}, \Omega_b)$ can be characterized as

$$H(\text{curl}, \Omega_b) = \{E \in \mathbb{H}(\text{curl}, \Omega_b) \mid e^{2\pi i \alpha_1} E(0, x_2, x_3) \times e_1 = E(2\pi, x_2, x_3) \times e_1,$$
$$e^{2\pi i \alpha_2} E(x_1, 0, x_3) \times e_2 = E(x_1, 2\pi, x_3) \times e_2\},$$

where $e_1 = (1, 0, 0)$ and $e_2 = (0, 1, 0)$.

For $x' = (x_1, x_2, b) \in \Gamma_b$, $s \in \mathbb{R}$ define

$$H_t^s(\Gamma_b) = \left\{ E(x') = \sum_{n \in \mathbb{Z}^2} E_n \exp(i\alpha_n \cdot x') \,|\, E_n \in \mathbb{C}^3,\ e_3 \cdot E = 0, \right.$$

$$\left. \|E\|^2_{H^s(\Gamma_b)} = \sum_{n \in \mathbb{Z}^2} (1 + |\alpha_n|^2)^s |E_n|^2 < +\infty \right\},$$

$$H_t^s(\mathrm{div}, \Gamma_b) = \left\{ E(x') = \sum_{n \in \mathbb{Z}^2} E_n \exp(i\alpha_n \cdot x') \,|\, E_n \in \mathbb{C}^3,\ e_3 \cdot E = 0, \right.$$

$$\left. \|E\|^2_{H^s(\mathrm{div}, \Gamma_b)} = \sum_{n \in \mathbb{Z}^2} (1 + |\alpha_n|^2)^s (|E_n|^2 + |E_n \cdot \alpha_n|^2) < +\infty \right\},$$

$$H_t^s(\mathrm{curl}, \Gamma_b) = \left\{ E(x') = \sum_{n \in \mathbb{Z}^2} E_n \exp(i\alpha_n \cdot x') \,|\, E_n \in \mathbb{C}^3,\ e_3 \cdot E = 0, \right.$$

$$\left. \|E\|^2_{H^s(\mathrm{curl}, \Gamma_b)} = \sum_{n \in \mathbb{Z}^2} (1 + |\alpha_n|^2)^s (|E_n|^2 + |E_n \times \alpha_n|^2) < +\infty \right\},$$

and write $L_t^2(\Gamma_b) = H_t^0(\Gamma_b)$. Recalling the trace theorem on $\mathbb{H}(\mathrm{curl}, \Omega_b)$, we have

$$H_t^{-1/2}(\mathrm{div}, \Gamma_b) = \{ e_3 \times E|_{\Gamma_b} \,|\, E \in H(\mathrm{curl}, \Omega_b) \}$$

and that the trace mapping from $H(\mathrm{curl}, \Omega_b)$ to $H_t^{-1/2}(\mathrm{div}, \Gamma_b)$ is continuous and surjective (see [24] and the references therein).

We assume throughout this article that q satisfies the following conditions:

(A1) $q \in C^1(\overline{\Omega_b})$ and $q(x) = 1$ when $x_3 > b$;
(A2) $\mathrm{Im}[q(x)] \geq 0$ for all $x \in \overline{\Omega_b}$ and $\mathrm{Im}[q(x_0)] > 0$ for some $x_0 \in \overline{\Omega_b}$;
(A3) $\mathrm{Re}[q(x)] \geq \gamma$ for all $x \in \overline{\Omega_b}$ for some positive constant γ.

3. A QPBVP

Before studying the original problem (2.2)–(2.6), we consider the following QPBVP in Ω_b:

$$\mathrm{curl}\,\mathrm{curl}\,E - k^2 q(x) E = 0 \quad \text{in } \Omega_b, \tag{3.1}$$

$$\nu \times E = 0 \quad \text{on } \Gamma_0, \tag{3.2}$$

$$\nu \times E = f \quad \text{on } \Gamma_b, \tag{3.3}$$

where $f \in H_{\mathrm{div}}^{-1/2}(\Gamma_b)$ with the norm

$$\|f\|_{H_{\mathrm{div}}^{-1/2}(\Gamma_b)} = \inf\{ \|W\|_{H(\mathrm{curl}, \Omega_b)} \,|\, \nu \times W = 0 \text{ on } \Gamma_0 \text{ and } \nu \times W = f \text{ on } \Gamma_b \}.$$

LEMMA 3.1 *If the conditions* (A1)–(A3) *are satisfied, then the problem* (3.1)–(3.3) *has a unique solution* $E \in H(\mathrm{curl}, \Omega_b)$ *such that*

$$\|E\|_{H(\mathrm{curl}, \Omega_b)} \leq C \|f\|_{H_{\mathrm{div}}^{-1/2}(\Gamma_b)}, \tag{3.4}$$

where C *is a positive constant independent of* f.

Proof We first prove the uniqueness part. Let $f=0$. Multiplying both sides of (3.1) by $\overline{E}$ it follows from Green's vector formula, the quasi-periodic property of E and the boundary conditions (3.2) and (3.3) that

$$\int_{\Omega_b}[|\mathrm{curl}\, E|^2 - k^2 q|E|^2]\mathrm{d}x = 0.$$

Take the imaginary part of the above equation and use the assumption on $q(x)$ to find that

$$\int_{B_\epsilon(x_0)} |E(x)|^2\, \mathrm{d}x = 0,$$

where $B_\epsilon(x_0)\subset\Omega_b$ is a small ball centred at x_0 with radius ϵ. Thus $E(x)\equiv 0$ in $B_\epsilon(x_0)$. By [25, Theorem 6] we have $E\in(H^1(\Omega_b))^3$. Thus, by the unique continuation principle [26, Theorem 2.3] we have $E\equiv 0$ in Ω_b.

We now use the variational method to prove the existence of solutions. To this end, for any $V\in H(\mathrm{curl},\, \Omega_b)$ such that $\nu\times V=0$ on $\Gamma_0\cup\Gamma_b$, multiplying both sides of (3.1) by $\overline{V}$ yields

$$\int_{\Omega_b}[\mathrm{curl}\, E\cdot \mathrm{curl}\,\overline{V} - k^2 qE\cdot\overline{V}]\mathrm{d}x = 0. \tag{3.5}$$

There exists at least one element $W\in H(\mathrm{curl},\, \Omega_b)$ satisfying that $\nu\times W=0$ on Γ_0 and $\nu\times W=f$ on Γ_b. Let $X:=\{U\in H(\mathrm{curl},\Omega_b)\,|\,\nu\times U=0 \text{ on } \Gamma_0\cup\Gamma_b\}$. Then $U:=E-W\in X$. Thus the problem (3.1)–(3.3) is equivalent to the following variational problem: find $U\in X$ such that

$$a(U,V) = F_W(V) \quad \forall V\in X, \tag{3.6}$$

where $a(U,V)=\int_{\Omega_b}[\mathrm{curl}\, U\cdot\mathrm{curl}\,\overline{V}-k^2qU\cdot\overline{V}]\mathrm{d}x$ and $F_W(V)=-\int_{\Omega_b}[\mathrm{curl}\, W\cdot \mathrm{curl}\,\overline{V}-k^2qW\cdot\overline{V}]\mathrm{d}x$. The proof is broken down into the following steps.

Step 1 Let $S=\{p\in H^1(\Omega_b)\,|\,p=0 \text{ on } \Gamma_0\cup\Gamma_b\}$ and let $X_0=\{\xi\in X\,|\,a(\xi,\nabla p)=0\ \forall p\in S\}$. Then it is easy to prove the Hodge decomposition:

$$X = X_0 \oplus \nabla S. \tag{3.7}$$

Step 2 To prove the existence of a unique solution $U\in X$ to the problem (3.6).

By (3.7) we may assume that $U=\xi+\nabla p$, $V=\eta+\nabla q$ with $\xi,\eta\in X_0$ and $p,q\in S$. Then the problem (3.6) becomes the following one: find $\xi\in X_0$ and $p\in S$ such that

$$a(\nabla p,\nabla q) + a(\xi,\eta) = F_W(\nabla q) + F_W(\eta).$$

Since $a(\cdot,\cdot)$ is coercive on ∇S, there exists a unique $p\in S$ such that

$$a(\nabla p,\nabla q) = F_W(\nabla q) \quad \forall\, q\in S$$

with the estimate $\|\nabla p\|_{H(\mathrm{curl},\Omega_b)}\le C\|W\|_{H(\mathrm{curl},\Omega_b)}$. It remains to find $\xi\in X_0$ such that $a(\xi,\eta)=F_W(\eta)$ for all $\eta\in X_0$. The bilinear form $a(\cdot,\cdot)$ can be decomposed into the sum of the following two forms:

$$a_1(\xi,\eta) = \int_{\Omega_b}\mathrm{curl}\,\xi\cdot\mathrm{curl}\,\overline{\eta} + \xi\cdot\overline{\eta}\,\mathrm{d}x,$$

$$a_2(\xi,\eta) = -k^2\int_{\Omega_b}(1+q)\xi\cdot\overline{\eta}\,\mathrm{d}x.$$

Obviously, $a_1(\cdot,\cdot)$ is coercive on X_0, and it follows from [27, Lemma 3.2] that X_0 is compactly embedded into $(L^2(\Omega_b))^3$. Thus, by the standard Fredholm alternative theory there exists a unique $\xi \in X_0$ satisfying that $a(\xi,\eta) = F_W(\eta)$ for all $\eta \in X_0$. Furthermore, $\|\xi\|_{H(\mathrm{curl},\Omega_b)} \le C\|W\|_{H(\mathrm{curl},\Omega_b)}$.

Step 3 By Steps 1 and 2 we know that $E = \xi + \nabla p + W \in H(\mathrm{curl},\,\Omega_b)$ is a solution to the problem (3.1)–(3.3) with the estimate

$$\|E\|_{H(\mathrm{curl},\,\Omega_b)} \le \|\xi\|_{H(\mathrm{curl},\,\Omega_b)} + \|\nabla p\|_{H(\mathrm{curl},\,\Omega_b)} + \|W\|_{H(\mathrm{curl},\,\Omega_b)} \le C\|W\|_{H(\mathrm{curl},\,\Omega_b)}. \tag{3.8}$$

From (3.8) and the definition of $\|f\|_{H^{-1/2}_{\mathrm{div}}(\Gamma_b)}$ it follows that $\|E\|_{H(\mathrm{curl},\,\Omega_b)} \le C\|f\|_{H^{-1/2}_{\mathrm{div}}(\Gamma_b)}$. ■

For $f \in H^{-1/2}_{\mathrm{div}}(\Gamma_b)$ define the operator T by

$$T(f) = \nu \times (\mathrm{curl}\,E \times \nu) \quad \text{on } \Gamma_b,$$

where E solves the QPBVP (3.1)–(3.3). By Lemma 3.1, the operator T is well-defined. Note that $T(f)$ belongs to the dual space $(H^{-1/2}_{\mathrm{div}}(\Gamma_b))' = H^{-1/2}_{\mathrm{curl}}(\Gamma_b)$ of $H^{-1/2}_{\mathrm{div}}(\Gamma_b)$ with the duality defined by

$$\langle T(f), g\rangle = \int_{\Omega_b} [\mathrm{curl}\,E \cdot \mathrm{curl}\,\overline{V} - k^2 qE \cdot \overline{V}]\mathrm{d}x$$

for $g \in H^{-1/2}_{\mathrm{div}}(\Gamma_b)$, where $V \in H(\mathrm{curl},\,\Omega_b)$ with $\nu \times V = g$ on Γ_b and $\nu \times V = 0$ on Γ_0. The operator T can be considered as a Dirichlet-to-Neumann map associated with the problem (3.1)–(3.3) and depending on the index $q(x)$. Under the assumptions (A1)–(A3), the above definition of $T(f)$ is independent of the choice of V and therefore $T: H^{-1/2}_{\mathrm{div}}(\Gamma_b) \to (H^{-1/2}_{\mathrm{div}}(\Gamma_b))' = H^{-1/2}_{\mathrm{curl}}(\Gamma_b)$ is well-defined. Moreover, it follows from the above equality and Lemma 3.1 that

$$\|T(f)\|_{H^{-1/2}_{\mathrm{curl}}(\Gamma_b)} \le C\|E\|_{H(\mathrm{curl},\,\Omega_b)} \le C\|f\|_{H^{-1/2}_{\mathrm{div}}(\Gamma_b)}$$

which implies the following result.

Corollary 3.2 *T is continuous from $H^{-1/2}_{\mathrm{div}}(\Gamma_b)$ to $H^{-1/2}_{\mathrm{curl}}(\Gamma_b)$.*

The continuity of the operator T plays an important role in the study of the IP.

4. Solvability of the scattering problem

In this section, we will establish the solvability of the scattering problem (2.2)–(2.6), employing the variational method. To this end, we propose a variational formulation of the scattering problem in a truncated domain by introducing a transparent boundary condition on Γ_b.

Let $x' = (x_1, x_2, b) \in \Gamma_b$ for $b > 0$. For $\widetilde{E} \in H_t^{-\frac{1}{2}}(\mathrm{div},\Gamma_b)$ with $\widetilde{E}(x') = \sum_{n\in\mathbb{Z}^2} \widetilde{E}_n \times \exp(i\alpha_n \cdot x')$, define $\mathcal{R}: H_t^{-\frac{1}{2}}(\mathrm{div},\Gamma_b) \to H_t^{-\frac{1}{2}}(\mathrm{curl},\Gamma_b)$ by

$$(\mathcal{R}\widetilde{E})(x') = (e_3 \times \mathrm{curl}\,E) \times e_3 \quad \text{on } \Gamma_b, \tag{4.1}$$

where E satisfying the Rayleigh expansion condition (2.6) is the unique quasi-periodic solution to the problem

$$\mathrm{curl}\,\mathrm{curl}\,E - k^2E = 0 \quad \text{for } x_3 > b, \quad \nu \times E = \widetilde{E}(x') \quad \text{on } \Gamma_b.$$

The map $\mathcal{R}$ is well-defined and can be used to replace the radiation condition (2.6) on Γ_b. Then the scattering problem (2.2)–(2.6) can be transformed into the following boundary value problem in the truncated domain Ω_b:

$$\mathrm{curl}\,\mathrm{curl}\,E - k^2qE = 0 \quad \text{in } \Omega_b, \tag{4.2}$$

$$\nu \times E = 0 \quad \text{on } \Gamma_0, \tag{4.3}$$

$$(\mathrm{curl}\,E)_T - \mathcal{R}(e_3 \times E) = (\mathrm{curl}\,E^i)_T - \mathcal{R}(e_3 \times E^i) \quad \text{on } \Gamma_b, \tag{4.4}$$

where, for any vector function U, $U_T = (\nu \times U) \times \nu$ denotes its tangential component on a surface. The variational formulation for the problem (4.2)–(4.4) is given as follows: find $E \in X := \{E \in H(\mathrm{curl}, \Omega_b) \mid \nu \times E = 0 \text{ on } \Gamma_0\}$ such that

$$\begin{aligned} B(E,\varphi) &:= \int_{\Omega_b} \left[\mathrm{curl}\,E \cdot \mathrm{curl}\,\overline{\varphi} - k^2qE \cdot \overline{\varphi}\right]\mathrm{d}x - \int_{\Gamma_b} \mathcal{R}(e_3 \times E) \cdot (e_3 \times \overline{\varphi})\mathrm{d}s \\ &= \int_{\Gamma_b} \left[(\mathrm{curl}\,E^i)_T - \mathcal{R}(e_3 \times E^i)\right] \cdot (e_3 \times \overline{\varphi})\mathrm{d}s \end{aligned} \tag{4.5}$$

for all $\varphi \in X$.

THEOREM 4.1 *Assume that the conditions* (A1)–(A3) *are satisfied. Then the problem* (2.2)–(2.6) *has a unique solution* $E \in H_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3_+)$ *such that*

$$\|E\|_{H(\mathrm{curl},\,\Omega_a)} \le C\|E^i\|_{H(\mathrm{curl},\,\Omega_b)}$$

for any $a > b$, *where* C *is a positive constant depending on the domain and* q.

Proof By using the properties of the map $\mathcal{R}$ [28] and arguing similarly as in the proof of Theorem 3.1 in [28] (cf. the proof of Lemma 3.1) it can be shown that the problem (4.5) has a unique solution $E \in X$ satisfying that $\|E\|_{H(\mathrm{curl},\Omega_b)} \le C\|E^i\|_{H(\mathrm{curl},\Omega_b)}$. It remains to extend $E(x)$ to be a function in $H_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3_+)$. Suppose $e_3 \times (E - E^i)|_{\Gamma_b} = \sum_{n\in\mathbb{N}\times\mathbb{N}} A_n e^{i\alpha_n\cdot x} \in H^{-1/2}(\mathrm{div}, \Gamma_b)$. Let

$$E^s(x) = \sum_{n\in\mathbb{N}\times\mathbb{N}} (A_n \times e_3 + B_n e_3) e^{i\alpha_n\cdot x + i\beta_n(x_3-b)}, \quad x_3 > b$$

and let E^s satisfy that $\mathrm{div}\,E^s(x) = 0$ for $x_3 > b$. Then we have $B_n = \frac{1}{\beta_n}(e_3 \times A_n)\cdot\alpha_n$. Thus

$$E^s(x) = \sum_{n\in\mathbb{N}\times\mathbb{N}} \left[A_n \times e_3 + \frac{1}{\beta_n}(e_3 \times A_n)\cdot\alpha_n e_3\right] e^{i\alpha_n\cdot x + i\beta_n(x_3-b)}, \quad x_3 > b.$$

Define $E(x) = E^i(x) + E^s(x)$ for $x_3 > b$. Then it is easy to prove that $E \in H(\mathrm{curl}, \Omega_a\backslash\Omega_b)$ with $\|E\|_{H(\mathrm{curl},\Omega_a\backslash\Omega b)} \le C\|E^i\|_{H(\mathrm{curl},\Omega_b)}$ for any $a > b$, so $E \in H(\mathrm{curl}, \Omega_a)$ for any $a > b$, that is, $E \in H_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3_+)$ with the required estimate (4.6). The proof is thus completed. ∎

5. The IP

Let $a>b$ and assume that there are two refractive index functions q_i $(i=1,2)$ satisfying the assumptions (A1)–(A3). For $g\in L_t^2(\Gamma_a)$ let the incident waves be $E^i(x,g)$ given in (2.7). Write the scattered electric field and the total electric field as $E_i^s(x,g)$ and $E_i(x,g)$, respectively, indicating their dependence on g and the refractive index function $q_i(i=1,\ 2)$.

For the refractive index q_i denote by T_i the corresponding Dirichlet-to-Neumann map associated with the problem (3.1)–(3.3) with q replaced by q_i $(i=1,\ 2)$, as defined at the end of Section 3.

Lemma 5.1 *If $T_1(f)=T_2(f)$ for all $f\in H_t^{-1/2}(\mathrm{div},\Gamma_b)$, then*

$$\int_{\Omega_b} E_1(x)\cdot\overline{E}_2(x)[q_1(x)-q_2(x)]\mathrm{d}x=0,$$

where $E_1, E_2\in H(\mathrm{curl},\ \Omega_b)$ solve the problem (3.1)–(3.3) with q replaced by q_1 and $\overline{q}_2$, respectively.

Proof Let E_1 and $F_2\in H(\mathrm{curl},\ \Omega_b)$ be the solution of the problems

$$\mathrm{curl}\,\mathrm{curl}\,E_1-k^2q_1E_1=0\quad\text{in }\Omega_b,\qquad \nu\times E_1=0\quad\text{on }\Gamma_0$$

and

$$\mathrm{curl}\,\mathrm{curl}\,F_2-k^2q_2F_2=0\quad\text{in }\Omega_b,\qquad \nu\times F_2=0\quad\text{on }\Gamma_0,\qquad \nu\times F_2=\nu\times E_1\quad\text{on }\Gamma_b,$$

respectively. Let $E=F_2-E_1$. Then it is easy to see that

$$\begin{aligned}\mathrm{curl}\,\mathrm{curl}\,E-k^2q_2E&=k^2(q_2-q_1)E_1\quad\text{in }\Omega_b,\\ \nu\times E&=0\quad\text{on }\Gamma_0\cup\Gamma_b,\\ \nu\times\mathrm{curl}\,E&=0\quad\text{on }\Gamma_b,\end{aligned}$$

where the last equality is obtained from the assumption $T_1=T_2$. Thus, it follows from the Green vector formula that

$$\begin{aligned}\int_{\Omega_b}(q_2-q_1)E_1\cdot\overline{E}_2\,\mathrm{d}x&=\frac{1}{k^2}\int_{\Omega_b}(\mathrm{curl}\,\mathrm{curl}\,E-k^2q_2E)\cdot\overline{E}_2\,\mathrm{d}x\\ &=\frac{1}{k^2}\int_{\Omega_b}(\mathrm{curl}\,E\cdot\mathrm{curl}\,\overline{E}_2-k^2q_2E\cdot\overline{E}_2)\mathrm{d}x\\ &=\frac{1}{k^2}\int_{\Omega_b}(E\cdot\mathrm{curl}\,\mathrm{curl}\,\overline{E}_2-k^2q_2E\cdot\overline{E}_2)\mathrm{d}x\\ &=\frac{1}{k^2}\int_{\Omega_b}(E\cdot k^2q_2\overline{E}_2-k^2q_2E\cdot\overline{E}_2)\mathrm{d}x=0.\end{aligned}$$

The proof is thus completed. ■

For $g\in L_t^2(\Gamma_a)$ appearing in the incident waves (2.7), we define an operator $F: L_t^2(\Gamma_a)\to H_t^{-1/2}(\mathrm{div},\Gamma_b)$ by

$$F(g)=e_3\times E(x,g)\quad\text{on }\Gamma_b,$$

where $E(x,\ g)$ solves the problem (2.2)–(2.5) with the incident wave $E^i(x,g)$. The operator F can be considered as an input–output operator mapping the sum of the electric dipoles to the tangential component of the corresponding total electric field on Γ_b. Moreover, for all $g \in L_t^2(\Gamma_a)$ the operator F has a dense range in $H_t^{-1/2}(\text{div}, \Gamma_b)$, as stated in the following lemma.

Lemma 5.2 *The operator F has a dense range in* $H_t^{-1/2}(\text{div}, \Gamma_b)$.

Proof We only need to prove that $F^* : H_t^{-1/2}(\text{curl}, \Gamma_b) \to L_t^2(\Gamma_a)$ is injective. First, we show that for any $f \in H_t^{-1/2}(\text{curl}, \Gamma_b)$, $F^*(f)$ is given by

$$F^*(f) = \left[\text{curl}_y \text{curl}_y \int_{\Gamma_b} \overline{G(x,y)} \text{curl}\,(\overline{V^+(x) - W(x)}) \times e_3 ds(x)\right]_T, \tag{5.1}$$

where the superscripts + and − indicate the limit obtained from $\mathbb{R}_3 \backslash \Omega_b$ and Ω_b, respectively, and for any $a > b$ the function $V \in H(\text{curl}, \Omega_b) \cap H(\text{curl}, \Omega_a \backslash \Omega_b)$ solves the problem

$$\text{curl}\,\text{curl}\, V - k^2 V = 0 \quad \text{for } x_3 > b, \tag{5.2}$$

$$\text{curl}\,\text{curl}\, V - k^2 q V = 0 \quad \text{in } \Omega_b, \tag{5.3}$$

$$\nu \times V = 0 \quad \text{on } \Gamma_0, \tag{5.4}$$

$$\nu \times V^+ - \nu \times V^- = 0 \quad \text{on } \Gamma_b, \tag{5.5}$$

$$\left[\text{curl}\, V^+ - \text{curl}\, V^-\right]_T = \bar{f} \quad \text{on } \Gamma_b \tag{5.6}$$

and satisfies the Rayleigh expansion condition (2.6) with α replaced by $-\alpha$ for $x_3 > b$, that is,

$$V(x) = \sum_{n \in \mathbb{Z}^2} V_n e^{i(\alpha_n' \cdot x + \beta_n' x_3)}, \quad x_3 \geq b \tag{5.7}$$

with $\alpha_n' = (-\alpha_1 + n_1, -\alpha_2 + n_2, 0) \in \mathbb{R}^3$, $V_n \in \mathbb{C}^3$ and

$$\beta_n' = \begin{cases} (k^2 - |\alpha_n'|^2)^{\frac{1}{2}} & \text{if } |\alpha_n'| < k, \\ i(|\alpha_n'|^2 - k^2)^{\frac{1}{2}} & \text{if } |\alpha_n'| > k. \end{cases}$$

In addition, the function W is given by

$$W(x) = \sum_{n \in \mathbb{Z}^2} V_n e^{i((\alpha_n' \cdot x + \beta_n'(2b - x_3))}, \quad x_3 \leq b. \tag{5.8}$$

In fact, for any $f \in H_t^{-1/2}(\text{curl}, \Gamma_b)$ and $g \in H_t^{-1/2}(\text{div}, \Gamma_b)$ we have

$$\begin{aligned} &\langle Fg, f\rangle_{H_t^{-1/2}(\text{div}, \Gamma_b) \times H_t^{-1/2}(\text{curl}, \Gamma_b)} \\ &= \int_{\Gamma_b} \nu \times E(\cdot, g) \cdot \bar{f} \mathrm{d}s \\ &= \int_{\Gamma_b} \nu \times E(\cdot, g) \cdot [\text{curl}\, V^+ - \text{curl}\, V^-] \mathrm{d}s \\ &= \int_{\Gamma_b} [(\nu \times E \cdot \text{curl}\, V^+ - \nu \times V^+ \cdot \text{curl}\, E) \\ &\quad - (\nu \times E \cdot \text{curl}\, V^- - \nu \times V^- \cdot \text{curl}\, E)] \mathrm{d}s, \end{aligned}$$

where the transmission conditions (5.5) and (5.6) have been used. It follows from the Maxwell equations (5.3) and (2.3) and the boundary conditions (5.4) and (2.4) that

$$\int_{\Gamma_b} [\nu \times E \cdot \operatorname{curl} V^- - \nu \times V^- \cdot \operatorname{curl} E] \mathrm{d}s = 0. \tag{5.9}$$

On the other hand, from the Rayleigh expansion conditions (2.6) and (5.7) it is derived that

$$\begin{aligned}
&\int_{\Gamma_b} [\nu \times E \cdot \operatorname{curl} V^+ - \nu \times V^+ \cdot \operatorname{curl} E] \mathrm{d}s \\
&\quad = \int_{\Gamma_b} [(\nu \times E^i \cdot \operatorname{curl} V^+ - \nu \times V^+ \cdot \operatorname{curl} E^i) \\
&\qquad + (\nu \times E^s \cdot \operatorname{curl} V^+ - \nu \times V^+ \cdot \operatorname{curl} E^s)] \mathrm{d}s \\
&\quad = \int_{\Gamma_b} [\nu \times E^i \cdot \operatorname{curl} V^+ - \nu \times V^+ \cdot \operatorname{curl} E^i] \mathrm{d}s.
\end{aligned} \tag{5.10}$$

Similarly, from the definition of E^i and the Rayleigh expansion condition (5.8) it follows that

$$\int_{\Gamma_b} [\nu \times E^i \cdot \operatorname{curl} W - \nu \times W \cdot \operatorname{curl} E^i] \mathrm{d}s = 0. \tag{5.11}$$

Equations (5.9)–(5.11) together with the fact that $V = W$ on Γ_b yield

$$\begin{aligned}
\langle Fg, f \rangle &= \int_{\Gamma_b} [\nu \times E^i \cdot \operatorname{curl} V^+ - \nu \times V^+ \cdot \operatorname{curl} E^i] \mathrm{d}s \\
&= \int_{\Gamma_b} [\nu \times E^i \cdot \operatorname{curl} V^+ - \nu \times W \cdot \operatorname{curl} E^i] \mathrm{d}s \\
&= \int_{\Gamma_b} [\nu \times E^i \cdot \operatorname{curl} V^+ - \nu \times E^i \cdot \operatorname{curl} W] \mathrm{d}s \\
&= \int_{\Gamma_b} \nu \times E^i \cdot (\operatorname{curl} V^+ - \operatorname{curl} W) \mathrm{d}s.
\end{aligned}$$

Substituting the expression (2.7) of E^i into the above equation and exchanging the order of integration we get

$$\langle Fg, f \rangle = \int_{\Gamma_a} g(y) \cdot \operatorname{curl}_y \operatorname{curl}_y \left[\int_{\Gamma_b} G(x,y) \operatorname{curl} [V^+(x) - W(x)] \times e_3 \, \mathrm{d}s(x) \right] \mathrm{d}s(y),$$

which implies (5.1).

We now prove that F^* is injective. Suppose $F^*(f) = 0$ for some $f \in H_t^{-1/2}(\operatorname{curl}, \Gamma_b)$. Define U by

$$U(y) := \operatorname{curl}_y \operatorname{curl}_y \left[\int_{\Gamma_b} \overline{G(x,y)} h(x) \mathrm{d}s(x) \right], \quad y \in \mathbb{R}^3 \backslash \Gamma_b,$$

where $h = \operatorname{curl} (\overline{V^+ - W}) \times e_3$. Then $e_3 \times U(y) = 0$ on Γ_a. It is clear that $U(y)$ is a $-\alpha$-quasi-periodic function satisfying the Rayleigh expansion condition (2.6) when $y_3 > a$. By the uniqueness of solutions to the exterior Dirichlet problem [7] we have

$U(y)=0$ when $y_3>a$, which together with the unique continuation principle [29] implies that $U(y)=0$ when $y_3>b$. Now from the jump relation $e_3\times U^+(y)-e_3\times U^-(y)=0$ on Γ_b and again the uniqueness of solutions for the exterior Dirichlet problem for $y_3<b$ we get that $U(y)=0$ when $y_3<b$. Thus, $h(y)=e_3\times \mathrm{curl}[U^+(y)-U^-(y)]=0$ on Γ_b, which, together with (5.7) and (5.8), implies that

$$e_3\times V^+=e_3\times W,\quad e_3\times \mathrm{curl}\, V^+=e_3\times \mathrm{curl}\, W\quad \text{on } \Gamma_b. \tag{5.12}$$

Since V and W satisfy the Maxwell equation curl curl $E-k^2E=0$ in the regions $x_3>b$ and $x_3<b$, respectively, then it easily follows from the transmission condition (5.12) and the Rayleigh expansion conditions (5.7) and (5.8) that $V=0$ for $x_3>b$ and $W=0$ for $x_3<b$. Thus, by (5.5) we have $\nu\times V^-=0$ on Γ_b, so $V\in H(\mathrm{curl},\ \Omega_b)$ satisfies the problem (3.1)–(3.3) with $f=0$. By Lemma 3.1 we have $V=0$ in Ω_b. Thus, $f=[\mathrm{curl}\,\overline{V}^+-\mathrm{curl}\,\overline{V}^-]_T=0$, which completes the proof of Lemma 5.2. ■

Combining Lemmas 5.1 and 5.2, we have the following orthogonality relation for two different functions q_i $(i=1,2)$.

LEMMA 5.3 *Let the incident waves $E^i(x,\ g)$ be defined by* (2.7). *If*

$$e_3\times E_1(x,g)=e_3\times E_2(x,g)\quad on\ \Gamma_a \tag{5.13}$$

for all $g\in L_t^2(\Gamma_a)$ and some $a>b$, then the following orthogonality relation holds:

$$\int_{\Omega_b} E_1(x)\cdot \overline{E}_2(x)(q_1(x)-q_2(x))\mathrm{d}x=0,$$

where $E_1,E_2\in H(\mathrm{curl},\ \Omega_b)$ solve the problem (3.1)–(3.3) *with q replaced by q_1 and $\overline{q}_2$, respectively.*

Proof From Equation (5.13), the uniqueness of solutions for the exterior Dirichlet problem and the unique continuation principle it follows that $E_1(x,\ g)=E_2(x,\ g)$ for all $x_3>b$. This implies that

$$e_3\times \mathrm{curl}\, E_1^+(x,g)=e_3\times \mathrm{curl}\, E_2^+(x,g)\quad \text{on } \Gamma_b.$$

Since $[e_3\times \mathrm{curl}\, E_j^+(x;g)]|_{\Gamma_b}=0$ for $j=1,2$, then we have

$$e_3\times \mathrm{curl}\, E_1^-(x,g)=e_3\times \mathrm{curl}\, E_2^-(x,g)\quad \text{on } \Gamma_b.$$

By the above two equalities and the definition of T_i we have

$$T_1(e_3\times E_1(x,g))=T_2(e_3\times E_2(x,g))$$

for all $g\in L_t^2(\Gamma_a)$. The continuity of T_j $(j=1,2)$ and Lemma 5.2 lead to

$$T_1(f)=T_2(f)\quad \forall f\in H_t^{-1/2}(\mathrm{div},\Gamma_b).$$

This together with Lemma 5.1 gives the desired result. ■

We are now ready to prove our main result for the inverse scattering problem.

THEOREM 5.4 *Let q_j $(j=1,2)$ satisfy the Assumptions* (A1)–(A3) *and let q_j depend on only one direction x_1 or x_2 with $j=1,2$. If*

$$e_3\times E_1(x,g)=e_3\times E_2(x,g)\quad on\ \Gamma_a$$

for all $g \in L_t^2(\Gamma_a)$ *with some* $a > b$, *where* $E_j(x, g)$ *solves the problem* (2.2)–(2.5) *with* $q = q_j (j = 1, 2)$ *corresponding to the incident wave* $E^i(x, g)$ *given by* (2.7), *then* $q_1 = q_2$.

Proof By Lemma 5.3 we have the orthogonality relation:

$$\int_{\Omega_b} E_1(x) \cdot \overline{E}_2(x)[q_1(x) - q_2(x)]\mathrm{d}x = 0, \tag{5.14}$$

where $E_1, E_2 \in H(\mathrm{curl}, \Omega_b)$ solve the problem (3.1)–(3.3) with q replaced by q_1 and $\overline{q}_2$, respectively.

We now look for solutions to the problem (3.1)–(3.3) in the following form:

$$E(x) = (0, 0, E_3(x_1, x_2)) = (0, 0, v(x_1)u(x_2))$$

with the scalar functions v and u satisfying the following quasi-periodic conditions:

$$v(x_1)e^{2i\alpha_1\pi} = v(x_1 + 2\pi), \quad u(x_2)e^{2i\alpha_2\pi} = v(x_2 + 2\pi).$$

It is clear that such a function E is α-quasi-periodic and satisfies the boundary condition (3.2). Without loss of generality, we may assume that $q_j(x) = q_j(x_1)$, that is, q_j depends only on the x_1-direction with $j = 1, 2$. Substituting such E into the Maxwell equation (3.1) and noting that curl, curl $= -\Delta + \nabla(\nabla\cdot)$, we find that

$$v''(x_1)u(x_2) + v(x_1)u''(x_2) + k^2q(x_1)v(x_1)u(x_2) = 0, \quad x_1, x_2 \in (0, 2\pi).$$

This implies that

$$\frac{v''(x_1)}{v(x_1)} + k^2q(x_1)v(x_1) = \frac{u''(x_2)}{u(x_2)} = \lambda$$

for some constant λ, where $x_1, x_2 \in (0, 2\pi)$. Following the idea of Kirsch [2], we construct a special kind of solutions v by considering the following quasi-periodic Sturm–Liouville eigenvalue problem:

$$(\mathrm{I}): \quad \begin{cases} v''(x_1) + k^2q(x_1)v(x_1) = \lambda v(x_1), & x_1 \in (0, 2\pi) \\ v(x_1)e^{2i\alpha_1\pi} = v(x_1 + 2\pi), \\ v'(x_1)e^{2i\alpha_1\pi} = v'(x_1 + 2\pi). \end{cases}$$

The eigenvalues λ_n and the corresponding eigenfunctions v_n, normalized to $v_n(0) = 1$, have the following asymptotic behaviours as $n \to \infty$ [30]:

$$\lambda_n^{\pm} = \left(n \pm \frac{\alpha_1}{2\pi}\right)^2 - \frac{k^2}{2\pi}\int_0^{2\pi} q(s)\mathrm{d}s + \mathcal{O}\left(\frac{1}{n}\right),$$
$$v_n^{\pm}(x_1) = \exp\left[i\left(\pm n + \frac{\alpha_1}{2\pi}\right)x_1\right] + \mathcal{O}\left(\frac{1}{n}\right)$$

which are uniform in $x_1 \in [0, 2\pi]$. We also consider the following quasi-periodic boundary problem for u:

$$(\mathrm{II}): \quad \begin{cases} u''(x_2) - \lambda_n u(x_2) = 0, & x_2 \in (0, 2\pi) \\ u(x_2)e^{2i\alpha_2\pi} = v(x_2 + 2\pi). \end{cases}$$

The non-trivial solutions to the problem (II) can be written explicitly as

$$u_n(x_2) = c_{n,1}e^{\sqrt{\lambda_n}x_2} + c_{n,1}e^{-\sqrt{\lambda_n}x_2}, \quad \lambda_n \neq 0,$$

where $c_{n,1}$ and $c_{n,2}$ are constants satisfying

$$c_{n,1} = c_{n,2}\left(e^{-2\pi\sqrt{\lambda_n}} - e^{i2\pi\alpha_2}\right)\Big/\left(e^{i2\pi\alpha_2} - e^{2\pi\sqrt{\lambda_n}}\right). \tag{5.15}$$

Now, let $E^{\pm}_{3,n} = v^{\pm}_n(x_1)u^{\pm}_n(x_2)$ be the third component of $E^{\pm}_n = (0, 0, E^{\pm}_{3,n})$ corresponding to $q_1(x_1)$ and let $E^{\pm}_{3,m} = v^{\pm}_m(x_1)u^{\pm}_m(x_2)$ be the third component of $E^{\pm}_n$ corresponding to $\overline{q_2}(x_1)$. It follows from (5.14) that

$$0 = \int_{\Omega_b} E_{3,n}(x_1, x_2) \cdot \overline{E}_{3,m}(x_1, x_2)[q_1(x_1) - q_2(x_1)]\mathrm{d}x = bA^{n,m}_1 A^{n,m}_2, \tag{5.16}$$

where

$$A^{n,m}_1 := \int_0^{2\pi} [q_1(x_1) - q_2(x_1)]e^{i(n-m)x_1}\mathrm{d}x_1 + \mathcal{O}\left(\frac{1}{n}\right) + \mathcal{O}\left(\frac{1}{m}\right),$$

$$A^{n,m}_2 := \int_0^{2\pi} \left(c_{n,1}e^{\sqrt{\lambda_n}x_2} + c_{n,2}e^{-\sqrt{\lambda_n}x_2}\right)\left(\overline{c_{m,1}e^{\sqrt{\lambda_m}x_2} + c_{m,2}e^{-\sqrt{\lambda_m}x_2}}\right)\mathrm{d}x_2$$

and $c_{n,j}$, $c_{m,j}$ satisfy (5.15) with $j = 1, 2$. For arbitrarily fixed $l \in \mathbb{N}$, letting $m = n - l$ gives

$$A^{m+l,m}_1 = \int_0^{2\pi} [q_1(x_1) - q_2(x_1)]e^{ilx_1}\mathrm{d}\,(x_1) + \mathcal{O}\left(\frac{1}{m}\right),$$

$$A^{m+l,m}_2 = \int_0^{2\pi} \left(c_{m+l,1}e^{\sqrt{\lambda_{m+l}}x_2} + c_{m+l,2}e^{-\sqrt{\lambda_{m+l}}x_2}\right)\left(\overline{c_{m,1}e^{\sqrt{\lambda_m}x_2} + c_{m,2}e^{-\sqrt{\lambda_m}x_2}}\right)\mathrm{d}x_2.$$

We can always choose appropriate constants $c_{m,2}$ and $c_{m,1}$ satisfying (5.15) such that $A^{m+l,m}_2 \neq 0$ for sufficiently large m. In fact, we may assume that l is a positive number since otherwise we can take $n = m - l'$ for some positive l' instead of l. Now choose $c_{m,2} = e^{2\pi\sqrt{\lambda_m}}$. Then, by (5.15), $|c_{m,1}| \geq C_1$ for large m with some positive constant C_1 independent of m and $\left|\int_0^{2\pi} c_{m,2}e^{-2\pi\sqrt{\lambda_m}x_2}\mathrm{d}x_2\right|$ tends to $+\infty$ as $m \to \infty$. This implies that $|A^{m+l,m}_2| \to +\infty$ as $m \to +\infty$. Letting $m \to +\infty$ we conclude from (5.16) and the above discussion that

$$\int_0^{2\pi} (q_1(x_1) - q_2(x_1))e^{ilx_1}\,\mathrm{d}x_1 = 0$$

for every $l \in \mathbb{N}$, which implies that $q_1 = q_2$. The proof is thus completed. ■

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