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Nearly cloaking the elastic wave fields

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A B S T R A C T

In this work, we develop a general mathematical framework on regularized approximate cloaking of elastic waves governed by the Lamé system via the approach of transformation elastodynamics. Our study is rather comprehensive. We first provide a rigorous justification of the transformation elastodynamics. Based on the blow-up-a-point construction, elastic material tensors for a perfect cloak are derived and shown to possess singularities. In order to avoid the singular structure, we propose to regularize the blow-up-a-point construction to be the blow-up-a-small-region construction. However, it is shown that without incorporating a suitable lossy layer, the regularized construction would fail due to resonant inclusions. In order to defeat the failure of the lossless construction, a properly designed lossy layer is introduced into the regularized cloaking construction. We derive sharp asymptotic estimates in assessing the cloaking performance. The proposed cloaking scheme is capable of nearly cloaking an arbitrary content with a high accuracy.
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R É S U M É

Dans cet article, on propose un cadre mathématique général pour la dissimulation d'une cible élastique dans le cadre des équations de Lamé via une approche de transformations élastodynamiques. Notre étude est exhaustive. Dans un premier temps on donne une justification rigoureuse des transformations élastodynamiques. En utilisant la construction d'un point singulier, on calcule les tenseurs élastiques nécessaires à l'invisibilité et on montre qu'ils possèent des singularités. Pour palier ce problème, on propose une méthode de régularisation. Cependant, on montre que sans l'incorporation d'une couche absorbante convenable, la structure régularisée présente des petites inclusions résonnantes qui la fragilise. Afin de contourner ce problème, on introduit une couche absorbante convenable lors de la construction. On quantifie les performances par des majorations asymptotiques optimales. La méthode proposée peut masquer quasi-complètement une cible arbitraire avec une bonne précision.
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1. Introduction

This paper concerns the cloaking of elastic waves. An elastic region is said to be cloaked if its content together with the cloak is "unseen" by the exterior elastic wave detections. In recent years, the study on elastic cloaking has gained growing interest in the physics literature (cf. [9,20–22,47,53,54]), much followed the development of transformation-optics cloaking of optical waves including the acoustic and electromagnetic waves. A proposal for cloaking for electrostatics using the invariance properties of the conductivity equation was pioneered in [26,27]. Blueprints for making objects invisible to electromagnetic (EM) waves were proposed in two articles in *Science* in 2006 [40,55]. The article by Pendry et al. uses the same transformation used in [26,27] while the work of Leonhardt uses a conformal mapping in two dimensions. The method based on the invariance properties of the equations modeling the optical wave phenomenon has been named *transformation optics* and has received a lot of attentions in the scientific community due to its significant practical importance. We refer to the survey articles [16,28,29,45] and the references therein for the theoretical and experimental progress on optical cloaking.

The Lamé system governing the elastic wave propagation also possesses a certain transformation property, in a more complicated manner than that for the optical wave equations. Using the transformation property, the transformation-elastodynamics approach can be developed for the construction of elastic cloaks, following a similar spirit to the transformation-optics construction of optical cloaks. In a rather heuristic way, an ideal invisibility cloak can be obtained by the blow-up-a-point construction proceeded as follows. One first selects a region Ω in the homogeneous space for constructing the cloaking device. Let $P \in \Omega$ be a single point and let F be a diffeomorphism which blows up P to a region D within Ω. Using transformation-elastodynamics, the ambient homogeneous medium around P is then 'compressed' via the push-forward to form the cloaking medium in $\Omega\backslash\overline{D}$, whereas the 'hole' D forms the cloaked region within which one can place the target object. The cloaking region $\Omega\backslash\overline{D}$ and the cloaked region D yield the cloaking device in the physical space, whereas the homogeneous background space containing the singular point P is referred to as the virtual space. Due to the transformation invariance of the elastic system, the exterior measurements corresponding to the cloaking device in the physical space are the same to those in the virtual space corresponding to a singular point. Intuitively speaking, the scattering information of the elastic cloak is 'hidden' in a singular point P.

However, the blow-up-a-point construction would yield server singularities for the cloaking elastic material tensors. Most of the physics literature accepts the singular structure and focuses more on the application side (cf. [9,20,21,53]). To our best knowledge, there is very little mathematical study on rigorously dealing with the singular elastic cloaking problem. On the other hand, there are a few mathematical works seriously dealing with the singular cloaking problems associated with the optical cloaks. Concurrently, there are two theoretical approaches in the literature: one approach is to accept the singularity and proposes to investigate the physically meaningful solutions, i.e. finite energy solutions, to the singular acoustic and electromagnetic wave equations (see [30,45]); the other approach is to regularize the singular ideal cloaking construction and investigate the near-invisibility instead; see [6,38] on the treatment of electrostatics, [4,7,37,41–43] on acoustics, and [8,10,11] on electromagnetism. In this work, we follow the latter approach to develop a general framework of constructing near-cloaks for elastic waves via the transformation-elastodynamics approach. Compared to the acoustic and electromagnetic cases, the elastic cloaking problem turns out to possess more complicated physical nature due to the coupling of shear and pressure waves that propagate at different speeds (see, e.g., [17–19,39]).

The present study on regularized approximate cloaking of elastic waves is rather comprehensive and includes several salient ingredients. First, we provide a rigorous justification of the transformation elastodynamics, which lacks in the physics literature. Particularly, we prove the well-posedness of the transformed Lamé system. This is presented in Section 2. In Section 3, we consider the elastic cloaking problem, and based on the blow-up-a-point transformation, we give the construction of an ideal elastic cloak and analyze the singularity of the cloaking elastic material parameters. In Section 4, we introduce the regularized construction based on a blow-up-a-small-region transformation. Then, we show the existence of resonant inclusions which defy any attempt in achieving near-cloaks. Section 5 is devoted to the development of our near-cloaking scheme by incorporating a properly designed lossy layer into the regularized construction. We derive sharp estimate in assessing the cloaking performance. The asymptotic estimate is independent of the elastic content in the cloaked region, which means the proposed cloaking scheme is capable of nearly cloaking an arbitrary elastic object. The estimate is based on the use of a variety of variational arguments and layer-potential techniques. We also verify that the proposed lossy layer is a finite realization of the a *traction-free* lining.

Finally, we would like to mention in passing that our study may find important applications in seismic metamaterials (cf. [46,35,36,12]) to construct feasible devices for protecting key structures from the catastrophic destruction of natural earthquake waves or terrorist attacks (e.g., nuclear blast). For instance, the elastic invisibility cloak could be of great significance in safeguarding nuclear power plants, electric pylons, oil refineries, nuclear reactors and old or fragile monuments as well as the important components within them.

2. Lamé system and transformation elastodynamics

Consider the time-harmonic elastic wave propagating through an anisotropic medium occupying a bounded Lipschitz domain $\Omega \subset \mathbb{R}^N$ $(N = 2, 3)$. In linear elasticity, the spatially-dependent displacement vector $u(x) = (u_1, \cdots, u_N)(x)$ is governed by the following boundary value problem of the reduced Lamé system

$$\begin{cases} \sum_{j,k,l=1}^{N} \frac{\partial}{\partial x_j}\left(C_{ijkl}(x)\frac{\partial u_k}{\partial x_l}\right) + \omega^2 \rho(x)\, u_i = 0, & \text{in } \Omega, \quad i = 1, 2, \cdots, N, \\ \mathcal{N}_{\mathcal{C}} u = \psi \in H^{-1/2}(\partial\Omega)^N, & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where ω denotes the frequency and the Neumann data $\mathcal{N}_{\mathcal{C}} u$ is defined as

$$\mathcal{N}_{\mathcal{C}} u := \left(\sum_{j,k,l=1}^{N} \nu_j C_{1jkl} \frac{\partial u_k}{\partial x_l}, \ \sum_{j,k,l=1}^{N} \nu_j C_{2jkl} \frac{\partial u_k}{\partial x_l}, \ \cdots, \ \sum_{j,k,l=1}^{N} \nu_j C_{Njkl} \frac{\partial u_k}{\partial x_l} \right)$$

with $\nu = (\nu_1, \nu_2, \ldots, \nu_N) \in \mathbb{S}^{N-1}$ denoting the exterior unit normal vector to $\partial\Omega$. In (1), $\mathcal{C} = (C_{ijkl})_{i,j,k,l=1}^{N}$ is a fourth-rank constitutive material tensor of the elastic medium which shall be referred to as the stiffness tensor. ρ is a complex-valued function with $\Re\rho > 0$ and $\Im\rho \geq 0$, respectively, denoting the density and damping parameter of the elastic medium. In this paper, we employ the notation $\{\Omega; \mathcal{C}, \rho\}$ to denote the elastic medium supported in Ω characterized by the stiffness tensor $\mathcal{C}$ with entries $C_{ijkl}(x) \in L^\infty(\Omega)$ and $\rho \in L^\infty(\Omega)$. The stiffness tensor satisfies the following symmetries for a generic anisotropic elastic material:

$$\text{major symmetry:} \quad C_{ijkl} = C_{klij}, \qquad \text{minor symmetries:} \quad C_{ijkl} = C_{jikl} = C_{ijlk}, \tag{2}$$

for all $i, j, k, l = 1, 2, \cdots, N$. By Hooke's law, the stress tensor σ relates with the stiffness tensor $\mathcal{C}$ via the identity $\sigma(u) := \mathcal{C} : \nabla u$, where the action of $\mathcal{C}$ on a matrix $A = (a_{ij})$ is defined as

$$\mathcal{C} : A = (\mathcal{C} : A)_{ij} = \sum_{k,l=1}^{N} C_{ijkl}\, a_{kl}.$$

Hence, the elliptic system in (1) can be restated as

$$\nabla \cdot (\mathcal{C} : \nabla u) + \omega^2 \rho u = 0 \quad \text{in} \quad \Omega.$$

Moreover, the boundary operator in (1) can be rewritten as $\mathcal{N}_{\mathcal{C}} u = \nu \cdot \sigma(u) = \nu \cdot (\mathcal{C} : \nabla u)$, which is exactly the *stress vector* or *traction* on $\partial\Omega$.

The equivalent variational formulation of (1) reads as follows: find $u = (u_1, \cdots, u_N) \in H^1(\Omega)^N$ such that

$$a_{\mathcal{C}}(u, v) := \int_{\Omega} \left\{ \sum_{i,j,k,l=1}^{N} C_{ijkl} \frac{\partial u_k}{\partial x_l} \frac{\partial \overline{v_i}}{\partial x_j} - \omega^2 \rho(x) u_i \bar{v}_i \right\} dx = \int_{\partial\Omega} \psi \cdot \bar{v} dx, \tag{3}$$

for any $v = (v_1, v_2, \cdots, v_N) \in H^1(\Omega)^N$. Suppose further that the elastic tensor $\mathcal{C}$ satisfies the uniform Legendre ellipticity condition

$$\sum_{i,j,k,l=1}^{N} C_{ijkl}(x)\, a_{ij} a_{kl} \geq c_0 \sum_{i,j=1}^{N} |a_{ij}|^2, \quad a_{ij} = a_{ji}, \tag{4}$$

for all $x \in \Omega$, i.e., $(\mathcal{C}(x) : A) : A \geq c_0 ||A||^2$ for all symmetry matrices $A = (a_{ij})_{i,j=1}^N \in \mathbb{R}^{N\times N}$. Then the sesquilinear form on the left hand side of (3) satisfies Gårding's inequality

$$a_{\mathcal{C}}(u, u) \geq c_0 ||\nabla u||_{L^2(\Omega)^{N\times N}} - \omega^2\, ||\rho||_{L^\infty(\Omega)} ||u||_{L^2(\Omega)^N} \quad \text{for all} \quad u \in H^1(\Omega)^N.$$

As a consequence, there exists a unique weak solution to (3) for all frequencies $\omega \in \mathbb{R}_+$ excluding possibly a discrete set $\mathcal{D}$ with the only accumulating point at infinity. The well-posedness of the boundary value problem (1) allows one to define the boundary Neumann-to-Dirichlet (NtD) map as follows

$$\Lambda_{\mathcal{C},q} : H^{-1/2}(\partial\Omega)^N \to H^{1/2}(\partial\Omega)^N, \quad \Lambda_{\mathcal{C},q}\psi = u|_{\partial\Omega},$$

where $u \in H^1(\Omega)^N$ is the unique solution to (1). Throughout the rest of the paper, we refer to an elastic medium $\{\Omega; \mathcal{C}, \rho\}$ as regular if it satisfies the major symmetry in (2) and the uniform Legendre ellipticity condition in (4), otherwise it is called singular. We note that for a regular elastic medium, the corresponding Lamé system is well-posed provided $\omega \notin \mathcal{D}$.

If an elastic material is isotropic and homogeneous, one has

$$\mathcal{C}(x) \equiv \mathcal{C}^{(0)}, \quad C^{(0)}_{ijkl} = \lambda \delta_{i,j}\delta_{k,l} + \mu(\delta_{i,k}\delta_{j,l} + \delta_{i,l}\delta_{j,k}). \tag{5}$$

That is, the stiffness tensor is constant throughout the material with the Lamé constants λ and μ satisfying $\mu > 0, N\lambda + 2\mu > 0$. For simplicity, the mass density is usually normalized to be one in an isotropic homogeneous medium, i.e., $\rho(x) \equiv 1$. Under these assumptions, the stress tensor takes the form

$$\sigma(u) = \lambda \mathbf{I} \operatorname{div} u + 2\mu\epsilon(u), \quad \epsilon(u) := \frac{1}{2}\left(\nabla u + \nabla u^\top\right),$$

where $\mathbf{I}$ stands for the $N \times N$ identity matrix. In this case, the Lamé system (1) reduces to the boundary value problem for Navier's equation,

$$\mathcal{L}u + \omega^2 u = 0 \quad \text{in} \quad \Omega, \quad Tu = \psi \quad \text{on} \quad \partial\Omega, \tag{6}$$

where $Tu = T_{\lambda,\mu}u := \nu \cdot (\mathcal{C}^{(0)} : \nabla u)$ stands for the traction on the boundary of the isotropic medium $\{\Omega; \mathcal{C}^{(0)}, 1\}$, and

$$\mathcal{L}u := \nabla \cdot (\mathcal{C}^{(0)} : \nabla u) = \mu\Delta u + (\lambda + \mu) \operatorname{grad} \operatorname{div} u.$$

In two dimensions, Tu can be explicitly written as

$$Tu := 2\mu\,\partial_\nu u + \lambda\,\nu\,\operatorname{div} u + \mu\tau(\partial_2 u_1 - \partial_1 u_2), \quad \tau := (-\nu_2, \nu_1),\ \nu = (\nu_1, \nu_2), \tag{7}$$

whereas in three dimensions,

$$Tu := 2\mu\,\partial_\nu u + \lambda\,\nu\,\operatorname{div} u + \mu\nu \times \operatorname{curl} u, \quad \nu = (\nu_1, \nu_2, \nu_3). \tag{8}$$

Here and also in what follows, we write $T_{\lambda,\mu}u = Tu$ to drop the dependence of $T_{\lambda,\mu}$ on the Lamé constants λ and μ. Moreover, we shall refer to $\{\Omega; \mathcal{C}^{(0)}, 1\}$ as the free space or reference space in our subsequent study on the invisibility cloaking.

Let $\tilde{x} = F(x) : \Omega \to \tilde{\Omega}$ be a bi-Lipschitz and orientation-preserving mapping. The push-forwards of $\mathcal{C}$ and ρ are defined respectively by

$$\begin{aligned} F_*\mathcal{C} := \tilde{\mathcal{C}} = \tilde{C}_{iqkp}(\tilde{x}) &= \frac{1}{\det(M)} \left\{ \sum_{l,j=1}^{N} C_{ijkl} \frac{\partial \tilde{x}_p}{\partial x_l} \frac{\partial \tilde{x}_q}{\partial x_j} \right\} \Bigg|_{x=F^{-1}(\tilde{x})}, \\ F_*\rho := \tilde{\rho} &= \left(\frac{\rho}{\det(M)} \right) \Bigg|_{x=F^{-1}(\tilde{x})}, \quad M = \left(\frac{\partial \tilde{x}_i}{\partial x_j} \right)_{i,j=1}^{N}. \end{aligned} \tag{9}$$

For notational convenience, we shall write $\tilde{\nabla} = \nabla_{\tilde{x}}$ and denote by $\{\tilde{\Omega}; \tilde{\mathcal{C}}, \tilde{\rho}\} = F_*\{\Omega; \mathcal{C}, \rho\}$ the push-forward defined in (9). We first show the following transformation invariance of the Lamé system (1), which shall play a crucial role in designing nearly cloaking devices to be exploited in subsequent sections.

Lemma 2.1. *(i) The function $u \in H^1(\Omega)^N$ is a solution to $\nabla \cdot (\mathcal{C} : \nabla u) + \omega^2 \rho u = 0$ in Ω if and only if $\tilde{u} = (F^{-1})^* u := u \circ F^{-1} \in H^1(\tilde{\Omega})^N$ is a solution to*

$$\tilde{\nabla} \cdot (\tilde{\mathcal{C}} : \tilde{\nabla} \tilde{u}) + \omega^2 \tilde{\rho} \tilde{u} = 0 \quad \text{in} \quad \tilde{\Omega}. \tag{10}$$

(ii) If the boundary $\partial\Omega$ remains fixed under the transformation, i.e., $F = \text{Identity}$ on $\partial\Omega$, then $\Lambda_{\mathcal{C},\rho} = \Lambda_{\tilde{\mathcal{C}},\tilde{\rho}}$.

Proof. By changing the variables $\tilde{x} = F(x)$ in the sesquilinear form of (3) and using Green's formula, one has

$$\begin{aligned} \int_{\partial\Omega} \psi \cdot \bar{v}\, ds &= \int_{\tilde{\Omega}} \left\{ \sum_{i,q,k,p=1}^{N} \tilde{C}_{iqkp} \frac{\partial \tilde{u}_k}{\partial \tilde{x}_p} \overline{\frac{\partial \tilde{v}_i}{\partial \tilde{x}_q}} - \omega^2 \tilde{\rho}(\tilde{x}) \tilde{u}_i \bar{\tilde{v}}_i \right\} d\tilde{x} \\ &= -\int_{\tilde{\Omega}} \left\{ \operatorname{div}(\tilde{\mathcal{C}} : \nabla \tilde{u}) + \omega^2 \tilde{\rho} \tilde{u} \right\} \cdot \tilde{v}\, d\tilde{x} + \int_{\partial\tilde{\Omega}} \mathcal{N}_{\mathcal{C}} \tilde{u} \cdot \bar{\tilde{v}}\, ds. \end{aligned} \tag{11}$$

By choosing $v \in C_0^\infty(\Omega)^N$, we see $\tilde{v} \in H_0^1(\tilde{\Omega})^N$ and hence (11) readily implies that $\operatorname{div}(\tilde{\mathcal{C}} : \nabla \tilde{u}) + \omega^2 \tilde{\rho} \tilde{u} = 0$ in $\tilde{\Omega}$.

Before proceeding to prove the second assertion, we verify the uniform Legendre elliptic condition for the transformed tensor $\tilde{\mathcal{C}}_{iqkp}$. Indeed, for any symmetric matrix $A = (a_{ij}) \in \mathbb{R}^{N\times N}$ it holds that

$$\begin{aligned} \sum_{i,q,k,p=1}^{N} \tilde{C}_{iqkp}\, a_{iq}\, a_{kp} &= \frac{1}{\det(M)} \sum_{i,q,k,p,l,j=1}^{N} C_{ijkl} \frac{\partial \tilde{x}_p}{\partial x_l} \frac{\partial \tilde{x}_q}{\partial x_j}\, a_{iq}\, a_{kp} \\ &= \frac{1}{\det(M)} \sum_{i,j,l,k=1}^{N} C_{ijkl}\, \tilde{a}_{ij}\, \tilde{a}_{kl} \end{aligned} \tag{12}$$

with

$$\tilde{a}_{ij} = \sum_{q=1}^{N} \frac{\partial \tilde{x}_q}{\partial x_j} a_{iq}, \quad i, j = 1, 2, \cdots, N.$$

In view of the Legendre elliptic condition for $\mathcal{C}$ and the bi-Lipschitz assumption on F, we deduce from (12) that

$$\sum_{i,q,k,p=1}^{N} \tilde{C}_{iqkp}\, \tilde{a}_{ij}\, \tilde{a}_{kl} \geq c_0 \sum_{i,j=1}^{N} |\tilde{a}_{ij}|^2 \geq \tilde{c}_0 \sum_{i,j=1}^{N} |a_{ij}|^2$$

for some constant $\tilde{c}_0 > 0$. That is, the transformed tensor $\tilde{\mathcal{C}}$ satisfies the uniform Legendre elliptic condition. Therefore, the transformed Lamé system is well-posed and particularly we have a well-defined NtD map $\Lambda_{\tilde{\mathcal{C}},\tilde{\rho}} : H^{-1/2}(\tilde{\Omega})^N \to H^{1/2}(\tilde{\Omega})^N$ associated with the transformed system.

Finally, suppose that F = Identity on $\partial\Omega$ and $\mathcal{N}_{\tilde{\mathcal{C}}}\tilde{u} = \mathcal{N}_{\mathcal{C}} u = \psi$ on $\partial\Omega$ for some $\psi \in H^{-1/2}(\tilde{\Omega})^N$. Then, one has

$$\Lambda_{\tilde{\mathcal{C}},\tilde{\rho}}\psi = \tilde{u}|_{\partial\Omega} = (u \circ F^{-1})|_{\partial\Omega} = u|_{\partial\Omega} = \Lambda_{\mathcal{C},\rho}\psi,$$

which readily implies that $\Lambda_{\tilde{\mathcal{C}},\tilde{\rho}} = \Lambda_{\mathcal{C},\rho}$.

The proof is complete. □

Remark 2.1. The transformed elastic tensor $\tilde{\mathcal{C}}$ possesses only the major symmetry, i.e.,

$$\begin{aligned}\tilde{C}_{iqkp} &= \frac{1}{\det(M)} \sum_{l,j=1}^{N} C_{ijkl} \frac{\partial \tilde{x}_p}{\partial x_l} \frac{\partial \tilde{x}_q}{\partial x_j} = \frac{1}{\det(M)} \sum_{l,j=1}^{N} C_{klij} \frac{\partial \tilde{x}_p}{\partial x_l} \frac{\partial \tilde{x}_q}{\partial x_j} \\ &= \frac{1}{\det(M)} \sum_{j,l=1}^{N} C_{kjil} \frac{\partial \tilde{x}_p}{\partial x_j} \frac{\partial \tilde{x}_q}{\partial x_l} = \tilde{C}_{kpiq},\end{aligned}$$

where the second equality follows from the major symmetry of $\mathcal{C}$. However, $\tilde{\mathcal{C}}$ does not possess the minor symmetry. In fact, it has been pointed out by Milton et al. [47] that the invariance of the Lamé system can be achieved only if one relaxes the assumption on the minor symmetry of the transformed elastic tensor. This has led Norris and Shuvalov [53] and Parnell [54] to explore the elastic cloaking by using Cosserat material or by employing non-linear pre-stress in a neo-Hookean elastomeric material. Design of transformation-elastodynamics-based Cosserat elastic cloaks (without the minor symmetry) has been numerically tested in the cylindrical case [9] as well as in the spherical case [20]. Note that the transformed equation (10) retains its original form of the Lamé system and avoids any coupling between stress and velocity. Furthermore, the transformed mass density is still isotropic. We refer to [47,53,20,9] for discussions and investigations of the form of the elastodynamic equations under general transformations.

3. Elastic cloaking and blowup construction

We are in a position to introduce the elastic cloaking for our study. Henceforth, we let $\Omega \subset \mathbb{R}^N$ and $D \Subset \Omega$ be bounded and connected smooth domains. It is further assumed that $\Omega\backslash\overline{D}$ is connected and D contains the origin. Let $h \in \mathbb{R}_+$ and $D_h := \{hx; x \in D\}$. Let $D_{1/2}$ represent the region which we intend to cloak and let

$$\{D_{1/2}; \mathcal{C}^{(a)}, \rho^{(a)}\}$$

be the target medium. From a practical viewpoint, throughout the present study, we assume that $\{D_{1/2}; \mathcal{C}^{(a)}, \rho^{(a)}\}$ is arbitrary but regular. Let

$$\{\Omega\backslash\overline{D}_{1/2}; \mathcal{C}^{(c)}, \rho^{(c)}\} \tag{13}$$

be a properly designed layer of elastic medium, which is referred to as the cloaking medium. Let

$$\{\Omega; \mathcal{C}, \rho\} = \begin{cases} \{\Omega\backslash\overline{D}_{1/2}; \mathcal{C}^{(c)}, \rho^{(c)}\} & \text{in } \Omega\backslash\overline{D}_{1/2}, \\ \{D_{1/2}; \mathcal{C}^{(a)}, \rho^{(a)}\} & \text{in } D_{1/2}, \end{cases} \tag{14}$$

be the extended medium occupying Ω and let $\Lambda_{\mathcal{C},\rho}$ be the associated NtD map. Next, we introduce the "free" NtD map as follows. Let v be the solution to the Navier equation in the free space $\{\Omega; \mathcal{C}^{(0)}, 1\}$ (cf. (6))

$$\mathcal{L}\, v + \omega^2 v = 0 \quad \text{in} \quad \Omega, \quad Tv = \psi \in H^{-1/2}(\partial\Omega)^N \quad \text{on} \quad \partial\Omega. \tag{15}$$

It is assumed that $-\omega^2$ is not an eigenvalue of the elliptic operator $\mathcal{L}$ with the traction-free boundary condition on $\partial\Omega$, and hence one has a well-defined "free" NtD map

$$\Lambda_0\psi = v|_{\partial\Omega}$$

where $v \in H^1(\Omega)^N$ solves (15). The solution v to (15) can be decomposed into its compressional and shear parts as $v = v_p + v_s$, where in three dimensions

$$v_p := -\frac{1}{k_p^2}\,\mathrm{grad}\,\mathrm{div}\, v\,, \quad k_p = \omega/\sqrt{(2\mu+\lambda)}\,, \tag{16}$$

$$v_s := \frac{1}{k_s^2}\,\mathrm{curl}\,\mathrm{curl}\, v\,, \quad k_s = \omega/\sqrt{\mu}, \tag{17}$$

and k_p, k_s are known as the compressional and shear wave numbers, respectively. It is straightforward to verify that the functions v_p and v_s satisfy the vectorial Helmholtz equations

$$(\Delta + k_p^2)\, v_p = 0, \quad \mathrm{curl}\, v_p = 0 \quad \text{in} \quad \Omega, \tag{18}$$

$$(\Delta + k_s^2)\, v_s = 0, \quad \mathrm{div}\, v_s = 0 \quad \text{in} \quad \Omega\,. \tag{19}$$

This implies that the compressional and shear waves propagate at different speeds. By the elliptic equations (18) and (19), one can define another two boundary NtD maps

$$\Lambda_0^{(p)}: \quad H^{-1/2}(\partial\Omega)^N \to H^{1/2}(\partial\Omega)^N, \quad \Lambda_0^{(p)}\psi = v_p|_{\partial\Omega},$$
$$\Lambda_0^{(s)}: \quad H^{-1/2}(\partial\Omega)^N \to H^{1/2}(\partial\Omega)^N, \quad \Lambda_0^{(s)}\psi = v_s|_{\partial\Omega},$$

where $v_p, v_s \in H^1(\Omega)^N$ are solutions to (18) and (19), respectively, prescribed with the boundary values

$$T_p v_p = \psi, \quad T_s v_s = \psi \quad \text{on} \quad \partial\Omega, \tag{20}$$

with the operators T_p and T_s given by (in three dimensions)

$$T_p v_p := 2\mu\, \partial_\nu v_p + \lambda\, \nu\, \mathrm{div}\, v_p, \qquad T_s v_s := 2\mu\, \partial_\nu v_s + \mu\nu \times \mathrm{curl}\, v_s.$$

One observes that (19) is equivalent to the Maxwell system $\mathrm{curl}\,\mathrm{curl}\, v^s - k_s^2 v^s = 0$ in Ω, hence the boundary data $\nu \times \mathrm{curl}\, v^s := \tilde{\psi} \in H^{-1/2}(\mathrm{Div}, \partial\Omega)$ is sufficient to uniquely determine $v_s \in H(\mathrm{curl}, \Omega)$; we refer to [13] for the definition of Sobolev spaces mentioned here. Since boundary value problems (18), (19) and (20) are not always solvable for general $\psi \in H^{-1/2}(\partial\Omega)^N$, we define the admissible sets of inputs by

$$\mathcal{P} := \{\psi \in H^{-1/2}(\partial\Omega)^N : \text{there exists a } v_p \text{ to (18) such that } T_p v_p = \psi\},$$
$$\mathcal{S} := \{\psi \in H^{-1/2}(\partial\Omega)^N : \text{there exists a } v_s \text{ to (19) such that } T_s v_s = \psi\}.$$

Then it is clear that (18), (19) and (20) are uniquely solvable for every $\psi \in \mathcal{P}$ (resp. $\psi \in \mathcal{S}$), provided ω^2 is not an eigenvalue of the operator $\mathcal{L}$ with the traction-free boundary condition.

Definition 3.1. The layer of elastic medium $\{\Omega\backslash\overline{D}_{1/2}; \mathcal{C}^{(c)}, \rho^{(c)}\}$ is said to be a full elastic cloak if $\Lambda_{\mathcal{C},\rho}(\psi) = \Lambda_0(\psi)$ for all $\psi \in H^{-1/2}(\partial\Omega)^N$; it is called a compressional elastic cloak if $\Lambda_0^{(p)}(\psi) = \Lambda_{\mathcal{C},\rho}(\psi)$ for all $\psi \in \mathcal{P}$; and it is called a shear elastic cloak if $\Lambda_0^{(s)}(\psi) = \Lambda_{\mathcal{C},\rho}(\psi)$ for all $\psi \in \mathcal{S}$.

We would like to emphasize that the shear and pressure waves are inherently coupled in the Lamé system and that an incident pure shear or pressure wave would incite the two kinds of waves simultaneously in general. An inverse problem of significant importance arising in practical applications is to infer information of the interior object $\{\Omega; \mathcal{C}, \rho\}$ by knowledge of the exterior elastic wave measurements. The boundary NtD map $\Lambda_{\mathcal{C},\rho}$ encodes the exterior measurements that one can obtain. We refer to [1,5,15,31–33,14,24,25,23,51,48–50] for the theoretical unique identifiability results and numerical reconstruction algorithms developed for these inverse problems. According to Definition 3.1, the cloaking layer $\{\Omega\backslash\overline{D}_{1/2}; \mathcal{C}^{(c)}, \rho^{(c)}\}$ makes itself and the elastic object $\{D_{1/2}; \mathcal{C}^{(a)}, \rho^{(a)}\}$ undetectable by the exterior elastic wave measurements.

In this paper we focus on the design of full elastic cloaks. In what follows, we show that the entire elastic waves diffracted by $\{D_{1/2}; \mathcal{C}^{(a)}, \rho^{(a)}\}$ can be cloaked if and only if both the compressional and shear waves can be cloaked.

Lemma 3.1. *Let $\{\Omega; \mathcal{C}, \rho\}$ be an elastic cloak as described above, which is assumed to be regular. Then, $\Lambda_{\mathcal{C},\rho} = \Lambda_0$ if and only if $\Lambda_{\mathcal{C},\rho} = \Lambda_0^{(p)}$ and $\Lambda_{\mathcal{C},\rho} = \Lambda_0^{(s)}$.*

Proof. The necessity follows directly from the fact that the equations in (18) and (19) can be reformulated as the Navier equation (15).

Next, we prove the sufficiency. Let v and u solve the boundary value problems (15) and (1), respectively. The function v can be decomposed as $v = v_p + v_s$ with $v_p, v_s \in H^1(\Omega)^N$ given by (16) and (17), respectively. Set $\psi_p := T v_p = T_p v_p$ and $\psi_s := T v_s = T_s v_s$ in Ω. Then $\psi_p + \psi_s = Tv = \psi$ on $\partial\Omega$. Consider the boundary value problems

$$\nabla \cdot (\mathcal{C} : \nabla u_p) + \omega^2 \rho u_p = 0 \quad \text{in} \quad \Omega, \quad \mathcal{N}_{\mathcal{C}} u_p = \psi_p \quad \text{on} \quad \partial\Omega,$$
$$\nabla \cdot (\mathcal{C} : \nabla u_s) + \omega^2 \rho u_s = 0 \quad \text{in} \quad \Omega, \quad \mathcal{N}_{\mathcal{C}} u_s = \psi_s \quad \text{on} \quad \partial\Omega.$$

By uniqueness of solutions to (1), we have $u = u_p + u_s$. On the other hand, it follows from the assumptions $\Lambda_0^{(p)} = \Lambda_{\mathcal{C},\rho}$ and $\Lambda_0^{(s)} = \Lambda_{\mathcal{C},\rho}$ that $u_p = v_p$ and $u_s = v_s$ on $\partial\Omega$. Therefore,

$$\Lambda_0 \psi = v = v_p + v_s = u_p + u_s = u = \Lambda_{\mathcal{C},\rho}\psi \quad \text{on} \quad \partial\Omega.$$

The proof is complete. $\square$

In the rest of this section, using the transformation-elastodynamics approach based on Lemma 2.1, we present the blow-up-a-point construction of an ideal full elastic cloak. This elastic cloak has been studied in the physics and engineering literature (cf. [9,20,21]), and we shall focus on analyzing the singular structure from a mathematical point of view. Henceforth, we denote by B_R the central ball of radius $R>0$, and S_R the boundary of B_R, i.e., $S_R=\{x;|x|=R\}$. We take $\Omega=B_2$ and $D_{1/2}=B_1$, following the notations introduced earlier in this section. Let $\{B_2;\mathcal{C}^{(0)},1\}$ be the isotropic homogeneous free space, and let $\Lambda_0=\Lambda_{\mathcal{C}^{(0)},1}$ on S_2 be the free NtD boundary operator. Consider the transformation

$$F:\begin{cases} B_2\backslash\{0\} & \to B_2\backslash\overline{B}_1, \\ x & \to y=(1+\frac{|x|}{2})\frac{x}{|x|}. \end{cases} \tag{21}$$

The transform F blows up the origin in the reference space to B_1 while maps $B_2\backslash\{0\}$ to $B_2\backslash\overline{B}_1$ and keeps the sphere S_2 fixed. Using the transformation F, the reference medium in $B_2\backslash\{0\}$ is then push-forwarded to form the transformation medium in $\{B_2\backslash\overline{B}_1;\mathcal{C}^{(c)},\rho^{(c)}\}$ as follows:

$$\mathcal{C}^{(c)}(y):=F_*(\mathcal{C}^{(0)})(x)|_{x=F^{-1}(y)},\quad \rho^{(c)}(y)=F_*(1)(x)|_{x=F^{-1}(y)},\quad y\in B_2\backslash\overline{B}_1. \tag{22}$$

Let us consider the boundary value problem (1) in $\Omega=B_2$ with

$$\{B_2;\mathcal{C},\rho\}=\begin{cases} \{B_2\backslash\overline{B}_1;\mathcal{C}^{(c)},\rho^{(c)}\} & \text{in } B_2\backslash\overline{B}_1, \\ \{B_1;\mathcal{C}^{(a)},\rho^{(a)}\} & \text{in } B_1, \end{cases} \tag{23}$$

which defines the NtD map $\Lambda_{\mathcal{C},\rho}$ on S_2. By Lemma 2.1, one may infer that $\Lambda_{\mathcal{C},\rho}=\widetilde{\Lambda}_0$ on S_2, where $\widetilde{\Lambda}_0$ is the NtD map associated with the elastic configuration $\{B_2\backslash\{0\};\mathcal{C}^{(0)},1\}$. Noting that the inhomogeneity of the elastic medium $\{B_2\backslash\{0\};\mathcal{C}^{(0)},1\}$ is supported in a singular point, one may infer that $\widetilde{\Lambda}_0=\Lambda_0$, which in turn implies that $\Lambda_{\mathcal{C},\rho}=\Lambda_0$. That is, the construction (23) yields an ideal full elastic cloak. However, the above argument is rather heuristic and intuitive. Indeed, we shall show that the cloaking elastic medium parameters $\mathcal{C}^{(c)}$ and $\rho^{(c)}$ possess singularities, which make the attempt to rigorously justify the ideal elastic cloak highly nontrivial; see [30,44] for the relevant discussions on the singular optical cloaking of acoustic and electromagnetic waves.

Next, let us determine the explicit expressions of the material parameters for the cloaking medium in (22). First, one can easily obtain that the Jacobian matrix of F in (21) and its determinant are given as follows:

$$M(y)=\frac{r}{2(r-1)}(\mathbf{I}-\hat{y}\otimes\hat{y})+\frac{1}{2}\hat{y}\otimes\hat{y},\quad \hat{y}:=y/r,\ r=|y|,$$
$$\det(M)=\begin{cases} \frac{r}{4(r-1)}, & \text{if } N=2, \\ \frac{r^2}{8(r-1)^2}, & \text{if } N=3. \end{cases}$$

Hence, by Lemma 2.1, the push-forwarded elastic tensor and density in $B_2\backslash\overline{B}_1$ are given by

$$\mathcal{C}^{(c)}(y)=\frac{M(y)\diamond\mathcal{C}^{(0)}(x)|_{x=F^{-1}(y)}\diamond M(y)^\top}{\det(M)},\quad \rho^{(c)}(y)=[\det(M)]^{-1}, \tag{24}$$

where the operator $\diamond$ denotes the multiplication between a matrix and a fourth-rank tensor. More precisely, in view of the definition of F_* in Lemma 2.1, we have for $\mathcal{C}^{(c)}=(C^{(c)}_{ijkl})^N_{i,j,k,l=1}$ that

$$\begin{pmatrix} C^{(c)}_{i1k1} & C^{(c)}_{i1k2} & \cdots & C^{(c)}_{i1kN} \\ C^{(c)}_{i2k1} & C^{(c)}_{i2k2} & \cdots & C^{(c)}_{i2kN} \\ \vdots & \vdots & \cdots & \vdots \\ C^{(c)}_{iNk1} & C^{(c)}_{iNk2} & \cdots & C^{(c)}_{iNkN} \end{pmatrix}=M\begin{pmatrix} C^{(0)}_{i1k1} & C^{(0)}_{i1k2} & \cdots & C^{(0)}_{i1kN} \\ C^{(0)}_{i2k1} & C^{(0)}_{i2k2} & \cdots & C^{(0)}_{i2kN} \\ \vdots & \vdots & \cdots & \vdots \\ C^{(0)}_{iNk1} & C^{(0)}_{iNk2} & \cdots & C^{(0)}_{iNkN} \end{pmatrix}M^\top/\det(M) \tag{25}$$

for $i, k = 1, 2, \cdots, N$. Writing the fourth-rank tensor $\mathcal{C}^{(0)}$ in the tensor product form $(C^{(0)}_{ijkl})^N_{i,j,k,l=1} = (C^{(0)}_{i\cdot k\cdot})^N_{i,k=1} \otimes (C^{(0)}_{\cdot j\cdot l})^N_{j,l=1}$, with the second-rank tensors $(C^{(0)}_{i\cdot k\cdot})^N_{i,k=1}$ and $(C^{(0)}_{\cdot j\cdot l})^N_{j,l=1}$, then the multiplication operator $\diamond$ in the first relation of (24) is understood as the two-mode tensor–matrix product in the sense of (25). It is easily seen that the push-forwarded density $\rho^{(c)}$ vanishes on the inner boundary of the cloaking device, namely S_1, in $\mathbb{R}^N$. To see the singularity of $\mathcal{C}^{(c)}$, we insert the expression of M into $\mathcal{C}^{(c)}$,

$$\begin{aligned} \mathcal{C}^{(c)}(y) = &\frac{r^2}{4(r-1)^2}(\mathbf{I} - \hat{y}\otimes\hat{y}) \diamond \mathcal{C}^{(0)} \diamond (\mathbf{I} - \hat{y}\otimes\hat{y})\ [\det(M)]^{-1} \\ &+ \frac{r}{4(r-1)}(\mathbf{I} - \hat{y}\otimes\hat{y}) \diamond \mathcal{C}^{(0)} \diamond (\hat{y}\otimes\hat{y})\ [\det(M)]^{-1} \\ &+ \frac{r}{4(r-1)}(\hat{y}\otimes\hat{y}) \diamond \mathcal{C}^{(0)} \diamond (\mathbf{I} - \hat{y}\otimes\hat{y})\ [\det(M)]^{-1} \\ &+ \frac{1}{4}(\hat{y}\otimes\hat{y}) \diamond \mathcal{C}^{(0)} \diamond (\hat{y}\otimes\hat{y})\ [\det(M)]^{-1}. \end{aligned} \tag{26}$$

Clearly, in two dimensions, the item in the first line in (26) has a singularity of the form $1/(r-1)$ as $r \to 1$, while the item in the fourth line vanishes on S_1. The 3D spherical cloak obtained by blowing up a single point turns out to be less singular than the 2D one, since there are no unbounded entries in the transformed elasticity tensor. Using the relations

$$(\hat{y}\otimes\hat{y})(\hat{y}\otimes\hat{y}) = (\hat{y}\otimes\hat{y}), \quad (\mathbf{I} - \hat{y}\otimes\hat{y})(\hat{y}\otimes\hat{y}) = 0,$$

one can deduce from (26) that

$$\mathcal{C}^{(c)}(y) : (\hat{y}\otimes\hat{y}) = \frac{1}{4\det(M)}\,\mathcal{C}^{(0)}(y) : (\hat{y}\otimes\hat{y}) \to 0$$

as $|y| \to 1$. This implies that the tensor $\mathcal{C}^{(c)}$ does not satisfy the uniform Legendre ellipticity condition (4) in B_2.

We have calculated the cloaking elastic medium parameters in the Cartesian coordinates using the identity (25). The derivation of the cloaking medium tensor in the 2D polar coordinates (r, θ) or 3D spherical coordinates (r, θ, φ) can be proceeded as follows. Noting the symmetric matrix $\hat{y}\otimes\hat{y}$ maps y to its radial direction, one can see that the Jacobian matrix M in the polar or spherical coordinates is of the form

$$M = \begin{pmatrix} 1/2 & 0 \\ 0 & \frac{r}{2(r-1)}\mathbf{I}_{N-1} \end{pmatrix} \tag{27}$$

where $\mathbf{I}_{N-1}$ denotes the $(N-1)\times(N-1)$ identify matrix. Here we have employed the following conventional correspondence between indexes: $1 \mapsto r$, $2 \mapsto \theta$, $3 \mapsto \varphi$ in $\mathbb{R}^N$, $N = 2, 3$. Recalling the *Voigt* notation for tensor indices,

$$11 \mapsto 1,\ 22 \mapsto 2,\ 33 \mapsto 3,\ 23, 32 \mapsto 4,\ 13, 31 \mapsto 5,\ 12, 21 \mapsto 6,$$

we may write the elasticity tensor (5) as

$$C_{\alpha\beta} = \begin{pmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{pmatrix}, \quad C_{\alpha\beta} = \begin{pmatrix} \lambda+2\mu & \lambda & 0 \\ \lambda & \lambda+2\mu & 0 \\ 0 & 0 & \mu \end{pmatrix} \tag{28}$$

in three and two dimensions, respectively, with $\alpha, \beta = 1, 2, \cdots, N$. Using the polar coordinates in 2D, one can deduce from (25), (27) and (28) the transformed elasticity tensor $\mathcal{C}^{(c)}$ with eight nontrivial entries (see also [9]):

$$
\begin{aligned}
&C^{(c)}_{rrrr} = (\lambda + 2\mu)(r-1)/r, \quad C^{(c)}_{\theta r\theta r} = \mu r/(r-1), \\
&C^{(c)}_{rr\theta\theta} = C^{(c)}_{\theta\theta rr} = \lambda, \quad C^{(c)}_{r\theta\theta r} = C^{(c)}_{\theta rr\theta} = \mu, \\
&C^{(c)}_{r\theta r\theta} = \mu(r-1)/r, \quad C^{(c)}_{\theta\theta\theta\theta} = (\lambda + 2\mu)r/(r-1).
\end{aligned}
$$

Physically, the vanishing of $C^{(c)}_{rrrr}$, $C^{(c)}_{r\theta r\theta}$ and the singularity of $C^{(c)}_{\theta\theta\theta\theta}$, $C^{(c)}_{\theta r\theta r}$ on S_1 imply an infinite velocity of the pressure and shear waves propagating along the inner side of the cloaking interface. The stress tensor for the 3D spherical cloak turns out to have 21 nontrivial entries in $B_2\backslash\overline{B}_1$ with ten of them vanishing on the sphere S_1; we refer to [20] for detailed discussions.

4. Regularized blowup construction and cloak-busting inclusions

It is seen from our earlier discussion that an elastic cloak can be obtained by using the transformation-elastodynamics approach through a blowup transformation. However, as shown in the last section, the blow-up-a-point construction produces singular cloaking medium parameters, which pose server difficulties not only to the corresponding mathematical analysis but also to the practical realization. The singular cloaking medium comes from the use of the singular blowup transformation (21). In order to avoid the singular structure, it is natural to regularize the singular blow-up-a-point transformation F as follows. Let $h \in \mathbb{R}_+$ be a sufficiently small regularization parameter, and consider the transformation $F_h : B_2\backslash\overline{B}_h \to B_2\backslash\overline{B}_1$ defined by

$$
F_h(x) := \begin{cases} \left(\frac{2-2h}{2-h} + \frac{|x|}{2-h}\right)\frac{x}{|x|} & \text{for } h \le |x| \le 2, \\ \frac{x}{h} & \text{for } |x| < h. \end{cases}
$$

It is easy to verify that $F_h : B_2 \to B_2$ is bi-Lipschitz, orientation-preserving and $F_h|_{\partial B_2} = \text{Identity}$. Moreover, F_h degenerates to the singular transformation F in (21) as $h \to +0$. Now we consider the cloaking construction similar to (23) of the form

$$
\{B_2; \mathcal{C}, \rho\} = \begin{cases} \{B_2\backslash\overline{B}_1; \mathcal{C}^{(c)}_h, \rho^{(c)}_h\} & \text{in } B_2\backslash\overline{B}_1, \\ \{B_1; \mathcal{C}^{(a)}, \rho^{(a)}\} & \text{in } B_1, \end{cases} \tag{29}
$$

with the cloaking medium given by

$$
\mathcal{C}^{(c)}_h(y) := (F_h)_*(\mathcal{C}^{(0)})(x)|_{x=F_h^{-1}(y)}, \quad \rho^{(c)}_h(y) = (F_h)_*(1)(x)|_{x=F_h^{-1}(y)}, \quad y \in B_2\backslash\overline{B}_1. \tag{30}
$$

We let $\Lambda^h_{\mathcal{C},\rho}$ denote the NtD map associated with the elastic configuration in (29). Since F_h degenerates to the singular blow-up-a-point transformation as $h \to +0$, one may expect that $\Lambda^h_{\mathcal{C},\rho} \to \Lambda_0$ as $h \to +0$. That is, (29) would produce an approximate elastic cloak, namely a near-cloak. However, in what follows, we shall show that no matter how small $h \in \mathbb{R}_+$ is, there always exists a certain elastic inclusion $\{B_1; \mathcal{C}^{(a)}, \rho^{(a)}\}$ depending on h, that defies any attempt to achieve the near-cloak. Indeed, we shall show that for any $h \in \mathbb{R}_+$, there exists a certain elastic inclusion $\{B_1; \mathcal{C}^{(a)}, \rho^{(a)}\}$ such that the corresponding $\Lambda^h_{\mathcal{C},\rho}$ is not even well-defined due to resonance. In doing so, we first note that by using Lemma 2.1, the NtD map associated with $\{B_2; \mathcal{C}, \rho\}$ in (29) is the same as the one associated with the virtual elastic configuration

$$\{B_2;\tilde{\mathcal{C}},\tilde{\rho}\} := (F_h^{-1})_*\{B_2;\mathcal{C},\rho\} = \begin{cases} \{B_2\backslash\overline{B}_h;\mathcal{C}^{(0)},1\} & \text{in } B_2\backslash\overline{B}_h, \\ \{B_h;\tilde{\mathcal{C}}^{(a)},\tilde{\rho}^{(a)}\} & \text{in } B_h, \end{cases} \tag{31}$$

where $\{B_h;\tilde{\mathcal{C}}^{(a)},\tilde{\rho}^{(a)}\} = (F_h^{-1})_*\{B_1;\mathcal{C}^{(a)},\rho^{(a)}\}$. That is, the NtD map $\Lambda^h_{\mathcal{C},\rho}$ characterizes the boundary effect due to the small inclusion $\{B_h;\tilde{\mathcal{C}}^{(a)},\tilde{\rho}^{(a)}\}$ supported in B_h. Since the target elastic medium $\{B_1;\mathcal{C}^{(a)},\rho^{(a)}\}$ is arbitrary but regular, we see that the content of the small inclusion $\{B_h;\tilde{\mathcal{C}}^{(a)},\tilde{\rho}^{(a)}\}$ is in principle also arbitrary but regular. Hence, in order to show the failure of the near elastic cloaking construction (29), it is sufficient to show that for any $h \in \mathbb{R}_+$, there always exists a certain $\{B_h;\tilde{\mathcal{C}}^{(a)},\tilde{\rho}^{(a)}\}$ such that the NtD map $\Lambda_{\tilde{\mathcal{C}},\tilde{\rho}}$ associated with the elastic configuration $\{B_2;\tilde{\mathcal{C}},\tilde{\rho}\}$ is not well-defined due to resonance.

We would like to appeal for a bit more general study in two dimensions only by considering the following Lamé system:

$$\begin{cases} \nabla\cdot(\mathcal{C}^{(0)}:\nabla u_1) + \omega^2\rho_1 u_1 = 0 & \text{in } r_0 < |x| < r_1, \\ \nabla\cdot(\mathcal{C}^{(0)}:\nabla u_0) + \omega^2\rho_0 u_0 = 0 & \text{in } |x| < r_0, \\ Tu_1 = 0 & \text{on } |x| = r_1, \\ u_1 = u_2, \quad Tu_1 = Tu_2 & \text{on } |x| = r_0. \end{cases} \tag{32}$$

Here, ρ_1, ρ_0 are two positive constants, $\omega \in \mathbb{R}_+$ is a fixed frequency, and the elastic tensor $\mathcal{C}^{(0)}$ is given by (5) with fixed Lamé constants λ, μ in $|x| < r_1$. In what follows we shall verify that, for any $r_0 < r_1$, there always exist constant densities $\rho_0, \rho_1 > 0$ for which the system (32) admits non-trivial solutions. This implies that resonance occurs and the boundary NtD map is not well-defined for the system. Clearly, this also indicates the failure of the near-cloaking construction (29) due to the existence of resonant inclusions.

Let (r,φ) be the polar coordinates of $x = (x_1,x_2) \in \mathbb{R}^2$. We look for special solutions to (32) of the form

$$u_j = c_j\,\nabla_x(J_0(k_p^{(j)}r)), \quad \nabla_x := (\partial_{x_1},\partial_{x_2}), \quad k_p^{(j)} := \omega^2\sqrt{\rho_j/(\lambda+2\mu)}, \quad c_j \in \mathbb{C},$$

i.e., u_j consists of spherically-symmetric compressional waves only. Here J_0 denotes the Bessel function of order zero. Similar examples can be constructed for general elastic waves by using the Bessel function of order n. Simple calculations show that

$$u_j = c_j\,k_p^{(j)}\,J_0'(k_p^{(j)}r)\begin{pmatrix}\cos\varphi\\ \sin\varphi\end{pmatrix}. \tag{33}$$

Since $(\Delta + (k_p^{(j)})^2)J_0(k_p^{(j)}r) = 0$, one can readily check that u_j satisfy the Navier equations in (32) and that

$$Tu_1 = (k_p^{(1)})^2\begin{pmatrix}\cos\varphi\\ \sin\varphi\end{pmatrix}\left[2\mu J_0''(k_p^{(1)}r_1) - \lambda J_0(k_p^{(1)}r_1)\right]c_1 \quad \text{on} \quad |x| = r_1.$$

On the other hand, the transmission conditions on $|x| = r_0$ in (32) are equivalent to (see Lemma 4.1 below)

$$u_1 = u_2, \quad \frac{\partial u_1}{\partial r} = \frac{\partial u_2}{\partial r} \quad \text{on} \quad |x| = r_0, \tag{34}$$

due to the invariance of the Lamé constants on both sides of the interface. Inserting (33) into (34) yields the linear system

$$\begin{pmatrix} (k_p^{(1)}r_0)\,J_0'(k_p^{(1)}r_0) & -(k_p^{(2)}r_0)\,J_0'(k_p^{(2)}r_0) \\ (k_p^{(1)}r_0)^2\,J_0''(k_p^{(1)}r_0) & -(k_p^{(2)}r_0)^2\,J_0''(k_p^{(2)}r_0) \end{pmatrix}\begin{pmatrix}c_1\\ c_2\end{pmatrix} = 0.$$

Introduce the functions

$$f(t) := 2\mu J_0''(t) - \lambda J_0(t), \quad g(t) := \frac{J_0'(t)}{t\, J_0''(t)}.$$

Since there are infinitely many positive zeros of f tending to infinity, we may choose a $\rho_1 > 0$ such that $f(k_p^{(1)} r_1) = 0$. Set $t_1 := k_p^{(1)} r_0$. Then we can find a $t_2 \in \mathbb{R}_+$, $t_2 \neq t_1$ such that

$$g(t_1) = g(t_2) \quad \text{if} \quad J_0''(t_1) \neq 0; \quad J_0''(t_2) = 0 \quad \text{if} \quad J_0''(t_1) = 0.$$

Now, the number $\rho_2 > 0$ is chosen such that the relation $k_p^{(2)} r_0 = t_2$ holds. With those choices of ρ_1 and ρ_2, we have the homogeneous Neumann boundary condition $Tu_1 = 0$ on $|x| = r_1$. Moreover, due to the vanishing of the determinant of the matrix, one can find a nontrivial solution (c_1, c_2) to the above linear system so that the transmission conditions hold true. Hence we have constructed a non-trivial pair of solutions (u_1, u_2) to (32) for any fixed $r_0 < r_1$ and $\omega \in \mathbb{R}_+$.

Finally, we give the proof of (34).

Lemma 4.1. *Let $D \subset \Omega$ be a C^2-smooth domain in $\mathbb{R}^2$. Assume that $u_1 \in H^1(\Omega\backslash\overline{D})^2$, $u_2 \in H^1(D)^2$ satisfy the transmission condition $u_1 = u_2$, $Tu_1 = Tu_2$ on ∂D. Then $\partial_\nu u_1 = \partial_\nu u_2$ on ∂D.*

Proof. We carry out the proof by making use of the definition (7). Let $\nu = (\nu_1, \nu_2)$ and $\tau = (-\nu_2, \nu_1)$ denote the normal and tangential directions on ∂D, respectively. Set $u_j = (u_j^{(1)}, u_j^{(2)})^\top$ for $j = 1, 2$. Using the formula

$$\partial_1 w = \nu_1 \partial_\nu w - \nu_2 \partial_\tau w, \quad \partial_2 w = \nu_2 \partial_\nu w + \nu_1 \partial_\tau w,$$

we separate the normal and tangential derivatives involved in the stress operator. Consequently, the stress operator Tu_j on ∂D can be rewritten as

$$\begin{aligned} Tu_j &= \begin{pmatrix} \mu + (\lambda+\mu)\nu_1^2 & (\lambda+\mu)\nu_1\nu_2 \\ (\lambda+\mu)\nu_1\nu_2 & \mu + (\lambda+\mu)\nu_2^2 \end{pmatrix} \begin{pmatrix} \partial_\nu u_j^{(1)} \\ \partial_\nu u_j^{(2)} \end{pmatrix} \\ &\quad + \begin{pmatrix} -(\lambda+\mu)\nu_1\nu_2 & \lambda\nu_1^2 - \mu\nu_2^2 \\ -\lambda\nu_2^2 + \mu\nu_1^2 & (\lambda+\mu)\nu_1\nu_2 \end{pmatrix} \begin{pmatrix} \partial_\tau u_j^{(1)} \\ \partial_\tau u_j^{(2)} \end{pmatrix} \\ &=: A(\lambda,\mu,\nu)\, \partial_\nu u_j + B(\lambda,\mu,\nu)\, \partial_\tau u_j. \end{aligned} \tag{35}$$

Set $U = u_1 - u_2$. From (35) and the assumptions $Tu_1 = Tu_2$, $u_1 = u_2$ on ∂D we see

$$0 = TU = A(\lambda,\mu,\nu)\, \partial_\nu U \quad \text{on} \quad \partial D. \tag{36}$$

Direct calculations yield that the determinant of $A(\lambda, \mu, \nu)$ is given by

$$\det(A) = \mu(\lambda + 2\mu) > 0.$$

Hence, by (36) we obtain $\partial_\nu u_1 = \partial_\nu u_2$ on ∂D.

The proof is complete. □

5. Nearly cloaking the elastic waves

5.1. Our near-cloaking scheme

Through the discussion in Section 4, we see that the regularized blow-up-a-small-ball construction fails due to the existence of resonant inclusions, namely the cloak-busting inclusions. We would like to mention that similar phenomena have been observed in regularized optical cloaks; see [10,11,37,43]. In order to defeat the resonance, a natural idea is to introduce a certain damping mechanism. This motivates us to develop a near-cloaking scheme by incorporating a suitable lossy layer right between the cloaked region and cloaking layer.

We are in a position to present the proposed near-cloaking scheme. Let Ω and D be as described in Section 3. Let $h \in \mathbb{R}_+$ be a small regularization parameter and let F_h be a bi-Lipschitz and orientation-preserving mapping such that

$$F_h : \overline{\Omega}\backslash D_h \to \overline{\Omega}\backslash D, \quad F_h(\partial\Omega) = \partial\Omega.$$

Introduce the mapping

$$F(x) = \begin{cases} F_h(x) & \text{for } x \in \overline{\Omega}\backslash D_h, \\ x/h & \text{for } x \in D_h. \end{cases}$$

Clearly, $F : \Omega \to \Omega$ is bi-Lipschitz and orientation-preserving and $F(\partial\Omega) = \partial\Omega$.

Our proposed regularized near-cloaking construction takes the following general form:

$$\{\Omega; \mathcal{C}, \rho\} = \begin{cases} \{\Omega\backslash\overline{D}_{1/2}; \mathcal{C}^{(c)}, \rho^{(c)}\} & \text{in } \Omega\backslash\overline{D}_{1/2}, \\ \{D_{1/2}; \mathcal{C}^{(a)}, \rho^{(a)}\} & \text{in } D_{1/2}, \end{cases} \tag{37}$$

where

$$\{\Omega\backslash\overline{D}_{1/2}; \mathcal{C}^{(c)}, \rho^{(c)}\} = \begin{cases} \{\Omega\backslash\overline{D}; \mathcal{C}^{(1)}, \rho^{(1)}\} & \text{in } \Omega\backslash\overline{D}, \\ \{D\backslash\overline{D}_{1/2}; \mathcal{C}^{(2)}, \rho^{(2)}\} & \text{in } D\backslash\overline{D}_{1/2}, \end{cases} \tag{38}$$

with

$$\begin{aligned} \{\Omega\backslash\overline{D}; \mathcal{C}^{(1)}, \rho^{(1)}\} &= (F_h)_*\{\Omega\backslash\overline{D}_h; \mathcal{C}^{(0)}, 1\}, \\ \{D\backslash\overline{D}_{1/2}; \mathcal{C}^{(2)}, \rho^{(2)}\} &= (F_h)_*\{D_h\backslash\overline{D}_{h/2}; \tilde{\mathcal{C}}^{(2)}, \tilde{\rho}^{(2)}\}. \end{aligned} \tag{39}$$

In (39), the elastic medium in $D_h\backslash\overline{D}_{h/2}$ is given by

$$\{D_h\backslash\overline{D}_{h/2}; \tilde{\mathcal{C}}^{(2)}, \tilde{\rho}^{(2)}\}, \quad \tilde{\mathcal{C}}^{(2)} = \gamma\, h^{2+\delta}\, \mathcal{C}^{(0)}, \quad \tilde{\rho}^{(2)} = \alpha + i\beta, \tag{40}$$

where α, β, γ and δ are fixed positive constants. Here, we note that in (40), we introduce a critical lossy layer $\{D_h\backslash\overline{D}_{h/2}; \tilde{\mathcal{C}}^{(2)}, \tilde{\rho}^{(2)}\}$, wherein β is the damping parameter of the elastic medium. We next present the main theorem in assessing the near-cloaking performance of the above proposed construction. Henceforth, for two Banach spaces $\mathscr{X}$ and $\mathscr{Y}$, we let $\mathscr{L}(\mathscr{X}, \mathscr{Y})$ denote the Banach space of the linear functionals from $\mathscr{X}$ to $\mathscr{Y}$. Moreover, we let C denote a generic positive constant, which may change in different estimates, but should be clear in the context. Then, we have:

Theorem 5.1. *Assume $-\omega^2$ is not an eigenvalue of the elliptic operator $\mathcal{L}$ on Ω with the traction-free boundary condition. Let $\Lambda_{\mathcal{C},\rho}$ be the NtD map corresponding to the elastic configuration (37)–(40), and let Λ_0 be the free NtD map for the Lamé system. Then there exists a constant $h_0 \in \mathbb{R}_+$ such that when $h < h_0$,*

$$||\Lambda_{\mathcal{C},\rho} - \Lambda_0||_{\mathscr{L}(H^{-1/2}(\partial\Omega)^N, H^{1/2}(\partial\Omega)^N)} \le C\, h^N, \tag{41}$$

where C is a positive constant independent of h, $\mathcal{C}^{(a)}$, $\rho^{(a)}$ and δ.

By Theorem 5.1, we see the construction (37)-(40) produces a near-cloak within h^N-accuracy of the ideal cloak in $\mathbb{R}^N$. Moreover, since the estimate (41) is independent of the content being cloak, namely $\{D_{1/2}; \mathcal{C}^{(a)}, \rho^{(a)}\}$, it is capable of nearly cloaking an arbitrary target elastic medium. We would like to remark that $\delta \geq 0$ in (40) is a free parameter and one may simply choose it to be 0.

In order to prove Theorem 5.1, we first note that by using Lemma 2.1

$$\Lambda_{\mathcal{C},\rho} = \Lambda_{\tilde{\mathcal{C}},\tilde{\rho}}, \tag{42}$$

where

$$\{\Omega; \tilde{\mathcal{C}}, \tilde{\rho}\} = (F^{-1})_*\{\Omega; C, \rho\} = \begin{cases} \mathcal{C}^{(0)} & \text{in } \Omega\backslash\overline{D}_h, \\ \tilde{\mathcal{C}}^{(2)}, \tilde{\rho}^{(2)} & \text{in } D_h\backslash\overline{D}_{h/2}, \\ \tilde{\mathcal{C}}^{(a)}, \tilde{\rho}^{(a)} & \text{in } D_{h/2}, \end{cases} \tag{43}$$

with

$$\{D_{h/2}; \tilde{\mathcal{C}}^{(a)}, \tilde{\rho}^{(a)}\} = (F^{-1})_*\{D_{1/2}; \mathcal{C}^{(a)}, \rho^{(a)}\}.$$

Let $u \in H^1(\Omega)^N$ be the solution to the Lamé system associated with the elastic configuration $\{\Omega; \mathcal{C}, \rho\}$; that is

$$\nabla \cdot (\mathcal{C} : \nabla u) + \omega^2 \rho u = 0 \quad \text{in} \quad \Omega, \quad \mathcal{N}_{\mathcal{C}}\, u = \psi \in H^{-1/2}(\partial\Omega)^N \quad \text{on} \quad \partial\Omega.$$

Let $\tilde{u} := u \circ F^{-1}$. Then by Lemma 2.1, $\tilde{u}$ solves the boundary value problem

$$\nabla \cdot (\tilde{\mathcal{C}} : \nabla \tilde{u}) + \omega^2 \tilde{\rho}\tilde{u} = 0 \quad \text{in} \quad \Omega, \quad \mathcal{N}_{\tilde{\mathcal{C}}}\, \tilde{u} = \psi \quad \text{on} \quad \partial\Omega, \tag{44}$$

and let $u_0 \in H^1(\Omega)^N$ be the solution in the free space; see (15). By (42), we see that Theorem 5.1 immediately follows from:

Theorem 5.2. *Assume $-\omega^2$ is not an eigenvalue of the elliptic operator $\mathcal{L}$ on Ω with the traction-free boundary condition. Let $\tilde{u}$ and u_0 be solutions to (44) and (15), respectively. Then there exists a constant $h_0 \in \mathbb{R}_+$ such that when $h < h_0$,*

$$||\tilde{u} - u_0||_{H^{1/2}(\partial\Omega)^N} \leq C\, h^N\, ||\psi||_{H^{-1/2}(\partial\Omega)^N}, \tag{45}$$

where C is a positive constant independent of h, ψ, $\tilde{\mathcal{C}}$, $\tilde{\rho}$ and δ.

5.2. Proof of Theorem 5.2

Before giving the proof of Theorem 5.2, we sketch the general structure of our argument. First, by using a variational argument together with the use of the lossy layer $\{D_h\backslash\overline{D}_{h/2}; \tilde{\mathcal{C}}^{(2)}, \tilde{\rho}^{(2)}\}$, one can control the energy of the elastic wave field in $D_h\backslash\overline{D}_{h/2}$. Next, by a duality argument, we control the trace of the traction of the elastic wave field on ∂D_h. In this step, we need derive a critical Sobolev extension result. Then, the study is reduced to estimating the boundary effect on $\partial\Omega$ due to a small elastic inclusion D_h with a prescribed traction trace on ∂D_h. We shall make use of a variety of layer potential techniques in this step. Finally, the sharpness of our estimate has been numerically verified and shall be reported in a forthcoming work.

Lemma 5.1. *The solutions to (44) and (15) satisfy the estimate*

$$\beta\,\omega^2\,||\tilde{u}||^2_{L^2(D_h\setminus\overline{D}_{h/2})} \le C\,||\psi||_{H^{-1/2}(\partial\Omega)^N}||\tilde{u}-u_0||_{H^{1/2}(\partial\Omega)^N},$$

where C is a positive constant depending only on Ω.

Proof. Multiplying $\bar{\tilde{u}}$ to (44) and integrating by parts yield

$$\begin{aligned}-\int_\Omega(\tilde{\mathcal{C}}:\nabla\tilde{u}):\nabla\bar{\tilde{u}}\,dx+\omega^2\int_\Omega\tilde{\rho}|\tilde{u}|^2\,dx &= -\int_{\partial\Omega}[(\tilde{\mathcal{C}}:\nabla\tilde{u})\cdot\nu]\cdot\bar{\tilde{u}}\,ds\\ &= -\int_{\partial\Omega}\psi\cdot\bar{\tilde{u}}\,ds.\end{aligned}$$

Similarly,

$$-\int_\Omega(\mathcal{C}^{(0)}:\nabla u_0):\nabla\bar{u}_0\,dx+\omega^2\int_\Omega|u_0|^2\,dx=-\int_{\partial\Omega}\psi\cdot\bar{u}_0\,ds.$$

Taking the imaginary parts of the above two identities and making use of the definition of $\tilde{\rho}^{(2)}$ in (40), we arrive at

$$\int_{D_{h/2}}\mathrm{Im}(\tilde{\rho}^{(a)})|\tilde{u}|^2dx+\beta\omega^2\int_{D_h\setminus D_{h/2}}|\tilde{u}|^2dx=-\,\mathrm{Im}\int_{\partial\Omega}\psi\cdot(\bar{\tilde{u}}-\bar{u}_0)\,ds. \tag{46}$$

Since $\mathrm{Im}(\tilde{\rho}^{(a)})\ge 0$, Lemma 5.1 follows easily from (46).

The proof is complete. □

In what follows, we employ the notation $T^{\pm}\tilde{u}$ to denote the traction operators on ∂D_h when limits are taken from the outside and inside of D_h, respectively. For simplicity we write $\Psi^{\pm}(x)=T^{\pm}\tilde{u}(hx)$ for $x\in\partial D$.

Lemma 5.2. *Let $\tilde{u}$ and u_0 be solutions to (44) and (15), respectively. We have the estimates*

$$||\Psi^-||^2_{H^{-3/2}(\partial D)^N}\le C\frac{(\gamma+\sqrt{\alpha^2+\beta^2}h^{-\delta}\omega^2)^2}{\beta\gamma^2\omega^2}h^{-N-2}\,||\tilde{u}-u_0||_{H^{1/2}(\partial\Omega)^N}\,||\psi||_{H^{-1/2}(\partial\Omega)^N},$$

$$||\Psi^+||^2_{H^{-3/2}(\partial D)^N}\le C\,\frac{(\gamma+\sqrt{\alpha^2+\beta^2}h^{-\delta}\omega^2)^2}{\beta\omega^2}h^{2(1+\delta)-N}\,||\tilde{u}-u_0||_{H^{1/2}(\partial\Omega)^N}\,||\psi||_{H^{-1/2}(\partial\Omega)^N},$$

where C is a positive constant depending only on D and Ω but independent of h and ψ.

Proof. By the definition of the norm $||\cdot||_{H^{-3/2}(\partial D)}$,

$$||\Psi||_{H^{-3/2}(\partial D)^N}=\sup_{||\phi||_{H^{3/2}(\partial D)^N}=1}\Big|\int_{\partial D}\Psi(x)\cdot\phi(x)\,ds\Big|.$$

For any $\phi\in H^{3/2}(\partial D)^N$, there exists $w\in H^2(D)^N$ such that (see Lemma A.1 in Appendix A)

(i) $w=\phi$ on ∂D and $Tw=0$ on ∂D;
(ii) $||w||_{H^2(D)^N}\le C\,||\phi||_{H^{3/2}(\partial D)^N}$;
(iii) $w=0$ in $D_{1/2}$.

Then we have

$$\int_{\partial D} \Psi^-(x) \cdot \phi(x)\, ds = \int_{\partial D} T^- \tilde{u}(hx) \cdot \phi(x)\, ds = \int_{\partial D} T^- \tilde{u}(hx) \cdot w(x)\, ds. \tag{47}$$

For $y \in D_h$, write $x = y/h \in D$. Set $v(x) := \tilde{u}(hx) = \tilde{u}(y)$ for $x \in D$. By the definitions of $\tilde{\mathcal{C}}^{(2)}$ and $\tilde{\rho}^{(2)}$ (see (40)), we know

$$\gamma\, h^{2+\delta}\, \mathcal{L}\tilde{u}(y) + \omega^2(\alpha + i\beta)\tilde{u}(y) = 0 \quad \text{in} \quad D_h\backslash \overline{D}_{h/2}. \tag{48}$$

Direct calculations show that

$$\gamma\, h^{\delta}\, \mathcal{L}v(x) + \omega^2(\alpha + i\beta)v(x) = 0 \quad \text{in} \quad D\backslash \overline{D}_{1/2}. \tag{49}$$

Using the fact $Tw = 0$ on ∂D, it is seen from (47) that

$$\begin{aligned} \int_{\partial D} \Psi^-(x) \cdot \phi(x)\, ds &= h^{-1} \int_{\partial D} T^- v(x) \cdot w(x)\, ds \\ &= h^{-1} \int_{\partial D} T^- v(x) \cdot w(x) - v(x) \cdot T^- w(x)\, ds \\ &= h^{-1} \int_{D\backslash D_{1/2}} \mathcal{L}v \cdot w - \mathcal{L}w \cdot v\, dx, \end{aligned}$$

where the third equality follows from Betti's formula and the fact that $w = 0$ in $D_{1/2}$. Recalling (49) and applying the Cauchy–Schwarz inequality yield

$$\begin{aligned} \Big|\int_{\partial D} \Psi^-(x) \cdot \phi(x)\, ds\Big| \le{}& h^{-1-\delta}\, \sqrt{\alpha^2 + \beta^2}\, \omega^2 \gamma^{-1}\, ||v||_{L^2(D\backslash \overline{D}_{1/2})^N}\, ||w||_{L^2(D)^N} \\ &+ h^{-1} ||v||_{L^2(D\backslash \overline{D}_{1/2})^N}\, ||\mathcal{L}w||_{L^2(D)^N}. \end{aligned} \tag{50}$$

In view of the relations

$$\begin{aligned} ||v||_{L^2(D\backslash \overline{D}_{1/2})^N} = ||\tilde{u}(h\,\cdot)||_{L^2(D\backslash \overline{D}_{1/2})^N} = h^{-N/2} ||\tilde{u}||_{L^2(D_h\backslash \overline{D}_{h/2})^N}, \\ ||\mathcal{L}w||_{L^2(D)^N} + ||w||_{L^2(D)^N} \le C\, ||\phi||_{H^{3/2}(\partial D)^N}, \end{aligned}$$

we derive from (50) that

$$\Big|\int_{\partial D} \Psi^-(x) \cdot \phi(x)\, ds\Big| \le C\, h^{-N/2-1} \left(1 + \sqrt{\alpha^2 + \beta^2}\, \omega^2 \gamma^{-1}\, h^{-\delta}\right)\, ||\tilde{u}||_{L^2(D_h\backslash \overline{D}_{h/2})^N}\, ||\phi||_{H^{3/2}(\partial D)^N}.$$

This implies that

$$||\Psi^-||_{H^{-3/2}(\partial D)^N} \le C\, h^{-N/2-1} \left(1 + \sqrt{\alpha^2 + \beta^2}\, \omega^2 \gamma^{-1}\, h^{-\delta}\right)\, ||\tilde{u}||_{L^2(D_h\backslash \overline{D}_{h/2})^N}, \tag{51}$$

which together with Lemma 5.1 leads to the first assertion of Lemma 5.2. By (48) and the transmission conditions on ∂D_h, we have

$$T^+\tilde{u} = \gamma\, h^{2+\delta} T^-\tilde{u} \quad \text{on} \quad \partial D_h.$$

Hence, $\Psi^+ = \gamma\, h^{2+\delta}\Psi^-$ on ∂D. Combining this with the estimate of $||\Psi^-||_{H^{-3/2}(\partial D)^N}$ in (51) proves the second assertion of Lemma 5.2.

The proof is complete. □

Lemma 5.3. *Assume that $-\omega^2$ is not an eigenvalue of the elliptic operator $\mathcal{L}$ on Ω with the traction-free boundary condition. Let $u_0 \in H^1(\Omega)^N$ be the solution of (15) with $\psi \in H^{-1/2}(\partial\Omega)^N$. For $h > 0$ and $\varphi \in H^{-1/2}(\partial D_h)^N$, consider the elliptic boundary value problem*

$$\begin{cases} \mathcal{L}v + \omega^2 v = 0 & \text{in } \Omega\backslash\overline{D}_h, \\ Tv = \varphi & \text{on } \partial D_h, \\ Tv = \psi & \text{on } \partial\Omega. \end{cases}$$

Then there exists $h_0 \in \mathbb{R}_+$ such that when $h < h_0$,

$$||v - u_0||_{H^{1/2}(\partial\Omega)^N} \leq C\, \left(h^N\, ||\psi||_{H^{-1/2}(\partial\Omega)^N} + h^{N-1}\, ||\varphi(h\,\cdot)||_{H^{-3/2}(\partial D)^N}\right), \tag{52}$$

where C is a positive constant independent of h, φ and ψ.

Proof. Set $w = v - u_0$ in $\Omega\backslash\overline{D}_h$. Then $w \in H^1(\Omega\backslash\overline{D}_h)^N$ satisfies

$$\begin{cases} \mathcal{L}w + \omega^2 w = 0 & \text{in } \Omega\backslash\overline{D}_h, \\ Tw = \varphi - Tu_0 & \text{on } \partial D_h, \\ Tw = 0 & \text{on } \partial\Omega. \end{cases}$$

Let $w_1 \in H^1(\Omega\backslash\overline{D}_h)^N$ be the unique solution of

$$\begin{cases} \mathcal{L}w_1 + \omega^2 w_1 = 0 & \text{in } \Omega\backslash\overline{D}_h, \\ Tw_1 = \varphi & \text{on } \partial D_h, \\ Tw_1 = 0 & \text{on } \partial\Omega. \end{cases}$$

Then $w_2 := w - w_1 \in H^1(\Omega\backslash\overline{D}_h)^N$ satisfies

$$\begin{cases} \mathcal{L}w_2 + \omega^2 w_2 = 0 & \text{in } \Omega\backslash\overline{D}_h, \\ Tw_2 = Tu_0 & \text{on } \partial D_h, \\ Tw_2 = 0 & \text{on } \partial\Omega. \end{cases}$$

By Lemma 5.4 in Section 5.3, we know

$$||w_2||_{H^{1/2}(\partial\Omega)^N} \leq C\, h^N\, ||\psi||_{H^{-1/2}(\partial\Omega)^N}.$$

Hence, in order to prove (52) we only need to verify

$$||w_1||_{H^{1/2}(\partial\Omega)^N} \leq C\, h^{N-1}\, ||\varphi(h\,\cdot)||_{H^{-3/2}(\partial D)^N}. \tag{53}$$

To that end, we consider the following elastic scattering problem in an unbounded domain:

$$\mathcal{L}\,W + \omega^2 W = 0 \quad \text{in} \quad \mathbb{R}^N\backslash\overline{D}_h, \quad T\,W = \varphi \quad \text{on} \quad \partial D_h,$$

where $W \in H^1_{loc}(\mathbb{R}^N\backslash\overline{D}_h)$ is additionally required to satisfy the Kupradze radiation condition (see (83) below) when $|x| \to \infty$. We shall show in Section 5.3 that (see Lemma 5.5)

$$||W||_{H^{1/2}(\partial\Omega)^N} + ||TW|_{\partial\Omega}||_{C(\partial\Omega)} \le C\, h^{N-1}\, ||\varphi(h\,\cdot)||_{H^{-3/2}(\partial D)^N}. \tag{54}$$

Obviously, the difference $P := w_1 - W \in H^1(\Omega\backslash\overline{D}_h)$ is the unique solution to

$$\begin{cases} \mathcal{L}P + \omega^2 P = 0 & \text{in } \Omega\backslash\overline{D}_h, \\ TP = 0 & \text{on } \partial D_h, \\ TP = -TW & \text{on } \partial\Omega. \end{cases}$$

Making use of layer potential techniques, one can show that

$$||P||_{H^{1/2}(\partial\Omega)^N} \le C\, ||TW|_{\partial\Omega}||_{C(\partial\Omega)}. \tag{55}$$

(55) can be proved in a completely similar manner to that of Lemma 5.4 in what follows. Finally, combining (54) and (55) yields the estimate (53), which completes the proof. □

Proof of Theorem 5.2. We set $\varphi = T^+\tilde{u}|_{\partial D_h}$ in Lemma 5.2, so that $v = \tilde{u}$ and $\Psi^+ = \varphi(h\,\cdot)$. By Lemma 5.2, it holds that

$$||\tilde{u} - u_0||_{H^{1/2}(\partial\Omega)^N} \le C_1\, \left(h^N\, ||\psi||_{H^{-1/2}(\partial\Omega)^N} + h^{N-1}\, ||\Psi^+||_{H^{-3/2}(\partial D)^N}\right). \tag{56}$$

Recalling from the second assertion of Lemma 5.2 that, for sufficiently small h,

$$||\Psi^+||_{H^{-3/2}(\partial D)^N} \le C_2\, h^{(2-N)/2}\, ||\tilde{u} - u_0||^{1/2}_{H^{1/2}(\partial\Omega)^N}\, ||\psi||^{1/2}_{H^{-1/2}(\partial\Omega)^N}. \tag{57}$$

Combining (56) and (57) and applying Young's inequality yield the desired estimate in (45).

The proof is complete. □

5.3. Estimates on small inclusions

5.3.1. Layer potentials for the Lamé system

We first recall the fundamental solution $\Pi(x, y)$ (Green's tensor) to the Navier equation (15) in $\mathbb{R}^N$. Let $G_k(x, y)$ denote the free-space fundamental solution to the scalar Helmholtz equation $(\Delta + k^2)u = 0$ in $\mathbb{R}^N$. In three dimensions, it takes the form

$$G_k(x, y) = \frac{\exp(ik|x - y|)}{4\pi\, |x - y|}, \quad x \ne y, \quad x, y \in \mathbb{R}^3,$$

while in two dimensions,

$$G_k(x, y) = \frac{i}{4}\, H_0^{(1)}(k|x - y|), \quad x \ne y, \quad x, y \in \mathbb{R}^2,$$

where $H_0^{(1)}(\cdot)$ is the Hankel function of the first kind of order zero. Then the Green's tensor $\Pi(x, y)$ for the Lamé system can be represented as

$$\Pi^{(\omega)}(x, y) = \frac{1}{\mu}\, G_{k_s}(x, y)\, \mathbf{I} + \frac{1}{\omega^2}\, \mathrm{grad}_x\, \mathrm{grad}_x^\top \left[G_{k_s}(x, y) - G_{k_p}(x, y)\right] \tag{58}$$

for $x, y \in \mathbb{R}^N$, $x \neq y$, where the compressional and shear wave numbers k_p and k_s are given respectively in (16) and (17), and $\mathbf{I}$ stands for the $N \times N$ identity matrix.

Let Q be a bounded simply connected domain in $\mathbb{R}^N$ with the smooth boundary ∂Q. In our subsequent applications, $Q = D_h$ or $Q = \Omega$. For surface densities $\varphi(x)$ with $x \in \partial Q$, define the single and double layer potential operators for the Navier equation by

$$(SL_Q\varphi)(x) := \int_{\partial Q} \Pi(x,y)\varphi(y)ds(y), \quad x \in \mathbb{R}^N \backslash \partial Q, \tag{59}$$

$$(DL_Q\varphi)(x) := \int_{\partial Q} \Xi(x,y)\varphi(y)ds(y), \quad x \in \mathbb{R}^N \backslash \partial Q, \tag{60}$$

where $\Xi(x, y)$ is a matrix function whose l-th column vector is defined as

$$[\Xi(x,y)]^\top \mathbf{e}_l := T_{\nu(y)} [\Pi(x,y)\mathbf{e}_l] = \nu(y) \cdot [\sigma(\Pi(x,y)\,\mathbf{e}_l)] \quad \text{on} \quad \partial Q,$$

for $x \neq y, l = 1, 2, N$. Here, $\mathbf{e}_l$, $1 \leq l \leq N$ are the standard Euclidean base vectors in $\mathbb{R}^N$, and $T_{\nu(y)}$ is the stress operator defined in (7) and (8). We also let

$$(S_Q\varphi)(x) := \int_{\partial Q} \Pi(x,y)\varphi(y)ds(y), \quad x \in \partial Q, \tag{61}$$

$$(K_Q\varphi)(x) := \int_{\partial Q} \Xi(x,y)\varphi(y)ds(y), \quad x \in \partial Q. \tag{62}$$

Using Taylor series expansion for exponential functions, one can rewrite the matrix $\Pi^{(\omega)}(x, y)$ in 3D as the series (see, e.g., [5])

$$\begin{aligned}\Pi^{(\omega)}(x,y) = {} & \frac{1}{4\pi}\sum_{n=0}^{\infty} \frac{(n+1)(\lambda+2\mu)+\mu}{\mu(\lambda+2\mu)} \frac{(i\omega)^n}{(n+2)n!}|x-y|^{n-1}\,\mathbf{I} \\ & - \frac{1}{4\pi}\sum_{n=0}^{\infty} \frac{\lambda+\mu}{\mu(\lambda+2\mu)} \frac{(i\omega)^n(n-1)}{(n+2)n!}|x-y|^{n-3}\,(x-y)\otimes(x-y),\end{aligned} \tag{63}$$

where $x \otimes x := x^\top x \in \mathbb{R}^{N\times N}$ for $x = (x_1, \cdots, x_N) \in \mathbb{R}^N$. Letting $x \to y$, we get

$$\begin{aligned}\Pi^{(\omega)}(x,y) = {} & \frac{\lambda+3\mu}{8\pi\mu(\lambda+2\mu)} \frac{1}{|x-y|}\,\mathbf{I} + i\omega \frac{2\lambda+5\mu}{12\pi\mu(\lambda+2\mu)}\,\mathbf{I} \\ & + \frac{\lambda+\mu}{8\pi\mu(\lambda+2\mu)} \frac{1}{|x-y|^3}\,(x-y)\otimes(x-y) + o(1)\omega^2.\end{aligned} \tag{64}$$

Taking $\omega \to +0$ in (64), we obtain the fundamental tensor of the Lamé system with $\omega = 0$ in $\mathbb{R}^3$:

$$\Pi^{(0)}(x,y) = \frac{\lambda+3\mu}{8\pi\mu(\lambda+2\mu)} \frac{1}{|x-y|}\,\mathbf{I} + \frac{\lambda+\mu}{8\pi\mu(\lambda+2\mu)} \frac{1}{|x-y|^3}\,(x-y)\otimes(x-y). \tag{65}$$

Analogously, in two dimensions we have the expression (see [34, Chapter 2.2])

$$\Pi^{(0)}(x,y) = \frac{1}{4\pi}\left[-\frac{3\mu+\lambda}{\mu(2\mu+\lambda)} \ln|x-y|\,\mathbf{I} + \frac{\mu+\lambda}{\mu(2\mu+\lambda)\,|x|^2}\,(x-y)\otimes(x-y)\right]. \tag{66}$$

Similar to the definitions of SL_Q, DL_Q, S_Q, D_Q, we define the operators $SL_Q^{(0)}$, $DL_Q^{(0)}$, $S_Q^{(0)}$, $D_Q^{(0)}$ in the same way as (59), (60), (61) and (62), but with the tensor $\Pi^{(\omega)}(x,y)$ replaced by $\Pi^{(0)}(x,y)$. It is well known that these operators all have weakly singular kernels; see, e.g., [34] and [39].

Using the asymptotic behavior of Bessel functions, it has been shown in two dimensions (see, e.g., [33, Lemma 2.1])

$$\Pi^{(\omega)}(x,y) = \Pi^{(0)}(x,y) + \eta\,\mathbf{I} + \mathcal{O}(|x-y|^2 \ln|x-y|) \tag{67}$$

as $x \to y$, where η is a constant given by

$$\eta = -\frac{1}{4\pi}\left[\frac{\lambda+3\mu}{\mu(\lambda+2\mu)}(\ln\frac{\omega}{2} + E - \frac{i\pi}{2}) + \frac{\lambda+\mu}{\mu(\lambda+2\mu)} - \frac{1}{2}(\frac{\ln\mu}{\mu} + \frac{\ln(\lambda+2\mu)}{\lambda+2\mu})\right],$$

with $E = 0.57721\cdots$ being Euler's constant. From the asymptotic behavior (67), it follows that for $x \in \overline{D}_h$,

$$\int\limits_{D_h} ||\Pi^{(\omega)}(x,y)||_{\max}\, dy = \mathcal{O}(h^2\ln h), \int\limits_{D_h} ||\partial_{x_j}\Pi^{(\omega)}(x,y)||_{\max}\, dy = \mathcal{O}(h), \tag{68}$$

for $j = 1,2,3$ as $h \to +0$, where $||\cdot||_{\max}$ denotes the maximum norm of a matrix. Analogously, we can deduce from (64) that in 3D,

$$\int\limits_{D_h} ||\Pi^{(\omega)}(x,y)||_{\max}\, dy = \mathcal{O}(h^2), \int\limits_{D_h} ||\partial_{x_j}\Pi^{(\omega)}(x,y)||_{\max}\, dy = \mathcal{O}(h),\ j=1,2,3, \tag{69}$$

as $h \to +0$. The relations in (68) and (69) remain valid for all $\omega \geq 0$. The difference $\Pi^{(\omega)}(x,y) - \Pi^{(0)}(x,y)$ is a continuous function in $\mathbb{R}^N \times \mathbb{R}^N$.

5.3.2. Estimates on small inclusions

Lemma 5.4. *Assume that $-\omega^2$ is not an eigenvalue of the elliptic operator $\mathcal{L}$ on Ω with the traction-free boundary condition. Let $u_0 \in H^1(\Omega)^N$ be the unique solution of (15) with $\psi \in H^{-1/2}(\partial\Omega)^N$. Consider the Lamé system*

$$\begin{cases} \mathcal{L}w + \omega^2 w = 0 & \text{in } \Omega\backslash\overline{D}_h, \\ Tw = Tu_0 & \text{on } \partial D_h, \\ Tw = 0 & \text{on } \partial\Omega. \end{cases} \tag{70}$$

Then there exists a constant $h_0 \in \mathbb{R}_+$ such that for all $h < h_0$, the above Lamé system admits a unique solution $w \in H^1(\Omega\backslash\overline{D}_h)^N$. Moreover, there holds the estimate

$$||w||_{H^{1/2}(\partial\Omega)^N} \leq C\, h^N\, ||\psi||_{H^{-1/2}(\partial\Omega)^N}, \tag{71}$$

where C is a positive constant independent of h and ψ.

Proof. For clarity we divide our proof into three steps.

Step 1. Show that the function

$$V(x) := \int\limits_{\partial D_h} \Pi(x,y)\, Tu_0(y)\, ds(y), \quad x \in \Omega\backslash D_h,$$

satisfies the estimates

$$||V(h\,\cdot)||_{C(\partial D)} \le C\,h||\psi||_{H^{-1/2}(\partial\Omega)^N}, \quad ||V||_{C(\partial\Omega)} \le C\,h^N\,||\psi||_{H^{-1/2}(\partial\Omega)^N}. \tag{72}$$

We first estimate $V(x)$ for $x \in \partial\Omega$. From Betti's formula, we rewrite the j-th component V_j of V as

$$\begin{aligned} V_j(x) &= \int\limits_{\partial D_h} [\Pi^\top(x,y)\mathbf{e}_j] \cdot Tu_0(y)\,ds(y) \\ &= \int\limits_{D_h} \mathcal{L}u_0(y) \cdot [\Pi^\top(x,y)\mathbf{e}_j]dy + \int\limits_{D_h} [\mathcal{C}^{(0)} : \nabla_y(\Pi^\top(x,y)\mathbf{e}_j)] : \nabla_y u_0\,dy \\ &= -\omega^2 \int\limits_{D_h} u_0(y) \cdot [\Pi^\top(x,y)\mathbf{e}_j]dy + \int\limits_{D_h} [\mathcal{C}^{(0)} : \nabla_y(\Pi^\top(x,y)\mathbf{e}_j)] : \nabla_y u_0\,dy. \end{aligned} \tag{73}$$

Since $||\Pi(x,y)||_{\max}$ and $||\partial_{y_j}\Pi(x,y)||_{\max}$ are uniformly bounded for all $x \in \partial\Omega$, $y \in \partial D_h$ and for all $j = 1,2,3$, we readily derive from (73) that

$$|V_j(x)| \le C\,h^N\,\left(\omega^2||u_0||_{L^\infty(D_h)^N} + ||\nabla u_0||_{L^\infty(D_h)^{N\times N}}\right) \le C\,h^N||\psi||_{H^{-1/2}(\partial\Omega)^N},$$

for all $j = 1,2,3$, where the last inequality follows from the stability of the boundary value problem (44). This proves the second estimate in (72). The first estimate when $x \in \partial D_h$ follows straightforwardly from (73), the relations in (68) and (69), together with the fact that both $||u_0||_{L^\infty(D)^N}$ and $||\nabla u_0||_{L^\infty(D)^{N\times N}}$ are bounded by $||\psi||_{H^{-1/2}(\partial\Omega)^N}$.

Step 2. Set $\phi_1 = w|_{\partial D_h}$, $\phi_2 = w|_{\partial\Omega}$. In this step, we shall verify

$$||\phi_1||_{L^2(\partial D_h)^N} \le C\,h^{(N+1)/2}\,||\psi||_{H^{-1/2}(\partial\Omega)^N}, \quad ||\phi_2||_{L^2(\partial\Omega)^N} \le C\,h^N\,||\psi||_{H^{-1/2}(\partial\Omega)^N}. \tag{74}$$

Again using Betti's formula, we represent the solution w to (70) as

$$\begin{aligned} w(x) &= \int\limits_{\partial\Omega\cup\partial D_h} \{\Pi(x,y)Tw(y) - \Xi(x,y)w(y)\}\,ds(y) \\ &= -\int\limits_{\partial\Omega} \Xi(x,y)w(y)\,ds(y) - \int\limits_{\partial D_h} \{\Pi(x,y)Tu_0(y) - \Xi(x,y)w(y)\}\,ds(y) \\ &= -DL_{\partial\Omega}(\phi_2)(x) + DL_{\partial D_h}(\phi_1)(x) - V(x), \end{aligned}$$

where the function V is defined in Step 1. Since Tu_0 is smooth on ∂D_h and the boundaries of Ω and D_h are both smooth, by the elliptic regularity w is smooth up to the boundary of $\Omega\backslash D_h$. Hence ϕ_1 and ϕ_2 are both smooth functions. Letting x tend to $\partial\Omega$ and ∂D_h, and applying the jump relations for double layer potentials, we have for $\phi_1 \in C(\partial D_h)$, $\phi_2 \in C(\partial\Omega)$ that

$$\begin{cases} \frac{1}{2}\phi_1(x) = (K_{\partial D_h}\phi_1)(x) - (DL_{\partial\Omega}\phi_2)(x) + V(x), & x \in \partial D_h, \\ \frac{1}{2}\phi_2(x) = (DL_{\partial D_h}\phi_1)(x) - (K_{\partial\Omega}\phi_2)(x) + V(x), & x \in \partial\Omega. \end{cases} \tag{75}$$

Since $-\omega^2$ is not an eigenvalue of the operator $\mathcal{L}$ on Ω with the traction-free boundary condition, the operator $\frac{1}{2}I + K_{\partial\Omega} : C(\partial\Omega)^N \to C(\partial\Omega)^N$ is continuously invertible. Thus it follows from (75) and (72) that

$$||\phi_2||_{C(\partial\Omega)^N} \leq C\left(||DL_{\partial D_h}\phi_1||_{C(\partial\Omega)^N} + ||V||_{C(\partial\Omega)^N}\right)$$
$$\leq C\left(h^{(N-1)/2}\,||\phi_1||_{L^2(\partial D_h)^N} + h^N\,||\psi||_{H^{-1/2}(\partial\Omega)^N}\right). \tag{76}$$

Since the L^2-norm of ϕ_2 can also be bounded by the left hand side of (76), we only need to verify the first estimate in (74). To that end, we rewrite the first equation in (75) as

$$\left[\left(\frac{1}{2}I + K^{(0)}_{\partial D} - \mathcal{R}\right)\phi_1(h\,\cdot)\right](hx) + \left(DL_{\partial\Omega}\phi_2(\cdot)\right)(hx) = V(hx), \quad x \in D, \tag{77}$$

where the kernel of the operator $\mathcal{R} : L^2(\partial D)^N \to L^2(\partial D)^N$ is given by the continuous matrix $\Pi^{(0)} - \Pi^{(\omega)}$. Further, it can be straightforwardly checked that

$$||\mathcal{R}||_{\mathcal{L}(L^2(\partial D)^N, L^2(\partial D)^N)} \leq C\,h^{N-1}.$$

On the other hand, the L^2-norm of $DL_{\partial\Omega}\phi_2(h\,\cdot)$ over ∂D can be bounded by the left hand side of (76) and that of $V(h\,\cdot)$ can be estimated as in the first relation of (72). Hence, by the boundedness of $(\frac{1}{2}I - K^{(0)}_{\partial D})^{-1} : L^2(\partial D)^N \to L^2(\partial D)^N$, we deduce from (77) that

$$||\phi_1(h\,\cdot)||_{L^2(\partial D)^N} \leq C\,h\,||\psi||_{H^{-1/2}(\partial\Omega)^N},$$

leading to the first relation of (74) on ∂D_h.

Step 3. By the second equality in (75) and the definition of ϕ_2 in Step 2, one has

$$w(x) = 2\left[\left(DL_{\partial D_h}\phi_1\right)(x) - \left(K_{\partial\Omega}\phi_2\right)(x) + V(x)\right], \quad x \in \partial\Omega. \tag{78}$$

By a similar argument to that for the proof of the second relation in (72), one can show that

$$||V||_{H^{1/2}(\partial\Omega)^N} \leq C\,||V||_{C^1(\partial\Omega)^N} \leq C\,h^N\,||\psi||_{H^{-1/2}(\partial\Omega)^N}. \tag{79}$$

Further, using the first estimate in (74) yields

$$||DL_{\partial D_h}\varphi_1||_{H^{1/2}(\partial\Omega)^N} \leq C\,||DL_{\partial D_h}\varphi_1||_{C^1(\partial\Omega)^N} \leq C\,h^N\,||\psi||_{H^{-1/2}(\partial\Omega)^N}. \tag{80}$$

Since $K_{\partial\Omega} : L^2(\Omega)^N \to H^1(\Omega)^N$ is bounded, by the second estimate of (74) we find

$$||K_{\partial\Omega}\phi_2||_{H^{1/2}(\partial\Omega)^N} \leq C\,h^N\,||\psi||_{H^{-1/2}(\partial\Omega)^N}. \tag{81}$$

Combining (78)–(81) yields (71).

The proof is completed. □

Consider the time-harmonic elastic scattering problem from a small cavity $D_h \subset \Omega$. This can be modeled by the following boundary value problem in the exterior of D_h: find $W \in H^1_{loc}(\mathbb{R}^N\backslash\overline{D}_h)^N$ such that

$$\mathcal{L}\,W + \omega^2 W = 0 \quad \text{in} \quad \mathbb{R}^N\backslash\overline{D}_h, \quad T\,W = \varphi \quad \text{on} \quad \partial D_h. \tag{82}$$

Since $\mathbb{R}^N\backslash\overline{D}_h$ is unbounded, W is required to satisfy the Kupradze radiation condition when $|x| \to \infty$ (see, e.g., [1]):

$$\lim_{r\to\infty}\left(\frac{\partial W_p}{\partial r}-ik_pW_p\right)=0,\quad \lim_{r\to\infty}\left(\frac{\partial W_s}{\partial r}-ik_sW_s\right)=0,\quad r=|x|, \tag{83}$$

which holds uniformly in all directions $\hat{x}=x/|x|\in\mathbb{S}^{N-1}$. The functions W_p and W_s denote the compressional and shear parts of W, respectively; see (16) and (17). It is well known that the above boundary value problem is well-posed for any $\varphi\in H^{-1/2}(\partial D_h)^N$.

Lemma 5.5. *Let $W\in H^1_{loc}(\mathbb{R}^N\backslash\overline{D}_h)^N$ be the unique solution of the system (82)–(83). Then, there exists $h_0\in\mathbb{R}_+$ such that when $h<h_0$,*

$$||W||_{H^{1/2}(\partial\Omega)^N}+||TW|_{\partial\Omega}||_{C(\partial\Omega)^N}\le C\,h^{N-1}\,||\varphi(h\,\cdot)||_{H^{-3/2}(\partial D)^N}, \tag{84}$$

where C is a positive constant independent of h and φ.

Proof. From Betti's formula, we have the expression

$$W(x)=(DL_{\partial D_h}\phi)(x)-(SL_{\partial D_h}\varphi)(x),\quad x\in\mathbb{R}^N\backslash\overline{D}_h, \tag{85}$$

with $\phi=W|_{\partial D_h}\in H^{1/2}(\partial D_h)^N$. Letting $x\to\partial D_h$ and applying the jump relations of layer potential operators, we obtain

$$\frac{1}{2}\phi(x)=(K_{\partial D_h}\phi)(x)-(S_{\partial D_h}\varphi)(x),\quad x\in\partial D_h. \tag{86}$$

Similar to (77), (86) can be equivalently formulated as

$$\left[\left(\frac{1}{2}I-K^{(0)}_{\partial D}-\mathcal{R}\right)\phi(h\,\cdot)\right](hx)=-(S_{\partial D_h}\varphi)(hx),\quad x\in\partial D, \tag{87}$$

where the kernel of the operator $\mathcal{R}:H^{-1/2}(\partial D)^N\to H^{-1/2}(\partial D)^N$ is given by the continuous matrix $\Pi^{(0)}-\Pi^{(\omega)}$, satisfying the estimate (see [52, Chapter 4.3])

$$||\mathcal{R}||_{\mathcal{L}(H^{-1/2}(\partial D)^N,H^{-1/2}(\partial D)^N)}\le C\,h^{N-1}. \tag{88}$$

Since $\frac{1}{2}I-K^{(0)}_{\partial D}$ is an isomorphism from $H^{-1/2}(\partial D)^N$ to $H^{-1/2}(\partial D)^N$, it follows from (87) and (88) that for $h\in\mathbb{R}_+$ sufficiently small

$$||\phi(h\,\cdot)||_{H^{-1/2}(\partial D)^N}\le C\,||(S_{\partial D_h}\varphi)(h\,\cdot)||_{H^{-1/2}(\partial D)^N}. \tag{89}$$

In order to estimate the left hand side of (89), we decompose $(S_{\partial D_h}\varphi)(hx)$ into

$$\begin{aligned}(S_{\partial D_h}\varphi)(hx)&=h^{N-1}(S_{\partial D}\varphi(h\,\cdot))(hx)\\&=h^{N-1}\,(S^{(0)}_{\partial D}\varphi(h\,\cdot))(hx)+(\mathcal{G}_{\partial D}\varphi(h\,\cdot))(hx),\end{aligned} \tag{90}$$

where the integral kernel of the integral operator $\mathcal{G}_{\partial D}$ is given by

$$\Pi'(x,y)=h^{N-1}\Pi^{(\omega)}(hx,hy)-h^{N-1}\Pi^{(0)}(x,y).$$

Using (64) and (65), together with straightforward calculations, one can show that when $N=3$

$$\Pi'(x,y) = h^2 \frac{1}{4\pi} \sum_{n\geq 2}^{\infty} \frac{(n+1)(\lambda+2\mu)+\mu}{\mu(\lambda+2\mu)} \frac{(i\omega)^n}{(n+2)n!} |hx-hy|^{n-1}\,\mathbf{I}$$
$$- h^2 \frac{1}{4\pi} \sum_{n\geq 2}^{\infty} \frac{\lambda+\mu}{\mu(\lambda+2\mu)} \frac{(i\omega)^n (n-1)}{(n+2)n!} |hx-hy|^{n-3}\,(hx-hy)\otimes(hx-hy)$$
$$= h^2\, i\omega \frac{2\lambda+5\mu}{12\pi\mu(\lambda+2\mu)}\,\mathbf{I} + h^2\omega^2\, A(h|x-y|),$$

where A is a real-analytic function satisfying $A(t) \to 0$ as $t \to 0$. In two dimensions, it follows from (66) and (67) that

$$\begin{aligned}\Pi'(x,y) &= h[\Pi^{(\omega)}(hx,hy) - \Pi^{(0)}(x,y)] \\ &= h\left[\Pi^{(0)}(hx,hy) - \Pi^{(0)}(x,y) + \eta\,\mathbf{I} + h^2 \ln h\,\mathcal{O}(|x-y|^2 \ln|x-y|)\right] \\ &= -\frac{3\mu+\lambda}{4\pi\mu(2\mu+\lambda)} h\ln h + \eta\,\mathbf{I}h + h^3 \ln h\,\mathcal{O}(|x-y|^2\ln|x-y|) \\ &= \mathcal{O}(h\ln h)\end{aligned}$$

as $h \to +0$. Hence, by the mapping properties presented in [52, Chapter 4.3],

$$||\,(\mathcal{G}_{\partial D}\varphi(hx))\,(h\,\cdot)||_{H^{-1/2}(\partial D)^N} \leq C\,e(h)\,||\varphi(h\,\cdot)||_{H^{-3/2}(\partial D)^N}, \tag{91}$$

with the dimensional constant

$$e(h) := \begin{cases} h^2 & \text{if } N=3, \\ h\,\ln h & \text{if } N=2. \end{cases}$$

Recalling the boundedness of the operator $S_{\partial D}^{(0)} : H^{-3/2}(\partial D)^N \to H^{-1/2}(\partial D)^N$, we see from (90) the estimate

$$||\,(S_{\partial D_h}\varphi)\,(h\,\cdot)||_{H^{-1/2}(\partial D)^N} \leq C\,\tilde{e}(h)\,||\varphi(h\,\cdot)||_{H^{-3/2}(\partial D)^N}$$

with

$$\tilde{e}(h) := \begin{cases} h & \text{if } N=3, \\ h\,\ln h & \text{if } N=2. \end{cases}$$

Hence, by (89),

$$||\phi(h\,\cdot)||_{H^{-1/2}(\partial D)^N} \leq C\,\tilde{e}(h)\,||\varphi(h\,\cdot)||_{H^{-3/2}(\partial D)^N}. \tag{92}$$

Let Ω_1 be a compact set of $\mathbb{R}^N\backslash\overline{D}$ containing $\partial\Omega$. For $x \in \Omega_1$, we see from (85) and (92) that

$$\begin{aligned}||W||_{L^2(\Omega_1)^N} &\leq C\,h^{N-1}\{||\phi(h\,\cdot)||_{H^{-1/2}(\partial D)^N} + ||\varphi(h\,\cdot)||_{H^{-3/2}(\partial D)^N}\} \\ &\leq C\,h^{N-1}\,||\varphi(h\,\cdot)||_{H^{-3/2}(\partial D)^N}.\end{aligned} \tag{93}$$

Finally, the estimate in (84) is a consequence of (93) and the interior estimate for elliptic boundary value problems.

The proof is complete. □

5.4. Finite realization of the traction-free lining

Finally, we present an interesting observation on the physical nature of the proposed lossy layer $\{D\backslash\overline{D}_{1/2};\mathcal{C}^{(2)},\ \rho^{(2)}\} = (F_h)_*\{D_h\backslash\overline{D}_{h/2};\tilde{\mathcal{C}}^{(2)},\tilde{\rho}^{(2)}\}$ in our near-cloaking construction (37)–(40). It can be shown to be a finite realization of the traction-free lining. Indeed, we have:

Lemma 5.6. *Suppose that $-\omega^2$ is not an eigenvalue of the elliptic operator $\mathcal{L}$ on $\Omega\backslash\overline{D}$ with the traction-free boundary condition. Let $U \in H^1(\Omega\backslash\overline{D})^3$ be the unique solution of*

$$\begin{cases} \nabla\cdot(\mathcal{C}^{(1)}:\nabla U)+\omega^2\rho^{(1)}\,U=0 & \text{in } \Omega\backslash\overline{D},\\ \mathcal{N}_{\mathcal{C}^{(1)}}\,U=\psi & \text{on } \partial\Omega,\\ \mathcal{N}_{\mathcal{C}^{(1)}}\,U=0 & \text{on } \partial D, \end{cases} \tag{94}$$

where $\psi\in H^{-1/2}(\partial\Omega)^N$ and $\{\Omega\backslash\overline{D};\mathcal{C}^{(1)},\rho^{(1)}\}$ is the elastic medium in (39). Let $u\in H^1(\Omega)^N$ be the solution to the boundary value problem

$$\nabla\cdot(\mathcal{C}:\nabla u)+\omega^2\rho\,u=0 \quad \text{in} \quad \Omega, \quad \mathcal{N}_{\mathcal{C}}\,u=\psi \quad \text{on} \quad \partial\Omega, \tag{95}$$

where $(\Omega;\mathcal{C},\rho)$ is given in (37)–(40). Then for sufficiently small $h\in\mathbb{R}_+$, we have

$$||U-u||_{H^{1/2}(\partial\Omega)^N}\le C\,h^N||\psi||_{H^{-1/2}(\partial\Omega)^N},$$

where C is a positive constant independent of h and ψ.

Proof. Set $\tilde{U}=F_*U$, $\tilde{u}=F_*u$ in Ω. By Lemma 2.1, we see $\tilde{u}$ satisfies (44) and $\tilde{U}$ is the solution of the boundary value problem

$$\mathcal{L}\tilde{U}+\omega^2\,\tilde{U}=0 \quad \text{in} \quad \Omega\backslash\overline{D}_h, \quad T\tilde{U}=\psi \quad \text{on} \quad \partial\Omega, \quad T\tilde{U}=0 \quad \text{on} \quad \partial D_h.$$

Moreover, we have $U=\tilde{U}$, $u=\tilde{u}$ on $\partial\Omega$. Let u_0 be the solution to the free-space boundary value problem (15). Then the difference $W:=u_0-\tilde{U}$ satisfies

$$\mathcal{L}W+\omega^2\,W=0 \quad \text{in} \quad \Omega\backslash\overline{D}_h, \quad TW=0 \quad \text{on} \quad \partial\Omega, \quad TW=Tu_0 \quad \text{on} \quad \partial D_h.$$

From Lemma 5.4, we see

$$||\tilde{U}-u_0||_{H^{1/2}(\partial\Omega)^N}=||W||_{H^{1/2}(\partial\Omega)^N}\le C\,h^N\,||\psi||_{H^{-1/2}(\partial\Omega)^N}. \tag{96}$$

On the other hand, it follows from Theorem 5.2 that

$$||\tilde{u}-u_0||_{H^{1/2}(\partial\Omega)^N}\le C\,h^N\,||\psi||_{H^{-1/2}(\partial\Omega)^N}. \tag{97}$$

Hence, combining the two estimates in (96) and (97), we finally have

$$\begin{aligned} ||U-u||_{H^{1/2}(\partial\Omega)^N} &= ||\tilde{U}-\tilde{u}||_{H^{1/2}(\partial\Omega)^N}\\ &\le ||\tilde{U}-u_0||_{H^{1/2}(\partial\Omega)^N}+||\tilde{u}-u_0||_{H^{1/2}(\partial\Omega)^N}\\ &\le C\,h^N\,||\psi||_{H^{-1/2}(\partial\Omega)^N}. \end{aligned}$$

The proof is complete. $\square$

6. Concluding remarks

In this work, we develop a general mathematical framework on approximate cloaking of elastic wave fields via the approach of transformation elastodynamics. This opens up many new and interesting problems for further investigation. Based on the so-called GPT-vanishing structure (where GPT stands for generalized polarization tensor), schemes of significantly enhancing the accuracy of the approximate cloaks of acoustic and electromagnetic waves were proposed in [4,6,7]. We would like to note that similar idea may be applied to the regularized approximate elastic cloaks by using the generalized *elastic moment tensors* (EMT) proposed in [3,2] to devise enhancement schemes. Moreover, as described in Section 3, it is of particular interest to devise compressional and shear elastic cloaks; that is, cloaking devices that are used for cloaking only compressional or shear waves. Finally, it is of practical importance to construct cloaking devices whose elastic materials retain both the major and minor symmetries.

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Appendix A

We derive the following Sobolev extension result which is required in the proof of Lemma 5.2.

Lemma A.1. *For any $\phi \in H^{3/2}(\partial D)^N$, there exists $w \in H^2(D)^N$ such that*

(i) *$w = \phi$ on ∂D and $Tw = 0$ on ∂D;*
(ii) *$||w||_{H^2(D)^N} \le C\,||\phi||_{H^{3/2}(\partial D)^N}$;*
(iii) *$w = 0$ in $D_{1/2}$.*

Proof. For $\psi \in H^{1/2}(\partial D)^N$, one can clearly find $w_1 \in H^2(D)^N$ such that

$$w_1 = 0 \quad \text{in } D_{1/2}, \quad \partial_\nu w_1 = \psi \quad \text{on } \partial D, \quad ||w_1||_{H^2(D)^N} \le C\,||\psi||_{H^{1/2}(\partial D)^N}.$$

By [56, Theorem 14.1], there exists $w_2 \in H^2(D)^N$ such that

(i) $w_2 = \phi - w_1$ on ∂D and $\partial_\nu w_2 = 0$ on ∂D;
(ii) $||w_2||_{H^2(D)^N} \le C\,||\phi - w_1||_{H^{3/2}(\partial D)^N}$;
(iii) $w_2 = 0$ in $D_{1/2}$.

Hence, the sum $w := w_1 + w_2$ satisfies

(a) $w = \phi$ on ∂D and $\partial_\nu w = \psi$ on ∂D;
(b) $||w||_{H^2(D)^N} \le C(||\phi||_{H^{3/2}(\partial D)^N} + ||\psi||_{H^{1/2}(\partial D)^N})$;
(c) $w = 0$ in $D_{1/2}$.

In order to conclude the proof of the lemma, it is sufficient to determine a $\psi = \psi(\phi) \in H^{1/2}(\partial D)^N$ such that

$$Tw = 0 \quad \text{on} \quad \partial D \quad \text{and} \quad ||\psi||_{H^{1/2}(\partial D)^N} \le C\, ||\phi||_{H^{3/2}(\partial D)^N}. \tag{98}$$

In two dimensions, we recall from (35) that the stress operator can be decomposed into

$$Tw = A(\lambda, \mu, \nu)\, \partial_\nu w + B(\lambda, \mu, \nu)\, \partial_\tau w \quad \text{on} \quad \partial D.$$

In particular, the matrix A is invertible, B is bounded and the tangential derivative $\partial_\tau w \in H^{1/2}(\partial D)^N$ is uniquely determined by $w = \phi$ on ∂D. Hence, choosing $\psi := -A^{-1} B \partial_\tau w \in H^{1/2}(\partial D)^N$, we see that the relations in (98) are both fulfilled.

In the 3D case we need the following identity:

$$\operatorname{grad} \varphi = \operatorname{Grad} \varphi + \frac{\partial \varphi}{\partial \nu}\, \nu, \quad \nu = (\nu_1, \nu_2, \nu_3), \tag{99}$$

where $\operatorname{Grad}(\cdot)$ denotes the surface gradient of a scalar function on ∂D. Write $w = (w_1, w_2, w_3)$ and $\operatorname{Grad} w_i = ([\operatorname{Grad} w_j]^1, [\operatorname{Grad} w_j]^2, [\operatorname{Grad} w_j]^3)$. By (99) and the definition (8), one can represent the three dimensional stress operator as

$$Tw = A(\lambda, \mu, \nu)\, \partial_\nu w + \mathbf{B}(\lambda, \mu, \nu, \operatorname{Grad} w),$$

where

$$\begin{aligned}
\mathbf{B} &:= \lambda \nu \, \left([\operatorname{Grad} w_1]^1 + [\operatorname{Grad} w_2]^2 + [\operatorname{Grad} w_3]^3\right) \\
&\quad + \mu\, \nu \times \left([\operatorname{Grad} w_3]^2 - [\operatorname{Grad} w_2]^3, [\operatorname{Grad} w_1]^3 - [\operatorname{Grad} w_3]^1, [\operatorname{Grad} w_2]^1 - [\operatorname{Grad} w_1]^2\right), \\
A &:= 2\mu + \lambda \nu(\nu \cdot\ ) + \mu \nu \times (\nu \times\ ) = \begin{pmatrix} \mu + (\lambda+\mu)\nu_1^2 & (\lambda+\mu)\nu_1\nu_2 & (\lambda+\mu)\nu_1\nu_3 \\ (\lambda+\mu)\nu_1\nu_2 & \mu + (\lambda+\mu)\nu_2^2 & (\lambda+\mu)\nu_2\nu_3 \\ (\lambda+\mu)\nu_1\nu_3 & (\lambda+\mu)\nu_2\nu_3 & \mu + (\lambda+\mu)\nu_3^2 \end{pmatrix}.
\end{aligned}$$

It is straightforward to verify that

$$||\mathbf{B}||_{H^{1/2}(\partial D)^N} \le C\, ||w||_{H^{1/2}(\partial D)^N} = C\, ||\phi||_{H^{1/2}(\partial D)^N},$$

since only the surface gradients are involved in $\mathbf{B}$. On the other hand, we have $\det(A) = \mu^2(\lambda + 2\mu) > 0$. Hence, similar to that in the 2D case, one can take $\psi := -A^{-1}\mathbf{B} \in H^{1/2}(\partial D)^N$. This verifies (98) in three dimensions and completes the proof. □

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