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Uniqueness to inverse acoustic scattering from coated polygonal obstacles with a single incoming wave

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Abstract

It is proved that a connected polygonal obstacle coated by thin layers together with its surface impedance function can be determined uniquely from the far field pattern of a single incident plane wave. As a by-product, we prove that the wave field cannot be real-analytic on each corner point lying on the convex hull of the scatterer. Our arguments are based on the Schwarz reflection principle for the Helmholtz equation satisfying the impedance boundary condition on a flat boundary.

Keywords: inverse acoustic scattering, impedance boundary condition, reflection principle, uniqueness

(Some figures may appear in colour only in the online journal)

1. Introduction and main results

Let $D \subset \mathbb{R}^2$ be a coated obstacle by a thin dielectric layer, which is embedded in an infinite homogeneous and isotropic medium. In this paper, D is supposed to be a bounded connected polygon such that its exterior $D^c := \mathbb{R}^2 \setminus \overline{D}$ is connected. The time-harmonic acoustic scattering of a plane wave $u^{\text{in}}(x) = e^{ikx \cdot d}$ from the impenetrable scatterer D is modelled by the following system of equations

$$\Delta u + k^2 u = 0, \quad \text{in } D^c := \mathbb{R}^2 \setminus \overline{D}, \quad (1)$$

$$u(x) = e^{ikx \cdot d} + u^{\text{sc}}(x), \quad x, d \in \mathbb{R}^2, \quad (2)$$

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$$\lim_{r \rightarrow \infty} \sqrt{r} (\partial_r u^{\text{sc}} - iku^{\text{sc}}) = 0, \quad r = |x|, \quad (3)$$

$$\partial_\nu u + i\lambda u = 0, \quad \text{on } \partial D, \quad (4)$$

where $k > 0$ is the wave number, $d \in \mathbb{S} := \{x \in \mathbb{R}^2 : |x| = 1\}$ is the incident direction, and u^{sc} is the scattered field. The normal direction $\nu \in \mathbb{S}$ is supposed to be outward. Equation (3) is known as the Sommerfeld radiation condition and the impedance coefficient $\lambda > 0$ is supposed to be constant. It is well known (see [3, 4, 8]) that there exists a unique solution u^{sc} to the system of equations (1)–(4) such that $u^{\text{sc}} \in H_{\text{loc}}^1(D^c)$ if ∂D is Lipschitz. Moreover, the Sommerfeld radiation condition of u^{sc} leads to the asymptotic expansion

$$u^{\text{sc}}(x) = \frac{e^{ikr}}{\sqrt{r}} \left(u^\infty(\hat{x}) + O\left(\frac{1}{r}\right) \right), \quad \hat{x} := x/|x|, \quad (5)$$

uniformly in all directions $\hat{x} \in \mathbb{S}$, where the function u^∞ defined on the unit circle is known as the far field pattern. Inverse scattering problem is to determine the shape and position of the scatterer D together with the impedance coefficient λ from knowledge of the far field pattern u^∞ . In this paper we prove the following uniqueness theorem with a single incoming wave.

Theorem 1.1. *Assume that (D_1, λ_1) and (D_2, λ_2) are two connected polygonal obstacles. Let u_j^∞ for $j = 1, 2$, denote the far-field patterns of the scattering problem (1)–(4) with the fixed incident direction $d \in \mathbb{S}$ when $D = D_j$ and $\lambda = \lambda_j$, respectively. Then the relation $u_1^\infty(\hat{x}) = u_2^\infty(\hat{x})$ over all observation directions $\hat{x} \in \mathbb{S}$ implies $D_1 = D_2$ and $\lambda_1 = \lambda_2$.*

As a by-product of the uniqueness proof to theorem 1.1, we can obtain

Theorem 1.2. *Let $\lambda = \lambda(x)$ ($x \in \mathbb{R}^2$) be an unknown function which can be continued to an entire function on \mathbb{C}^2 such that $\lambda \geq 0$ on ∂D . Then the far-field pattern $u^\infty(\hat{x})$ for all $\hat{x} \in \mathbb{S}$ with one incident direction uniquely determines ∂D and $\lambda(x)|_{\partial D}$. Moreover, the total field cannot be analytically extended into D across any corner point lying on the convex hull of D .*

The unique identification of an obstacle from knowledge of far-field pattern goes back to Schiffer [27] in 1967, where he proved that a sound-soft obstacle can be uniquely recovered using infinite number of plane waves with distinct directions. In [7], Colton and Sleeman proved the same uniqueness with a finite number of plane waves, provided *a priori* information on the size of the sound-soft obstacle is available. In particular, one plane wave is enough if the underlying obstacle is sufficiently small compared to the wave length; see also [19, 24] for related discussions on the bound. Global uniqueness results with a single wave were obtained within the class of sound-soft/sound-hard/penetrable polygonal and polyhedral scatterers [1, 6, 12, 13, 15, 16, 21, 29]. In addition, sound-soft balls/disks can be uniquely determined by a single far-field pattern of a plane wave or point source wave [20, 28].

For coated obstacles, Isakov [25, theorem 2.4] proved that far-field patterns excited at multi frequencies uniquely determine the shape as well as the impedance function. Two plane waves with different directions were used in [5] to determine a convex-polygonal obstacle. Determination of non-convex coated polygons was discussed early in [30]. However, the proof of [30, theorem 2.1, page 302] contains a serious gap in using normal directions of the gap domain between two different obstacles. Hence, the unique determination of a non-convex polygon of impedance type from a single far field pattern remains open up to now, which is the aim of this paper.

We remark that the idea of using the Schwarz reflection principle [1, 6, 12, 13, 29] for recovering sound-soft/sound-hard polygonal obstacles cannot be carried over to the impedance case straightforwardly. This is due to the fact that the ‘point-to-point’ reflection principle for

the Helmholtz equation subject to the Dirichlet/Neumann boundary condition on a flat surface is no longer valid under the Robin boundary condition. In [10], Diaz and Ludford established a ‘non-point-to-point’ reflection principle under the impedance boundary conditions given on a subset of a hyperplane if the domain of the Helmholtz equation fulfills an additional geometric condition in \mathbb{R}^n ($n \geq 2$) (see also remark 2.3 (b)).

Such kind of reflection principle turns out to be sufficient in uniquely determining coated convex-polygonal obstacles with a constant surface impedance; see e.g. [22]. However, for non-convex polygonal obstacles, relaxing or removing the geometric condition of [10] seems necessary. Our arguments for treating non-convex polygons essentially consist of two ingredients. (i) A non-local extension formula for the two-dimensional Helmholtz equation which applies to general connected domains with the Robin boundary condition enforcing on a flat subboundary (see section 3). This is mostly motivated by the reflection principles for the harmonic (resp. Helmholtz) equation subject to the Robin (resp. Dirichlet and Neumann) boundary condition(s) on a real-analytic subboundary [2, 34–36]. (ii) A novel path argument for applying ‘non-point-to-point’ reflection principles to prove the analytical extension of wave fields in polygonal domains.

Our method for proving theorem 1.1 is inspired by the uniqueness proof in inverse conductivity problems with a single measurement [17] and the path argument proposed in [14] where the ‘non-point-to-point’ reflection principle for the Navier equation was applied. This paper also provides new uniqueness proofs in determining sound-soft/sound-hard polygonal obstacles with a single incoming wave. In particular, theorems 1.1 and 1.2 remain valid when the surface impedance function is piecewise holomorphic (for instance, piecewise constant) and under mixed Dirichlet *et al* boundary conditions; see remark 4.1 at the end of the paper.

The remaining part of this paper is organized as follows. In the subsequent two sections, we recall the reflection principle for the Laplacian equation and derive the extension formula for the Helmholtz equation satisfying the Robin boundary condition on a flat subboundary. The proofs of theorems 1.1 and 1.2 will be carried out in section 4.

2. Reflection principle for Laplacian equation

Consider a non-singular real-analytic curve $\Gamma \subset \mathbb{R}^2$ given by $\Gamma := \{x = (x_1, x_2) \in \mathbb{R}^2 : f(x) = 0, \nabla f(x) \neq 0\}$ where f is an algebraic function defined on \mathbb{R}^2 (which means that f is polynomial in x_1 and x_2 with real coefficients). Below we define the mapping R_Γ with respect to Γ by the *Schwarz function* of Γ [32]. Let $U \subset \mathbb{R}^2$ be an open connected set separated into two parts U^+ and U^- by Γ . Now consider a complex domain $V \subset \mathbb{C}^2$ such that $V \cap \mathbb{R}^2 = U$. Using the bicharacteristic coordinates $z = x_1 + ix_2$, $\omega = x_1 - ix_2$, we obtain the equation of the complexified curve $\Gamma_{\mathbb{C}}$ in the form

$$\Gamma_{\mathbb{C}} := \left\{ (z, \omega) \in \mathbb{C}^2 : f\left(\frac{z+\omega}{2}, \frac{z-\omega}{2i}\right) = 0 \right\}. \quad (6)$$

Note that $\Gamma_{\mathbb{C}}$ coincides with Γ on those points such that $z = \bar{\omega}$, where the bar denotes the complex conjugate. Since $\nabla f \neq 0$ on Γ , the above equation (6) is uniquely solvable for z and ω in a neighborhood of Γ in \mathbb{C}^2 . We denote the corresponding solutions by $\omega = S(z)$ and $z = \tilde{S}(\omega)$. Here the function $S(z)$ is called the *Schwarz function* of the curve Γ and $z = \tilde{S}(\omega)$ represents the inverse of $S(z)$. The mapping R_Γ is given by the formula (see [9])

$$R_\Gamma(x) = R_\Gamma(z) := \{y = (y_1, y_2) \in \mathbb{R}^2 : y_1 + iy_2 = \overline{S(z)}\}.$$

It is well known that $R_\Gamma : U \rightarrow U$ is a conformal mapping permuting U^+ and U^- (that is, if $x \in U^\pm$, then $R_\Gamma(x) \in U^\mp$). For a geometric interpretation of R_Γ we refer to [35, 37]. With these notations, we state the reflection principle for harmonic functions with the Robin boundary condition given on an algebraic curve, which was verified in [2]. For notational simplicity we write $\partial_j w(x) = \partial_{x_j} w(x)$ for $x = (x_1, x_2) \in \mathbb{R}^2$.

Proposition 2.1. [2] *Let $\gamma \subset \Gamma$ be a subset of Γ and let $U = U^+ \cup \gamma \cup U^-$ be defined as above with $\gamma := \Gamma \cap U$. Let $w(x)$ be a solution to*

$$\Delta w(x) = 0 \quad \text{in } U^+, \quad \alpha(x)\partial_\nu w + \beta(x)w = 0 \quad \text{on } \gamma, \quad (7)$$

where α, β are holomorphic functions not vanishing simultaneously and ∂_ν denotes the unit normal to γ pointing into U^- . Then w can be extended from U^+ to U by defining $\tilde{w} = w$ in U^+ and

$$\begin{aligned} \tilde{w}(x) := & w(R_\Gamma(x)) + \frac{1}{2i} \int_\gamma^{R_\Gamma(x)} V(y_1, y_2; R_\Gamma(x)) [\partial_2 w(y_1, y_2) dy_1 - \partial_1 w(y_1, y_2) dy_2] \\ & - \frac{1}{2i} \int_\gamma^{R_\Gamma(x)} w(y_1, y_2) [\partial_2 V(y_1, y_2; R_\Gamma(x)) dy_1 - \partial_1 V(y_1, y_2; R_\Gamma(x)) dy_2], \end{aligned} \quad (8)$$

for $x \in U^-$, where the integral $\int_\gamma^{R_\Gamma(x)}$ is calculated on any path joining an arbitrary point on γ with the point $R_\Gamma(x)$. The function $V(y; \tilde{x})$ ($y, \tilde{x} \in \mathbb{R}^2$) is given by $V = V_1 - V_2$, where

$$\begin{aligned} V_1(y; \tilde{x}) &= 1 - 2 \exp \left[-i \int_{S(z_0)}^\omega \frac{\beta(\tilde{S}(\tau), \tau) \sqrt{\tilde{S}'(\tau)}}{\alpha(\tilde{S}(\tau), \tau)} d\tau \right], \\ V_2(y; \tilde{x}) &= 1 - 2 \exp \left[i \int_{\tilde{S}(\omega_0)}^z \frac{\beta(\tau, S(\tau)) \sqrt{S'(\tau)}}{\alpha(\tau, S(\tau))} d\tau \right] \end{aligned} \quad (9)$$

with $z = y_1 + iy_2$, $\omega = y_1 - iy_2$, $z_0 = \tilde{x}_1 + i\tilde{x}_2$, $\omega_0 = \tilde{x}_1 - i\tilde{x}_2$. The integrals in (9) are complex integrals between two points in the complex plane \mathbb{C} .

We remark that, since Γ is an algebraic curve, the Schwarz function S and its inverse \tilde{S} are complex analytic functions on \mathbb{C} with algebraic singularities only. The function V is a multi-valued function over \mathbb{C}^2 whose singularities coincide with S and \tilde{S} . It is supposed in proposition 2.1 that the domain U does not contain such singularities. On the other hand, the constructed function V ensures that the integrand appearing in (8) vanishes on $\Gamma_\mathbb{C}$ (see [2]). Hence, the integral on the right-hand side of (8) is independent on the path of integration and the choice of the point on γ . Although the mapping R_Γ is originally constructed in a small neighborhood of $\Gamma_\mathbb{C}$, by the uniqueness theorem for analytic functions the resulting extension formula is valid in the large domain. In the special case that Γ is a straight line, one can obviously get a non-local extension formula in the real space \mathbb{R}^2 .

Proposition 2.1 will be used later to prove theorem 1.2 for recovering holomorphic impedance coefficients. For the purpose of clarity, from now on we shall restrict our discussions to the case when $\alpha \equiv 1$, $\beta \equiv i\lambda$ for some constant $\lambda > 0$ and when the curve $\Gamma \subset \mathbb{R}^2$ coincides with the real axis. In these settings, we can get a single-valued extension formula over \mathbb{R}^2 with a more explicit integral form than (8). Let $\Omega \subset \mathbb{R}^2$ be a connected open set which is symmetric with respect to the real axis $L := \{(x_1, x_2) \in \mathbb{R}^2 : x_2 = 0\}$, i.e., $(x_1, x_2) \in \Omega$ if and only if $(x_1, -x_2) \in \Omega$. Set $\tilde{L} := L \cap \Omega$, and denote by Ω^+ and Ω^- the two symmetric parts of Ω into which $\Omega \setminus \tilde{L}$ is divided by L (see figure 1, below).

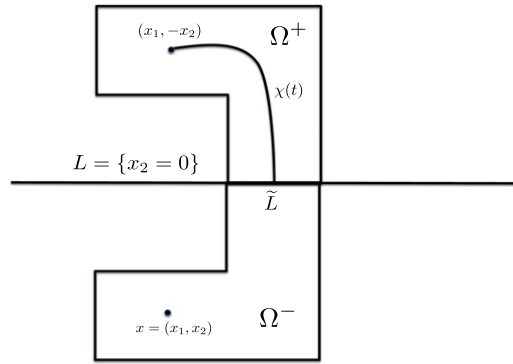


Figure 1. Illustration of the reflection principle over the domain $\Omega := \Omega^+ \cup \tilde{L} \cup \Omega^-$ for the impedance boundary condition given on the straight line segment $\tilde{L} \subset L$.

Corollary 2.2. Suppose that w is a solution to

$$\Delta w = 0 \quad \text{in } \Omega^+, \quad \partial_2 w + i\lambda w = 0 \quad \text{on } \tilde{L}. \quad (10)$$

Then w can be extended as a harmonic function from Ω^+ to Ω . If $\chi(t) := \{(\chi_1(t), \chi_2(t)) : t \in [0, 1]\} \subset \Omega^+$ is a piecewise-smooth curve joining $\chi(1) := x \in \Omega$ to an arbitrary point $\chi(0) \in \tilde{L}$, then the extended function \tilde{w} is given by

$$\tilde{w}(x) = \begin{cases} w(x) & \text{if } x \in \Omega^+ \cup \tilde{L}, \\ (\tilde{\mathcal{D}}w)(x_1, -x_2) & \text{if } x \in \Omega^-, \end{cases} \quad (11)$$

where the operator $\tilde{\mathcal{D}}$ is defined by

$$\begin{aligned} (\tilde{\mathcal{D}}w)(x) := & w(x) + 2i\lambda \int_0^1 e^{-i\lambda\chi_2(t)} \cosh[\lambda(\chi_1(t) - x_1)] \chi_2'(t) \\ & \times w(\chi_1(t), \chi_2(t)) dt - 2\lambda \int_0^1 e^{-i\lambda\chi_2(t)} \sinh[\lambda(\chi_1(t) - x_1)] \chi_1'(t) \\ & \times w(\chi_1(t), \chi_2(t)) dt + 2i \int_0^1 e^{-i\lambda\chi_2(t)} \sinh[\lambda(\chi_1(t) - x_1)] \\ & \times [\partial_2 w(\chi_1(t), \chi_2(t)) \chi_1'(t) - \partial_1 w(\chi_1(t), \chi_2(t)) \chi_2'(t)] dt. \end{aligned} \quad (12)$$

Here $\sinh(t) := (e^t - e^{-t})/2$ and $\cosh(t) := (e^t + e^{-t})/2$ are hyperbolic functions.

Proof. Let $x = (x_1, x_2) \in \Omega^+$, so that $(x_1, -x_2) \in \Omega^-$. By proposition 2.1,

$$\begin{aligned} w(x_1, -x_2) = & w(x) + \frac{1}{2i} \int_{\tilde{L}}^x V(y; x) [\partial_2 w(y) dy_1 - \partial_1 w(y) dy_2] \\ & - \frac{1}{2i} \int_{\tilde{L}}^x w(y) [\partial_2 V(y; x) dy_1 - \partial_1 V(y; x) dy_2], \end{aligned} \quad (13)$$

where the integral $\int_{\tilde{L}}^x$ is independent on the path joining an arbitrary point on \tilde{L} with x , and the function $V = V_1 - V_2$ is given by (9) with $\alpha = 1, \beta = i\lambda$ and the Schwarz functions

$S(z) = \tilde{S}(z) = z$. Obviously,

$$V_1(y_1, y_2; x_1, x_2) = 1 - \exp \left[-i \int_{S(z_0)}^{y_1 - iy_2} i \lambda d\tau \right],$$

$$V_2(y_1, y_2; x_1, x_2) = 1 - \exp \left[i \int_{\tilde{S}(w_0)}^{y_1 + iy_2} i \lambda d\tau \right].$$

Since $S(z_0) = z_0 = x_1 + ix_2$ and $\tilde{S}(w_0) = w_0 = x_1 - ix_2$, we get

$$V_1(y; x) = 1 - 2 \exp [\lambda ((y_1 - x_1) - i(y_2 + x_2))],$$

$$V_2(y; x) = 1 - 2 \exp [-\lambda ((y_1 - x_1) + i(y_2 + x_2))],$$

implying that

$$V(y_1, y_2; x_1, x_2) = -4 e^{-i\lambda(y_2+x_2)} \sinh[\lambda(y_1 - x_1)],$$

$$\partial_{y_1} V(y_1, y_2; x_1, x_2) = -4\lambda e^{-i\lambda(y_2+x_2)} \cosh[\lambda(y_1 - x_1)],$$

$$\partial_{y_2} V(y_1, y_2; x_1, x_2) = 4i\lambda e^{-i\lambda(y_2+x_2)} \sinh[\lambda(y_1 - x_1)].$$

Now let $\chi(t) := (\chi_1(t), \chi_2(t))$ be a path connecting an arbitrary point $(\bar{x}_1, 0) \in \tilde{L}$ with the point $x := (x_1, x_2) \in \Omega^+$ such that $\chi(0) = (\bar{x}_1, 0) \in \tilde{L}$ and $\chi(1) = x$. With this choice of $\chi(t)$ and the expression for V , the right-hand side of (13) can be rewritten as

$$\begin{aligned} \tilde{w}(x_1, -x_2) &= w(x_1, x_2) - \frac{1}{2i} \int_0^1 w(\chi_1(t), \chi_2(t)) \\ &\quad \times (4i\lambda e^{-i\lambda(\chi_2(t)+x_2)} \sinh \lambda(\chi_1(t) - x_1)) \chi_1'(t) dt \\ &\quad + \frac{1}{2i} \int_0^1 w(\chi_1(t), \chi_2(t)) (-4\lambda e^{-i\lambda(\chi_2(t)+x_2)} \cosh \lambda(\chi_1(t) - x_1)) \\ &\quad \times \chi_2'(t) dt + \frac{1}{2i} \int_0^1 (-4 e^{-i\lambda(\chi_2(t)+x_2)} \sinh \lambda(\chi_1(t) - x_1)) \\ &\quad \times (\partial_2 w(\chi_1(t), \chi_2(t)) \chi_1'(t) - \partial_1 w(\chi_1(t), \chi_2(t)) \chi_2'(t)) dt. \end{aligned}$$

Simple calculations show that for $x \in \Omega^+$,

$$\begin{aligned} \tilde{w}(x_1, -x_2) &= w(x) - 2\lambda e^{-i\lambda x_2} \int_0^1 e^{-i\lambda \chi_2(t)} w(\chi_1(t), \chi_2(t)) \\ &\quad \times \sinh[\lambda(\chi_1(t) - x_1)] \chi_1'(t) dt + 2i\lambda e^{-i\lambda x_2} \int_0^1 e^{-i\lambda \chi_2(t)} w(\chi_1(t), \chi_2(t)) \\ &\quad \times \cosh[\lambda(\chi_1(t) - x_1)] \chi_2'(t) dt + 2ie^{-i\lambda x_2} \int_0^1 e^{-i\lambda \chi_2(t)} \sinh[\lambda(\chi_1(t) - x_1)] \\ &\quad \times (\partial_2 w(\chi_1(t), \chi_2(t)) \chi_1'(t) - \partial_1 w(\chi_1(t), \chi_2(t)) \chi_2'(t)) dt =: (\tilde{D}w)(x). \quad (14) \end{aligned}$$

Hence the function \tilde{w} given by (11) is well defined in Ω^- . Below we show that \tilde{w} is indeed the extension from Ω^+ to Ω . For this purpose, we first prove that $\Delta \tilde{w} = 0$ in Ω^- . For $x \in \Omega^-$, we

have

$$\begin{aligned}\tilde{w}(x) &= (\tilde{\mathcal{D}}w)(x_1, -x_2) := w(x_1, -x_2) + 2i\lambda e^{i\lambda x_2} \\ &\quad \times \int_0^1 e^{-i\lambda\chi_2(t)} w(\chi_1(t), \chi_2(t)) \cosh[\lambda(\chi_1(t) - x_1)] \chi_2'(t) dt \\ &\quad - 2\lambda e^{i\lambda x_2} \int_0^1 e^{-i\lambda\chi_2(t)} w(\chi_1(t), \chi_2(t)) \sinh[\lambda(\chi_1(t) - x_1)] \chi_1'(t) dt \\ &\quad + 2i e^{i\lambda x_2} \int_0^1 e^{-i\lambda\chi_2(t)} \sinh[\lambda(\chi_1(t) - x_1)] \\ &\quad \times (\partial_2 w(\chi_1(t), \chi_2(t)) \chi_1'(t) - \partial_1 w(\chi_1(t), \chi_2(t)) \chi_2'(t)) dt.\end{aligned}$$

Taking Laplacian on both sides of the above equation and using $v''(\lambda t) = \lambda^2 v(\lambda t)$ for $v = \sinh, \cosh$, we have

$$\Delta \tilde{w}(x_1, x_2) = \Delta w(x_1, -x_2) = 0 \quad \text{for all } x \in \Omega^-.$$

Next we show that \tilde{w} and $\partial_2 \tilde{w}$ are continuous on \tilde{L} . For $x = (x_1, x_2) \in \Omega^-$ with x_2 close to zero, we can choose the finite line segment $\chi(t) = (\chi_1(t), \chi_2(t)) := (x_1, -tx_2) \subset \Omega^+$ for $t \in [0, 1]$, which is perpendicular to the interface \tilde{L} at $(x_1, 0)$. This is due to the reason that the integral does not depend on the path of integration. Using this curve in (11), we get

$$\tilde{w}(x) = w(x_1, -x_2) - 2i\lambda e^{-i\lambda x_2} \int_0^1 e^{i\lambda t x_2} w(x_1, -tx_2) x_2 dt,$$

for $x \in \Omega^-$ with x_2 sufficiently small. Changing the variable $-tx_2 = s$ in the above expression, we get (cf [2, remark 4.1])

$$\tilde{w}(x_1, x_2) = w(x_1, -x_2) + 2i\lambda e^{-i\lambda x_2} \int_0^{-x_2} e^{-i\lambda s} w(x_1, s) ds, \quad (15)$$

which implies the continuity of \tilde{w} on \tilde{L} . Taking the derivative with respect to x_2 , we obtain

$$\begin{aligned}\partial_2 \tilde{w}(x_1, x_2) &= -\partial_2 w(x_1, -x_2) - 2\lambda^2 e^{i\lambda x_2} \\ &\quad \times \int_0^{-x_2} e^{-i\lambda s} w(x_1, s) ds - 2i\lambda e^{2i\lambda x_2} w(x_1, -x_2),\end{aligned}$$

for $x \in \Omega^-$ close to \tilde{L} , and thus

$$\lim_{x_2 \rightarrow 0^+} \partial_2 \tilde{w}(x_1, x_2) = -\partial_2 w(x_1, 0) - 2i\lambda w(x_1, 0), \quad \text{if } (x_1, x_2) \in \Omega^-.$$

Using equation (10), we conclude that $\partial_2 \tilde{w} \in C(\Omega)$. Hence, the Cauchy data of \tilde{w} are continuous on \tilde{L} , implying that \tilde{w} is harmonic over Ω . In particular,

$$\lim_{x_2 \rightarrow 0^-} (\partial_2 \tilde{w} + i\lambda \tilde{w}) = -\partial_2 w(x_1, 0) - i\lambda w(x_1, 0) = 0.$$

This proves that \tilde{w} given by (11) is indeed the extension formula for w . \square

Remark 2.3.

- (a) Note that the operator $\tilde{\mathcal{D}}$ defined in (14) maps analytic functions in Ω^+ to itself and it can be extended to an operator defined on a class of functions over Ω . From the proof of corollary 2.2, it holds that $\tilde{w}(R_L x) = \tilde{\mathcal{D}}\tilde{w}(x)$ for all $x \in \Omega$, where R_L denotes the reflection with respect to L .
- (b) Suppose that Ω^+ fulfills the following geometric condition: the projection onto L of any line segment lying in Ω^+ is a subset of \tilde{L} , that is, $(x_1, x_2) \in \Omega^+$ implies $(x_1, 0) \in \tilde{L}$. Then the extension formula (12) can be reduced to (15) by choosing the path $\chi_1(t) \equiv x_1$ and $\chi_2(t) = tx_2$ for $t \in [0, 1]$.

3. Reflection principle for Helmholtz equation

There have been extensive works in the literature related to reflection principles for elliptic partial differential equations of second order with constant or real analytic coefficients. We refer to [2, 9, 11, 18, 26] for references related to harmonic functions. In [35, 36] the reflection principle for the Helmholtz equation with the Dirichlet boundary condition given on a real analytic curve was derived, which was later extended to the Neumann boundary condition in [34]. In two dimensions, we believe that the reflection principle for the Helmholtz equation subject to the Robin boundary condition can be established analogously, following the approaches for harmonic functions [2] and the ideas in [34–36]. In fact, for this purpose it is only required to redefine the function V appearing in (8) subject to the Helmholtz equation. However, in this paper we prefer an alternative method presented in our previous paper [14], where a non-local extension formula for the Navier equation with the Dirichlet boundary condition was established in 2D. By [14], the existence of an extension operator for the Helmholtz equation relies on the case of the harmonic function.

Let $\tilde{L} \subset L$ and $\Omega = \Omega^+ \cup \tilde{L} \cup \Omega^-$ be defined in the same way as before. We remark that the domain Ω^+ can be any connected open set lying on one side of L and having the flat subboundary \tilde{L} on its boundary. Consider the Helmholtz equation subject to the impedance boundary condition

$$\Delta u + k^2 u = 0 \quad \text{in } \Omega^+, \quad \partial_2 u + i\lambda u = 0 \quad \text{on } \tilde{L}, \quad (16)$$

where $k^2 > 0$ and $\lambda > 0$ are constants. Our aim is to extend the solution u of equation (16) from Ω^+ to Ω^- . The following theorem is sufficient in our uniqueness proofs.

Theorem 3.1. *The solution u to equation (16) can be extended from Ω^+ to Ω^- by some extension operator \mathcal{D} [see (19) below], that is, defining*

$$u(x_1, -x_2) := (\mathcal{D}u)(x_1, x_2) \quad \text{for } (x_1, x_2) \in \Omega^+,$$

we have $\Delta u + k^2 u = 0$ in Ω .

Proof. Let $\Phi(x; y) = -\frac{1}{2\pi} \log |x - y|$, with $x, y \in \mathbb{R}^2$ and $x \neq y$, be the fundamental solution to $-\Delta$ in \mathbb{R}^2 . Denote by $G_0(x; y)$ the Green's function to $-\Delta$ in the half plane $\mathbb{R}_+^2 := \{x : x_2 > 0\}$ subject to the Robin boundary condition (16) on $x_2 = 0$. By [14, corollary 2.2], $G_0(x; y)$ can be constructed from $\Phi(x; y)$ by

$$G_0(x; y) = \Phi(x; y) + \tilde{\mathcal{D}}_x \Phi(R_L x; y), \quad x, y \in \mathbb{R}_+^2, \quad x \neq y,$$

where $\tilde{\mathcal{D}}$ is the extension operator for $-\Delta$ defined in (12). Here we write $\tilde{\mathcal{D}} = \tilde{\mathcal{D}}_x$ to indicate the action of the integral operator with respect to the variable x . For a solution u to (16), we introduce a new function

$$w(x) := u(x) - k^2 \int_{\Omega^+} G_0(x; y) u(y) dy, \quad x \in \Omega^+. \quad (17)$$

It then follows that w satisfies the boundary value problem

$$\Delta w = 0 \quad \text{in } \Omega^+, \quad \partial_2 w + i\lambda w = 0 \quad \text{on } \tilde{L}. \quad (18)$$

Applying corollary 2.2, w can be extended as a harmonic function defined in all of Ω . Denoting by w the extended function, we have

$$w(x_1, x_2) = (\tilde{\mathcal{D}}w)(x_1, -x_2), \quad w(x_1, -x_2) = (\tilde{\mathcal{D}}w)(x_1, x_2) \quad \text{for all } x \in \Omega.$$

Inserting (17) into the second identity of the previous relations, we get

$$\begin{aligned} u(x_1, -x_2) &= (\tilde{\mathcal{D}}u)(x) - k^2 \int_{\Omega^+} \tilde{\mathcal{D}}_x G_0(x; y) u(y) dy + k^2 \int_{\Omega^+} G_0(x_1, -x_2; y) u(y) dy \\ &:= (\mathcal{D}u)(x) \end{aligned} \quad (19)$$

for all $x \in \Omega^+$. Since the right-hand side of (19) depends only on $u|_{\Omega^+}$, the operator \mathcal{D} can be regarded as an extension operator for the Helmholtz equation. This finishes the proof of theorem 3.1. \square

In the special case that $\lambda \equiv 0$ (Neumann boundary condition), we have $\tilde{\mathcal{D}} = \tilde{\mathcal{D}}_x \equiv I$ and $G_0(x; y) = \Phi(x_1, x_2; y) + \Phi(x_1, -x_2; y)$ for $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{R}_+^2$. This implies that $\tilde{\mathcal{D}}_x G_0(x; y) = G_0(x_1, -x_2; y)$. Hence, we obtain from (19) that $\mathcal{D}u = \tilde{\mathcal{D}}u = u$.

Remark 3.2. The integral domain Ω^+ in (19) can be replaced by any connected domain $K \subset \Omega^+$ containing x such that $\partial K \cap \tilde{L} \neq \emptyset$. In fact, for such K we can always choose a curve $\chi(t) \subset K$ connecting a point on $\partial K \cap \tilde{L}$ with x , so that $\tilde{\mathcal{D}}$ can be well defined with the path $\chi(t)$. Since $\tilde{\mathcal{D}}_x G_0(x; y) = G_0(x_1, -x_2; y)$ for all $y \in \Omega^+ \setminus \bar{K}$, we have

$$\begin{aligned} &-k^2 \int_{\Omega^+} \tilde{\mathcal{D}}_x G_0(x; y) u(y) dy + k^2 \int_{\Omega^+} G_0(x_1, -x_2; y) u(y) dy \\ &= -k^2 \int_K \tilde{\mathcal{D}}_x G_0(x; y) u(y) dy + k^2 \int_K G_0(x_1, -x_2; y) u(y) dy, \end{aligned}$$

from which the assertion follows. Note that both $\tilde{\mathcal{D}}_x G_0(x; y)$ and $G_0(x_1, -x_2; y)$ are singular for $y \in \Omega^+$ getting close to the curve $\chi(t) \subset \Omega^+$.

Remark 3.3. Theorem 3.1 is valid for all $k, \lambda \in \mathbb{C}$ in two dimensions. However, in 3D the analogue of theorem 3.1 does not hold if $\Omega^+ \subset \mathbb{R}^3$ is nonconvex and $k = ia$, $\lambda = -ic$ with $0 < c < a$; see the example constructed in [10]. Under the geometric condition of Ω^+ stated in remark 2.3 (b), the extension operator takes the same form as the Laplacian equation for all $k, \lambda \in \mathbb{C}$ in \mathbb{R}^n ($n \geq 2$) (see [10]).

As an application of the extension operator, we prove that solutions of the Helmholtz equation in a sector with the Robin boundary condition can be extended to the whole space. To state this result, we denote by (r, θ) the polar coordinates in \mathbb{R}^2 . Given $\theta_0 \in (0, \pi/2]$, set

$$\Sigma_j := \{(r, \theta) : r > 0, \theta \in (j\theta_0, (j+1)\theta_0)\}, \quad L_j := \{(r, \theta) : r > 0, \theta = j\theta_0\} \quad (20)$$

for $j = 0, 1, 2, \dots$. Obviously, we have $\partial\Sigma_j = L_j \cup L_{j+1}$. Suppose that the normal direction ν at L_j is directed into Σ_j .

Lemma 3.4. *Let $K \supset \Sigma_0$ be an unbounded domain. Suppose that u is a solution to the Helmholtz equation $(\Delta + k^2)u = 0$ in K satisfying the impedance boundary condition $\partial_\nu u + i\eta_j u = 0$ with the constant $\eta_j \in \mathbb{C}$ on L_j for $j = 0, 1$. Then u can be analytically extended onto the whole space \mathbb{R}^2 .*

Proof. By theorem 3.1, u can be analytically extended from Σ_0 to Σ_1 by

$$u(x) = (\mathcal{D}_1 u)(R_{L_1} x) \quad \text{for all } x \in \Sigma_1,$$

where \mathcal{D}_1 is the extension operator for the Helmholtz equation corresponding to the boundary condition $\partial_\nu u + i\eta_1 u = 0$ on L_1 . Moreover, it holds that $u(R_{L_1} x) = (\mathcal{D}_1 u)(x)$ for all $x \in \Sigma_0 \cup L_1 \cup \Sigma_1$. The function $v(x) := u(R_{L_1} x)$ for $x \in \Sigma_1$ satisfies the boundary value problem

$$\Delta v + k^2 v = 0 \quad \text{in } \Sigma_1, \quad \partial_\nu v - i\eta_1 v = 0 \quad \text{on } L_1, \quad \partial_\nu v - i\eta_0 v = 0 \quad \text{on } L_2.$$

Note that the boundary conditions of v on L_1 and L_2 are derived from those of u on L_1 and L_0 , respectively. Applying theorem 3.1 again, we can extend v from Σ_1 to Σ_2 by

$$v(x) = \mathcal{D}_2 v(R_{L_2} x) = \mathcal{D}_2 u(R_{L_1} R_{L_2} x) \quad \text{for all } x \in \Sigma_2,$$

where \mathcal{D}_2 is the extension operator corresponding to the boundary operator $\partial_\nu - i\eta_0$ enforcing on L_2 . Since $u = \mathcal{D}_1 v$ in Σ_1 , one can extend u from Σ_1 to Σ_2 by

$$u(x) = \mathcal{D}_1 v(x) = \mathcal{D}_1 \mathcal{D}_2 v(R_{L_2} x) = \mathcal{D}_1 \mathcal{D}_2 u(R_{L_1} R_{L_2} x)$$

for all $x \in \Sigma_2$.

For any $\theta_0 \in (0, \pi/2]$, one may find a positive integer $N \geq 2$ such that $N\theta_0 \geq \pi$. Repeating the previous process $N - 1$ times, we may extend u to the upper half plane $x_2 \geq 0$. In particular, this implies that $\partial_\nu u + i\eta_0 u = 0$ on $x_2 = 0$, because the same condition holds on the right half of the line $x_2 = 0$. Applying theorem 3.1 again we can extend u from \mathbb{R}_+^2 to the whole space \mathbb{R}^2 . \square

Remark 3.5.

- The results in theorem 3.1 and lemma 3.4 are well known in the Dirichlet and Neumann cases. The Dirichlet boundary condition corresponds to the odd reflection $u(R_L x) = -u(x)$ (i.e., $\mathcal{D}u = -u$), whereas the Neumann boundary condition gives the even reflection $u(R_L x) = u(x)$ (i.e., $\mathcal{D}u = u$). Hence, lemma 3.4 remains valid under mixed Dirichlet *et al* boundary conditions imposed on $L_0 \cup L_1$.
- If θ_0/π is irrational, using the series expansion of the Helmholtz equation around $O = (0, 0)$ (see [16]), one can easily prove that u must vanish identically in K . Hence, u can be extended onto the whole space by zero extension.

Lemma 3.4 will be used later for recovering convex polygons (see remark 4.1). For non-convex polygonal obstacles of impedance type, we need to repeatedly apply the results in the following lemma 3.6. It follows from theorem 3.1 and lemma 3.4 with slight modifications.

Lemma 3.6. *Let $\Omega \subset \mathbb{R}^2$ be a bounded connected polygonal domain and suppose that $\lambda(x)$ is a piecewise constant function on $\partial\Omega$, satisfying $\lambda \equiv \text{Const.}$ on the line segment $l \subset \partial\Omega$. Denote by L the straight line containing l , $R_L(\Omega)$ the symmetric domain of Ω with respect to L and by \mathcal{D} the constructed extension operator corresponding to the boundary condition on l in theorem 3.1. Consider the Helmholtz equation*

$$\Delta u + k^2 u = 0 \quad \text{in } \Omega \cup l \cup U, \quad \partial_\nu u + i\lambda(x)u = 0 \quad \text{on } \partial\Omega, \quad (21)$$

where $U \cap \Omega = \emptyset$ is an unbounded connected domain with $l \subset \partial U$. We have

(a) *The function $v := \mathcal{D}u$ is a well-defined analytic function in $U \cup R_L(\Omega)$ and satisfies*

$$\Delta v + k^2 v = 0 \quad \text{in } U \cup R_L(\Omega), \quad \partial_\nu v + i\lambda(R_L x)v = 0 \quad \text{on } \partial R_L(\Omega).$$

Here the normal direction on $\partial R_L(\Omega)$ is obtained by reflecting the counterpart on $\partial\Omega$ with respect to the straight line parallel to L and passing through the origin $(0, 0) \in \mathbb{R}^2$.

(b) *The functions v and u can be analytically extended to the whole space under one of the following conditions:*

- (1) *One side of $\partial R_L(\Omega)$ extends to a whole straight line in U . Moreover, one half plane divided by this straight line is a subset of U .*
- (2) *There exist two neighboring sides of $\partial R_L(\Omega)$, each of them extending to a half-line in U . Moreover, the infinite sector formed by these two half lines is a subset of U .*

Proof. (a) By theorem 3.1, v satisfies the Helmholtz equation in U and $u(R_L x) = \mathcal{D}u(x)$ at least for $x \in U \cup \Omega$ close to the interface l . Hence, $v(x) = u(R_L x)$ for $x \in U$ near l . Since the function $x \rightarrow u(R_L x)$ can be extended to $R_L(\Omega)$, v can also be extended from U to $R_L(\Omega)$, if $R_L(\Omega)$ is not a subset of U . In the case $R_L(\Omega) \subset U$, the assertion follows directly from theorem 3.1.

(b) In case (1), v is a solution to the Helmholtz equation in an unbounded domain containing the half plane and satisfies the Robin boundary condition on the boundary. Applying theorem 3.1, v can be analytically extended onto \mathbb{R}^2 . The global extension also applies to u , since $u(x) = v(R_L x)$ for x near l . The case (2) can be proved analogously using lemma 3.4. \square

Note that in lemma 3.6 the function v is not the extension of u , therefore v and u are not the same when extended to the whole space. The extension of v to the whole space guarantees the extension of u to the whole space. In our uniqueness proof (see case (b) in theorem 1.1 below), the connected polygon Ω of lemma 3.6 is taken as the gap domain between two different obstacles in the initial step, u the total field corresponding to one of these two obstacles, U a subset of the complement of this obstacle in \mathbb{R}^2 and l a Robin level set of u (a curve on which the Robin boundary condition is satisfied). We refer to figure 3 for two simple examples corresponding to cases (1) and (2) in lemma 3.6 (b). In (21) we can neglect the direction of the normal vector on $\partial\Omega$, since λ can be taken as an arbitrary constant on each side of $\partial\Omega$. To determine the normal direction on $\partial R_L(\Omega)$, we note that the reflection of a unit vector with respect to a straight line makes sense only if this line passes through the origin.

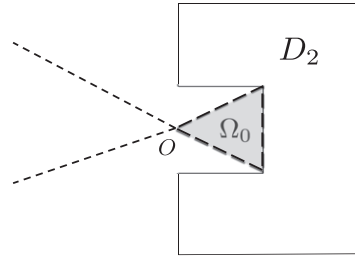


Figure 2. Illustration of two different polygonal obstacles D_2 and D_1 , where Ω_0 denotes the gap domain between D_1 and D_2 . There are two sides of D_1 around the corner O , each of them can be extended to a half-line in D_2^c .

4. Proof of main results

To prove theorem 1.1, we shall combine the arguments in [12] for treating the Neumann boundary condition and those in elasticity [14] where a ‘non-point-to-point’ reflection principle for the Navier equation was applied to determine a connected rigid polygon. In contrast with the Dirichlet case [1, 6, 14, 29], the Robin level set of the total field can be unbounded. We will adapt the arguments of [12] by investigating two Robin half-lines starting from a corner point in our uniqueness proof.

4.1. Proof of theorem 1.1

Theorem 1.1 will be proved by contradiction argument. We only need to consider the determination of the shape, because the recovery of the Robin coefficients follows directly from Holmgren’s uniqueness theorem. Suppose that $D_1 \neq D_2$, but the corresponding far-field patterns are identical, i.e., $u_1^\infty(\hat{x}, d) = u_2^\infty(\hat{x}, d)$ for all $\hat{x} \in \mathbb{S}$ and for fixed $d \in \mathbb{S}$. Using the Rellich’s

$$u_1 = u_2 \quad \text{in the unbounded component } E \quad \text{of } \mathbb{R}^2 \setminus \overline{D_1 \cup D_2}. \quad (22)$$

Since both D_1 and D_2 are polygons, using (22) and interchanging u_1 and u_2 if necessary, we only need to consider the following cases (see [12, lemma 7] for the details):

Case (i). There are two half-lines L_0 and L_1 starting from a corner point $O \in \partial D_1 \cap D_2^c \cap \partial E$ such that $\partial_\nu u_2 + i\lambda_2 u_2 = 0$ on $L_0 \cup L_1$; see figure 2. Here $D_j^c := \mathbb{R}^2 \setminus \overline{D_j}$ for $j = 1, 2$ and we may neglect the normal directions at L_0 and L_1 .

Since the Helmholtz equation is rotational invariant, we may suppose that the corner O is located at the origin and that the half-lines L_0 and L_1 coincide with those defined in (20). By the connectness of D_2 , the infinite sector Σ_0 formed by $L_0 \cup L_1$ must lie completely in $E \cap D_2^c$. This implies that the total field $u_2 = \exp(ikx \cdot d) + u_2^{sc}$ is a solution to the Helmholtz equation in a neighboring area of the sector Σ_0 . Applying lemma 3.4, we can extend u_2 and thus u_2^{sc} to the whole space, which gives $u_2^{sc} \equiv 0$. Consequently, $u_2(x) = e^{ikx \cdot d}$ fulfills the Robin boundary condition on ∂D_2 . It then follows that

$$\nu(x) \cdot d + \lambda/k = 0 \quad \text{for all } x \in \partial D_2 \quad (23)$$

with fixed $d \in \mathbb{S}$ and $k, \lambda > 0$. However, this is impossible, since the boundary ∂D_2 has at least three unit normal directions $\nu^{(j)} \in \mathbb{S}$ ($j = 1, 2, 3$) pointing into D_2^c such that $\nu^{(2)} - \nu^{(1)}$ and $\nu^{(3)} - \nu^{(1)}$ are linearly independent.

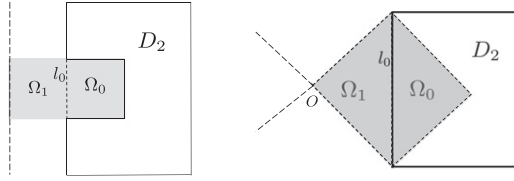


Figure 3. Illustration of two different polygonal obstacles D_2 and D_1 , where Ω_0 denotes the gap domain between D_1 and D_2 . Ω_1 denotes the reflection of Ω_0 with respect to l_0 . Left: one side of $\partial\Omega_1$ extends to a whole straight line in D_2^c . Right: two neighboring sides of $\partial\Omega_2$ extend to half-lines in D_2^c starting from the corner O , forming an infinite sector in D_2^c .

Case (ii). There exists a finite line segment $l_0 \subset E \cap D_2^c$ with both the end points lying on $\partial D_2 \cap \partial E$ such that $\partial_\nu u_2 + i\lambda_2 u_2 = 0$ on l_0 ; see figure 3.

In case (ii), we shall apply the approach of [14] to prove that the total field u_2 can be analytically extended onto \mathbb{R}^2 , leading to the same contradiction as in case (i). Choose a point $P_0 \in l_0$ and a continuous injective path $\gamma(t)$, $t \geq 0$ which starts from $\gamma(0) = P_0$ and connects to infinity in the unbounded component of $D_2^c \setminus l_0$. Denote by $\Omega_0 \subset D_2^c$ the bounded connected component of $D_2^c \setminus l_0$. Since $\partial\Omega_0 \subset \partial D_2 \cup l_0$, u_2 is solution to

$$\Delta u_2 + k^2 u_2 = 0 \quad \text{in } \Omega_0, \quad \mathcal{B}_0 u_2 = 0 \quad \text{on } \partial\Omega_0,$$

where $\mathcal{B}_0 := \partial_\nu + i\lambda_0(x)$ is a Robin boundary operator with some piecewise constant function λ_0 defined on $\partial\Omega_0$. Note that $\lambda_0 \equiv \text{Const.}$ on any side of $\partial\Omega_0$ not containing l_0 and may be piecewise constant on the side of $\partial\Omega_0$ containing l_0 .

We proceed with several notations to be used below. Denote by R_n ($n \geq 0$) the reflection with respect to the straight line L_n containing the line segment l_n . Let \mathcal{D}_n denote the extension operator for the Helmholtz equation with respect to the line segment $l_n \subset L_n$ and subject to the Robin boundary operator \mathcal{B}_n on l_n . This can be obtained from equation (19) after translation and rotation.

Applying lemma 3.6 (a), the function $u_2^{(0)} := u_2$ can be extended to the symmetric domain $\Omega_1 := R_0(\Omega_0)$ of Ω_0 with respect to L_0 by the extension operator \mathcal{D}_0 . Setting $u_2^{(1)}(x) := \mathcal{D}_0 u_2^{(0)}(x)$ in $\Omega_1 \cup D_2^c$, we find that $u_2^{(1)}(x) = u_2^{(0)}(R_0 x)$ in Ω_1 and it satisfies the boundary value problem

$$\Delta u_2^{(1)} + k^2 u_2^{(1)} = 0 \quad \text{in } \Omega_1, \quad \mathcal{B}_1 u_2^{(1)} = 0 \quad \text{on } \partial\Omega_1,$$

for some Robin boundary operator \mathcal{B}_1 . By lemma 3.6, the Robin coefficient is piecewise constant on $\partial\Omega_1$. Since Ω_1 is bounded, we have $\partial\Omega_1 \cap \{\gamma(t) : t > 0\} \neq \emptyset$. Set $t_1 := \sup\{t : \Omega_1 \cap \gamma(t) \neq \emptyset\}$. By [14] we can assume without loss of generality that $P_1 := \gamma(t_1)$ is not a corner point of $\partial\Omega_1$. Using the continuity and injectivity of path $\gamma(t)$, we have $P_1 \neq P_0$ and $\Omega_1 \cap \{\gamma(t) : t > t_1\} = \emptyset$. Let $l_1 \subset \partial\Omega_1$ denote the line segment containing point P_1 and define $\Omega_2 := R_1(\Omega_1)$. Using lemma 3.6 and repeating the previous step, we can define the function $u_2^{(2)}(x) := \mathcal{D}_1 u_2^{(1)}(x)$ for $x \in \Omega_2 \cup D_2^c$. Then we have $u_2^{(2)} = \mathcal{D}_1 \mathcal{D}_0 u_2$ in D_2^c and $u_2^{(2)}(x) = u_2^{(1)}(R_1 x)$ for $x \in \Omega_2$. Hence, it is a solution to the boundary value problem

$$\Delta u_2^{(2)} + k^2 u_2^{(2)} = 0 \quad \text{in } \Omega_2, \quad \mathcal{B}_2 u_2^{(2)} = 0 \quad \text{on } \partial\Omega_2.$$

Following the previous argument, we choose a point $P_2 := \gamma(t_2) \neq P_1$ for some $t_2 > t_1$ and a line segment $l_2 \subset \partial\Omega_2$ such that $P_2 \in l_2$ and $\Omega_2 \cap \{\gamma(t) : t > t_2\} = \emptyset$. In general, we can find

a polygonal domain $\Omega_N := [R_{N-1}R_{N-2} \cdots R_1](\Omega_1)$, $N \geq 1$ and a function

$$u_2^{(N)}(x) := [\mathcal{D}_{N-1}\mathcal{D}_{N-2} \cdots \mathcal{D}_0 u_2](x), \quad x \in D_2^c \cup \Omega_N$$

such that

$$\Delta u_2^{(N)} + k^2 u_2^{(N)} = 0 \quad \text{in } \Omega_N, \quad \mathcal{B}_N u_2^{(N)} = 0 \quad \text{on } \partial\Omega_N. \quad (24)$$

To proceed, we suppose that $\{\gamma(t) : t > t_N\} \cap \Omega_N = \emptyset$ for some $t_N > t_{N-1}$ and that $P_N = \gamma(t_N) \in l_N$ where $l_N \subset \partial\Omega_N$ is a line segment. Since the path $\gamma(t)$ is connected to infinity in D_2^c , by [14, lemma 3.3] we have $t_N \rightarrow \infty$ as $N \rightarrow \infty$. After a finite number of steps, we have that either L_N lies completely in D_2^c or l_N together with one of its neighbouring side of $\partial\Omega_N$ extends to two half lines in D_2^c . Using the connectness of D_2 and lemma 3.6 (b), we can extend $u_2^{(N)}$ to the whole space. This implies that $u_2^{(N-1)}, u_2^{(N-2)}, \dots, u_2^{(0)} = u_2$ can also be extended to \mathbb{R}^2 , which is impossible. \square

4.2. Proof of theorem 1.2

If $\lambda = \lambda(x)$ is the restriction of a holomorphic function over \mathbb{C}^2 to \mathbb{R}^2 , the extension formula (12) for the harmonic equation should be replaced by (8) with $\alpha(z, w) \equiv 1$, $\beta = i\lambda(z, w)$ over \mathbb{C}^2 and $S(\tau) = \tilde{S}(\tau) = \tau$ for $\tau \in \mathbb{C}$. Existence of the non-local reflection principle for the Helmholtz equation in theorem 3.1 can be established in the same manner. Lemma 3.4 carries over to the case that η_j ($j = 0, 1$) are the restriction of holomorphic functions over \mathbb{C}^2 to \mathbb{R}^2 , and lemma 3.6 still holds true for impedance coefficients that are piecewise holomorphic on the boundary of a polygon. Arguing the same as in the proof of theorem 1.1, we can arrive at the same contradiction if $D_1 \neq D_2$.

To prove $\lambda_1(x) = \lambda_2(x)$ on ∂D , where $D = D_1 = D_2$, we observe that $u_1 = u_2 =: u$ in $D^c := \mathbb{R}^2 \setminus \overline{D}$ if the corresponding far-field patterns are identical over all observation directions. This yields the coincidence of the Cauchy data of u_1 and u_2 on ∂D . Hence,

$$0 = (\partial_\nu - i\lambda_1(x))u_1 - (\partial_\nu - i\lambda_2(x))u_2 = -i[\lambda_1(x) - \lambda_2(x)]u(x) \quad \text{on } \partial D. \quad (25)$$

If $\lambda_1(x_0) \neq \lambda_2(x_0)$ at some $x_0 \in \partial D$, by the continuity of λ_j , there exists a neighborhood of x_0 at ∂D such that they are not identical there. Using (25), we get the vanishing of $u = u_1 = u_2$ in an open set of ∂D . In view of the impedance boundary condition of u_j , we also get the vanishing of the Neumann data on this open set. Now the Holmgren's theorem gives $u \equiv 0$ in D^c which is impossible in the area far away from D .

Let $O \in \partial D$ be a corner point lying on the convex hull of D . Assume that the total field u extends to $B_\epsilon(O)$ for some $\epsilon > 0$ as a solution to the Helmholtz equation $\Delta u + k^2 u = 0$. By the real-analyticity of λ on each side of ∂D and the analytical extension of any line segment to infinity, the total field satisfies all conditions in lemma 3.4, where the infinite sector Σ_0 is a subset of $\mathbb{R}^2 \setminus \overline{D}$ with the vertex located at O . By lemma 3.4, u can be analytically extended into the whole space \mathbb{R}^2 , which is impossible by the proof of theorem 1.1. This proves that the total field cannot be analytically extended into D across any corner point of the convex hull of ∂D . \square

We end up the paper with several remarks.

Remark 4.1. Theorem 1.2 holds true even if the impedance coefficient $\lambda(x)$ is piecewise holomorphic in the sense that the restriction of λ to each side of ∂D extends to an entire holomorphic function defined on \mathbb{C}^2 . In particular, $\lambda(x)$ can be a piecewise-constant function whose values remain the same on each side of ∂D . For convex polygons, lemma 3.4 and the arguments in the first case of the proof of theorem 1.1 are sufficient to imply uniqueness. For non-convex

polygons, lemma 3.6 should be applied to handle the second case. Further, one can generalize the results of theorem 1.2 to the case of mixed Dirichlet *et al* boundary conditions. On each side of the polygon, there should be only one type of these boundary conditions. The ‘singularity’ of the total field at corner points might be helpful in designing numerical schemes for imaging an arbitrary convex-polygonal obstacle; see [14, 22, 23, 31] and [33, chapter 15.4].

Remark 4.2. The uniqueness results are valid for other form of non-vanishing incoming waves u^{in} that are solutions to the Helmholtz equation in a neighboring area of D . In place of using (23) for a plane wave, one needs to consider the Robin boundary value problem

$$\Delta u^{\text{in}} + k^2 u^{\text{in}} = 0 \quad \text{in } D, \quad \partial_\nu u^{\text{in}} + i\lambda(x)u^{\text{in}} = 0 \quad \text{on } \partial D,$$


where the normal is directed into outward. Applying integration by part, it is easy to prove $u^{\text{in}} \equiv 0$ in \mathbb{R}^2 if $\lambda(x) \geq 0$ on ∂D and $\lambda(x) \geq \lambda_0 > 0$ on an open set of ∂D .

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