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UNIQUENESS IN INVERSE ELASTIC SCATTERING FROM UNBOUNDED RIGID SURFACES OF RECTANGULAR TYPE

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Abstract. Consider the two-dimensional inverse elastic scattering problem of recovering a piecewise linear rigid rough or periodic surface of rectangular type for which the neighboring line segments are always perpendicular. We prove the global uniqueness with at most two incident elastic plane waves by using near-field data. If the Lamé constants satisfy a certain condition, then the data of a single plane wave is sufficient to imply the uniqueness. Our proof is based on a transcendental equation for the Navier equation, which is derived from the expansion of analytic solutions to the Helmholtz equation. The uniqueness results apply also to an inverse scattering problem for non-convex bounded rigid bodies of rectangular type.

1. **Introduction.** This paper is concerned with the inverse scattering of time-harmonic elastic waves from rigid unbounded periodic and rough surfaces of rectangular type (see Sections 2.1 and 3 for a precise description), which has a wide field of applications, particularly in geophysics, seismology and nondestructive testing. For instance, identifying fractures in sedimentary rocks has significant impact on the production of underground gas and liquids by employing controlled explosions. The sedimentary rock under consideration can be regarded as a homogeneous transversely isotropic elastic medium with periodic vertical fractures which can be extended to infinity in one of the horizontal directions. Using an elastic plane wave as an incoming source, we thus obtain a two-dimensional inverse problem of recovering a rectangular interface from the knowledge of near-field data measured above the periodic structure (diffraction grating); see [17]. The associated direct scattering problem is formulated as a Dirichlet boundary value problem for the time-harmonic Navier equation in the unbounded domain above the surface, which can be considered as a simple model problem in linear elasticity.

We refer to [2] for the first uniqueness result in inverse elastic scattering from rigid periodic surfaces. It was proved that a smooth (C^2) surface can be uniquely determined from incident pressure waves for one incident angle and an interval of wave numbers. Furthermore, a finite set of wave numbers is enough if a priori information about the height of the grating curve is known. This extends the periodic version of Schiffer's theorem by Hettlich and Kirsch (see [11]) to the case

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of inverse elastic diffraction problems. The application of the Kirsch-Kress optimization scheme with one or several incident elastic plane waves can be found in [8], where the reconstruction of rectangular rigid surfaces was also treated. The factorization method established in [13] gives rise to uniqueness results by utilizing only the compressional or shear components of the scattered field corresponding to all quasi-periodic incident plane waves with a common phase-shift.

Other studies on the uniqueness have been carried out within the class of piecewise linear periodic and rough surfaces using a single plane or point source wave. Global uniqueness results for the Helmholtz equation were first shown in [10] within the rectangular periodic structures under the Dirichlet or Neumann condition. Relying on the reflection principles for the Helmholtz, Navier and Maxwell equations, one can find out and classify several extremely rare sets of unidentifiable polygonal or polyhedral periodic structures by one incident plane wave. Thus, the global uniqueness with one incoming wave holds within the piecewise linear periodic structures excluding all unidentifiable sets; see [6, 1, 7]. In particular, sending a single incident point source wave always leads to the uniqueness of the inverse problem within polygonal periodic or rough surfaces; see [12] for the Helmholtz equation. However, such an argument applies so far only to the third or fourth kind boundary value problems of the Navier equation, and it still remains a challenging problem to prove the uniqueness under the more practical Dirichlet or Neumann-type boundary conditions, due to the lack of corresponding reflection principles.

In this paper, we restrict our discussions to the unbounded rigid periodic and rough surfaces of rectangular type in $\mathbb{R}^2$. Instead of using reflection principles, our approach to the uniqueness in the inverse scattering problem is based on the expansion of analytic solutions to the Navier equation with zero Dirichlet data on two perpendicular lines. The main ingredient in the uniqueness proof is the study of a transcendental equation for the Navier equation, which has been already used in [3, 14, 18] to analyze corner singularities of the Lamé equation (i.e., Navier equation without the zeroth order term) in a sector. We show the uniqueness with a single incident plane wave in the case of no integer roots to the resulting transcendental equation. If an integer root exists, then we further verify that the dimension of the solution space to the Navier equation is at most one, giving rise to a uniqueness result with at most two incident angles for both periodic and non-periodic scattering surfaces. We conjecture that non-rectangular piecewise linear surfaces can be uniquely determined by sending a finite number of incident plane waves, provided some a priori information on the angles of the interface is available. Moreover, our uniqueness results are extended to non-convex bounded rigid bodies of rectangular type by using far-field measurements of at most two incident directions.

The rest of the paper is organized as follows. In Section 2, we state and prove the uniqueness results for diffraction gratings. The transcendental equation with a general angle is studied in Section 2.2, and the equation in the case of the right angle is utilized for justifying our uniqueness with at most two incident directions in Section 2.1. Finally in Section 3, the proof of the uniqueness in periodic structures is carried over to the case of rough surfaces.

2. Uniqueness in periodic structures.

2.1. **Mathematical formulation and main result.** Consider the elastic scattering problem from a rigid diffraction grating Λ in $\mathbb{R}^2$. It is supposed that Λ is of rectangular type, i.e., the neighboring line segments are always perpendicular.

More precisely, we assume that for some $b > 0$ the scattering surface Λ belongs to the following admissible class:

$$\mathcal{A} = \left\{ \Lambda : \begin{array}{l} \Lambda \text{ is a piecewise linear curve in } |x_2| < b \text{ which is } 2\pi\text{-periodic in } x_1. \\ \text{The angle between any two neighboring line segments is } \pi/2. \end{array} \right\}.$$

We emphasize that Λ is allowed to be a non-graph profile, and the line segments of Λ are not necessarily parallel or perpendicular to the coordinate axes; see Figure 1 (right). We formulate the direct scattering problem following the lines in [15] for the

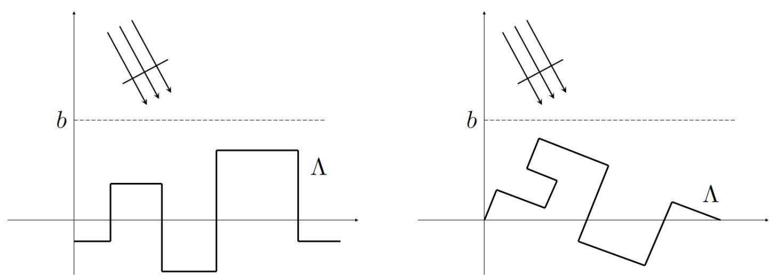

FIGURE 1. Examples of rectangular diffraction gratings.

Helmholtz equation and [5] for the Navier equation. Denote by Ω_Λ the unbounded periodic region above Λ and assume, for simplicity, that Ω_Λ is occupied by a linear isotropic and homogeneous elastic material with mass density one. Suppose an incident pressure wave (with the incident angle $\theta \in (-\pi/2, \pi/2)$) given by

$$(1) \quad u_p^{in} = u_p^{in}(\theta) = \hat{\theta} \exp(ik_p x \cdot \hat{\theta}), \qquad \hat{\theta} := (\sin\theta, -\cos\theta)^T, \;\; x = (x_1, x_2) \in \mathbb{R}^2,$$

is incident on Λ from the region above. Here, $k_p := \omega/\sqrt{2\mu+\lambda}$ is the compressional wave number, λ and μ denote the Lamé constants satisfying $\mu > 0$ and $\lambda + \mu > 0$, $\omega > 0$ is the angular frequency of the harmonic motion, and the symbol $(\cdot)^T$ stands for the transpose of a vector in $\mathbb{R}^2$. The shear wave number is defined as $k_s := \omega/\sqrt{\mu}$.

Recall that a function v is called quasi-periodic with phase-shift α (or α-quasi-periodic) in Ω_Λ, if $\exp(-i\alpha x_1)\, v(x_1, x_2)$ is 2π-periodic with respect to x_1, or equivalently,

$$(2) \qquad v(x_1 + 2\pi, x_2) = \exp(2i\alpha\pi)\, v(x_1, x_2)\,, \quad (x_1, x_2) \in \Omega_\Lambda.$$

Obviously, the incident pressure wave u_p^{in} is α-quasi-periodic with $\alpha = k_p \sin\theta$ in Ω_Λ. If the scattered field u^{sc} is supposed to be quasi-periodic with the same phase-shift as that of u^{in}, then the direct scattering problem, due to the incident pressure wave (1), aims to find the quasi-periodic scattered field $u^{sc} \in H^1_{loc}(\Omega_\Lambda)^2$ such that

$$(3) \qquad (\Delta^* + \omega^2)\, u^{sc} = 0 \quad \text{in} \quad \Omega_\Lambda, \quad \Delta^* := \mu\Delta + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,,$$

$$(4) \qquad u^{sc} = -u_p^{in} \quad \text{on} \quad \Lambda,$$

and that satisfies the Rayleigh expansion ([5])

$$(5) \quad u^{sc}(x;\theta) = \sum_{n\in\mathbb{Z}} \left\{ A_{p,n} \begin{pmatrix} \alpha_n \\ \beta_n \end{pmatrix} e^{i\alpha_n x_1 + i\beta_n x_2} + A_{s,n} \begin{pmatrix} \gamma_n \\ -\alpha_n \end{pmatrix} e^{i\alpha_n x_1 + i\gamma_n x_2} \right\}$$

for all $x_2 \geq \Lambda^+ := \max_{(x_1,x_2)\in\Lambda} x_2$. Here, the constants $A_{p,n}$, $A_{s,n} \in \mathbb{C}$ are called the Rayleigh coefficients, $\alpha_n := \alpha + n$ and

$$\beta_n := \begin{cases} \sqrt{k_p^2 - \alpha_n^2} & \text{if} \quad |\alpha_n| \leq k_p\,, \\ i\sqrt{\alpha_n^2 - k_p^2} & \text{if} \quad |\alpha_n| > k_p, \end{cases} \tag{6}$$
$$\gamma_n := \begin{cases} \sqrt{k_s^2 - \alpha_n^2} & \text{if} \quad |\alpha_n| \leq k_s\,, \\ i\sqrt{\alpha_n^2 - k_s^2} & \text{if} \quad |\alpha_n| > k_s. \end{cases}$$

Since β_n and γ_n are real for at most a finite number of indices $n \in \mathbb{Z}$, only a finite number of plane waves in (5) propagate into the far field, with the remaining evanescent waves (or surface waves) decaying exponentially as $x_2 \to +\infty$. The above expansion (5) converges uniformly with all derivatives in the half-plane $\{x \in \mathbb{R}^2 : x_2 \geq \Lambda^+\}$ and the Rayleigh coefficients $\{A_{p,n}\}_{n\in\mathbb{Z}}, \{A_{s,n}\}_{n\in\mathbb{Z}} \in \ell^2$.

The uniqueness and the existence of quasi-periodic solutions to (3)-(5) were verified in [5] by the variational argument for grating profiles given by step functions (see Figure 1 (Left)) or Lipschitz functions. If the scattering surface is given by a general Lipschitz curve, existence can always be proved at arbitrary incident frequencies, although there is no uniqueness in general. The solvability results for pressure wave incidence extend directly to the incident shear wave

$$u_s^{in} = u_s^{in}(\theta) = \hat{\theta}^\perp \exp(ik_s x \cdot \hat{\theta}), \tag{7}$$

with $\hat{\theta} := (\sin\theta, -\cos\theta)^T$ and $\hat{\theta}^\perp := (\cos\theta, \sin\theta)^T$, for which the phase-shift of the scattered field is $\alpha = k_s \sin\theta$. This differs from the case of pressure wave incidence given in (1). The incident wave in our paper is also allowed to be a general elastic plane wave of the form

$$u^{in}(\theta) = c_p u_p^{in}(\theta) + c_s u_s^{in}(\theta), \quad c_p, c_s \in \mathbb{C}, \tag{8}$$

for which the unique solution belongs to the sum of a $k_p \sin\theta$ and a $k_s \sin\theta$-quasi-periodic Sobolev space, since the scattered field depends linearly on the incident field.

In this paper we are interested in the inverse problem of recovering an unknown periodic scattering surface $\Lambda \in \mathcal{A}$ from the knowledge of the scattered near-field measured on $\Gamma_b := \{(x_1, x_2) : x_2 = b, 0 < x_1 < 2\pi\}$, where $b > \Lambda^+$is given as in the definition of the admissible class $\mathcal{A}$. We state the uniqueness results with at most two incident angles as follows:

Theorem 2.1. *Let the incident elastic wave be given by (8).*

(i): *If the Lamé constants satisfy*

$$\frac{\lambda + \mu}{\lambda + 3\mu} \neq \frac{1}{n} \qquad \textit{for all odd numbers } n \in \mathbb{N}, \tag{9}$$

then Λ can be uniquely determined by $u^{sc}(x;\theta)|_{\Gamma_b}$ with a single incident angle $\theta \in (-\pi/2, \pi/2)$.

(ii): *If*

$$\frac{\lambda + \mu}{\lambda + 3\mu} = \frac{1}{n_0} \qquad \textit{for some odd number } n_0 \in \mathbb{N}, \tag{10}$$

then Λ can be uniquely determined by $u^{sc}(x;\theta_j)|_{\Gamma_b}$ $(j = 1, 2)$ corresponding to two distinct incident angles $\theta_1, \theta_2 \in (-\pi/2, \pi/2)$.

We shall carry out the proof of Theorem 2.1 in Section 2.3, relying on some lemmas to be established in Section 2.2.

2.2. **Key lemmas.** For $x = (x_1, x_2)$, let (r, φ) be the polar coordinates of x in $\mathbb{R}^2$. For notational convenience, we set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. We first derive the power series expansion of analytic solutions to the Helmholtz equation around the origin.

Lemma 2.2. *Assume $(\Delta + k^2)u = 0$ in a neighborhood of the origin. Then we can expand $u = u(r, \varphi)$ into a convergent power series*

$$u(r,\varphi) = \sum_{n,m\in\mathbb{N}_0} r^{n+2m} \left(u^+_{n,m}\cos(n\varphi) + u^-_{n,m}\sin(n\varphi)\right), \tag{11}$$

around the origin, where $u^{\pm}_{n,m} \in \mathbb{C}$ satisfy the recurrence relations

$$u^{\pm}_{n,m+1} = -\frac{k^2}{4(m+1)(n+m+1)} u^{\pm}_{n,m}, \quad \text{for all} \quad n, m \in \mathbb{N}_0. \tag{12}$$

Remark 1. The expansion (11) is nothing else than the reformulation of the corresponding expansion in terms of Bessel functions (see e.g., [4, Chapter 3.4]). Note that (11) reduces to the power series for harmonic functions if $k = 0$.

Proof of Lemma 2.2. We begin with the Taylor expansion of u around the origin

$$u(x_1, x_2) = \sum_{n,m\in\mathbb{N}_0} A_{n,m}\, x_1^n\, x_2^m, \quad A_{n,m} \in \mathbb{C}.$$

Performing the change of variables $z_1 = x_1 + ix_2 = re^{i\varphi}$, $z_2 = x_1 - ix_2 = re^{-i\varphi}$, the above expression can be transformed into

$$\begin{aligned}
u(x_1,x_2) &= \sum_{n,m\in\mathbb{N}_0} A_{n,m} \left(\frac{z_1+z_2}{2}\right)^n \left(\frac{z_1-z_2}{2i}\right)^m = \sum_{n,m\in\mathbb{N}_0} B_{n,m}\, z_1^n\, z_2^m \\
&= \sum_{n,m\in\mathbb{N}_0} B_{n,m}\, r^{m+n}\, e^{i(n-m)\varphi} \\
&= \sum_{m\in\mathbb{N}_0, n\in\mathbb{Z}: n+2m\geq 0} B_{m+n,m}\, r^{2m+n}\, e^{in\varphi}
\end{aligned}$$

for some $B_{n,m} \in \mathbb{C}$. Moreover, u can be reformulated in the form (11) with some $u^{\pm}_{n,m} \in \mathbb{C}$. Applying the Laplace operator to u, we have

$$\begin{aligned}
\Delta u &= \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi}\right) u \\
&= \sum_{n\in\mathbb{N}_0, m\in\mathbb{N}} 4m(n+m)\, r^{n+2m-2} \left(u^+_{n,m}\cos(n\varphi) + u^-_{n,m}\sin(n\varphi)\right) \\
&= \sum_{n,m\in\mathbb{N}_0} 4(m+1)(n+m+1)\, r^{n+2m} \left(u^+_{n,m+1}\cos(n\varphi) + u^-_{n,m+1}\sin(n\varphi)\right).
\end{aligned}$$

Since u is a solution of the Helmholtz equation, the coefficients $u^{\pm}_{n,m}$ have to satisfy the recurrence relations (12). □

In the following we study a transcendental equation for the Navier equation with the Dirichlet boundary condition. This equation has been used to compute corner singularities of solutions to the Lamé equation; see e.g., [3, 14, 18].

Lemma 2.3. *Suppose* $(\Delta^* + \omega^2)u = 0$ *in* $\mathbb{R}^2$ *and* $u = 0$ *on* $\varphi = \varphi_1, \varphi_2$, *where* $-\pi < \varphi_2 < \varphi_1 \le \pi$. *Suppose further that the transcendental equation in* $z \in \mathbb{C}$,

$$\sin^2(z\psi) - z^2 \sin^2\psi \left(\frac{\lambda+\mu}{\lambda+3\mu}\right)^2 = 0, \quad \psi = \varphi_1 - \varphi_2, \tag{13}$$

has no integer roots $z = n \in \mathbb{N}$. *Then it holds that* $u \equiv 0$ *in* $\mathbb{R}^2$.

Proof. Since the Navier equation is rotationally invariant and u is analytic in a neighborhood of the corner point, without loss of generality we may assume the positive x_1-axis coincides with the half-line $\{(r,\varphi) : \varphi = (\varphi_2 + \varphi_1)/2\}$ and $\psi = 2\varphi_0$ for some $\varphi_0 \in (0, \pi/2)$. This implies that $\varphi_1 = \varphi_0$ and $\varphi_2 = -\varphi_0$. For $x = r(\cos\varphi, \sin\varphi)$, set $\hat{x} = x/r = (\cos\varphi, \sin\varphi)$, and $\hat{x}^\perp = (-\sin\varphi, \cos\varphi)$. We decompose u into its compressional and shear parts by

$$u = \nabla v + \overrightarrow{\mathrm{curl}}\, w, \quad \text{with} \quad v = -\frac{1}{k_p^2}\mathrm{div}\, u,\ w = \frac{1}{k_s^2}\mathrm{curl}\, u, \tag{14}$$

where the two curl operators in $\mathbb{R}^2$ are defined by

$$\mathrm{curl}\, u := \partial_1 u_2 - \partial_2 u_1, \quad \overrightarrow{\mathrm{curl}}\, w := (\partial_2 w, -\partial_1 w)^T,$$

and the two scalar functions v and w satisfy the Helmholtz equations

$$(\Delta + k_p^2)v = 0, \quad (\Delta + k_s^2)w = 0 \quad \text{in} \quad \mathbb{R}^2. \tag{15}$$

It is easy to check that

$$\hat{x}\cdot\nabla v = \frac{\partial v}{\partial r}, \quad \hat{x}^\perp \cdot \nabla v = \frac{1}{r}\frac{\partial v}{\partial\varphi}, \quad \hat{x}\cdot\overrightarrow{\mathrm{curl}}\, w = \frac{1}{r}\frac{\partial w}{\partial\varphi}, \quad \hat{x}^\perp\cdot\overrightarrow{\mathrm{curl}}\, w = -\frac{\partial w}{\partial r}.$$

This, together with (14), enables us to define the functions

$$F(r,\varphi) := \hat{x}\cdot u = \frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial w}{\partial\varphi}, \quad G(r,\varphi) := \hat{x}^\perp\cdot u = \frac{1}{r}\frac{\partial v}{\partial\varphi} - \frac{\partial w}{\partial r}, \tag{16}$$

with the vanishing data

$$F(r, \pm\varphi_0) = G(r, \pm\varphi_0) = 0, \tag{17}$$

since $u = 0$ on $\varphi = \pm\varphi_0$. Observing that v and w are solutions to the homogeneous Helmholtz equation in $\mathbb{R}^2$, by Lemma 2.2 we may expand them into the series

$$\begin{aligned} v(r,\varphi) &= \sum_{n,m\in\mathbb{N}_0} r^{n+2m}\left(v^+_{n,m}\cos(n\varphi) + v^-_{n,m}\sin(n\varphi)\right), \\ w(r,\varphi) &= \sum_{n,m\in\mathbb{N}_0} r^{n+2m}\left(w^+_{n,m}\cos(n\varphi) + w^-_{n,m}\sin(n\varphi)\right), \end{aligned} \tag{18}$$

in a small neighborhood of the origin, where $v^\pm_{n,m}, w^\pm_{n,m} \in \mathbb{C}$ satisfy the recurrence relations

$$\begin{aligned} v^\pm_{n,m+1} &= -\frac{k_p^2}{4(m+1)(n+m+1)}v^\pm_{n,m}, \\ w^\pm_{n,m+1} &= -\frac{k_s^2}{4(m+1)(n+m+1)}w^\pm_{n,m}, \end{aligned} \tag{19}$$

for all $n, m \in \mathbb{N}_0$. By unique continuation, it is now sufficient to prove $v^\pm_{n,m} = w^\pm_{n,m} = 0$ for all $n, m \in \mathbb{N}_0$, if the transcendental equation (13) has no integer roots.

Inserting (18) into the definitions of F and G in (16) yields

$$
\begin{aligned}
F(r,\varphi) &= \sum_{n\in\mathbb{N},m\in\mathbb{N}_0} r^{n+2m-1}\left(f^+_{n,m}\cos(n\varphi)+f^-_{n,m}\sin(n\varphi)\right)\\
&=: \sum_{N\in\mathbb{N}_0} r^N F_N(\varphi),\\
G(r,\varphi) &= \sum_{n\in\mathbb{N},m\in\mathbb{N}_0} r^{n+2m-1}\left(g^+_{n,m}\cos(n\varphi)+g^-_{n,m}\sin(n\varphi)\right)\\
&=: \sum_{N\in\mathbb{N}_0} r^N G_N(\varphi),
\end{aligned}
\tag{20}
$$

with

$$
\begin{aligned}
f^+_{n,m} &= (n+2m)\,v^+_{n,m}+n\,w^-_{n,m}, \quad & f^-_{n,m} &= (n+2m)\,v^-_{n,m}-n\,w^+_{n,m},\\
g^-_{n,m} &= -n\,v^+_{n,m}-(n+2m)\,w^-_{n,m}, \quad & g^+_{n,m} &= n\,v^-_{n,m}-(n+2m)\,w^+_{n,m}
\end{aligned}
\tag{21}
$$

and

$$
\begin{aligned}
F_N(\varphi) &= \sum_{n\geq 1,m\geq 0:n+2m-1=N}\left(f^+_{n,m}\cos(n\varphi)+f^-_{n,m}\sin(n\varphi)\right),\\
G_N(\varphi) &= \sum_{n\geq 1,m\geq 0:n+2m-1=N}\left(g^+_{n,m}\cos(n\varphi)+g^-_{n,m}\sin(n\varphi)\right).
\end{aligned}
$$

Obviously, $f^+_{n,0}=-g^-_{n,0}$, $f^-_{n,0}=g^+_{n,0}$ for all $n\geq 1$. Taking into account the Dirichlet condition (17), we deduce from the relations $F_N(\pm\varphi_0)=G_N(\pm\varphi_0)=0$ that

$$
\begin{aligned}
\sum_{n\geq 1,m\geq 0:n+2m-1=N} f^+_{n,m}\cos(n\varphi_0) &= \sum_{n\geq 1,m\geq 0:n+2m-1=N} f^-_{n,m}\sin(n\varphi_0)=0,\\
\sum_{n\geq 1,m\geq 0:n+2m-1=N} g^-_{n,m}\sin(n\varphi_0) &= \sum_{n\geq 1,m\geq 0:n+2m-1=N} g^+_{n,m}\cos(n\varphi_0)=0,
\end{aligned}
\tag{22}
$$

for all $N\in\mathbb{N}_0$.

We proceed by equating coefficients of r^N in (20). If $N=0$, then we have the indexes $n=1, m=0$. Hence, it follows from (22) and (21) that

$$
0=f^+_{1,0}=-g^-_{1,0}=v^+_{1,0}+w^-_{1,0},\quad 0=f^-_{1,0}=g^+_{1,0}=v^-_{1,0}-w^+_{1,0},
$$

implying that $v^+_{1,0}=-w^-_{1,0}$, $v^-_{1,0}=w^+_{1,0}$.

If $N=1$, then $n=2$ and $m=0$. By arguing as the previous case we find

$$
0=f^+_{2,0}=-g^-_{2,0}=2\left(v^+_{2,0}+w^-_{2,0}\right),\quad 0=f^-_{2,0}=g^+_{2,0}=2\left(v^-_{2,0}-w^+_{2,0}\right),
$$

leading to $v^+_{2,0}=-w^-_{2,0}$, $v^-_{2,0}=w^+_{2,0}$.

When $N=2$, it holds that $n=3, m=0$ or $n=1, m=1$. Consequently, it is seen from (22) that

$$
\left\{\begin{array}{l} f^+_{3,0}\cos(3\varphi_0)+f^+_{1,1}\cos\varphi_0=0,\\ g^-_{3,0}\sin(3\varphi_0)+g^-_{1,1}\sin\varphi_0=0,\end{array}\right.\qquad
\left\{\begin{array}{l} f^-_{3,0}\sin(3\varphi_0)+f^-_{1,1}\sin\varphi_0=0,\\ g^+_{3,0}\cos(3\varphi_0)+g^+_{1,1}\cos\varphi_0=0.\end{array}\right.
\tag{23}
$$

Making use of the recurrence relations (19), the equalities $v^\pm_{1,0}=\mp w^\mp_{1,0}$ and the definitions of $f^\pm_{1,1}$ and $g^\pm_{1,1}$ (see (21)), we represent $f^\pm_{1,1}$ and $g^\mp_{1,1}$ in terms of $v^\pm_{1,0}$ as (see also (28) with $j=0$)

$$
f^\pm_{1,1}=v^\pm_{1,0}\,(k_s^2-3k_p^2)/8,\quad g^\mp_{1,1}=v^\pm_{1,0}\,(k_p^2-3k_s^2)/8.
\tag{24}
$$

Combining (23) and (24), and using the fact that $g^-_{3,0} = -f^+_{3,0}, g^+_{3,0} = f^-_{3,0}$, we may transform the equations in (23) into

$$0 = \begin{pmatrix} \cos(3\varphi_0) & (k_s^2 - 3k_p^2)\cos\varphi_0 \\ -\sin(3\varphi_0) & (k_p^2 - 3k_s^2)\sin\varphi_0 \end{pmatrix} \begin{pmatrix} f^+_{3,0} \\ v^+_{1,0}/8 \end{pmatrix} =: A_0^+ \begin{pmatrix} f^+_{3,0} \\ v^+_{1,0}/8 \end{pmatrix},$$
$$0 = \begin{pmatrix} \sin(3\varphi_0) & (k_s^2 - 3k_p^2)\sin\varphi_0 \\ \cos(3\varphi_0) & -(k_p^2 - 3k_s^2)\sin\varphi_0 \end{pmatrix} \begin{pmatrix} f^-_{3,0} \\ v^-_{1,0}/8 \end{pmatrix} =: A_0^- \begin{pmatrix} f^-_{3,0} \\ v^-_{1,0}/8 \end{pmatrix}.$$

Simple calculations yield that the determinant of $A_0^\pm$ takes the form

$$\mathrm{Det}(A_0^\pm) = \mp(k_p^2 + k_s^2)\sin(4\varphi_0) \pm 2(k_s^2 - k_p^2)\sin(2\varphi_0).$$

Thus, $\mathrm{Det}(A_0^\pm) \neq 0$ if and only if

$$\pm\sin(2\psi) \neq 2\frac{k_s^2 - k_p^2}{k_s^2 + k_p^2}\sin\psi = 2\frac{\lambda+\mu}{\lambda+3\mu}\sin\psi, \quad \text{with} \quad \psi = 2\varphi_0.$$

This can be guaranteed by assuming that the number $z = 2$ is not an integer root of (13). Therefore, we obtain $v^\pm_{1,0} = f^\pm_{3,0} = 0$. Consequently, it holds that $w^\pm_{1,0} = g^\pm_{3,0} = 0$, and thus $w^\pm_{1,m} = v^\pm_{1,m} = 0$ for all $m \in \mathbb{N}_0$, $v^+_{3,0} = -w^-_{3,0}$, $v^-_{3,0} = w^+_{3,0}$. In summary, we have proved that for $j = 1$,

$$\begin{aligned} & w^\pm_{n,m} = v^\pm_{n,m} = 0 \quad \text{for all} \quad 1 \le n \le j,\ m \in \mathbb{N}_0, \\ & v^+_{n,0} = -w^-_{n,0}, \quad v^-_{n,0} = w^+_{n,0}, \quad n = j+1, j+2. \end{aligned} \tag{25}$$

Now, assuming that (25) is valid for some fixed $j \in \mathbb{N}$, we show that (25) also holds with j replaced by $j+1$.

Consider $N = j+2$. From (21) and (25), we see $f^\pm_{n,m} = g^\pm_{n,m} = 0$ for all $n \le j,\ m \in \mathbb{N}_0$. Hence, it follows from (22) with $N = j+2$ that

$$\left\{ \begin{array}{l} f^+_{j+3,0}\cos((j+3)\varphi_0) + f^+_{j+1,1}\cos((j+1)\varphi_0) = 0, \\ g^-_{j+3,0}\sin((j+3)\varphi_0) + g^-_{j+1,1}\sin((j+1)\varphi_0) = 0, \end{array} \right. \tag{26}$$

$$\left\{ \begin{array}{l} f^-_{j+3,0}\sin((j+3)\varphi_0) + f^-_{j+1,1}\sin((j+1)\varphi_0) = 0, \\ g^+_{j+3,0}\cos((j+3)\varphi_0) + g^+_{j+1,1}\cos((j+1)\varphi_0) = 0. \end{array} \right. \tag{27}$$

By the definition of $f^+_{n,m}$ and the recurrence relations (19) with $n = j+1, m = 0$, it follows that

$$\begin{aligned} f^+_{j+1,1} &= (j+3)\, v^+_{j+1,1} + (j+1)\, w^-_{j+1,1} \\ &= \frac{1}{4(j+2)}\left[-(j+3)\, k_p^2\, v^+_{j+1,0} - (j+1)\, k_s^2\, w^-_{j+1,0}\right] \\ &= \frac{v^+_{j+1,0}}{4(j+2)}\left[(j+1)\, k_s^2 - (j+3)\, k_p^2\right], \end{aligned} \tag{28}$$

where in the last equality we have used the relation $v^+_{j+1,0} = -w^-_{j+1,0}$ from (25). Analogously, we have

$$g^-_{j+1,1} = \frac{v^+_{j+1,0}}{4(j+2)}\left[(j+1)\, k_p^2 - (j+3)\, k_s^2\right].$$

Arguing in the same manner with the relation $v^-_{j+1,0} = w^-_{j+1,0}$, we find

$$f^-_{j+1,1} = \frac{v^-_{j+1,0}}{4(j+2)}\left[(j+1)\,k_s^2 - (j+3)\,k_p^2\right],$$

$$g^+_{j+1,1} = \frac{v^-_{j+1,0}}{4(j+2)}\left[-(j+1)\,k_p^2 + (j+3)\,k_s^2\right].$$

Inserting the previous expressions of $f^\pm_{j+1,1}$, $g^\pm_{j+1,1}$ into (26) and (27) yields the algebraic equations

$$A_j^+ \begin{pmatrix} f^+_{j+3,0} \\ v^+_{j+1,0}/[4(j+2)] \end{pmatrix} = 0, \tag{29}$$

$$A_j^- \begin{pmatrix} f^-_{j+3,0} \\ v^-_{j+1,0}/[4(j+2)] \end{pmatrix} = 0, \tag{30}$$

with

$$A_j^+ := \begin{pmatrix} \cos((j+3)\varphi_0) & [(j+1)k_s^2 - (j+3)k_p^2]\cos((j+1)\varphi_0) \\ -\sin((j+3)\varphi_0) & [(j+1)k_p^2 - (j+3)k_s^2]\sin((j+1)\varphi_0) \end{pmatrix},$$

$$A_j^- := \begin{pmatrix} \sin((j+3)\varphi_0) & [(j+1)k_s^2 - (j+3)k_p^2]\sin((j+1)\varphi_0) \\ \cos((j+3)\varphi_0) & -[(j+1)k_p^2 - (j+3)k_s^2]\cos((j+1)\varphi_0) \end{pmatrix}.$$

Note that $f^+_{j+3,0} = -g^-_{j+3,0}$, $f^-_{j+3,0} = g^+_{j+3,0}$ by definition. It can be readily checked that $\mathrm{Det}(A_j^\pm) \neq 0$ if and only if

$$\pm\sin((j+2)\psi) \neq (j+2)\frac{\lambda+\mu}{\lambda+3\mu}\sin\psi.$$

By the assumption of the lemma, we obtain $v^\pm_{j+1,0} = f^\pm_{j+3,0} = 0$, which in turn proves the relations in (25) with j replaced by $j+1$. Thus, by induction (25) is true for any $j \geq 1$. The proof of the lemma is complete. □

Based on the proof of Lemma 2.3, we now present the corresponding results when $\varphi_1 - \varphi_2 = \pi/2$, which will be used subsequently to prove our uniqueness results in inverse diffraction by rectangular rigid surfaces.

Lemma 2.4. *Suppose $(\Delta^* + \omega^2)u = 0$ in $\mathbb{R}^2$ and $u = 0$ on $\varphi = \varphi_1, \varphi_2$, where $\varphi_1 - \varphi_2 = \pi/2$. Then, we have either (i) $u \equiv 0$ under the condition (9), or (ii) $u = c\,u_0$ for some $c \in \mathbb{C}$ if (10) holds, where u_0 is some fixed real-analytic function.*

Remark 2. Lemma 2.4 implies that the dimension of the solution space to the Navier equation in $\mathbb{R}^2$ with vanishing data on two perpendicular straight lines is at most one.

Proof of Lemma 2.4. (i) In the case of $\psi = \varphi_1 - \varphi_2 = \pi/2$, the positive integer roots to (13) must be odd numbers satisfying the condition (10). Hence, the transcendental equation (13) has no integer roots under the condition (9). Applying Lemma 2.3 gives $u \equiv 0$.

(ii) If (10) holds, then $n_0 = (\lambda + 3\mu)/(\lambda + \mu) \in \mathbb{N}$ is the unique positive integer root to (13) with $\psi = \pi/2$. Let the matrices $A_j^\pm$ be defined as in the proof of Lemma 2.3 with $\varphi_0 = \pi/4$. Set $j = n_0 - 2$. Without loss of generality, we may suppose $\sin(n_0\pi/2) = 1$ so that

$$\mathrm{Det}(A_j^+) = 0,\ \mathrm{Det}(A_j^-) \neq 0, \quad \text{and} \quad \mathrm{Det}(A_n^\pm) \neq 0 \text{ for all } n \neq j.$$

The case $\sin(n_0\pi/2) = -1$ can be treated analogously. In view of the proof of Lemma 2.3, we see that the relations in (25) hold with the selected $j = n_0 - 2$. Consider again the coefficient of r^N in (20) and (22), where $\varphi_0 = \pi/4$. For clarity we divide our proof into three steps.

Step 1. Prove $v^{\pm}_{n,m} = w^{\pm}_{n,m} = 0$ for all $n = j+2, j+4, \cdots$, and $m \in \mathbb{N}_0$.

By (25), it holds that

$$w^{\pm}_{n,m} = v^{\pm}_{n,m} = 0, \quad n = j, j-2, \cdots, \; m \in \mathbb{N}_0, \tag{31}$$

$$v^{+}_{j+2,0} = -w^{-}_{j+2,0}, \quad v^{-}_{j+2,0} = w^{+}_{j+2,0}. \tag{32}$$

Hence, $f^{\pm}_{n,m} = g^{\pm}_{n,m} = 0$ for all $n = j, j-2, \cdots, m \in \mathbb{N}_0$. Consider $N = j+3$. It follows from (22) that (cf. (29), (30) in the case $N = j+2$)

$$A^{\pm}_{j+1}\begin{pmatrix} f^{\pm}_{j+4,0} \\ v^{\pm}_{j+2,0}/[4(j+3)] \end{pmatrix} = 0.$$

Since $\mathrm{Det}(A^{\pm}_{j+1}) \neq 0$, we get $f^{\pm}_{j+4,0} = v^{\pm}_{j+2,0} = 0$. This implies that (31) and (32) are valid with j replaced by $j+2$. By induction we finish the proof in Step 1.

Step 2. Prove $v^{-}_{n,m} = w^{+}_{n,m} = 0$ for all $n = j+1, j+3, \cdots$, and $m \in \mathbb{N}_0$.

Again using (25), we see

$$\begin{aligned} &w^{+}_{n,m} = v^{-}_{n,m} = 0, \quad \text{for all } n = j-1, j-3, \cdots, \;\; m \in \mathbb{N}_0, \\ &v^{-}_{j+1,0} = w^{+}_{j+1,0}. \end{aligned} \tag{33}$$

Then, the relations (27) and (30) can be proved again following the lines in the proof of Lemma 2.3. Since $\mathrm{Det}(A^{-}_{j}) \neq 0$, one can verify that $f^{+}_{j+3,0} = v^{-}_{j+1,0} = 0$, leading to the relations in (33) with j replaced by $j+2$. This implies the desired results in Step 2.

Step 3. Prove that $v^{+}_{n,m}, w^{-}_{n,m}$ depend linearly on some constant $c \in \mathbb{C}$ for all $n = j+1, j+3, \cdots$, and $m \in \mathbb{N}_0$.

Since $j = n_0 - 2$, $\sin(n_0\pi/2) = 1$, there hold

$$\sin((j+3)\pi/4) = \cos((j+1)\pi/4) \neq 0, \quad (j+1)k_s^2 = (j+3)k_p^2,$$

where the second equality follows from (10). While (30) is only trivially solvable, the equation (29) has non-trivial solutions given by

$$f^{+}_{j+3,0} = 0, \quad v^{+}_{j+1,0} = c, \tag{34}$$

for some constant $c \in \mathbb{C}$. By (34), we have

$$v^{+}_{j+3,0} = -w^{-}_{j+3,0}, \quad w^{-}_{j+1,0} = -v^{+}_{j+1,0} = -c. \tag{35}$$

The second equality in (35), together with (19) and the definition of $f^{+}_{j+1,m}$, implies

$$v^{+}_{j+1,m} = \tilde{v}^{+}_{j+1,m}\, c, \quad w^{-}_{j+1,m} = \tilde{w}^{-}_{j+1,0}\, c, \quad f^{+}_{j+1,m} = \tilde{f}^{+}_{j+1,m}\, c, \quad m \geq 0,$$

with some $\tilde{v}^{+}_{j+1,m}, \tilde{w}^{-}_{j+1,0}, \tilde{f}^{+}_{j+1,m} \in \mathbb{C}$. Now, set $N = j+4$. Making use of the first equality in (35), one can derive from $F_{j+4}(\pm\varphi_0) = G_{j+4}(\pm\varphi_0) = 0$ that (cf. (26) and (29) in the case $N = j+2$)

$$A^{+}_{j+2}\begin{pmatrix} f^{+}_{j+5,0} \\ v^{+}_{j+3,0}/[4(j+4)] \end{pmatrix} = -\begin{pmatrix} \tilde{f}^{+}_{j+1,3}\cos((j+1)\pi/4) \\ \tilde{g}^{-}_{j+1,3}\sin((j+1)\pi/4) \end{pmatrix} c.$$

The above equation is uniquely solvable, with the solution pair $(f^{+}_{j+5,0}, v^{+}_{j+3,0})$ depending linearly on c. This in turn implies that $v^{+}_{j+3,m}$, $m \in \mathbb{N}_0$, depend linearly on

c. Since $f^+_{j+3,0} = 0$, we also get the linear dependence of $w^-_{j+3,0}$ and that of $w^-_{j+3,m}$, $m \in \mathbb{N}_0$ on c. Repeating the above procedure, we finally conclude that

$$v^+_{n,m} = \tilde{v}^+_{n,m}\, c, \quad w^-_{n,m} = \tilde{w}^-_{n,m}\, c, \quad \text{for all} \quad m \in \mathbb{N}_0,\ n = n_0 - 1, n_0 + 1, n_0 + 3, \cdots .$$

In order to prove Lemma 2.4, we need to introduce the function $u_0 = \nabla v_0 + \overrightarrow{\operatorname{curl}}\, w_0$, where

$$v_0(r,\varphi) := \sum_{n=n_0-1,n_0+1,\cdots,m\in\mathbb{N}_0} \left[r^{n+2m}\, \tilde{v}^+_{n,m} \cos(n\varphi)\right],$$
$$w_0(r,\varphi) := \sum_{n=n_0-1,n_0+1,\cdots,m\in\mathbb{N}_0} \left[r^{n+2m}\, \tilde{w}^-_{n,m} \sin(n\varphi)\right].$$

Since $\tilde{v}^+_{n,m}$ and $\tilde{w}^-_{n,m}$ satisfy the recurrence relation (19), v_0 and w_0 are solutions to the Helmholtz equations in (15). Hence u_0 satisfies the Navier equation and $u = cu_0$. The proof of Lemma 2.4 is complete. □

2.3. **Proof of Theorem 2.1.** Relying on the properties of the Navier equation shown in Lemma 2.4, we prove the uniqueness results in Theorem 2.1 for diffraction gratings by contradiction. Let the incident elastic plane wave be given as in (8) with the incident angle θ. Assume there are two distinct scattering surfaces $\Lambda_1, \Lambda_2 \in \mathcal{A}$ generating the same near-field data on Γ_b:

$$u_1(x;\theta) = u_2(x;\theta), \quad x \in \Gamma_b.$$

By the well-posedness of the direct scattering problem for a flat profile, we get the coincidence of u_1 and u_2 in $x_2 > b$, and the unique continuation of solutions to the Navier equation leads to

$$u_1(x;\theta) = u_2(x;\theta) =: u(x), \quad x \in \Omega, \tag{36}$$

where Ω denotes the unbounded connected component of $\Omega_{\Lambda_1} \cap \Omega_{\Lambda_2}$. We consider two cases.

Case 1. The corners of Λ_1 and Λ_2 coincide.

Since the convex hull of the corner points coincides with a strip and both profiles are bounded in the x_2-direction, the line segments lying on them must be parallel to the coordinate axes in Case 1. Therefore, the horizontal line segments of Λ_j ($j = 1, 2$) lie on two straight lines Γ_{b_1} and Γ_{b_2} for some $-b < b_2 < b_1 < b$, whereas the vertical segments are identical (see Figure 2).

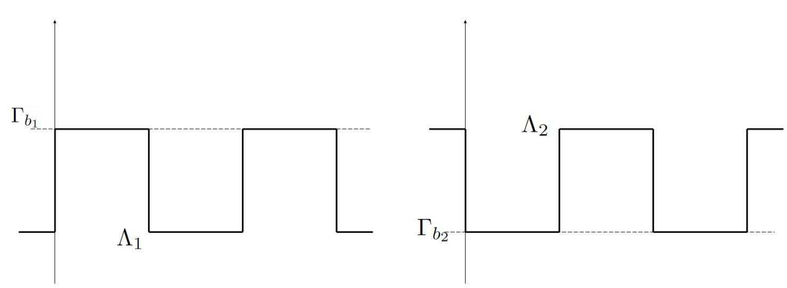

FIGURE 2. Examples of rectangular diffraction gratings sharing the same corners.

Without loss of generality, we suppose Γ_{b_1} to be the x_1-axis, i.e., $b_1 = 0$. Recalling the Dirichlet boundary conditions on Λ_1 and Λ_2, we get $u = 0$ on Γ_0. This suggests that u is the total field corresponding to the rigid scattering surface $x_2 = 0$ due to the incident plane wave (8). By linear supposition, it is not difficult to get the explicit expression of u in $x_2 \geq 0$ as follows: $u = (c_p/k_p)\, U_p + (c_s/k_s) U_s$, where c_p and c_s are the coefficients attached to the incident plane pressure and shear waves, respectively, and

$$
\begin{aligned}
U_p &= \begin{pmatrix} \alpha_p \\ -\beta_p \end{pmatrix} e^{i(\alpha_p x_1 - \beta_p x_2)} - \frac{\alpha_p^2 - \beta_p\gamma_p}{\alpha_p^2 + \beta_p\gamma_p} \begin{pmatrix} \alpha_p \\ \beta_p \end{pmatrix} e^{i(\alpha_p x_1 + \beta_p x_2)} \\
&\quad - \frac{2\alpha_p\beta_p}{\alpha_p^2 + \beta_p\gamma_p} \begin{pmatrix} \gamma_p \\ -\alpha_p \end{pmatrix} e^{i(\alpha_p x_1 + \gamma_p x_2)}, \\
U_s &= \begin{pmatrix} \gamma_s \\ \alpha_s \end{pmatrix} e^{i(\alpha_s x_1 - \gamma_s x_2)} - \frac{2\alpha_s\gamma_s}{\alpha_s^2 + \beta_s\gamma_s} \begin{pmatrix} \alpha_s \\ \beta_s \end{pmatrix} e^{i(\alpha_s x_1 + \beta_s x_2)} \\
&\quad - \frac{\beta_s\gamma_s - \alpha_s^2}{\alpha_s^2 + \beta_s\gamma_s} \begin{pmatrix} \gamma_s \\ -\alpha_s \end{pmatrix} e^{i(\alpha_s x_1 + \gamma_s x_2)},
\end{aligned}
\tag{37}
$$

with

$$
\begin{aligned}
\alpha_p &= k_p \sin\theta, \quad \beta_p = k_p \cos\theta, \quad \gamma_p = \sqrt{k_s^2 - \alpha_p^2}, \\
\alpha_s &= k_s \sin\theta, \quad \gamma_s = k_s \cos\theta, \quad \beta_s = \sqrt{k_p^2 - \alpha_s^2}.
\end{aligned}
$$

Since u consists of finitely many terms only, it extends analytically to the whole space $\mathbb{R}^2$. Hence, u must also vanish on at least one vertical straight line, for instance $\{x_1 = 0\}$, which can be extended to infinity in the x_2-direction. This implies that $c_p = c_s = 0$, which is a contradiction. Hence, $\Lambda_1 = \Lambda_2$.

Case 2. The corners of Λ_1 and Λ_2 do not coincide.

First we consider Case (a): there exists a corner point O_j of Λ_j in $\Omega_{\Lambda_{j+1}}$ for $j = 1$ or $j = 2$, where $\Omega_{\Lambda_3} = \Omega_{\Lambda_1}$. Without loss of generality, we suppose that Case (a) occurs with $j = 1$; see Figure 3 (left). It follows from (36) and the Dirichlet boundary condition of u_1 on Λ_1 that u_2 vanishes on the two perpendicular line segments of Λ_1 meeting at O_1 in Ω_{Λ_2}. Moreover, u_2 satisfies the Navier equation in a small neighborhood $D_1 \subset \Omega_{\Lambda_2}$ of O_1. Applying Lemma 2.4 to u_2 yields:

- **(i):** $u_2(x;\theta) \equiv 0$ under the condition (9). This contradiction implies $\Lambda_1 = \Lambda_2$, and thus uniqueness with a single incident plane wave holds.
- **(ii):** $u_2(x;\theta) = c\, u_0(x), x \in D_1$, under the condition (10). By arguing in the same manner we get $u_2(x;\theta') = c'\, u_0(x), x \in D_1$, if $u_2(x;\theta') = u_1(x;\theta')$ on Γ_b, where $\theta' \neq \theta$ is another incident angle. Hence, $u_2(x;\theta) = c/c'\, u_2(x;\theta')$ in D_1 and by unique continuation also in $x_2 > b$. This contradicts the linear independence of $u_2(x;\theta)$ and $u_2(x;\theta')$ in $x_2 > b_2$ which can be readily justified using the Rayleigh expansions of $u_2^{sc}(x;\theta)$ and $u_2^{sc}(x;\theta')$. Now we conclude that $\Lambda_1 = \Lambda_2$ if the near-field data coincide for two distinct incident angles.

If Case (a) is excluded, we may suppose the existence of a corner point O_1 of Λ_1 lying on a certain line segment $l \subset \Lambda_2$; see Figure 3 (right). In this case, l must be perpendicular to a line segment of Λ_1 passing through O_1, and l coincides partly with another line segment of Λ_1. Since l is an analytic boundary part of Ω_{Λ_2} and $u_2 = 0$ on l, u_2 is analytic in Ω_{Λ_2} up to l (see [16, Theorem A]) and thus u_2 has analytic Cauchy data on l. Applying the Cauchy-Kowalewski theorem, we

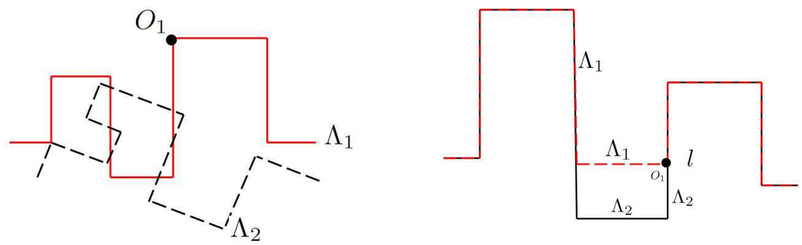

Figure 3. Two rectangular diffraction gratings whose corners are not identical.

can extend u_2 to a small neighborhood of O_1 as a solution to the Navier equation. Repeating the arguments in Case (a), we complete the proof of Theorem 2.1. $\square$

3. **Uniqueness for non-periodic rough surfaces.** The aim of this section is to remove the periodicity assumption imposed on the rectangular grating profiles from the admissible class $\mathcal{A}$. Define a new admissible class $\tilde{\mathcal{A}}$ by

$$\tilde{\mathcal{A}} = \left\{ \Lambda : \begin{array}{l} \Lambda \text{ is a piecewise linear curve in } |x_2| < b. \text{ Any two} \\ \text{neighboring line segments of } \Lambda \text{ are perpendicular.} \end{array} \right\}.$$

Before carrying over the proof of Theorem 2.1 to the non-periodic case, we give a brief sketch of the well-posedness of the forward elastic scattering from rigid rough surfaces for incident plane waves in 2D. Instead of the Rayleigh expansion radiation condition (5), the scattered field is now required to satisfy a more general upward radiation condition (which is usually referred to as the upward angular spectrum representation):

$$(38)\quad u^{sc}(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left(e^{i\gamma_p(\xi)\,(x_2-b)} M_p(\xi) + e^{i\gamma_s(\xi)\,(x_2-b)} M_s(\xi) \right) \hat{u}_b^{sc}(\xi)\, e^{ix_1\xi}\, d\xi$$

for $x_2 > b$, where M_p and M_s are two matrices given by

$$M_p(\xi) = \frac{1}{\xi^2 + \gamma_p\gamma_s} \begin{pmatrix} \xi^2 & \xi\gamma_s \\ \xi\gamma_p & \gamma_p\gamma_s \end{pmatrix}, \quad M_s(\xi) = \frac{1}{\xi^2 + \gamma_p\gamma_s} \begin{pmatrix} \gamma_p\gamma_s & -\xi\gamma_s \\ -\xi\gamma_p & \xi^2 \end{pmatrix},$$

respectively, with $\gamma_p(\xi) := \sqrt{k_p^2 - \xi^2}, \gamma_s(\xi) := \sqrt{k_s^2 - \xi^2}$. The notation $\hat{u}_b^{sc}(\xi)$ in (38) stands for the Fourier transform of $u^{sc}(x_1, b)$, given by

$$\hat{u}_b^{sc}(\xi) = (2\pi)^{-1/2} \int_{\mathbb{R}} \exp(-it\,\xi)\, u^{sc}(t,b)\, dt\,,\ \xi \in \mathbb{R}\,,$$

Let the incident plane wave be given as in (8), and define $S_h := \Omega_\Lambda \backslash \{x_2 \geq h\}$. It was shown in [9] that the forward two-dimensional scattering problem admits a unique total field $u = u^{in} + u^{sc}$ in the following weighted Sobolev space

$$(39)\qquad V_{h,\varrho} := (1 + x_1^2)^{-\varrho/2} H_0^1(S_h)^2 \quad \text{for all} \quad h \geq b,\ -1 < \varrho < -1/2,$$

provided the scattering surface Λ is given by the graph of a bounded and uniformly Lipschitz continuous function. Note that the space $H_0^1(S_h)$ denotes the functions in the standard Sobolev space $H^1(S_h)$ with vanishing trace on Λ, and that $V_{h,\varrho}$ is defined as the closure of $\{u|_{S_h} : u \in C_0^\infty(S_h)\}$ in the norm

$$||u||_{V_{h,\varrho}} := \left(\int_{S_h} (1+x_1^2)^\varrho \left(\left|u\right|^2 + \left|\nabla u\right|^2\right) dx\right)^{1/2}, \quad u \in V_{h,\varrho}.$$

Since the above mentioned uniqueness and existence results do not cover the non-graph rectangular surfaces from $\tilde{\mathcal{A}}$, we suppose the forward scattering problem for any $\Lambda \in \tilde{\mathcal{A}}$ is always solvable in the weighted Sobolev space (39). In particular, if $\Lambda = \{x_2 = 0\}$, the explicit solution takes the same form as that constructed in the proof of Theorem 2.1 for diffraction gratings (see (37)). Below we state the uniqueness result for the inverse scattering problem.

Theorem 3.1. *Let the incident elastic plane wave $u^{in}(x;\theta)$ be given by (8), and set $I = \{(x_1, b) : x_1 \in (c_1, c_2)\}$ for some $c_1 < c_2$. Then, $\Lambda \in \tilde{\mathcal{A}}$ can be uniquely determined by the scattered near field data $\{u^{sc}(x;\theta) : x \in I\}$ with a single angle θ under the condition (9), whereas the data from two distinct incident angles are sufficient if the condition (10) holds.*

Proof. Assume there are two scattering surfaces $\Lambda_1, \Lambda_2 \in \tilde{\mathcal{A}}$ generating the same near-field data on I, i.e., $u_1^{sc}(x) = u_2^{sc}(x)$ for $x \in I$. From the analyticity of u_1^{sc}, u_2^{sc} in $x_2 \geq b$, we see $u_1^{sc} = u_2^{sc}$ on $x_2 = b$. To adapt the proof of Theorem 2.1 to the non-periodic case, we only need to verify the linear independence of the total fields $u(x;\theta)$ and $u(x;\theta')$ in $x_2 > b$ for different incident angles θ and θ'. Here, $u(x;\theta) = u^{in}(x;\theta) + u_j^{sc}(x;\theta)$ for $j = 1, 2$. Assume $u(x;\theta) = a u(x;\theta')$ with some $a \in \mathbb{C}$. We then obtain

$$w(x) := u^{in}(x;\theta) - a\, u^{in}(x;\theta') = -\left(u_j^{sc}(x;\theta) - a\, u_j^{sc}(x;\theta')\right), \tag{40}$$

for all $x_2 \geq b$, which satisfies the upward radiation condition. From (40), we conclude that $w(x)$ can be regarded as the scattered field reflected from the rigid surface $\{x_2 = b\}$ with the incident field $U^{in} = -(u^{in}(x;\theta) - a\, u^{in}(x;\theta'))$. We observe that U^{in} cannot vanish identically, because $u^{in}(x;\theta)$ and $u^{in}(x;\theta')$ are linearly independent. The explicit form of w can be computed analogously to (37). On the other hand, w is a linear combination of scattered waves travelling upwards. Therefore, it is a contradiction that $w = -U^{in}$ is an incoming wave for $x_2 > b$, as shown in the first relation of (40). Hence, $u(x;\theta)$ and $u(x;\theta')$ are linearly independent in $x_2 > b$. Arguing analogously to the proof of Theorem 2.1, we complete the proof of Theorem 3.1. $\square$

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