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Uniqueness in the inverse scattering problem in a piecewise homogeneous medium

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Abstract
The scattering of time-harmonic acoustic plane waves by an impenetrable obstacle in a piecewise homogeneous medium is considered. Having established the well posedness of the direct problem by the variational method, we prove a uniqueness result for the inverse problem, that is, the unique determination of the obstacle and its boundary condition from a knowledge of the far-field pattern for incident plane waves. The proof is based on a generalization of the mixed reciprocity relation.

1. Introduction

In this paper, we consider the problem of scattering of time-harmonic acoustic plane waves by an impenetrable obstacle surrounded by a piecewise homogeneous medium. In practical applications, the background might not be homogeneous and then must be modeled as a layered medium. A medium of this type that is a nested body consisting of a finite number of homogeneous layers occurs in various areas of applications such as radar, remote sensing, geophysics and nondestructive testing.

Let Ω denote the piecewise homogeneous medium which is a bounded and closed subset of $\mathbb{R}^n$ $(n = 2, 3)$ with a C^2 boundary S_0. Let Ω_0 be the exterior region of Ω, that is, $\Omega_0 = \mathbb{R}^n\backslash\overline{\Omega}$ $(n = 2, 3)$. The interior of Ω is divided by means of closed and nonintersecting C^2 surfaces S_j $(j = 1, 2, \ldots, N)$ into subsets (layers) Ω_j $(j = 1, 2, \ldots, N + 1)$ with $\partial\Omega_{j-1}\bigcap\partial\Omega_j = S_{j-1}$ $(j = 1, 2, \ldots, N + 1)$. The regions Ω_j $(j = 0, 1, \ldots, N)$ are homogeneous media. The region Ω_{N+1} is the impenetrable obstacle.

We now give a brief description of the direct and inverse scattering problem.

The propagation of time-harmonic acoustic waves in a piecewise homogeneous isotropic medium in $\mathbb{R}^n$ $(n = 2, 3)$ is modeled by the reduced wave equation or Helmholtz equation with boundary conditions on their interfaces:

$$\Delta u + k_j^2 u = 0 \qquad \text{in} \quad \Omega_j, \qquad j = 0, 1, \ldots, N, \tag{1.1}$$

$$u_+ = u_-, \qquad \frac{\partial u_+}{\partial \nu} = \lambda_j \frac{\partial u_-}{\partial \nu} \qquad \text{on} \quad S_j, \qquad j = 0, 1, \ldots, N-1, \tag{1.2}$$

where ν is the unit outward normal to the boundary S_j, u_+, $\frac{\partial u_+}{\partial \nu}$ $\left(u_-, \frac{\partial u_-}{\partial \nu}\right)$ denote the limit of u, $\frac{\partial u}{\partial \nu}$ on the surface S_j from the exterior (interior) of S_j and λ_j represents the nonnegative constant across the surface S_j $(j = 0, 1, \ldots, N-1)$. Here, u denotes the complex-valued space-dependent part of the time-harmonic acoustic wave $u(x)\mathrm{e}^{-\mathrm{i}\omega t}$ and k_j is the positive wave number given by $k_j = \omega_j / c_j$ in terms of the frequency ω_j and the sound speed c_j in the corresponding region Ω_j $(j = 0, 1, \ldots, N)$. The distinct wave numbers k_j $(j = 0, 1, \ldots, N)$ correspond to the fact that the background medium consists of several physically different materials. On these surfaces S_j $(j = 0, 1, \ldots, N-1)$, the so-called transmission conditions (1.2) are imposed, which represent the continuity of the medium and equilibrium of the forces acting on it. On the boundary S_N of the obstacle Ω_{N+1} the total wave u has to satisfy a boundary condition of the form

$$B(u) = 0 \qquad \text{on} \quad S_N. \tag{1.3}$$

For a *sound-soft* obstacle the pressure of the total wave vanishes on the boundary, so a Dirichlet boundary condition

$$B(u) := u \qquad \text{on} \quad S_N$$

is imposed. Similarly, the scattering from a *sound-hard* obstacle leads to a Neumann boundary condition

$$B(u) := \frac{\partial u}{\partial \nu} \qquad \text{on} \quad S_N$$

since the normal velocity of the total acoustic wave vanishes on the boundary. More general and realistic boundary conditions are to allow that the normal velocity on the boundary is proportional to the excess pressure on the boundary, which leads to an impedance boundary condition of the form

$$B(u) := \frac{\partial u}{\partial \nu} + \mathrm{i}\lambda u \qquad \text{on} \quad S_N$$

with a nonnegative continuous function λ. Henceforth, we shall use $B(u) = 0$ to represent either of the above three types or mixed type of boundary conditions on S_N. In the region $\mathbb{R}^n \backslash \Omega_{N+1}$ $(n = 2, 3)$, the total wave u is the superposition of the given incident plane wave $u^i(x) = \mathrm{e}^{\mathrm{i}k_0 x \cdot d}$ and the scattered wave $u^s(x)$ which is required to satisfy the Sommerfeld radiation condition

$$\lim_{r \to \infty} r^{\frac{n-1}{2}} \left(\frac{\partial u^s}{\partial r} - \mathrm{i}k_0 u^s \right) = 0 \tag{1.4}$$

uniformly in all directions $x/|x|$, where $r = |x|$. It physically implies that energy is transported to infinity and it is an important ingredient in ensuring that the physically correct solution of the scattering problem is selected. The well posedness (existence, uniqueness and stability) of the direct problem for a *sound-soft* obstacle using the theory of generalized solutions has been studied by Athanasiadis and Stratis [2]. However, it is not suitable for our later use, that is, the proof of the uniqueness result in the inverse problem. Therefore, following Cakoni and Colton [3] and Mclean [19], we will give a new proof and consider a general mixed boundary value problem in section 2. Moreover, it is known that $u^s(x)$ has the following asymptotic representation:

$$u^s(x,d) = \frac{e^{ik_0|x|}}{|x|^{\frac{n-1}{2}}} \left\{ u^\infty(\widehat{x}, d) + O\left(\frac{1}{|x|}\right) \right\} \qquad \text{as} \quad |x| \to \infty \tag{1.5}$$

uniformly for all directions $\widehat{x} := x/|x|$, where the function $u^\infty(\widehat{x}, d)$ defined on the unit sphere S is known as the far-field pattern with $\widehat{x}$ and d denoting, respectively, the observation direction and the incident direction. By analyticity, the far-field pattern is completely determined on the whole unit sphere S by only knowing it on some open subset S^* of S [8]. Therefore, all the uniqueness results carry over to the case of limited aperture problems where the far-field pattern is only known on some open subset S^* of S. Without loss of generality, we can assume that the far-field data are given on the whole unit sphere S, that is, in every possible observation direction.

The inverse problem we consider in this paper is, given the wave numbers k_j ($j = 0, 1, \ldots, N$), the nonnegative constants λ_j ($j = 0, 1, \ldots, N-1$) and the far-field pattern $u^\infty(\widehat{x}, d)$ for all incident plane waves with incident direction $d \in S$, to determine the location and shape of the obstacle Ω_{N+1} and its boundary condition. As usual in most of the inverse problems, the first question to ask in this context is the identifiability, that is, whether an obstacle can be identified from a knowledge of the far-field pattern. Mathematically, the identifiability is the uniqueness issue which is of theoretical interest and is required in order to proceed to efficient numerical methods of solutions.

In the last 30 years, both the inverse scattering problem in a homogeneous medium (i.e. $N = 0$) and the inverse medium problem have obtained great development in the theoretical and numerical aspects. We refer to the monographs [8, 14, 23] and the references therein for a comprehensive discussion. As far as we know, there are few uniqueness results for inverse obstacle scattering in a piecewise homogeneous medium. When the obstacle is penetrable (with transmission boundary conditions), Athanasiadis, Ramm and Stratis [1] and Yan [25] proved that the obstacle is determined uniquely by the corresponding far-field pattern based on an orthogonality result [23]. Yan and Pang [26] gave a proof of uniqueness of the *sound-soft* obstacle based on Schiffer's idea. But their method cannot be extended to other boundary conditions. They also gave a result for the case of a *sound-hard* obstacle in a two-layered background medium in [22] using a generalization of Schiffer's method. However, their method is hard to be extended to the case of a multilayered background medium and seems unreasonable to require the interior wave number to be in an interval.

There are few results on uniqueness in determining an obstacle or a medium buried in an inhomogeneous medium. In 1998, Kirsch and Päivärinta [15] proved that a sound-soft obstacle or a penetrable inhomogeneous medium can be uniquely determined if the outside inhomogeneity is known in advance. In the same year, Hähner [13] showed that both the sound-soft obstacle and the outside inhomogeneous medium in 2D can be uniquely determined by the far-field patterns corresponding to all incident plane waves with an interval of wave numbers. Recently in [20], the authors showed that an obstacle buried in a known inhomogeneous medium can be determined from measurements of the far field at a fixed wave number without *a priori* knowledge of the boundary condition.

In recent years a new version of the linear sampling method based on the reciprocity gap functional has been developed for the numerical recovery of the shape of an obstacle or an inhomogeneous medium immersed in a two-layered background medium in the case when the nonnegative constants $\lambda_0 = 1$ (see, e.g. [4–7, 10, 11] and the references therein). In particular, in [10], Cristo and Sun also proved that the obstacle and the surface impedance can be uniquely determined by the near field on $\partial\Omega$ for all point sources on the boundary of a box containing Ω. It should be pointed out that, recently in [24], Yaman presented a numerical method based on

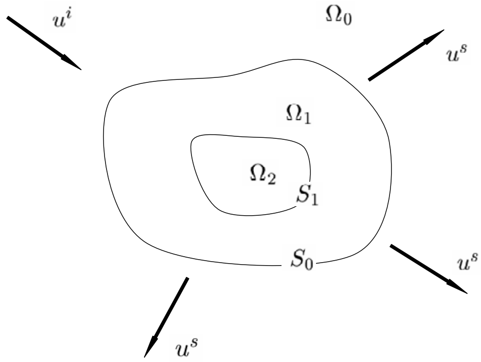

Figure 1. Scattering in a two-layered background medium.

Newton iterations and integral equations to reconstruct the location and shape of a *sound-soft* obstacle buried in a two-layered background medium in 2D using the far-field patterns.

Our contribution in this paper is to provide a uniqueness result for the inverse obstacle scattering problem in a known layered background medium with arbitrary nonnegative constants λ_j using the far-field patterns corresponding to incident plane waves.

This paper is organized as follows. In the next section, we will establish the well posedness of the direct scattering problem by the variational method. Section 3 is devoted to the unique determination of the obstacle and its boundary condition from a knowledge of the far-field pattern for incident plane waves based on a generalization of the mixed reciprocity relation. We will not assume that we know the boundary condition for the obstacle. This seems to be appropriate for a number of applications where the physical nature of the obstacle is unknown.

For simplicity, and without loss of generality, in this paper we only consider the case $N = 1$, that is, the case where the obstacle is buried in a two-layered background medium, as shown in figure 1. In this case, Ω_2 is the impenetrable obstacle (see figure 1). The results obtained in this paper are also available for the case of general N and can be proved similarly.

2. The direct scattering problem

We only consider the three-dimensional case. We remark that all the results of this section remain valid in two dimensions after appropriate modifications of the fundamental solution, the radiation condition and the spherical wavefunctions. The scattering of an incident field in a two-layered background medium is depicted in figure 1.

We will focus on the general case where mixed boundary conditions are imposed on the boundary S_1 of the obstacle Ω_2. More precisely, the boundary S_1 consists of two parts, that is, $S_1 = \overline{S}_{1,D} \cup \overline{S}_{1,I}$, where $S_{1,D}$ and $S_{1,I}$ are two disjoint, relatively open subsets (possibly disconnected) of S_1.

Following Cakoni and Colton [3] and Mclean [19], we shall use the variational method to find a solution of the problem (1.1)–(1.4). To this end, let D denote a bounded domain and let $B_R := \{x : |x| < R\}$. Define the Sobolev spaces

$$H^1(D) := \{u : u \in L^2(D), |\nabla u| \in L^2(D)\},$$
$$H^1_{\mathrm{loc}}(\mathbb{R}^3 \backslash D) := \{u : u \in H^1(B_R \backslash D) \text{ for every } R > 0 \text{ such that } B_R \backslash D \neq \emptyset\}.$$

Recall that $H^{\frac{1}{2}}(\partial D)$ is the trace space of $H^1(D)$ and $H^{-\frac{1}{2}}(\partial D)$ is the dual space of $H^{\frac{1}{2}}(\partial D)$.

In order to study the mixed boundary value problem, we need the following Sobolev spaces on an open part of the boundary. We refer the reader to [19] for a systematic treatment.

Let Γ be an open subset of the boundary ∂D. Define

$$H^{\frac{1}{2}}(\Gamma) := \left\{u|_\Gamma : u \in H^{\frac{1}{2}}(\partial D)\right\}, \qquad \widetilde{H}^{\frac{1}{2}}(\Gamma) := \left\{u \in H^{\frac{1}{2}}(\partial D) : \mathrm{supp}(u) \subseteq \overline{\Gamma}\right\}.$$

Both $H^{\frac{1}{2}}(\Gamma)$ and $\widetilde{H}^{\frac{1}{2}}(\Gamma)$ are Hilbert spaces equipped with the restriction of the inner product of $H^{\frac{1}{2}}(\partial D)$. Hence, we can define the corresponding dual spaces

$$H^{-\frac{1}{2}}(\Gamma) := \left(\widetilde{H}^{\frac{1}{2}}(\Gamma)\right)' = \text{the dual space of } \widetilde{H}^{\frac{1}{2}}(\Gamma),$$
$$\widetilde{H}^{-\frac{1}{2}}(\Gamma) := \left(H^{\frac{1}{2}}(\Gamma)\right)' = \text{the dual space of } H^{\frac{1}{2}}(\Gamma).$$

It can be shown (cf theorem A4 in [19]) that there exists a bounded extension operator $\tau : H^{\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\partial D)$. An important property of $\widetilde{H}^{\frac{1}{2}}(\Gamma)$ is that the extension by zero to the whole boundary ∂D of $u \in \widetilde{H}^{\frac{1}{2}}(\Gamma)$ is in $H^{\frac{1}{2}}(\partial D)$ and the extension operator is bounded from $\widetilde{H}^{\frac{1}{2}}(\Gamma)$ to $H^{\frac{1}{2}}(\partial D)$. Based on these results, we can identify the dual spaces as follows:

$$H^{-\frac{1}{2}}(\Gamma) := \{u|_\Gamma : u \in H^{-\frac{1}{2}}(\partial D)\}, \qquad \widetilde{H}^{-\frac{1}{2}}(\Gamma) := \{u \in H^{-\frac{1}{2}}(\partial D) : \mathrm{supp}(u) \subseteq \overline{\Gamma}\}.$$

Thus, the duality pairing can be explained as

$$_{H^{-\frac{1}{2}}(\Gamma)}\langle v, u\rangle_{\widetilde{H}^{\frac{1}{2}}(\Gamma)} = {}_{H^{-\frac{1}{2}}(\partial D)}\langle v, \widetilde{u}\rangle_{H^{\frac{1}{2}}(\partial D)},$$
$$_{\widetilde{H}^{-\frac{1}{2}}(\Gamma)}\langle v, u\rangle_{H^{\frac{1}{2}}(\Gamma)} = {}_{H^{-\frac{1}{2}}(\partial D)}\langle \widetilde{v}, u\rangle_{H^{\frac{1}{2}}(\partial D)},$$

where $\widetilde{u} \in H^{\frac{1}{2}}(\partial D)$ is the extension by zero of $u \in \widetilde{H}^{\frac{1}{2}}(\Gamma)$ and $\widetilde{v} \in H^{-\frac{1}{2}}(\partial D)$ is the extension by zero of $v \in \widetilde{H}^{-\frac{1}{2}}(\Gamma)$.

Consider the mixed boundary value problem: given $h \in L^2(\Omega_1)$, $g \in H^{-\frac{1}{2}}(S_0)$, $f \in H^{\frac{1}{2}}(S_{1,D})$ and $p \in H^{-\frac{1}{2}}(S_{1,I})$, find $u \in H^1(\Omega_1) \cap H^1_{\mathrm{loc}}(\Omega_0)$ such that

$$\Delta u + k_0^2 u = 0 \quad \text{in} \quad \Omega_0, \tag{2.1}$$
$$\Delta u + k_1^2 u = h \quad \text{in} \quad \Omega_1, \tag{2.2}$$
$$u^+ = u^-, \qquad \frac{\partial u^+}{\partial \nu} - \lambda_0 \frac{\partial u^-}{\partial \nu} = g \quad \text{on} \quad S_0, \tag{2.3}$$
$$u = f \quad \text{on} \quad S_{1,D}, \tag{2.4}$$
$$\frac{\partial u}{\partial \nu} + \mathrm{i}\lambda u = p \quad \text{on} \quad S_{1,I}, \tag{2.5}$$
$$\lim_{r\to\infty} r\left(\frac{\partial u}{\partial r} - \mathrm{i}k_0 u\right) = 0 \tag{2.6}$$

where $r = |x|$, k_j $(j = 0, 1)$ are positive wave numbers, λ_0 is a nonnegative constant and λ is a nonnegative continuous impedance function. Here, equations (2.1) and (2.2) are understood in a distributional sense and the boundary conditions (2.3)–(2.5) are understood in the trace sense.

Remark 2.1. The case $S_{1,I} = \emptyset$ corresponds to a *sound-soft* obstacle, and the case $S_{1,D} = \emptyset$, $\lambda = 0$ corresponds to a *sound-hard* obstacle.

Remark 2.2. The acoustic scattering of the incident plane wave $u^i = \mathrm{e}^{\mathrm{i}k_0 x\cdot d}$ is a particular case of the problem (2.1)–(2.6). In particular, the scattered field u^s satisfies the problem (2.1)–(2.6) with $u = u^s$, $h = \left(k_0^2 - k_1^2\right)u^i|_{\Omega_1}$, $g = (\lambda_0 - 1)\frac{\partial u^i}{\partial \nu}|_{S_0}$, $f = -u^i|_{S_{1,D}}$ and $p = \left(-\frac{\partial u^i}{\partial \nu} - \mathrm{i}\lambda u^i\right)\big|_{S_{1,I}}$.

Theorem 2.3. *The boundary value problem (2.1)–(2.6) admits at most one solution.*

Proof. Clearly, it is enough to show that u vanishes identically for the homogeneous boundary value problem (2.1)–(2.6), that is, $u = 0$ if $f = g = h = p = 0$. Choose a large ball B_R centered at the origin such that $\Omega \subset B_R$. Applying Green's first theorem over $B_R\backslash\Omega$, we obtain that

$$\int_{\partial B_R} u\frac{\partial \overline{u}}{\partial \nu}\,\mathrm{d}s = \int_{B_R\backslash\Omega} (u\Delta\overline{u} + |\nabla u|^2)\,\mathrm{d}x + \int_{S_0} u\frac{\partial \overline{u}}{\partial \nu}\,\mathrm{d}s.$$

Using Green's first theorem over Ω_1 again and taking into account the transmission conditions (2.3) and the boundary conditions (2.4) and (2.5), we have

$$\int_{\partial B_R} u\frac{\partial \overline{u}}{\partial \nu}\,\mathrm{d}s = \int_{B_R\backslash\Omega} (u\Delta\overline{u} + |\nabla u|^2)\,\mathrm{d}x + \lambda_0\int_{\Omega_1} (u\Delta\overline{u} + |\nabla u|^2)\,\mathrm{d}x + \mathrm{i}\lambda_0\int_{S_{1,I}} \lambda|u|^2\,\mathrm{d}s. \quad (2.7)$$

Using equations (2.1) and (2.2) and taking the imaginary part of (2.7) we have, on noting that $k_0^2,\ k_1^2,\ \lambda_0$ are nonnegative real numbers and λ is a nonnegative continuous function, that

$$\Im\int_{\partial B_R} u\frac{\partial \overline{u}}{\partial \nu}\,\mathrm{d}s = \lambda_0\int_{S_{1,I}} \lambda|u|^2\,\mathrm{d}s \geqslant 0.$$

Thus, by Rellich's lemma [8], it follows that $u = 0$ in $\mathbb{R}^3\backslash B_R$. By the unique continuation principle, we have $u = 0$ in Ω_0. Holmgren's uniqueness theorem [16] implies that $u = 0$ in $\mathbb{R}^3\backslash\Omega_2$, which completes the proof of the theorem. □

The boundary value problems arising in scattering theory are formulated in unbounded domains. In order to solve such problems by the variational method, we need to write them as an equivalent problem in a bounded domain. Choose a ball B_R centered at the origin large enough such that the domain Ω is contained in the ball and define the Dirichlet to Neumann operator

$$T : w \to \frac{\partial \widetilde{w}}{\partial \nu} \qquad \text{on} \quad \partial B_R$$

which maps w to $\frac{\partial \widetilde{w}}{\partial \nu}$ where $\widetilde{w}$ solves the exterior Dirichlet problem for the Helmholtz equation $\Delta\widetilde{w} + k_0^2\widetilde{w} = 0$ in $\mathbb{R}^3\backslash B_R$ with the Dirichlet boundary data $\widetilde{w}|_{\partial B_R} = w$. Since B_R is a ball, then, by separating variables, we can find a solution to the exterior Dirichlet problem outside B_R in the form of a series expansion involving Hankel functions. Based on this result the following important properties of the Dirichlet to Neumann operator can be established (see [8, p 116–117] or [3, theorem 5.20] for details).

Lemma 2.4. *The Dirichlet to Neumann operator T is a bounded linear operator from $H^{\frac{1}{2}}(\partial B_R)$ to $H^{-\frac{1}{2}}(\partial B_R)$. Furthermore, there exists a bounded operator $T_0 : H^{\frac{1}{2}}(\partial B_R) \to H^{-\frac{1}{2}}(\partial B_R)$ satisfying that*

$$-\int_{\partial B_R} T_0 w\overline{w}\,\mathrm{d}s \geqslant C\|w\|^2_{H^{\frac{1}{2}}(\partial B_R)}$$

for some constant $C > 0$ such that $T - T_0 : H^{\frac{1}{2}}(\partial B_R) \to H^{-\frac{1}{2}}(\partial B_R)$ is compact.

We now reformulate the problem (2.1)–(2.6) as follows: given $h \in L^2(\Omega_1)$, $g \in H^{-\frac{1}{2}}(S_0)$, $f \in H^{\frac{1}{2}}(S_{1,D})$ and $p \in H^{-\frac{1}{2}}(S_{1,I})$, find $u \in H^1(\Omega_1) \cap H^1(B_R\backslash\overline{\Omega})$ satisfying (2.1)–(2.5) and the equation

$$\frac{\partial u}{\partial \nu} = Tu \qquad \text{on} \quad \partial B_R. \quad (2.8)$$

In exactly the same way as in the proof of lemma 5.22 in [3] one can show that a solution u to the problem (2.1)–(2.5) and (2.8) can be extended to a solution to the scattering problem (2.1)–(2.6) and conversely, for a solution u to the scattering problem (2.1)–(2.6), u, restricted to $B_R\backslash\overline{\Omega}_2$, solves the problem (2.1)–(2.5) and (2.8). Therefore, by theorem 2.3, the problem (2.1)–(2.5) and (2.8) has at most one solution. We now have the following result on the well posedness of the problem (2.1)–(2.5) and (2.8).

Theorem 2.5. *Let* $h \in L^2(\Omega_1)$, $g \in H^{-\frac{1}{2}}(S_0)$, $f \in H^{\frac{1}{2}}(S_{1,D})$ *and* $p \in H^{-\frac{1}{2}}(S_{1,I})$. *Then the problem (2.1)–(2.5) and (2.8) has a unique solution* $u \in H^1(\Omega_1) \cap H^1(B_R\backslash\overline{\Omega})$ *satisfying that*

$$\|u\|_{H^1(\Omega_1)} + \|u\|_{H^1(B_R\backslash\overline{\Omega})} \leqslant C\big(\|f\|_{H^{\frac{1}{2}}(S_{1,D})} + \|p\|_{H^{-\frac{1}{2}}(S_{1,I})} + \|h\|_{L^2(\Omega_1)} + \|g\|_{H^{-\frac{1}{2}}(S_0)}\big) \tag{2.9}$$

with a positive constant C *independent of* h, g, p *and* f.

Proof. Let $\widetilde{f} \in H^{\frac{1}{2}}(S_1)$ be the extension of the Dirichlet data $f \in H^{\frac{1}{2}}(S_{1,D})$ satisfying that $\|\widetilde{f}\|_{H^{\frac{1}{2}}(S_1)} \leqslant C\|f\|_{H^{\frac{1}{2}}(S_{1,D})}$ with C independent of f and let $u_0 \in H^1(B_R\backslash\Omega_2)$ be such that $u_0 = \widetilde{f}$ on S_1, $u_0 = 0$ on ∂B_R and $\|u_0\|_{H^1(B_R\backslash\Omega_2)} \leqslant C\|\widetilde{f}\|_{H^{\frac{1}{2}}(S_1)}$ (this is possible in view of theorem 3.37 in [19]). Then for every solution u to the problem (2.1)–(2.5) and (2.8), $w = u - u_0$ is in the Sobolev space $H_0^1(B_R\backslash\overline{\Omega}_2, S_{1,D})$ defined by

$$H_0^1(B_R\backslash\overline{\Omega}_2, S_{1,D}) := \{w \in H^1(B_R\backslash\overline{\Omega}_2) : w = 0 \text{ on } S_{1,D}\}.$$

Multiplying equations (2.1) and (2.2) by a test function $\varphi \in H_0^1(B_R\backslash\overline{\Omega}_2, S_{1,D})$, integrating by parts and using the boundary conditions on S_1, S_0, ∂B_R, we obtain the following variational formulation for the problem (2.1)–(2.5) and (2.8): find $w \in H_0^1(B_R\backslash\overline{\Omega}_2, S_{1,D})$ such that

$$a(w, \varphi) = -a(u_0, \varphi) + b(\varphi) \qquad \forall \varphi \in H_0^1(B_R\backslash\overline{\Omega}_2, S_{1,D}) \tag{2.10}$$

where, for $v, \varphi \in H_0^1(B_R\backslash\overline{\Omega}_2, S_{1,D})$,

$$a(v, \varphi) = \lambda_0 \int_{\Omega_1} \big(\nabla v \cdot \nabla\overline{\varphi} - k_1^2 v\overline{\varphi}\big)\,\mathrm{d}x + \int_{B_R\backslash\overline{\Omega}} \big(\nabla v \cdot \nabla\overline{\varphi} - k_0^2 v\overline{\varphi}\big)\,\mathrm{d}x$$
$$\qquad - \int_{\partial B_R} T v\overline{\varphi}\,\mathrm{d}s - \mathrm{i}\lambda_0 \int_{S_{1,I}} \lambda v\overline{\varphi}\,\mathrm{d}s,$$

$$b(\varphi) = -\lambda_0 \int_{\Omega_1} h\overline{\varphi}\,\mathrm{d}x - \int_{S_0} g\overline{\varphi}\,\mathrm{d}s - \lambda_0 \int_{S_{1,I}} p\overline{\varphi}\,\mathrm{d}s.$$

We write $a = a_1 + a_2$ with

$$a_1(v, \varphi) = \lambda_0 \int_{\Omega_1} (\nabla v \cdot \nabla\overline{\varphi} + v\overline{\varphi})\,\mathrm{d}x + \int_{B_R\backslash\overline{\Omega}} (\nabla v \cdot \nabla\overline{\varphi} + v\overline{\varphi})\,\mathrm{d}x$$
$$\qquad - \int_{\partial B_R} T_0 v\overline{\varphi}\,\mathrm{d}s - \mathrm{i}\lambda_0 \int_{S_{1,I}} \lambda v\overline{\varphi}\,\mathrm{d}s$$

and

$$a_2(v, \varphi) = -\lambda_0\big(1 + k_1^2\big) \int_{\Omega_1} v\overline{\varphi}\,\mathrm{d}x - \big(1 + k_0^2\big) \int_{B_R\backslash\overline{\Omega}} v\overline{\varphi}\,\mathrm{d}x - \int_{\partial B_R} (T - T_0)v\overline{\varphi}\,\mathrm{d}s,$$

where T_0 is the operator defined in lemma 2.4. By the boundedness of T_0 and the trace theorem, a_1 is bounded. On the other hand, for all $w \in H_0^1(B_R\backslash\overline{\Omega}_2, S_{1,D})$,

$$\begin{aligned}\Re a_1(w, w) &\geqslant \min(\lambda_0, 1)\|w\|^2_{H_0^1(B_R\backslash\overline{\Omega}_2, S_{1,D})} - \int_{\partial B_R} T_0 w\overline{w}\,\mathrm{d}s \\ &\geqslant \min(\lambda_0, 1)\|w\|^2_{H_0^1(B_R\backslash\overline{\Omega}_2, S_{1,D})} + C\|w\|^2_{H^{\frac{1}{2}}(\partial B_R)} \\ &\geqslant \min(\lambda_0, 1)\|w\|^2_{H_0^1(B_R\backslash\overline{\Omega}_2, S_{1,D})},\end{aligned}$$

that is, a_1 is strictly coercive. By the compactness of $T - T_0$ and Rellich's selection theorem (that is, the compact imbedding of $H_0^1(B_R\backslash\overline{\Omega}_2, S_{1,D})$ into $L^2(B_R\backslash\Omega_2)$), it follows that a_2 is compact. The Lax–Milgram theorem implies that there exists a bijective bounded linear operator $A : H_0^1(B_R\backslash\overline{\Omega}_2, S_{1,D}) \to H_0^1(B_R\backslash\overline{\Omega}_2, S_{1,D})$ satisfying that

$$a_1(w, \varphi) = (Aw, \varphi) \qquad \text{for all} \quad \varphi \in H_0^1(B_R\backslash\overline{\Omega}_2, S_{1,D}).$$

By the Riesz representation theorem, there exists a bounded linear operator $B: H_0^1(B_R\backslash\overline{\Omega}_2, S_{1,D}) \to H_0^1(B_R\backslash\overline{\Omega}_2, S_{1,D})$ such that

$$a_2(w, \varphi) = (Bw, \varphi) \qquad \text{for all} \quad \varphi \in H_0^1(B_R\backslash\overline{\Omega}_2, S_{1,D}).$$

By the Riesz representation theorem again, one can find $v \in H_0^1(B_R\backslash\overline{\Omega}_2, S_{1,D})$ such that

$$F(\varphi) := -a(u_0, \varphi) + b(\varphi) = (v, \varphi) \qquad \text{for all} \quad \varphi \in H_0^1(B_R\backslash\overline{\Omega}_2, S_{1,D}).$$

Thus, the variational formulation (2.10) is equivalent to the problem:

$$\text{Find } w \in H_0^1(B_R\backslash\overline{\Omega}_2, S_{1,D}) \qquad \text{such that} \quad Aw + Bw = v, \tag{2.11}$$

where A is bounded and strictly coercive and B is compact. The Riesz–Fredholm theory and the uniqueness result (theorem 2.3) imply that the problem (2.11) or equivalently the problem (2.10) has a unique solution. The estimate (2.9) follows from the fact that, by a duality argument, $\|v\|_{H_0^1(B_R\backslash\overline{\Omega}_2, S_{1,D})} = \|F\|$ is bounded by $\|p\|_{H^{-\frac{1}{2}}(S_{1,I})}$, $\|h\|_{L^2(\Omega_1)}$, $\|g\|_{H^{-\frac{1}{2}}(S_0)}$ and $\|u_0\|_{H^1(B_R\backslash\Omega_2)}$ which in turn is bounded by $\|f\|_{H^{\frac{1}{2}}(S_{1,D})}$. □

3. The inverse scattering problem

To establish the uniqueness result for the inverse scattering problem as mentioned in the introduction, we need a generalization of the mixed reciprocity relation.

The scattering of incident plane waves in a two-layered background medium can be formulated as follows:

$$\Delta u + k_0^2 u = 0 \qquad \text{in} \quad \Omega_0, \tag{3.1}$$

$$\Delta u + k_1^2 u = 0 \qquad \text{in} \quad \Omega_1, \tag{3.2}$$

$$u_+ = u_-, \quad \frac{\partial u_+}{\partial \nu} = \lambda_0 \frac{\partial u_-}{\partial \nu} \qquad \text{on} \quad S_0, \tag{3.3}$$

$$B(u) = 0 \qquad \text{on} \quad S_1, \tag{3.4}$$

$$u = u^i + u^s \qquad \text{in} \quad \Omega_0 \cup \Omega_1, \tag{3.5}$$

$$\lim_{r\to\infty} r^{\frac{n-1}{2}} \left(\frac{\partial u^s}{\partial r} - \mathrm{i}k_0 u^s \right) = 0, \qquad r = |x|, \tag{3.6}$$

with positive wave numbers k_j $(j = 0, 1)$ and the nonnegative constant λ_0.

Recall that the fundamental solution of the Helmholtz equation is given by

$$\Phi(x, y) = \begin{cases} \dfrac{\mathrm{e}^{\mathrm{i}k_0|x-y|}}{4\pi|x-y|} & \text{for} \quad x, y \in \mathbb{R}^3, \qquad x \neq y, \\ \dfrac{\mathrm{i}}{4} H_0^1(k_0|x-y|) & \text{for} \quad x, y \in \mathbb{R}^2, \qquad x \neq y, \end{cases}$$

where H_0^1 is the *Hanker function* of the first kind of order zero.

As incident fields u^i, plane waves and point sources are of special interest. Denote by $u^s(\cdot, d)$ the scattered field for an incident plane wave $u^i(\cdot, d)$ with the incident direction $d \in S$

and by $u^\infty(\cdot, d)$ the corresponding far-field pattern. The scattered field for an incident point source $\Phi(\cdot, z)$ with the source point $z \in \mathbb{R}^n$ is denoted by $u^s(\cdot; z)$ and the corresponding far-field pattern by $\Phi^\infty(\cdot, z)$.

Remark 3.1. The trivial field $u = 0$ is not a solution of the scattering problem in the case when the incident field is a point source $\Phi(\cdot, z)$ with the source point $z \in \mathbb{R}^n$—although it satisfies the radiation condition (since the incident field does). This is because in this case the scattered field $u^s(\cdot; z) = -\Phi(\cdot, z)$ has a singularity at the source point $z \in \mathbb{R}^n$, contradicting to the fact that the scattered field has no singularities. In fact, in this case, equation (3.1) or (3.2) above should hold in $\Omega_0\backslash\{z\}$ or $\Omega_1\backslash\{z\}$ depending on where the source point z is (cf (3.17) in lemma 3.5 below).

For the mixed reciprocity relation we need the constant

$$\gamma_n = \begin{cases} \dfrac{1}{4\pi}, & n = 3, \\ \dfrac{e^{i\pi/4}}{\sqrt{8k_0\pi}}, & n = 2 \end{cases}$$

depending on the dimension n.

Lemma 3.2. *(Mixed reciprocity relation.) For the scattering of plane waves $u^i(\cdot, d)$ with $d \in S$ and point sources $\Phi(\cdot, z)$ from an obstacle Ω_2 we have*

$$\Phi^\infty(\widehat{x}, z) = \begin{cases} \gamma_n u^s(z, -\widehat{x}), & z \in \Omega_0, \widehat{x} \in S, \\ \lambda_0\gamma_n u^s(z, -\widehat{x}) + (\lambda_0 - 1)\gamma_n u^i(z, -\widehat{x}), & z \in \Omega_1, \widehat{x} \in S. \end{cases}$$

Remark 3.3. The proof for the case of an obstacle in a homogeneous medium (i.e. the case $\Omega = \Omega_2$) can be found in [17] or [21].

Proof. By Green's second theorem and the Sommerfeld radiation condition we have that

$$\int_{S_0} \left(u_+^s(y; z)\frac{\partial u_+^s(y, d)}{\partial \nu(y)} - u_+^s(y, d)\frac{\partial u_+^s(y; z)}{\partial \nu(y)} \right) ds(y) = 0 \tag{3.7}$$

for $z \in \Omega_0 \cup \Omega_1$ and $d \in S$. Using theorem 2.5 in [8], we obtain the representation

$$\Phi^\infty(\widehat{x}, z) = \gamma_n \int_{S_0} \left(u_+^s(y; z)\frac{\partial e^{-ik_0\widehat{x}\cdot y}}{\partial \nu(y)} - e^{-ik_0\widehat{x}\cdot y}\frac{\partial u_+^s(y; z)}{\partial \nu(y)} \right) ds(y) \tag{3.8}$$

for $z \in \Omega_0 \cup \Omega_1$ and $\widehat{x} \in S$.

We first consider the case $z \in \Omega_0$. Adding equation (3.8) to equation (3.7) multiplied by γ_n and with d replaced by $-\widehat{x}$ to equation (3.8) we obtain, with the help of the boundary condition on S_0 and Green's second theorem, that for $z \in \Omega_0$, $\widehat{x} \in S$,

$$\begin{aligned} \Phi^\infty(\widehat{x}, z) &= \gamma_n \int_{S_0} \left(u_+^s(y; z)\frac{\partial u_+(y, -\widehat{x})}{\partial \nu(y)} - u_+(y, -\widehat{x})\frac{\partial u_+^s(y; z)}{\partial \nu(y)} \right) ds(y) \\ &= \lambda_0\gamma_n \int_{S_1} \left(u_+^s(y; z)\frac{\partial u_+(y, -\widehat{x})}{\partial \nu(y)} - u_+(y, -\widehat{x})\frac{\partial u_+^s(y; z)}{\partial \nu(y)} \right) ds(y) \\ &\quad + \lambda_0\gamma_n \int_{\Omega_1} \left(\left(k_1^2 - k_0^2\right)\Phi(y, z)u(y, -\widehat{x}) \right) dy \\ &\quad + (1 - \lambda_0)\gamma_n \int_{S_0} \left(u_-(y, -\widehat{x})\frac{\partial \Phi(y, z)}{\partial \nu(y)} \right) ds(y). \end{aligned} \tag{3.9}$$

Green's second theorem gives that for $z \in \Omega_0$ and $\widehat{x} \in S$

$$\gamma_n \int_{S_0} \left(u^i(y, -\widehat{x}) \frac{\partial \Phi(z, y)}{\partial \nu(y)} - \Phi(z, y) \frac{\partial u^i(y, -\widehat{x})}{\partial \nu(y)} \right) \mathrm{d}s(y) = 0. \tag{3.10}$$

Using Green's formula (see theorem 2.4 in [8]) we have

$$\gamma_n u^s(z, -\widehat{x}) = \gamma_n \int_{S_0} \left(u_+^s(y, -\widehat{x}) \frac{\partial \Phi(z, y)}{\partial \nu(y)} - \Phi(z, y) \frac{\partial u_+^s(y, -\widehat{x})}{\partial \nu(y)} \right) \mathrm{d}s(y) \tag{3.11}$$

for $z \in \Omega_0$ and $\widehat{x} \in S$. Adding (3.10) to equation (3.11) we deduce, with the help of the boundary condition on S_0 and Green's second theorem, that

$$\begin{aligned}
\gamma_n u^s(z, -\widehat{x}) &= \gamma_n \int_{S_0} \left(u_+(y, -\widehat{x}) \frac{\partial \Phi(z, y)}{\partial \nu(y)} - \Phi(z, y) \frac{\partial u_+(y, -\widehat{x})}{\partial \nu(y)} \right) \mathrm{d}s(y) \\
&= \lambda_0 \gamma_n \int_{S_1} \left(u_+(y, -\widehat{x}) \frac{\partial \Phi(z, y)}{\partial \nu(y)} - \Phi(z, y) \frac{\partial u_+(y, -\widehat{x})}{\partial \nu(y)} \right) \mathrm{d}s(y) \\
&\quad + \lambda_0 \gamma_n \int_{\Omega_1} \left(\left(k_1^2 - k_0^2\right) \Phi(y, z) u(y, -\widehat{x}) \right) \mathrm{d}y \\
&\quad + (1 - \lambda_0) \gamma_n \int_{S_0} \left(u_-(y, -\widehat{x}) \frac{\partial \Phi(y, z)}{\partial \nu(y)} \right) \mathrm{d}s(y)
\end{aligned} \tag{3.12}$$

for $z \in \Omega_0$ and $\widehat{x} \in S$. It is easy to see that the last two terms on the right-hand side (rhs) of (3.9) and (3.12) coincide. By the boundary condition on S_1, the first terms on the rhs of (3.9) and (3.12) coincide, so $\Phi^\infty(\widehat{x}, z) = \gamma_n u^s(z, -\widehat{x})$ for all $z \in \Omega_0$, $\widehat{x} \in S$.

We now consider the case $z \in \Omega_1$. Adding equation (3.8) to equation (3.7) multiplied by γ_n and with d replaced by $-\widehat{x}$ to equation (3.8) and using the boundary condition on S_0 and Green's second theorem yield that

$$\Phi^\infty(\widehat{x}, z) = \gamma_n \int_{S_0} \left(u_+^s(y; z) \frac{\partial u_+(y, -\widehat{x})}{\partial \nu(y)} - u_+(y, -\widehat{x}) \frac{\partial u_+^s(y; z)}{\partial \nu(y)} \right) \mathrm{d}s(y).$$

We circumscribe the point $z \in \Omega_1$ with a sphere $\Omega(z; \epsilon) := \{y \in \mathbb{R}^n : |y - z| = \epsilon\}$ contained in Ω_1. Applying Green's second theorem in the domain $\Omega_{1,\epsilon} := \{y \in \Omega_1 : |y - z| > \epsilon\}$ and taking into account the boundary condition on S_0 we obtain that

$$\begin{aligned}
\Phi^\infty(\widehat{x}, z) &= \lambda_0 \gamma_n \int_{S_1} \left(u_+^s(y; z) \frac{\partial u_+(y, -\widehat{x})}{\partial \nu(y)} - u_+(y, -\widehat{x}) \frac{\partial u_+^s(y; z)}{\partial \nu(y)} \right) \mathrm{d}s(y) \\
&\quad + \lambda_0 \gamma_n \int_{\Omega(z;\epsilon)} \left(u^s(y; z) \frac{\partial u(y, -\widehat{x})}{\partial \nu(y)} - u(y, -\widehat{x}) \frac{\partial u^s(y; z)}{\partial \nu(y)} \right) \mathrm{d}s(y) \\
&\quad + \lambda_0 \gamma_n \int_{\Omega_{1,\epsilon}} \left(\left(k_1^2 - k_0^2\right) \Phi(y, z) u(y, -\widehat{x}) \right) \mathrm{d}y \\
&\quad + (1 - \lambda_0) \gamma_n \int_{S_0} \left(u_-(y, -\widehat{x}) \frac{\partial \Phi(y, z)}{\partial \nu(y)} \right) \mathrm{d}s(y)
\end{aligned} \tag{3.13}$$

for $z \in \Omega_1$ and $\widehat{x} \in S$. By the well posedness of the direct problem and the interior elliptic regularity (see section 6.3.1 in [12]), $u(\cdot, -\widehat{x}) \in C^\infty(\Omega_1)$ and $u^s(\cdot; z) \in H^2(V)$ for any compact subset V of Ω_1. Thus, there is a sequence ϵ_j such that $\epsilon_j \to 0$ and

$$\int_{\Omega(z;\epsilon_j)} \left(|u^s(y; z)|^2 + |\nabla u^s(y; z)|^2 \right) \mathrm{d}s(y) \to 0$$

as $j \to \infty$. This together with the Cauchy–Schwarz inequality implies that the integral on $\Omega(z; \epsilon)$ with $\epsilon = \epsilon_j$ tends to 0 as $j \to \infty$. By passing to the limit $j \to \infty$ in (3.13) with

$\epsilon = \epsilon_j$ we have

$$\Phi^\infty(\widehat{x}, z) = \lambda_0\gamma_n \int_{S_1} \left(u_+^s(y;z)\frac{\partial u_+(y,-\widehat{x})}{\partial\nu(y)} - u_+(y,-\widehat{x})\frac{\partial u_+^s(y;z)}{\partial\nu(y)}\right) \mathrm{d}s(y)$$
$$+\lambda_0\gamma_n \int_{\Omega_1} \left(\left(k_1^2 - k_0^2\right)\Phi(y,z)u(y,-\widehat{x})\right) \mathrm{d}y$$
$$+(1-\lambda_0)\gamma_n \int_{S_0} \left(u_-(y,-\widehat{x})\frac{\partial\Phi(y,z)}{\partial\nu(y)}\right) \mathrm{d}s(y). \tag{3.14}$$

The volume integral exists as an improper integral since its integrand is weakly singular.

On the other hand, with the help of Green's formula (see theorem 2.1 in [8]), we have

$$u(z,-\widehat{x}) = \int_{S_0} \left(\Phi(z,y)\frac{\partial u_-(y,-\widehat{x})}{\partial\nu(y)} - u_-(y,-\widehat{x})\frac{\partial\Phi(z,y)}{\partial\nu(y)}\right) \mathrm{d}s(y)$$
$$-\int_{S_1} \left(\Phi(z,y)\frac{\partial u_+(y,-\widehat{x})}{\partial\nu(y)} - u_+(y,-\widehat{x})\frac{\partial\Phi(z,y)}{\partial\nu(y)}\right) \mathrm{d}s(y)$$
$$+\int_{\Omega_1} \left(\left(k_1^2 - k_0^2\right)\Phi(z,y)u(y,-\widehat{x})\right) \mathrm{d}y. \tag{3.15}$$

It follows from (3.14) and (3.15) together with the boundary conditions on S_0 and S_1 and Green's second theorem that

$$\Phi^\infty(\widehat{x}, z) - \lambda_0\gamma_n u(z,-\widehat{x}) = \gamma_n \int_{S_0} \left(\Phi(z,y)\frac{\partial u_+(y,-\widehat{x})}{\partial\nu(y)} - u_+(y,-\widehat{x})\frac{\partial\Phi(z,y)}{\partial\nu(y)}\right) \mathrm{d}s(y)$$
$$= \gamma_n \int_{S_0} \left(\Phi(z,y)\frac{\partial u^i(y,-\widehat{x})}{\partial\nu(y)} - u^i(y,-\widehat{x})\frac{\partial\Phi(z,y)}{\partial\nu(y)}\right) \mathrm{d}s(y)$$
$$+\gamma_n \int_{S_0} \left(\Phi(z,y)\frac{\partial u_+^s(y,-\widehat{x})}{\partial\nu(y)} - u_+^s(y,-\widehat{x})\frac{\partial\Phi(z,y)}{\partial\nu(y)}\right) \mathrm{d}s(y)$$
$$= -\gamma_n u^i(z,-\widehat{x}).$$

This implies that $\Phi^\infty(\widehat{x}, z) = \lambda_0\gamma_n u^s(z,-\widehat{x}) + (\lambda_0 - 1)\gamma_n u^i(z,-\widehat{x})$ for all $z \in \Omega_1$, $\widehat{x} \in S$. □

Remark 3.4. For the case of an $(N+1)$-layered background medium, it is easy to deduce that

$$\Phi^\infty(\widehat{x}, z) = \begin{cases} \gamma_n u^s(z,-\widehat{x}), & z \in \Omega_0, \widehat{x} \in S, \\ \lambda_0\lambda_1\cdots\lambda_{N-1}\gamma_n u^s(z,-\widehat{x}) + (\lambda_0\lambda_1\cdots\lambda_{N-1} - 1)\gamma_n u^i(z,-\widehat{x}), & z \in \Omega_N, \widehat{x} \in S. \end{cases}$$

Lemma 3.5. *For $n = 2$ or 3 and for Ω_2, $\widetilde{\Omega}_2 \subset \Omega$ let G be the unbounded component of $\mathbb{R}^n\backslash\overline{(\Omega_2 \cup \widetilde{\Omega}_2)}$ and let $u^\infty(\hat{x}, d) = \widetilde{u}^\infty(\hat{x}, d)$ for all $\hat{x}, d \in S$ with $\widetilde{u}^\infty(\hat{x}, d)$ being the far-field pattern of the scattered field $\widetilde{u}^s(x, d)$ corresponding to the obstacle $\widetilde{\Omega}_2$ and the same incident plane wave $u^i(x, d)$. For $z_1 \in \Omega \cap G$ let $u^s = u^s(x; z_1)$ be the unique solution of the problem*

$$\Delta u^s + k_0^2 u^s = 0 \qquad \text{in} \quad \Omega_0, \tag{3.16}$$

$$\Delta u^s + k_1^2 u^s = \left(k_0^2 - k_1^2\right)\Phi(x, z_1) \qquad \text{in} \quad \Omega_1\backslash\{z_1\}, \tag{3.17}$$

$$u_+^s = u_-^s, \qquad \frac{\partial u_+^s}{\partial\nu} - \lambda_0\frac{\partial u_-^s}{\partial\nu} = (\lambda_0 - 1)\frac{\partial\Phi(x, z_1)}{\partial\nu} \qquad \text{on} \quad S_0, \tag{3.18}$$

$$B(u^s) = -B(\Phi(x, z_1)) \qquad \text{on} \quad S_1, \tag{3.19}$$

$$\lim_{r\to\infty} r^{\frac{n-1}{2}}\left(\frac{\partial u^s}{\partial r} - ik_0 u^s\right) = 0. \tag{3.20}$$

Assume that $\widetilde{u}^s = \widetilde{u}^s(x; z_1)$ *is the unique solution of the problem (3.16)–(3.20) with* Ω_2 *replaced by* $\widetilde{\Omega_2}$ *and* Ω_1 *replaced by* $\widetilde{\Omega_1} := \Omega\backslash\overline{\widetilde{\Omega_2}}$. *Then we have*

$$u^s(x; z_1) = \widetilde{u}^s(x; z_1), \qquad x \in \overline{G}. \tag{3.21}$$

Remark 3.6. By theorem 2.5, the problem (3.16)–(3.20) has a unique solution.

Proof. By Rellich's lemma [8], the assumption $u^\infty(\hat{x}, d) = \widetilde{u}^\infty(\hat{x}, d)$ for all $\hat{x},\ d \in S$ implies that

$$u^s(x, d) = \widetilde{u}^s(x, d), \qquad \frac{\partial u^s(x, d)}{\partial \nu} = \frac{\partial \widetilde{u}^s(x, d)}{\partial \nu}, \qquad x \in S_0, \qquad d \in S.$$

Using Holmgren's uniqueness theorem [16], we obtain that $u^s(z_1, d) = \widetilde{u}^s(z_1, d)$ for $z_1 \in \Omega \cap G,\ d \in S$. For the far-field pattern of incident point sources we have by lemma 3.2 that

$$\Phi^\infty(d, z_1) = \widetilde{\Phi}^\infty(d, z_1), \qquad z_1 \in \Omega \cap G, \qquad d \in S.$$

Thus, Rellich's lemma [8] gives

$$u^s(x; z_1) = \widetilde{u}^s(x; z_1), \qquad \frac{\partial u^s(x; z_1)}{\partial \nu} = \frac{\partial \widetilde{u}^s(x; z_1)}{\partial \nu}, \qquad x \in S_0, \qquad z_1 \in \Omega \cap G.$$

From Holmgren's uniqueness theorem [16] it is derived that

$$u^s(x; z_1) = \widetilde{u}^s(x; z_1), \qquad x \in \Omega \cap G, \qquad z_1 \in \Omega \cap G,$$

which yields the desired result (3.21). □

Using the generalized mixed reciprocity relation (lemma 3.2) and lemma 3.5, the following uniqueness result can be proved following Colton and Kress [9] and Kress [18].

Theorem 3.7. *Assume that* Ω_2 *and* $\widetilde{\Omega}_2$ *are two obstacles with boundary conditions* B *and* $\widetilde{B}$, *respectively, for the same piecewise homogeneous background medium. If the far-field patterns of the scattered fields for the same incident plane wave* $u^i(x) = e^{ik_0 x\cdot d}$ *coincide at a fixed frequency for all incident direction* $d \in S$ *and observation direction* $\widehat{x} \in S$, *then* $\Omega_2 = \widetilde{\Omega}_2$, $B = \widetilde{B}$.

Proof. Let G be the unbounded component of $\mathbb{R}^n\backslash\overline{(\Omega_2 \cup \widetilde{\Omega}_2)}$. Assume that $\Omega_2 \neq \widetilde{\Omega}_2$. Then, without loss of generality, we may assume that there exists $z_0 \in \partial\Omega_2 \cap (\mathbb{R}^n\backslash\overline{\widetilde{\Omega}_2})$. Choose $h > 0$ such that the sequence

$$z_j := z_0 + \frac{h}{j}\nu(z_0), \qquad j = 1, 2, \ldots,$$

is contained in $\Omega \cap G$, where $\nu(z_0)$ is the outward normal to $\partial\Omega_2$ at z_0. Consider the solution to the problem (3.16)–(3.20) with z_1 replaced by z_j. By lemma 3.5 we have that $u^s(x; z_j) = \widetilde{u}^s(x; z_j),\ x \in \overline{G}\backslash\{z_j\}$. Since z_0 has a positive distance from $\overline{\widetilde{\Omega}_2}$, we conclude from the well posedness of the direct scattering problem that there exists $C > 0$ such that

$$|B(\widetilde{u}^s(z_0; z_j))| \leqslant C$$

uniformly for $j \geqslant 1$. On the other hand, by the boundary condition on $\partial\Omega_2$,

$$|B(\widetilde{u}^s(z_0; z_j))| = |B(u^s(z_0; z_j))| = |-B(\Phi(z_0, z_j))| \to \infty$$

as $j \to \infty$. This is a contradiction, which implies that $\Omega_2 = \widetilde{\Omega}_2$.

We now assume that the boundary conditions are different, that is $B \neq \widetilde{B}$. First, for the case of impedance boundary conditions we assume that we have two different continuous impedance functions $\lambda \neq \widetilde{\lambda}$. Then from the conditions

$$\frac{\partial u}{\partial \nu} + \mathrm{i}\lambda u = 0, \qquad \frac{\partial u}{\partial \nu} + \mathrm{i}\widetilde{\lambda} u = 0 \qquad \text{on} \quad S_1,$$

we obtain that

$$(\lambda - \widetilde{\lambda})u = 0 \qquad \text{on} \quad S_1.$$

Therefore, on the open set $\Gamma := \{x \in S_1 : \lambda \neq \widetilde{\lambda}\}$ we have that $\frac{\partial u}{\partial \nu} = u = 0$. Then Holmgren's uniqueness theorem [16] implies that the total field $u = u^i + u^s = 0$. The scattered field u^s tends to zero uniformly at infinity while the incident plane wave has modulus 1 everywhere. Thus, the modulus of the total field tends to 1. This leads to a contradiction, giving that $\lambda = \widetilde{\lambda}$. The case for other boundary conditions can be dealt with similarly. □

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