Finite Element Method to Fluid–Solid Interaction Problems with Unbounded Periodic Interfaces

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Consider a time-harmonic acoustic plane wave incident onto a doubly periodic (biperiodic) surface from above. The medium above the surface is supposed to be filled with a homogeneous compressible inviscid fluid of constant mass density, whereas the region below is occupied by an isotropic and linearly elastic solid body characterized by its Lamé constants. This article is concerned with a variational approach to the fluid–solid interaction problems with unbounded biperiodic Lipschitz interfaces between the domains of the acoustic and elastic waves. The existence of quasiperiodic solutions in Sobolev spaces is established at arbitrary frequency of incidence, while uniqueness is proved only for small frequencies or for all frequencies excluding a discrete set. A finite element scheme coupled with Dirichlet-to-Neumann mappings is proposed and the convergence analysis is performed. The Dirichlet-to-Neumann mappings are approximated by truncated Rayleigh series expansions. Finally, numerical tests in 2D are presented to confirm the convergence of solutions and the energy balance formula. In particular, the frequency spectrum of normally reflected signals is plotted for water–brass and water–brass–water interfaces. Numer Methods Partial Differential Eq 32: 5–35, 2016

Keywords: convergence analysis; fluid–solid interaction; periodic structure; Helmholtz equation; Lamé system; Rayleigh expansion; variational approach

I. INTRODUCTION

Consider a time-harmonic acoustic plane wave incident onto an unbounded doubly periodic (biperiodic) surface from above (cf. Fig. 1). The medium above the surface is supposed to be filled with a homogeneous compressible inviscid fluid with a constant mass density, whereas the region below is occupied by an isotropic and linearly elastic solid body characterized by its Lamé constants. Due to the external incident acoustic field, an elastic wave propagating downward is

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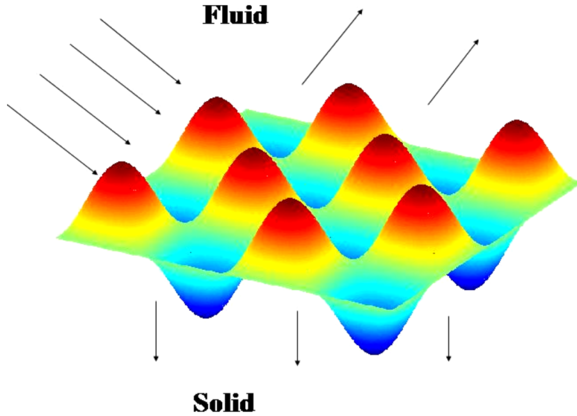


FIG. 1. Scattering of plane waves from an egg-crate shaped biperiodic interface separating the regions of fluid (above) and solid (below) in $\mathbb{R}^3$. [Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com.]

incited inside the solid, while the incident acoustic wave is scattered back into the fluid. This leads to the fluid–solid interaction (FSI) problem with unbounded biperiodic interfaces separating the domains of acoustic and elastic waves.

The FSI problem in periodic structures has many applications in underwater acoustics, sonic, and photonic crystals as well as in the field of ultrasonic nondestructive evaluation (cf. [1–4] and the references therein). In immersion testing, objects are always put in a tank of water to minimize the energy loss of the ultrasound beam transmitting from a transducer into a medium and vice versa. The investigation of surface (Rayleigh) waves generated by periodic interfaces can be important in developing new surface acoustic wave devices and planar actuators [1]. Manifold applications of periodic interfaces, for example, in grain structure, lamination, and fiber reinforcement as well as in the manufacturing of material surfaces, motivate us to rigorously investigate FSI problems in periodic structures. Conversely, fluid–solid interfaces are also important in seismology and exploration geophysics for the acoustical characterization of the first layer of the sea floor. This is because the propagation of Stoneley–Scholte waves, another type of surface waves that exit on a fluid–solid interface (cf. [3] for a detailed discussion), is strongly related to the shear wave velocity of the sedimentary bottom [5]. Note that, so far, a vast literature has come from the engineering and physical communities only. In this work, we develop a general mathematical framework for the FSI problem with a 1D or 2D periodic interface via variational arguments.

Since Lord Rayleigh's original work [6], grating diffraction problems have received much attention. In many publication of physical journals, the diffraction on periodic surfaces is simulated by the *Rayleigh–Fourier* approach, and numerical solutions are obtained by solving the classical grating equations under the so-called *Lippmann* condition [1–3, 7], which requires that the height of the corrugations of the interface is small compared with the wavelength and that the latter is comparable with the periods. From the mathematical point of view, the diffraction of pure acoustic, elastic, or electromagnetic waves has been studied extensively including theoretical analysis and numerical approximation, using integral equation methods (e.g., [8–12]), variational methods (e.g., [13–21]) or the coupling scheme [22, 23]. In particular, the variational approach appears to be well adapted to the analytical and numerical treatment of rather general two-dimensional and three-dimensional periodic diffractive structures involving complex materials and nonsmooth interfaces (cf., e.g., the adaptive finite element method (FEM) [24] and the mortar technique combined with Nitsche's method [19, 20, 25] for diffraction gratings).

The contributions of this work are twofold. First, we establish an equivalent variational formulation in a bounded periodic cell involving two nonlocal transparent boundary operators. Relying on properties of the Dirichlet-to-Neumann (DtN) maps for the Helmholtz and Navier equations, we show the existence of solutions in quasiperiodic Sobolev spaces by establishing the Fredholmness of the operator generated by the corresponding sesquilinear form. Moreover, uniqueness is proved at least for small frequencies or for all frequencies excluding a discrete set. A nonuniqueness example in Lemma 4.3 shows that uniqueness does not hold in general, even if the interface is given by the graph of some smooth biperiodic function. This is in sharp contrast to the result in [21] for the pure Helmholtz equation and that in [26] for the pure Lamé system, where the uniqueness is proved via periodic Rellich's identities for a scattering interface given by the graph of some function. This suggests the possible existence of surface (Rayleigh or evanescent) waves in general settings, and a corresponding search for eigensolutions may help to design new surface wave devices.

Second, based on the variational formulation, a finite element scheme with approximated Dirichlet-to-Neumann mappings in form of truncated Rayleigh series expansions is proposed. The scheme is capable of treating general piecewise smooth Lipschitz interfaces whose height is comparable with the periods and thus relaxes the *Lippmann* condition. The convergence analysis of the proposed algorithm is performed in terms of the mesh size and truncation order of the DtN mappings. Numerical examples in 2D are tested to confirm the convergence analysis and the energy balance formula. In particular, we plot and compare the frequency spectrum of normally reflected signals for the water–brass and water–brass–water interfaces. The frequency dips imply the existence of Wood anomalies and show the ability of periodic interfaces to incite surface waves, which have been verified in many papers of the physical literature [1–3, 7].

In the case of a bounded (nonperiodic) elastic body emerged in a fluid, the scattered acoustic field decays at infinity according to Sommerfeld's radiation condition (cf. the pioneering work [27]), which of course is different from the radiation condition imposed here by the Rayleigh series. Various systems of boundary integral equations and variational formulations were derived and analysed (cf., e.g., [28, 29]). It is well known that uniqueness for a bounded elastic body does not hold at *Jones* frequencies. In this article, we also show the existence of such irregular frequencies in periodic structures (cf. Section IV). We refer to [30] for a FEM-based ultraweak variational formulation for solving the 2D FSI problem with a bounded elastic body.

The article is organized as follows. In Section II, we rigorously formulate the interaction problems with biperiodic Lipschitz interfaces separating the domains of acoustic and elastic waves. In Section III, we propose an equivalent variational formulation in a truncated periodic cell by introducing two nonlocal transparent operators. Section IV is devoted to the solvability of the FSI problem through the variational approach. An energy balance formula will be stated in Section V and the numerical analysis of the FEM is given in Section VI. In the final Sections VII and VIII we introduce the corresponding two-dimensional setting and present numerical tests.

We end up this section by introducing some notation that will be used throughout the article. Denote by $(\cdot)^\top$ the transpose of a vector or a matrix, and by $(\cdot)^*$ the adjoint of an operator. The symbols e_j, $j = 1, 2, 3$ denote the Cartesian unit vectors in $\mathbb{R}^3$. For $a \in \mathbb{C}$, let $|a|$ denote its modulus, and, for $\mathbf{a} \in \mathbb{C}^3$, let $|\mathbf{a}|$ denote its Euclidean norm. The notation $\mathbf{a} \cdot \mathbf{b}$ stands for the bilinear inner product $\sum_{j=1}^{3} a_j b_j$ of $\mathbf{a} = (a_1, a_2, a_3)^\top, \mathbf{b} = (b_1, b_2, b_3)^\top \in \mathbb{C}^3$. For $x = (x_1, x_2, x_3)^\top \in \mathbb{R}^3$, we write $\tilde{x} = (x_1, x_2)^\top$ so that $x = (\tilde{x}^\top, x_3)^\top$.

II. MATHEMATICAL FORMULATIONS

We assume that an acoustic wave is incident onto a biperiodic Lipschitz surface $\Gamma \subset \mathbb{R}^3$ from above. Without loss of generality we suppose that Γ is 2π-periodic in x_1 and x_2, that is, $x = (\tilde{x}^\top, x_3)^\top \in \Gamma$ implies $(\tilde{x}^\top + 2\pi\, n^\top, x_3)^\top \in \Gamma$ for all $n = (n_1, n_2)^\top \in \mathbb{Z}^2$. Denote by Ω^+ the region above Γ, which is filled with a homogeneous compressible inviscid fluid with the constant mass density $\rho_f > 0$. The incident wave is supposed to be a time-harmonic plane wave of the form $v^{\text{in}}(x)\exp(-i\omega t)$ with frequency $\omega > 0$ and speed of sound $c_0 > 0$, where the spatially dependent function v^{in} takes the form

$$v^{\text{in}}(x) = \exp(ik\hat{\theta}\cdot x), \hat{\theta} = (\sin\theta_1\cos\theta_2, \sin\theta_1\sin\theta_2, -\cos\theta_1)^\top \in \mathbb{S}^2 := \{x \in \mathbb{R}^3 : |x| = 1\}. \tag{2.1}$$

In (2.1), the vector $\hat{\theta}$ denotes the incident direction with the incident angles $\theta_1 \in (-\pi/2, \pi/2), \theta_2 \in [0, 2\pi)$, and $k = \omega/c_0$ is the wave number in the fluid. We assume the region below Γ, denoted by Ω^-, is occupied by an isotropic and linearly elastic solid body characterized by the real valued constant mass density $\rho > 0$ and the Lamé constants $\lambda, \mu \in \mathbb{R}$ satisfying $\mu > 0$, $3\lambda + 2\mu > 0$.

Under the hypothesis of small amplitude oscillations both in the solid and the fluid, the direct or forward scattering problem looks for the total acoustic field $v = v^{in} + v^{sc}$ and the transmitted elastic field u generated from a known (prescribed) incident wave v^{in} such that (cf., e.g., [4, 28, 29])

$$\begin{cases} (\Delta + k^2)\, v = 0 & \text{in } \Omega^+, \\ (\Delta^* + \omega^2\rho)\, u = 0 & \text{in } \Omega^-, \quad \Delta^* := \mu\Delta + (\lambda+\mu)\,\text{grad}\,\text{div}, \\ \eta\, u\cdot\nu = \partial_\nu v & \text{on } \Gamma, \quad \eta := \rho_f\omega^2 > 0, \\ Tu = -v\,\nu & \text{on } \Gamma. \end{cases} \tag{2.2}$$

Here, $\nu = (\nu_1, \nu_2, \nu_3)^\top \in \mathbb{S}^2$ denotes the unit normal vector on Γ pointing into Ω^- and $\partial_\nu u := \nu\cdot\text{grad}\, u$. As a convention we shall use the symbol $\partial_j u$ to denote $\partial u/\partial x_j$. In (2.2), Tu stands for the three-dimensional *stress vector* or *traction* having the form

$$Tu = T(\lambda,\mu)u := 2\mu\,\partial_\nu u + \lambda(\text{div}\, u)\,\nu + \mu\,\nu\times\text{curl}\, u \tag{2.3}$$

on Γ. Due to Betti's formula (cf., e.g., [31]), the role of the above stress operator in the Lamé equation is the same as that of the normal derivative in the scalar Helmholtz equation.

Throughout the article, we write $\alpha = (\alpha_1, \alpha_2)^\top := k(\sin\theta_1\cos\theta_2, \sin\theta_1\sin\theta_2)^\top \in \mathbb{R}^2$. Now the incoming wave is α-quasiperiodic in the sense that $v^{\text{in}}(x)\exp(-i\alpha\cdot\tilde{x})$ is 2π-periodic with respect to x_1 and x_2. The geometry is 2π-periodic. So it is natural to restrict our considerations to α-quasiperiodic solutions of our transmission problem, that is, for $w = v$ in Ω^+ and $w = u$ in Ω^- it holds that

$$w(\tilde{x} + 2\pi\, n, x_3) = \exp(2\pi i\alpha\cdot n)\, w(x_1, x_2, x_3) \quad \text{for all } n = (n_1, n_2)^\top \in \mathbb{Z}^2. \tag{2.4}$$

Note that a mathematically rigorous argument for this restriction would require a uniqueness result in nonperiodic weighted Sobolev spaces covering plane waves, which is still open. The Rellich type identities derived in [32] and [33] for the pure Helmholtz and Navier equations do not apply to the FSI problem, even if the interface is given by the graph of a smooth function in two dimensions (cf. the nonuniqueness example in Section IV below).

As the domain $\Omega^{\pm}$ is unbounded in the $\pm x_3$-direction, a radiation condition must be imposed at infinity to ensure well-posedness of the boundary value problem (2.2). Let $\Gamma^+ := \max_{x\in\Gamma}\{x_3\}$ and $\Gamma^- := \min_{x\in\Gamma}\{x_3\}$. Following [21], we require that the scattered acoustic field v^{sc} admits an upward Rayleigh expansion (cf. also [15, 34, 35])

$$v^{sc}(\tilde{x}, x_3) = \sum_{n\in\mathbb{Z}^2} v_n \exp(i\alpha_n \cdot \tilde{x} + i\eta_n x_3), \quad x_3 > \Gamma^+, \tag{2.5}$$

with the Rayleigh coefficients $v_n \in \mathbb{C}$. The parameters $\alpha_n = (\alpha_n^{(1)}, \alpha_n^{(2)})^\top \in \mathbb{R}^2$ and $\eta_n \in \mathbb{C}$ in (2.5) are given by

$$\alpha_n := \alpha + n \in \mathbb{R}^2, \quad \eta_n := \begin{cases} (k^2 - |\alpha_n|^2)^{\frac{1}{2}} & \text{if } |\alpha_n| \le k, \\ i(|\alpha_n|^2 - k^2)^{\frac{1}{2}} & \text{if } |\alpha_n| > k, \end{cases} \quad \text{for } n \in \mathbb{Z}^2. \tag{2.6}$$

To see the corresponding expansion of the elastic field, we decompose it into the compressional and shear parts,

$$u = \frac{1}{i}(\operatorname{grad}\varphi + \operatorname{curl}\psi) \quad \text{with} \quad \varphi := -\frac{i}{k_p^2}\operatorname{div} u\,, \quad \psi := \frac{i}{k_s^2}\operatorname{curl} u\,, \tag{2.7}$$

where the scalar function φ and the vector function ψ satisfy the homogeneous Helmholtz equations $(\Delta + k_p^2)\,\varphi = 0$ and $(\Delta + k_s^2)\,\psi = 0$ in Ω^- with the compressional and shear wave numbers defined as $k_p := \omega\sqrt{\rho/(2\mu+\lambda)}$ and $k_s := \omega\sqrt{\rho/\mu}$. Applying the downward Rayleigh expansion for the scalar Helmholtz equation to φ and the components of ψ, that is, setting $\varphi(\tilde{x}, x_3) = \sum_{n\in\mathbb{Z}^2} \varphi_{p,n} \exp(i\alpha_n \cdot \tilde{x} - i\beta_n x_3)$ and $\psi(\tilde{x}, x_3) = \sum_{n\in\mathbb{Z}^2} \Psi_{s,n} \exp(i\alpha_n \cdot \tilde{x} - i\gamma_n x_3)$ with the orthogonality condition $\Psi_{s,n} \cdot (\alpha_n^\top, -\gamma_n)^\top = 0$, we finally obtain the corresponding expansion of u into downward propagating plane elastic waves

$$u(x) = \sum_{n\in\mathbb{Z}^2} \left\{ A_{p,n} \begin{pmatrix} \alpha_n \\ -\beta_n \end{pmatrix} \exp(i\alpha_n \cdot \tilde{x} - i\beta_n x_3) + \mathbf{A}_{s,n} \exp(i\alpha_n \cdot \tilde{x} - i\gamma_n x_3) \right\}, \quad x_3 < \Gamma^-. \tag{2.8}$$

In (2.8), the Rayleigh coefficients are given as $A_{p,n} := \varphi_{p,n} \in \mathbb{C}$ and $\mathbf{A}_{s,n} := (\alpha_n^\top, -\gamma_n)^\top \times \Psi_{s,n} \in \mathbb{C}^3$. In particular, we have the orthogonality $\mathbf{A}_{s,n} \cdot (\alpha_n^\top, -\gamma_n)^\top = 0$ for all $n \in \mathbb{Z}^2$. The parameters β_n and γ_n occurring in (2.8) are defined analogously to η_n in (2.6) with k replaced by k_p and k_s, respectively. By u_p and u_s we denote the compressional and shear parts of u, respectively, that is, for $x_3 < \Gamma^-$,

$$u_p(x) = \sum_{n\in\mathbb{Z}^2} A_{p,n} \begin{pmatrix} \alpha_n \\ -\beta_n \end{pmatrix} \exp(i\alpha_n \cdot \tilde{x} - i\beta_n x_3), \quad u_s(x) = \sum_{n\in\mathbb{Z}^2} \mathbf{A}_{s,n} \exp(i\alpha_n \cdot \tilde{x} - i\gamma_n x_3).$$

Then it is obvious that $u = u_p + u_s$ with $(\Delta + k_p^2)\,u_p = 0$, $\operatorname{curl} u_p = 0$ as well as $(\Delta + k_s^2)\,u_s = 0$, $\operatorname{div} u_s = 0$ in Ω^-. As η_n, β_n and γ_n are real for at most finitely many indices $n \in \mathbb{Z}^2$, we observe that only the finite number of plane waves in (2.5) corresponding to $|\eta_n| \le k$ and those in (2.8) corresponding to $|\beta_n| \le k_p$ and $|\gamma_n| \le k_s$ propagate into the far field. These plane waves are referred to as the upward and downward outgoing plane waves, respectively. The remaining part consists of evanescent (or surface) waves decaying exponentially as $|x_3| \to +\infty$. Thus, the Rayleigh expansion (2.5) converges uniformly with all derivatives in the upper half-space $\{x : x_3 > b\}$ for any $b > \Gamma^+$, while (2.8) converges in the lower half-space $\{x : x_3 < a\}$ for any $a < \Gamma^-$.

Now, we can formulate our FSI problem as the following boundary value problem, in which the interface Γ is not necessarily the graph of a biperiodic function.

Boundary value problem (BVP): Given a biperiodic Lipschitz surface $\Gamma \subset \mathbb{R}^3$, which is 2π-periodic in x_1 and x_2 and which splits $\mathbb{R}^3$ into an upper and lower half space, and an incident field v^{in} of the form (2.1), find a scalar function $v = v^{\text{in}} + v^{sc} \in H^1_{\text{loc}}(\Omega^+)$ and a vector function $u \in H^1_{\text{loc}}(\Omega^-)^3$ satisfying the equations and transmission conditions in (2.2), the quasi-periodic boundary condition (2.4) and the radiation conditions, that is, that u and v admit Rayleigh expansions like in (2.5) and (2.8), respectively.

III. VARIATIONAL FORMULATION IN A TRUNCATED DOMAIN

In this section, we propose a variational formulation equivalent to BVP, based on the approach of [21, 35] and [18, 26] for the scattering of acoustic and elastic waves by diffraction gratings. Thanks to the periodicity of the unbounded domains $\Omega^\pm$, we can restrict our discussions to one single periodic cell $\{x : 0 < x_j < 2\pi, j = 1, 2\}$ such that after a truncation in the x_3-direction the compact imbedding of Sobolev spaces can be applied. This together with Friedrich's inequality for the Helmholtz equation and Korn's inequality for the Navier equation, enables us to justify the strong ellipticity of the sesquilinear form generated by the variational formulation.

We begin with introducing artificial boundaries $\Gamma_b^\pm := \{(x_1, x_2, \pm b) : 0 \le x_1, x_2 \le 2\pi\}$ with the x_3-coordinates $b \ge \Gamma^+$ and $-b \le \Gamma^-$ and define the two adjacent bounded domains $\Omega_b^+ := \{x \in \Omega^+ : 0 < x_1, x_2 < 2\pi,\ x_3 < +b\}$ and $\Omega_b^- := \{x \in \Omega^- : 0 < x_1, x_2 < 2\pi,\ x_3 > -b\}$. For simplicity, we still use Γ to denote one period of the unbounded periodic grating surface Γ (cf. Fig. 2). As Γ is a Lipschitz surface, we may restrict our considerations to the case that $\Omega_b^\pm$ are bounded Lipschitz domains in $\mathbb{R}^3$. Let $H^1_\alpha(\Omega_b^\pm)$ denote the Sobolev space of scalar functions on $\Omega_b^\pm$, which are α-quasiperiodic with respect to x_1 and x_2.

Introduce $V_t^+ := H^t_\alpha(\Omega_b^+)$, $V_t^- := H^t_\alpha(\Omega_b^-)^3$ and the family of product spaces $V_t = V_t(\alpha) := V_t^+ \times V_t^-$, equipped with the norm in the usual product space of $H^t(\Omega_b^+) \times H^t(\Omega_b^-)^3$. In particular, V_1 is the energy space. Using the transmission conditions in (2.2), it follows from Green's and Betti's formulas that, for $(\varphi, \psi) \in V_1$,

$$-\int_{\Omega_b^+} (\Delta + k^2) v\, \overline{\varphi}\, dx = \int_{\Omega_b^+} \left[\operatorname{grad} v \cdot \operatorname{grad} \overline{\varphi} - k^2 v \overline{\varphi}\right] dx - \eta \int_\Gamma u \cdot \nu \overline{\varphi}\, ds - \int_{\Gamma_b^+} \partial_\nu v\, \overline{\varphi}\, ds\,,$$

$$-\int_{\Omega_b^-} (\Delta^* + \omega^2 \rho) u \cdot \overline{\psi}\, dx = \int_{\Omega_b^-} [\mathcal{E}(u, \overline{\psi}) - \omega^2 \rho u \cdot \overline{\psi}\,] dx - \int_\Gamma v\, \nu \cdot \overline{\psi} ds - \int_{\Gamma_b^-} T u \cdot \overline{\psi}\, ds\,, \tag{3.1}$$

where the bar indicates complex conjugation, T is the stress vector defined by (2.3) and

$$\mathcal{E}(u, \overline{\psi}) := 2\mu \sum_{i,j=1}^{3} \partial_i u_j\, \partial_i \overline{\psi}_j + \lambda\, (\operatorname{div} u)(\operatorname{div} \overline{\psi}) - \mu \operatorname{curl} u \cdot \operatorname{curl} \overline{\psi}. \tag{3.2}$$

Now we introduce the DtN maps $\mathcal{T}^\pm$ on the artificial boundaries $\Gamma_b^\pm$.

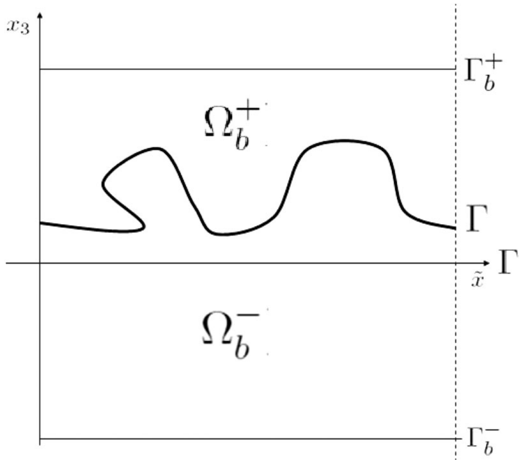


FIG. 2. The geometry settings in one periodic cell. Here $\Gamma_b^\pm := \{(x_1, x_2, \pm b)^\top : 0 < x_1, x_2 < 2\pi\}$ and $\Omega_b^\pm$ denotes the domain between $\Gamma_b^\pm$ and Γ.

Definition 3.1. *For any $w \in H_\alpha^s(\Gamma_b^+)$, $s > 0$, the DtN operator $\mathcal{T}^+$ applied to w is defined as $\partial_\nu v^{sc}|_{\Gamma_b^+}$, where v^{sc} is the unique α-quasiperiodic solution of the homogeneous Helmholtz equation in $x_3 > b$, which satisfies the upward radiation condition (2.5) and has the Dirichlet boundary value $v^{sc} = w$ on Γ_b^+.*

Analogously, for any $w \in H_\alpha^s(\Gamma_b^-)^3$, $s > 0$, the DtN operator $\mathcal{T}^-$ applied to w is defined as $Tu|_{\Gamma_b^-}$, where u is the unique α-quasiperiodic solution of the homogeneous Navier equation in $x_3 < -b$, which satisfies the downward radiation condition (2.8) and takes the Dirichlet boundary value $u = w$ on Γ_b^-.

In this article, we use the following equivalent norm on $H_\alpha^s(\mathbb{R}^2)$:

$$||w||_{H_\alpha^s(\mathbb{R}^2)} := \left(\sum_{n \in \mathbb{Z}^2} (1 + |n|)^{2s} \, |\hat{w}_n|^2 \right)^{1/2}, \quad s \in \mathbb{R}, \tag{3.3}$$

where $\hat{w}_n \in \mathbb{C}$ are the Fourier coefficients of the function $\exp(-i\alpha \cdot \tilde{x})\, w(\tilde{x})$, that is, $w(\tilde{x}) = \sum_{n \in \mathbb{Z}^2} \hat{w}_n \exp(i\alpha_n \cdot \tilde{x})$. Letting $w \in H_\alpha^s(\Gamma_b^+)$, $s > 0$ be given as above, one can readily derive an explicit expression of the DtN map $\mathcal{T}^+$ from its definition as follows:

$$(\mathcal{T}^+ w)(\tilde{x}) = \sum_{n \in \mathbb{Z}^2} i\eta_n \, \hat{w}_n \exp(i\alpha_n \cdot \tilde{x}), \tag{3.4}$$

where η_n is defined as in (2.6). Analogously, for $w \in H_\alpha^s(\Gamma_b^-)^3$, $s > 0$ with Fourier coefficients $\hat{w}_n \in \mathbb{C}^3$, we can represent the DtN map $\mathcal{T}^-$ as

$$(\mathcal{T}^- w)(\tilde{x}) = \sum_{n \in \mathbb{Z}^2} i W_n \, \hat{w}_n \exp(i\alpha_n \cdot \tilde{x}), \tag{3.5}$$

where W_n is the 3×3 matrix given by

$$W_n = W_n(\omega, \rho, \alpha) := \frac{1}{|\alpha_n|^2 + \beta_n \gamma_n} \begin{pmatrix} a_n & b_n & -c_n \\ b_n & d_n & -e_n \\ c_n & e_n & f_n \end{pmatrix}, \tag{3.6}$$

$$a_n := \mu[(\gamma_n - \beta_n)(\alpha_n^{(2)})^2 + k_s^2 \beta_n], \quad b_n := -\mu \alpha_n^{(1)} \alpha_n^{(2)} (\gamma_n - \beta_n),$$

$$c_n := (2\mu\alpha_n^2 - \omega^2 \rho + 2\mu\gamma_n\beta_n)\alpha_n^{(1)}, \quad e_n := (2\mu\alpha_n^2 - \omega^2\rho + 2\mu\gamma_n\beta_n)\alpha_n^{(2)},$$

$$d_n := \mu[(\gamma_n - \beta_n)(\alpha_n^{(1)})^2 + k_s^2\beta_n], \quad f_n := \gamma_n \omega^2 \rho.$$

The expression of $\mathcal{T}^+$ is well-known (cf. [21, 35]), whereas that of $\mathcal{T}^-$ can be derived following the way in [26] for upward propagating elastic waves. Throughout the article we assume ω is not an exceptional frequency, that is,

$$\omega \notin \mathcal{D}_0 := \left\{\omega : \exists n \in \mathbb{Z}^2 \text{ s.t. } |\alpha_n(\omega)|^2 + \beta_n(\omega)\gamma_n(\omega) = 0\right\}, \tag{3.7}$$

so that the denominator of (3.6) never vanishes. The condition (3.7) can be guaranteed if ω is sufficiently small or if the relation $\lambda + 2\mu \le \rho c_0^2$ (equivalently $k \le k_p$) holds [cf. Theorem 4.4 (ii)]. The condition $\omega \notin \mathcal{D}_0$ is a technical assumption only. If $\omega \in \mathcal{D}_0$ is an exceptional frequency, then the DtN mapping and the subsequent variational form (3.12) is to be modified in accordance with Remark 3.3.

Remark 3.2. Suppose that w satisfies the upward α-quasiperiodic Rayleigh expansion

$$w(x) = \sum_{n \in \mathbb{Z}^2} \left\{ A_{p,n} \begin{pmatrix} \alpha_n \\ \beta_n \end{pmatrix} \exp(i\alpha_n \cdot \tilde{x} + i\beta_n x_3) + \mathbf{A}_{s,n} \exp(i\alpha_n \cdot \tilde{x} + i\gamma_n x_3) \right\}, x_3 > \Gamma^+,$$

as a solution to the Navier equation in Ω^+, with the Rayleigh coefficients $A_{p,n} \in \mathbb{C}, \mathbf{A}_{s,n} \in \mathbb{C}^3$ such that $\mathbf{A}_{s,n} \cdot (\alpha_n^\top, \gamma_n)^\top = 0$. Then one can prove $(Tw)|_{\Gamma_b^+} = \sum_{n\in\mathbb{Z}^2} i\, W_n^\top\, \hat{w}_n \exp(i\alpha_n \cdot \tilde{x})$ (cf. [[26], Lemma 1]), where $\hat{w}_n$ denotes the Fourier coefficient of $\exp(-i\alpha \cdot \tilde{x})\, w(\tilde{x}, b)$ of order n. Hence, the matrix in (3.6) differs from that in [26] only in the signs of the entries c_n and e_n.

Making use of the norm (3.3) and the asymptotic behavior

$$\eta_n, \beta_n, \gamma_n \sim i|n|, \quad |\beta_n - \gamma_n| \sim \frac{1}{|n|^2}\frac{k_s^2 - k_p^2}{2}, \quad |\alpha_n|^2 + \beta_n\gamma_n \sim \frac{k_p^2 + k_s^2}{2} \quad \text{as } |n| \to \infty,$$

one can straightforwardly verify that $\mathcal{T}^+ : H_\alpha^s(\mathbb{R}^2) \to H_\alpha^{s-1}(\mathbb{R}^2)$ and $\mathcal{T}^- : H_\alpha^s(\mathbb{R}^2)^3 \to H_\alpha^{s-1}(\mathbb{R}^2)^3$ with $s > 0$ are both bounded operators. Moreover, the operator $-\mathrm{Re}\,\mathcal{T}^+$ is positive semidefinite over $H_\alpha^s(\Gamma_b^+)$, that is,

$$-\mathrm{Re} \int_{\Gamma_b^+} \mathcal{T}^+ w\, \bar{w}\, ds = 4\pi^2 \sum_{|\alpha_n| \ge k} |\eta_n|\, |\hat{w}_n|^2 \ge 0 \quad \text{for all} \quad w \in H_\alpha^s(\Gamma_b^+). \tag{3.8}$$

Unfortunately, the positive semidefiniteness of $-\mathrm{Re}\,\mathcal{T}^-$ over $H_\alpha^s(\Gamma_b^-)^3$ does not hold in general (cf. [18, 26]). With the definitions of $\mathcal{T}^\pm$, we can substitute the terms $\partial_\nu v$ and Tu on the right-hand sides of (3.1) by

$$(\partial_\nu v)|_{\Gamma_b^+} = f_0 + \mathcal{T}^+(v|_{\Gamma_b^+}), \quad (Tu)|_{\Gamma_b^-} = \mathcal{T}^-(u|_{\Gamma_b^-}), \tag{3.9}$$

$$f_0 := (\partial_\nu v^{\mathrm{in}})|_{\Gamma_b^+} - \mathcal{T}^+(v^{\mathrm{in}}|_{\Gamma_b^+}), \quad f_0(\tilde{x}) = -2i\eta_0 \exp(i\alpha\cdot\tilde{x} - i\eta_0 b) \in H_\alpha^{-1/2}(\Gamma_b^+), \tag{3.10}$$

which follows from the expression of v^{in} in (2.1). Combining (3.9) and (3.1), we obtain the following variational formulation of BVP: Find $(v,u) \in V_1$ such that

$$A((v,u),(\varphi,\psi)) = \int_{\Gamma_b^+} f_0\overline{\varphi}\, ds \quad \text{for all } (\varphi,\psi) \in V_1, \tag{3.11}$$

where the sesquilinear form $A : V_1 \times V_1 \to \mathbb{C}$ is defined as

$$\begin{aligned} A((v,u),(\varphi,\psi)) := &\int_{\Omega_b^+} \left[\operatorname{grad} v \cdot \operatorname{grad}\overline{\varphi} - k^2 v\overline{\varphi}\right] dx - \eta \int_\Gamma u\cdot\nu\,\overline{\varphi}\, ds - \int_{\Gamma_b^+} \mathcal{T}^+ v\,\overline{\varphi}\, ds \\ &+ \eta\left[\int_{\Omega_b^-} [\mathcal{E}(u,\overline{\psi}) - \omega^2\rho u\cdot\overline{\psi}\,]\, dx - \int_\Gamma v\,\nu\cdot\overline{\psi} ds - \int_{\Gamma_b^-} \mathcal{T}^- u\cdot\overline{\psi} ds\right] \end{aligned} \tag{3.12}$$

for all $(\varphi,\psi) \in V_1$. The above sesquilinear form obviously generates a continuous linear operator $\mathcal{A} : V_1 \to V_1'$ such that

$$A((v,u),(\varphi,\psi)) = \langle \mathcal{A}(v,u),(\varphi,\psi)\rangle \quad \text{for all } (\varphi,\psi) \in V_1. \tag{3.13}$$

Here V_1' denotes the dual space of V_1 with respect to the duality $\langle\cdot,\cdot\rangle$ extending the L^2 scalar product in $L^2(\Omega_b^+) \times L^2(\Omega_b^-)^3$.

Remark 3.3. Now we consider the case that the condition $\omega \notin \mathcal{D}_0$ is violated. For simplicity, we assume that the condition $|\alpha_n(\omega)|^2 + \beta_n(\omega)\gamma_n(\omega) = 0$ is satisfied if and only if $n = n_\#$ for a fixed $n_\# \in \mathbb{Z}^2$. To introduce a modified DtN map and a modified variational form, we need some notation. For the three downward Rayleigh modes and the two auxiliary modes

$$\begin{aligned} w_1^-(x) &:= \vec{w}_1^{\,-} \exp(i(\alpha_{n_\#}\cdot\tilde{x} - \beta_{n_\#}[x_3+b])), \quad \vec{w}_1^{\,-} := (\alpha_{n_\#}^\top, -\beta_{n_\#})^\top \\ w_2^-(x) &:= \vec{w}_2^{\,-} \exp(i(\alpha_{n_\#}\cdot\tilde{x} - \gamma_{n_\#}[x_3+b])), \quad \vec{w}_2^{\,-} := (\alpha_{n_\#}^{(2)}, -\alpha_{n_\#}^{(1)}, 0)^\top, \\ w_3^-(x) &:= \vec{w}_3^{\,-} \exp(i(\alpha_{n_\#}\cdot\tilde{x} - \gamma_{n_\#}[x_3+b])), \quad \vec{w}_3^{\,-} := (\gamma_{n_\#}\alpha_{n_\#}^\top, |\alpha_{n_\#}|^2)^\top, \\ w_0^\perp(x) &:= \vec{w}_0^{\,\perp} \exp(i(\alpha_{n_\#}\cdot\tilde{x} - \gamma_{n_\#}[x_3+b])), \quad \vec{w}_0^{\,\perp} := (\beta_{n_\#}\alpha_{n_\#}^\top, |\alpha_{n_\#}|^2)^\top, \\ w_0^-(x) &:= \gamma_{n_\#} w_1^- - w_3^-, \end{aligned}$$

we observe $w_0^-(x)|_{\Gamma_b^+} = 0$ and that the $w_j^-(x)$ are orthogonal to $w_0^\perp(x)$ over Γ_b^-. The tractions of w_j^- over Γ_b^- take the form $Tw_j(x) = \vec{w}_j^{\,t} \exp(\alpha_{n_\#}\cdot\tilde{x})$ with the vectors

$$\begin{aligned} &\vec{w}_1^{\,t} := i(-2\mu\beta_{n_\#}\alpha_{n_\#}^\top, [2\mu\beta_{n_\#}^2 + \lambda(\beta_{n_\#}^2 + |\alpha_{n_\#}|^2)])^\top, \quad \vec{w}_2^{\,t} := -i\mu\gamma_{n_\#}\vec{w}_2^{\,-}, \\ &\vec{w}_3^{\,t} := i\mu([-\gamma_{n_\#}^2 + |\alpha_{n_\#}|^2]\alpha_{n_\#}^\top, -2\gamma_{n_\#}|\alpha_{n_\#}|^2)^\top, \quad \vec{w}_0^{\,t} := \gamma_{n_\#}\vec{w}_1^{\,t} - \vec{w}_3^{\,t}. \end{aligned}$$

It is not hard to prove that $\vec{w}_0^{\,t} \neq 0$. Now, using the definition of matrices $\vec{w}_j^{\,t} \otimes \vec{w}_j^{\,-} := ([\vec{w}_j^{\,t}]_l [\vec{w}_j^{\,-}]_m)_{l,m=1}^3$ (tensor product of vectors $\vec{w}_j^{\,t}$ and $\vec{w}_j^{\,-}$), we define the modified DtN map by setting $(\mathcal{T}_\#^- w)(\tilde{x}) = \sum_{n\in\mathbb{Z}^2} i W_n^\# \hat{w}_n \exp(i\alpha_n \cdot \tilde{x})$ with $W_{n_\#}^\# := W_n$ for $n \neq n_\#$ and

$$W_{n_\#}^\# := \frac{1}{|\alpha_{n_\#}|^2}\, \vec{w}_2^{\,t} \otimes \vec{w}_2^{\,-} + \frac{1}{|\alpha_{n_\#}|^2(\gamma_{n_\#}^2 + |\alpha_{n_\#}|^2)}\, \vec{w}_3^{\,t} \otimes \vec{w}_3^{\,-}.$$

Note that $\mathcal{T}_\#^-(w_j^-|_{\Gamma_b^-}) = \mathcal{T} w_j^-|_{\Gamma_b^-}$, $j = 2, 3$ and $\mathcal{T}_\#^-(w_0^\perp|_{\Gamma_b^-}) = 0$. The modified variational form of (3.13) is obtained by adding a new unknown parameter ξ and by replacing the last term $-\int_{\Gamma_b^-} \mathcal{T}^- u \cdot \overline{\psi}\, ds$ in the square bracket by $-\int_{\Gamma_b^-} \{\mathcal{T}_\#^- u + \xi \vec{w}_0^{\,t}\} \cdot \overline{\psi}\, ds$. To get the solution (v, u, ξ), we have to solve the corresponding Eq. (3.11) together with the additional scalar equation $\int_{\Gamma_b^-} u \cdot \overline{w_0^\perp} ds = 0$.

It is not hard to show that the triple (v, u, ξ) with (v, u) the solution of the BVP and with $\xi := A_{p,n_\#}/\gamma_{n_\#}$ [cf. (2.8)] is a solution of the modified variational equation. Moreover, if BVP has no trivial solution, then it is not hard to see that the only solution of the homogeneous modified variational equation is trivial. In other words, BVP is equivalent to the modified variational equation.

IV. SOLVABILITY RESULTS

Having established the equivalent variational formulation in a truncated domain in Section III, the purpose of this section is to derive uniqueness and existence of weak solutions to the variational Eq. (3.13). We first prove the strong ellipticity of the sesquilinear form A.

Lemma 4.1. *The sesquilinear form A defined in (3.12) is strongly elliptic over V_1, and the operator $\mathcal{A}$ defined by (3.13) is always a Fredholm operator with index zero.*

Proof. As the matrix $-\mathrm{Re}\,(i W_n^\top)$ is positive for large $|n|$ (cf. [26, Lemma 2]), the operator $-\mathrm{Re}\,(\mathcal{T}^-)$ can be decomposed into the sum of a positive definite operator $\mathcal{T}_1$ and a finite rank operator $\mathcal{T}_2$ from $H_\alpha^{1/2}(\Gamma_b^-)^3$ to $H_\alpha^{-1/2}(\Gamma_b^-)^3$. We split the sesquilinear form A into the sum $A = A_1 + A_2$, where

$$\begin{aligned} A_1((v,u),(\varphi,\psi)) := {} & \int_{\Omega_b^+} \left[\mathrm{grad}\, v \cdot \mathrm{grad}\, \overline{\varphi} + v\overline{\varphi}\right] dx - \int_{\Gamma_b^+} \mathcal{T}^+ v\, \overline{\varphi}\, ds \\ & + \eta \left[\int_{\Omega_b^-} [\mathcal{E}(u, \overline{\psi}) + u \cdot \overline{\psi}\,]\, dx + \int_{\Gamma_b^-} \mathcal{T}_1\, u \cdot \overline{\psi} ds\right], \\ A_2((v,u),(\varphi,\psi)) := {} & -\int_{\Omega_b^+} \left[(1+k^2) v\overline{\varphi}\right] dx - \eta \int_\Gamma u \cdot \nu\, \overline{\varphi}\, ds \\ & + \eta \left[\int_{\Omega_b^-} [-(1+\omega^2\rho) u \cdot \overline{\psi}\,]\, dx - \int_\Gamma v\, \nu \cdot \overline{\psi}\, ds + \int_{\Gamma_b^-} \mathcal{T}_2 u \cdot \overline{\psi}\, ds\right]. \end{aligned}$$

Recalling (3.8) and Korn's inequality (cf., e.g., [28, Chap. 5.4] or [18]), we have

$$\mathrm{Re}\, A_1((v,u),(v,u)) \geq c_1\, (||v||_{V_1^+}^2 + ||u||_{V_1^-}^2) \quad \text{for all } (v,u) \in V_1,$$

with some constant $c_1 > 0$. Moreover, applying the Cauchy-Schwarz inequality yields

$$\operatorname{Re} A_2((v,u),(v,u)) \geq -c_2\,(||v||^2_{L^2(\Omega_b^+)} + ||v||^2_{L^2(\Gamma)} + ||u||^2_{L^2(\Omega_b^-)^3} + ||u||^2_{L^2(\Gamma)^3}) + \eta \operatorname{Re}\,(\mathcal{T}_2 u, u)_{L^2(\Gamma_b^-)^3},$$

for some constant $c_2 > 0$. From the compact imbeddings $H^1(\Omega_b^{\pm}) \hookrightarrow L^2(\Omega_b^{\pm})$, $H^{1/2}(\Gamma) \hookrightarrow L^2(\Gamma)$ and the compactness of $\mathcal{T}_2$, we conclude that the sesquilinear form A is strongly elliptic over $V_1 \times V_1$. Consequently, the operator $\mathcal{A}$ defined by (3.13) is always a Fredholm operator with index zero. ∎

From Lemma 4.1 and the Fredholm alternative, it follows that the variational formulation (3.11) is uniquely solvable provided the homogeneous operator equation $\mathcal{A}(v,u) = 0$ has only the trivial solutions $v = 0, u = 0$. However, uniqueness cannot be proved in the general case. It will be shown below that only the upward outgoing modes of v^{sc} and the downward outgoing modes of u can be uniquely determined, whereas the other evanescent modes may be nonunique.

Lemma 4.2. *Assume $(v^{sc}, u) \in V_1$ is a solution pair to the variational problem (3.11) with $v^{in} = 0$ or, equivalently, $f_0 = 0$. Then there holds $v_n = 0$ for $|\alpha_n| < k$, $A_{p,n} = 0$ for $|\alpha_n| < k_p$, and $|\mathbf{A}_{s,n}| = 0$ for $|\alpha_n| < k_s$, where v_n, $A_{p,n}$ and $\mathbf{A}_{s,n}$ denote the Rayleigh coefficients of v^{sc} and u [cf. (2.5) and (2.8)].*

Proof. Taking the imaginary part of (3.11) with $\varphi = v^{sc}, \psi = u, v^{in} = 0$ and using the fact that $\eta > 0$, we get

$$-\operatorname{Im}\,(\mathcal{T}^+ v^{sc}, v^{sc})_{L^2(\Gamma_b^+)} - \eta \operatorname{Im}\,(\mathcal{T}^- u, u)_{L^2(\Gamma_b^-)^3} = 0. \tag{4.1}$$

From the explicit expressions for $\mathcal{T}^+$ and $\mathcal{T}^-$, we can derive that

$$\operatorname{Im}\,(\mathcal{T}^+ v^{sc}, v^{sc})_{L^2(\Gamma_b^+)} = 4\pi^2 \sum_{n:|\alpha_n|<k} \eta_n\,|v_n|^2,$$

$$\operatorname{Im}\,(\mathcal{T}^- u, u)_{L^2(\Gamma_b^-)} = 4\pi^2 \left(\sum_{n:|\alpha_n|<k_p} \beta_n\,|A_{p,n}|^2\,\omega^2\rho + \sum_{n:|\alpha_n|<k_s} \gamma_n\,|\mathbf{A}_{s,n}|^2\mu \right), \tag{4.2}$$

where the second equality follows from the arguments in proving [26, Lemma 3]. As $\eta_n > 0$ for $|\alpha_n| < k$, $\beta_n > 0$ for $|\alpha_n| < k_p$ and $\gamma_n > 0$ for $|\alpha_n| < k_s$, we complete the proof of Lemma 4.2 by combining (4.1) and (4.2). ∎

Using the arguments of the above proof, we cannot prove uniqueness of solutions to (3.11) for general biperiodic Lipschitz interfaces separating domains of the fluid and solid. Moreover, uniqueness does not hold in general, even if Γ is the graph of a smooth biperiodic function. To see this, we construct a nonuniqueness example where Γ is a flat surface parallel to the x_1x_2-plane.

Lemma 4.3. *Assume that $\Gamma = \Gamma_0 := \{x : x_3 = 0\}$ is a flat interface, the incident angle $\theta_2 = 0$ and that $k = k_p = k \sin\theta_1 + m_0$ for some $m_0 \in \mathbb{Z}$. Then there exists at least one nontrivial solution pair $(v^{sc}, u) \in V_1$ to the homogeneous variational problem $A((v^{sc}, u), (\varphi, \psi)) = 0$ for all $(\varphi, \psi) \in V$.*

Proof. Observing that the interface Γ_0 is invariant in x_2 and the incident direction $\hat{\theta} = (\sin\theta_1, 0, -\cos\theta_1)$ is orthogonal to the x_2-axis, the original three-dimensional scattering problem reduces to a two-dimensional problem in the x_1x_3-plane. Consequently, we look for upward and downward Rayleigh expansion solutions v^{sc} and u of the special form

$$v^{sc}(x) = \sum_{m\in\mathbb{Z}} v_m\, e^{i(\tilde{\alpha}_m x_1 + \eta_m x_3)}, \quad x_3 > 0,$$

$$u(x) = \sum_{m\in\mathbb{Z}} \left(A_{p,m} \begin{pmatrix} \tilde{\alpha}_m \\ 0 \\ -\beta_m \end{pmatrix} e^{i(\tilde{\alpha}_m x_1 - \beta_m x_3)} + A_{s,m} \begin{pmatrix} \gamma_m \\ 0 \\ \tilde{\alpha}_m \end{pmatrix} e^{i(\tilde{\alpha}_m x_1 - \gamma_m x_3)} \right), \quad x_3 < 0,$$

with $v_m, A_{p,m}, A_{s,m} \in \mathbb{C}$, $\tilde{\alpha}_m := \alpha_1 + m = \alpha_n^{(1)}$ for $n = (m, 0)$. Here, $\alpha_1 = k\sin\theta_1$ due to the assumption that $\theta_2 = 0$. The parameters $\eta_m, \beta_m, \gamma_m$ for $m \in \mathbb{Z}$ are defined in the same way as $\eta_n, \beta_n, \gamma_n$ [cf. (2.6)] with $n = (m, 0)$ and $\alpha = (\alpha_1, 0)^\top$. Note that the solution pair (v^{sc}, u) does not depend on x_2.

Elementary calculations show that, using $\nu = (0, 0, -1)$ on Γ_0,

$$(Tu)(x)|_{\Gamma_0} = i \sum_{m\in\mathbb{Z}} \begin{pmatrix} 2\mu\tilde{\alpha}_m\beta_m & \omega^2\rho - 2\mu\tilde{\alpha}_m^2 \\ 2\mu\tilde{\alpha}_m^2 - \omega^2\rho & 2\mu\tilde{\alpha}_m\gamma_m \end{pmatrix} \begin{pmatrix} A_{p,m} \\ A_{s,m} \end{pmatrix} e^{i\tilde{\alpha}_m x_1},$$

$$\nu \cdot u(x)|_{\Gamma_0} = \sum_{m\in\mathbb{Z}} \left(A_{p,m}\beta_m - A_{s,m}\tilde{\alpha}_m\right) e^{i\tilde{\alpha}_m x_1}, \quad (\partial_\nu v^{sc})(x)|_{\Gamma_0} = \sum_{m\in\mathbb{Z}} -i\, v_m \eta_m e^{i\tilde{\alpha}_m x_1}.$$

Hence, the coupling conditions between $v = v^{sc}$ and u on Γ_0 are equivalent to the algebraic equations

$$D_m \begin{pmatrix} v_m \\ iA_{p,m} \\ iA_{s,m} \end{pmatrix} = 0, \quad D_m := \begin{pmatrix} 0 & 2\mu\tilde{\alpha}_m\beta_m & \omega^2\rho - 2\mu\tilde{\alpha}_m^2 \\ -1 & 2\mu\tilde{\alpha}_m^2 - \omega^2\rho & 2\mu\tilde{\alpha}_m\gamma_m \\ -\eta_m/(\rho_f\omega^2) & \beta_m & -\tilde{\alpha}_m \end{pmatrix} \tag{4.3}$$

The determinant of D_m is given by

$$\mathrm{Det}\,(D_m) = -\frac{\eta_m}{\rho_f\omega^2} \begin{vmatrix} 2\mu\tilde{\alpha}_m\beta_m & \omega^2\rho - 2\mu\tilde{\alpha}_m^2 \\ 2\mu\tilde{\alpha}_m^2 - \omega^2\rho & 2\mu\tilde{\alpha}_m\gamma_m \end{vmatrix} - \omega^2\rho\,\beta_m.$$

Under the assumption that $k_p = k = k\sin\theta_1 + m_0 = \tilde{\alpha}_{m_0}$ for some $m_0 \in \mathbb{Z}$, we have $\eta_{m_0} = \beta_{m_0} = 0$. Thus, the linear system (4.3) has the nontrivial solution $(v_{m_0}, A_{p,m_0}, A_{s,m_0})$, if this vector satisfies the relations $v_{m_0} + i\lambda k^2 A_{p,m_0} = 0$ and $A_{s,m_0} = 0$. These imply that one of the nontrivial solutions (v^{sc}, u) is of the form $v^{sc}(x) = c\, e^{ikx_1}$ for $x_3 > 0$ and $u(x) = -ic/(\lambda k^2)\,(k, 0, 0)^\top e^{ikx_1}$ for $x_3 < 0$ with a constant $c \in \mathbb{C}$. ∎

Next, we show the existence of Jones frequencies for the FSI problem in periodic structures. A frequency $\omega \in \mathbb{R}_+$ is called a Jones frequency with the quasiperiodic parameter $\alpha = (\alpha_1, \alpha_2)^\top \in \mathbb{R}^2$, if there exists at least one nontrivial α-quasiperiodic solution to the boundary value problem

$$(\Delta^* + \omega^2\rho)u = 0 \text{ in } \Omega^-, \quad Tu = 0,\ \nu \cdot u = 0 \text{ on } \Gamma, \quad u \text{ admits an expansion (2.8).} \tag{4.4}$$

Obviously, the solution $(0, u)$ satisfies the homogeneous transmission problem (2.2) with $v^{in} = 0$, provided u is a solution of (4.4). This implies that the FSI problem is not uniquely solvable at Jones frequencies. To find a nontrivial solution to (4.4), we suppose that $\gamma_n = \sqrt{k_s^2 - |\alpha_n|^2} = 0$ for some $n \in \mathbb{Z}^2$ and that $\Gamma := \{x : x_3 = 0\}$ is a flat surface. Then the α-quasiperiodic function $u(x) = (-\alpha_n^{(2)}, \alpha_n^{(1)}, 0)^\top e^{i(\alpha_n^{(1)} x_1 + \alpha_n^{(2)} x_2)}$ is a solution of (4.4).

Although there is no uniqueness in general, we can verify the existence of solutions to BVP at any frequency $\omega \in \mathbb{R}$ and the unique solvability for all frequencies excluding possibly a discrete set. This exceptional set does not necessarily include the values $\omega \in \mathcal{D}_0$, for which there is an $n \in \mathbb{Z}^2$ with $|\alpha_n|^2 + \beta_n \gamma_n \neq 0$ (cf. Remark 3.3). The main results of this section are stated in the following theorem, where the number c_0 denotes the speed of sound in the fluid.

Theorem 4.4.

i. For the incident plane wave v^{in} of the form (2.1), there always exists a solution $(v, u) \in V_1$ to the variational problem (3.11) and hence to BVP.

ii. Assume $\lambda + 2\mu \le \rho c_0^2$. There exists a small frequency $\omega_0 > 0$ such that for all $\omega \in (0, \omega_0]$ the solution to (3.11) is unique. Moreover, the variational problem (3.11) admits a unique solution for all frequencies excluding a discrete set $\mathcal{D}$ with the only possible accumulation point at infinity.

Proof. (i) The variational problem (3.11) can be formulated as the equivalent operator equation $\mathcal{A}(v, u) = \mathcal{F}_0$, where $\mathcal{F}_0 \in V_1'$ is defined as the right-hand side of (3.11). By the Fredholm alternative and Lemma 4.2, this operator equation (3.11) is solvable provided $\mathcal{F}_0$ is orthogonal to all solutions $(\tilde{v}, \tilde{u})$ of the homogeneous adjoint equation $\mathcal{A}^*(\tilde{v}, \tilde{u}) = 0$, that is, $\langle \mathcal{F}_0, (\tilde{v}, \tilde{u}) \rangle = 0$. Note that the component $\tilde{v}$ of such a pair can always be extended to a solution of the Helmholtz equation in the unbounded domain Ω^+ by setting $\tilde{v}(x) = \sum_{n \in \mathbb{Z}^2} \tilde{v}_n \exp(i\, \alpha_n \cdot \tilde{x} - i\overline{\eta}_n [x_3 - b])$ for $x_3 > b$, where the Rayleigh coefficients $\tilde{v}_n$ are determined as the n-th Fourier coefficient of $(e^{-i\alpha \cdot \tilde{x}} \tilde{v})|_{\Gamma_b^+}$. The above $\tilde{v}$ has a finite number of incoming plane waves that propagate downward, while the others terms in the sum are exponentially growing modes as $x_3 \to \infty$. Conversely, by arguing as in the proof of Lemma 4.2, it can be derived by taking the imaginary part of the equation $0 = \langle \mathcal{A}^*(\tilde{v}, \tilde{u}), (\varphi, \psi) \rangle = \langle (\tilde{v}, \tilde{u}), \mathcal{A}(\varphi, \psi) \rangle = \overline{A((\varphi, \psi), (\tilde{v}, \tilde{u}))}$ with $(\varphi, \psi) = (\tilde{v}, \tilde{u})$ that $\tilde{v}$ has vanishing Rayleigh coefficients of the incoming modes, that is, $\tilde{v}_n = 0$ for $|\alpha_n| < k$. In particular, we have $\tilde{v}_0 = 0$ and hence $\langle \mathcal{F}_0, (\tilde{v}, \tilde{u}) \rangle = \int_{\Gamma_b^+} f_0\, \overline{\tilde{v}} ds = \int_{\Gamma_b^+} f_0\, \overline{\tilde{v}_0} \exp(-i\, \alpha_0 \cdot \tilde{x}) ds(\tilde{x}) = 0$, with f_0 given in (3.10). Applying the Fredholm alternative yields the existence of a solution to BVP.

(ii) We first prove uniqueness for small frequencies. The assumption $\lambda + 2\mu \le \rho c_0^2$ implies that $k \le k_p$. If $\mathcal{A}(v^{sc}, u) = 0$ for some $(v^{sc}, u) \in V$, we conclude from $k \le k_p$ and Lemma 4.2 that the zero-order Rayleigh coefficients of v^{sc} and u vanish, that is, $v_0 = 0$, $A_{p,0} = 0$ and $\mathbf{A}_{s,0} = 0$. This together with the asymptotic behavior $|\eta_n| \ge C_0\, (1 + |n|^2)^{1/2}$ for $k = \omega / c_0 \to 0^+$ valid for $|n| \neq 0$ and a constant $C_0 > 0$, leads to the estimate [cf. (3.8)]

$$-\mathrm{Re} \int_{\Gamma_b^+} \bar{v}^{sc}\, \mathcal{T}^+ v^{sc} ds = 4\pi^2 \sum_{|n| \neq 0} |\eta_n|\, |v_n\, e^{i\eta_n b}|^2 = 4\pi^2 \sum_{n \in \mathbb{Z}^2} |\eta_n|\, |v_n\, e^{i\eta_n b}|^2 \ge C_1 ||v^{sc}||^2_{H_\alpha^{1/2}(\Gamma_b^+)}, \tag{4.5}$$

for some $C_1 > 0$ and $\omega \in (0, \omega_1]$ with $\omega_1 > 0$ being sufficiently small. In a completely similar manner, from the asymptotic properties of the matrix W_n as $\omega \to 0^+$ (cf. [18, Lemma 2]) we obtain

$$-\mathrm{Re} \int_{\Gamma_b^-} \bar{u} \cdot \mathcal{T}^- u\, ds \ge C_2 ||u||^2_{H_\alpha^{1/2}(\Gamma_b^-)^3}. \tag{4.6}$$

Inserting (4.5) into (3.11) and setting $(\varphi, \psi) = (v^{sc}, 0)$, $v^{\text{in}} = 0$, we arrive at

$$
\begin{aligned}
0 &= \operatorname{Re} A((v^{sc}, u), (v^{sc}, 0)) \\
&\geq ||\operatorname{grad} v^{sc}||^2_{L^2(\Omega_b^+)} + C_1 ||v^{sc}||^2_{H_\alpha^{1/2}(\Gamma_b^+)} - \omega^2/c_0^2\, ||v^{sc}||^2_{L^2(\Omega_b^+)} - \omega^2 \rho_f \int_\Gamma u \cdot \nu \overline{v^{sc}}\, ds.
\end{aligned}
$$

Applying Friedrich's inequality and the Cauchy-Schwarz inequality, it follows that

$$0 \geq C_3 ||v^{sc}||^2_{H_\alpha^1(\Omega_b^+)} - C_4\, \omega^2 ||u||^2_{L^2(\Gamma)^3}, \quad \omega \in (0, \omega_1], \tag{4.7}$$

for some constants $C_3, C_4 > 0$ uniformly in all $\omega \in (0, \omega_1]$. Similarly, inserting (4.6) into (3.11) with $(\varphi, \psi) = (0, u)$ and $f_0 = 0$ and applying Korn's inequality (cf., e.g., [28, Chap. 5.4] or [18]), we obtain

$$0 = \operatorname{Re} A((v^{sc}, u), (0, u)) \geq C_5 ||u||^2_{H_\alpha^1(\Omega_b^-)^3} - C_6\, ||v^{sc}||^2_{L^2(\Gamma)}, \quad \omega \in (0, \omega_1], \tag{4.8}$$

where $C_5, C_6 > 0$ are independent of $\omega \in (0, \omega_1]$. Now, combining (4.7), (4.8) and using the trace lemma we arrive at $v^{sc} = 0, u = 0$ for all $\omega \in (0, \omega_0]$ with some small frequency $\omega_0 > 0$. The existence follows directly from uniqueness by the Fredholm alternative.

With respect to the frequency parameter ω, the operators corresponding to the variational form (3.12) form an analytic operator family. Indeed, ω enters the coefficients of the volume integrals in a quadratic way, and the operators $\mathcal{T}^\pm$ are block diagonal with entries analytic over $\{\omega \in \mathbb{C} : \varepsilon_0 \operatorname{Re} \omega > \operatorname{Im} \omega, |\alpha_n(\omega)|^2 + \beta_n(\omega)\gamma_n(\omega) \neq 0\}$ for a fixed small $\varepsilon_0 > 0$ [cf. (3.7)]. In accordance with the proof of Lemma 4.1, the values of this analytic function of ω are Fredholm operators of index zero and, for $0 < \omega < \omega_0$, even invertible. Consequently, in view of the analytic Fredholm theory (cf., e.g., [36, Theorem 8.26] or [37, Theorem I. 5. 1]) there exists a discrete set $\mathcal{D}$ with $\mathcal{D}_0 \subseteq \mathcal{D} \subseteq \mathbb{R}^+$ such that the operators of the variational form are invertible for $\omega \in \mathbb{R}^+ \backslash \mathcal{D}$.

Moreover, we conclude from the arguments in [18, Theorem 6] or [35, Theorem 3.3] that $\mathcal{D}$ cannot have a finite accumulation point. The proof is completed. ∎

Remark 4.5. Theorem 4.4 (i) remains valid for a broad class of incident waves of the form $v^{\text{in}}(x) = \sum_{n \in \mathbb{Z}^2 : |\alpha_n| < k} q_n \exp(i\alpha_n \cdot \tilde{x} - i\eta_n x_3)$ with $q_n \in \mathbb{C}$.

V. ENERGY BALANCE

The energy balance in the FSI problem asserts that the sum of the reflected energy in the fluid and the transmitted energy in the solid should be equal to the energy of the incident wave. Let the incident plane wave $v^{\text{in}} = \exp(i\alpha \cdot x' - \eta_0 x_3)$ be given by (2.1), with $\eta_0 = k \cos \theta_1$. Define the efficiency of the reflected acoustic wave of order n as $E_n^+ := \frac{\eta_n}{\eta_0} |v_n|^2$. This is the ratio of the energy flux of the reflected mode of order n over the energy flux of the incoming mode. The energy flux is measured over a unit of time period on a unit square parallel to the $x_1 x_2$-plane. In the FSI problem, the efficiencies of the transmitted compressional and shear elastic waves in the fluid are defined as $E_{p,n}^- := \frac{\beta_n}{\eta_0} |A_{p,n}|^2\, \omega^2 \rho\, \eta$ and $E_{s,n}^- := \frac{\gamma_n}{\eta_0} |A_{s,n}|^2\, \mu\, \eta$, respectively. The energy balance formula, which can be used as an indicator of the validity of the numerical solution, is formulated as follows.

Theorem 5.1. *It holds that* $\sum_{n\in\mathbb{Z}^2:\eta_n>0} E_n^+ + \sum_{n\in\mathbb{Z}^2:\beta_n>0} E_{p,n}^- + \sum_{n\in\mathbb{Z}^2:\gamma_n>0} E_{s,n}^- = 1.$

Proof. It follows from (3.1) that

$$0 = \int_{\Omega_b^+} \left[\operatorname{grad} v \cdot \operatorname{grad} \overline{\varphi} - k^2 v \overline{\varphi}\right] dx - \eta \int_\Gamma u \cdot \nu \overline{\varphi}\, ds - \int_{\Gamma_b^+} \partial_\nu v\, \overline{\varphi}\, ds$$

$$+ \eta \left[\int_{\Omega_b^-} [\mathcal{E}(u, \overline{\psi}) - \omega^2 \rho u \cdot \overline{\psi}\,]\, dx - \int_\Gamma v\, \nu \cdot \overline{\psi} ds - \int_{\Gamma_b^-} \mathcal{T}^- u \cdot \overline{\psi} ds \right]$$

for all $(\varphi, \psi) \in H^1(\Omega_b^+) \times H^1(\Omega_b^-)^3$, where $v = v^{in} + v^{sc}$ denotes the total acoustic field in the fluid. Choosing $(\varphi, \psi) = (v, u)$ and taking the imaginary part of the above expression yields [cf. (4.1)]

$$\operatorname{Im} (\partial_\nu v, v)_{L^2(\Gamma_b^+)} + \eta \operatorname{Im} (\mathcal{T}^-\, u, u)_{L^2(\Gamma_b^-)^3} = 0, \tag{5.1}$$

It can be readily checked that

$$\operatorname{Im} (\partial_\nu v, v)_{L^2(\Gamma_b^+)} = \operatorname{Im} (\partial_\nu v^{in}, v^{in})_{L^2(\Gamma_b^+)} + \operatorname{Im} (\mathcal{T}^+\, v^{sc}, v^{sc})_{L^2(\Gamma_b^+)}. \tag{5.2}$$

Indeed, by the definition of $\mathcal{T}^+$ [cf. (3.4)] we observe

$$\operatorname{Im} \left[(\partial_\nu v^{\mathrm{in}}, v^{sc})_{L^2(\Gamma_b^+)} + (\mathcal{T}^+\, v^{sc}, v^{\mathrm{in}})_{L^2(\Gamma_b^+)} \right] = 4\pi^2 \operatorname{Im} \left[-i\eta_0 \bar{v}_0\, e^{-2i\eta_0 b} + i\eta_0 v_0 e^{2i\eta_0 b} \right] = 0$$

where v_0 denotes the zeroth-order Rayleigh coefficient of v^{sc} [cf. (2.5)]. Conversely, we get $\operatorname{Im} (\partial_\nu\, v^{in}, v^{in})_{L^2(\Gamma_b^+)} = -4\pi^2\, \eta_0$. Inserting this together with (4.2) and (5.1) into (5.1) yields the desired result of the lemma. ∎

Remark 5.2. If the Rayleigh expansion of u takes the following form equivalent to (2.8):

$$u(x) = \sum_{n\in\mathbb{Z}^2} \left\{ A_{p,n} \begin{pmatrix} \alpha_n \\ -\beta_n \end{pmatrix} \exp(i\alpha_n \cdot \tilde{x} - i\beta_n x_3) + \begin{pmatrix} \alpha_n \\ -\gamma_n \end{pmatrix} \times \tilde{\mathbf{A}}_{s,n} \exp(i\alpha_n \cdot \tilde{x} - i\gamma_n x_3) \right\} \tag{5.3}$$

for $x_3 < \Gamma^-$ with $\tilde{\mathbf{A}}_{s,n} \in \mathbb{C}^3$ such that $\tilde{\mathbf{A}}_{s,n} \cdot (\alpha_n, -\gamma_n)^\top = 0$, then it holds that [cf. (4.2)]

$$\operatorname{Im} \int_{\Gamma_b^-} \bar{u} \cdot \mathcal{T}^- u\, ds = 4\pi^2 \omega^2 \rho \left(\sum_{n:|\alpha_n|<k_p} \beta_n\, |A_{p,n}|^2 + \sum_{n:|\alpha_n|<k_s} \gamma_n\, |\tilde{\mathbf{A}}_{s,n}|^2 \right). \tag{5.4}$$

In this case, the definition of the efficiency $E_{s,n}^-$ in Theorem 5.1 should be replaced by $E_{s,n}^- := \frac{\gamma_n}{\eta_0} |\tilde{A}_{s,n}|^2\, \omega^2 \rho\, \eta$. The quantity in (5.4) denotes the energy flux through Γ_b^- for the transmitted elastic wave of the form (5.3).

VI. DISCRETIZATION VIA TRUNCATED DTN MAPPINGS AND FINITE ELEMENT METHOD

A. Truncation of DtN mappings

Clearly, for the numerical treatment of the infinite number of terms in the definition of the DtN maps (3.4) and (3.5), we have to truncate the sums. We choose an integer $N > 0$ and introduce the truncated DtN maps

$$(\mathcal{T}_N^+ w)(\tilde{x}) := \sum_{n\in\mathbb{Z}^2:|n|\leq N} i\eta_n\, \hat{w}_n \exp(i\alpha_n\cdot\tilde{x}), (\mathcal{T}_N^- w)(\tilde{x}) := \sum_{n\in\mathbb{Z}^2:|n|\leq N} iW_n\, \hat{w}_n \exp(i\alpha_n\cdot\tilde{x}). \quad (6.1)$$

We suppose that N is sufficiently large s.t. all the propagating plane wave modes have indices with $|n| \leq N$. Replacing the DtN maps in (3.12), we arrive at the approximate sesquilinear form

$$\begin{aligned} A_N((v,u),(\varphi,\psi)) := &\int_{\Omega_b^+} \left[\operatorname{grad} v\cdot\operatorname{grad}\overline{\varphi} - k^2 v\overline{\varphi}\right] dx - \eta\int_\Gamma u\cdot\nu\,\overline{\varphi}\, ds - \int_{\Gamma_b^+} \mathcal{T}_N^+ v\,\overline{\varphi}\, ds \\ &+ \eta\left[\int_{\Omega_b^-} [\mathcal{E}(u,\overline{\psi}) - \omega^2\rho u\cdot\overline{\psi}\,]\, dx - \int_\Gamma v\,\nu\cdot\overline{\psi} ds - \int_{\Gamma_b^-} \mathcal{T}_N^- u\cdot\overline{\psi} ds\right]. \end{aligned} \quad (6.2)$$

Using this, Eq. (3.11) turns to

$$A_N((v_N,u_N),(\varphi,\psi)) = \mathcal{F}_0((\varphi,\psi)) := \int_{\Gamma_b^+} f_0\overline{\varphi}\, ds \quad \text{for all } (\varphi,\psi)\in V_1, \quad (6.3)$$

which is equivalent to the operator equation $\mathcal{A}_N(v_N,u_N) = \mathcal{F}_0$. Here, $\mathcal{A}_N : V_1 \to V_1'$ is the approximate operator of $\mathcal{A}$ appearing in the operator equation $\mathcal{A}(v,u) = \mathcal{F}_0$ corresponding to (3.11). Now the exponential decay of the Rayleigh coefficients imply the following truncation error estimate.

Lemma 6.1.

i. *Suppose $(v,u)\in V_1$ is the solution of $\mathcal{A}(v,u) = \mathcal{F}_0$ with $\mathcal{F}_0$ as in (6.3), then the Rayleigh coefficients of (v,u) satisfy*

$$v(x) = \sum_{n\in\mathbb{Z}^2} v_n^+ \exp(i\alpha_n\cdot\tilde{x} + i\eta_n[x_3 - b]) + v^{in}(x), x_3 > \Gamma^+, \quad |v_n^+| \leq c||v||_{H^1_\alpha(\Omega_b^+)} q^{|n|}, \quad (6.4)$$

$$u(x) = \sum_{n\in\mathbb{Z}^2}\left\{u_{p,n}^-\begin{pmatrix}\alpha_n\\ \beta_n\end{pmatrix}\exp(i\alpha_n\cdot\tilde{x} - i\beta_n[x_3+b]) + \mathbf{u}_{s,n}^-\exp(i\alpha_n\cdot\tilde{x} - i\gamma_n[x_3+b])\right\},$$

$$x_3 < \Gamma^-,$$

$$|u_{p,n}^-| \leq c||u||_{H^1_\alpha(\Omega_b^-)^3} q^{|n|}, |\mathbf{u}_{s,n}^-| \leq c||u||_{H^1_\alpha(\Omega_b^-)^3} q^{|n|} \quad (6.5)$$

for any n . Here, c and q are constants independent of (u,v) s.t. $c > 0$ and $0 < q < 1$. Recall that $\mathbf{u}_{s,n}^-\cdot(\alpha_n^\top, -\gamma_n)^\top = 0$.

ii. *Suppose* $(v_N, u_N) \in V_1$ *is the solution of* $\mathcal{A}_N(v, u) = \mathcal{F}_0$ *with* $\mathcal{F}_0$ *as in (6.3), then the Rayleigh coefficients of* (v_N, u_N) *satisfy*

$$v_N(x) = \sum_{n\in\mathbb{Z}^2} \{v^+_{N,n} \exp(i\alpha_n \cdot \tilde{x} + i\eta_n[x_3 - b]) + v^-_{N,n} \exp(i\alpha_n \cdot \tilde{x} - i\eta_n[x_3 - b])\} + v^{in}(x), \tag{6.6}$$

$$x_3 > \Gamma^+,$$

$$v^-_{N,n} = 0 \; if\, |n| \le N, \quad v^-_{N,n} = v^+_{N,n} \; if\, |n| > N, \quad |v^\pm_{N,n}| \le c||v_N||_{H^1_\alpha(\Omega^+_b)} q^{|n|}, \tag{6.7}$$

$$u_N(x) = \sum_{n\in\mathbb{Z}^2} \{u^+_{N,p,n} \begin{pmatrix}\alpha_n\\ \beta_n\end{pmatrix} \exp(i\alpha_n \cdot \tilde{x} + i\beta_n[x_3 + b]) + \mathbf{u}^+_{N,s,n} \exp(i\alpha_n \cdot \tilde{x} + i\gamma_n[x_3 + b])$$

$$+ u^-_{N,p,n} \begin{pmatrix}\alpha_n\\ -\beta_n\end{pmatrix} \exp(i\alpha_n \cdot \tilde{x} - i\beta_n[x_3 + b]) + \mathbf{u}^-_{N,s,n} \exp(i\alpha_n \cdot \tilde{x} - i\gamma_n[x_3 + b])\},$$

$$x_3 < \Gamma^-,$$

$$u^+_{N,p,n} = 0, \quad \mathbf{u}^+_{N,p,n} = 0 \; if\, |n| \le N, \quad u^+_{N,p,n} = u^-_{N,p,n}, \quad \mathbf{u}^+_{N,p,n} = \mathbf{u}^-_{N,p,n} \; if\, |n| > N,$$

$$|u^\pm_{N,p,n}| \le c||u_N||_{H^1_\alpha(\Omega^-_b)^3} q^{|n|}, \; |\mathbf{u}^\pm_{N,s,n}| \le c||u_N||_{H^1_\alpha(\Omega^-_b)^3} q^{|n|} \tag{6.8}$$

for any n . Here, c and q are constants independent of N and (u, v) *s.t.* $c > 0$ *and* $0 < q < 1$. *Note that* $\mathbf{u}^\pm_{N,s,n} \cdot (\alpha_n^\top, \pm\gamma_n)^\top = 0$.

iii. *Suppose the operator* $\mathcal{A} : V_1 \to V_1'$ *is invertible. Then, of course, the problem BVP is uniquely solvable. Moreover, there is an integer* $N_0 > 0$ *s.t.* $\mathcal{A}_N : V_1 \to V_1'$ *is invertible for* $N \ge N_0$ *and* $\sup_{N\ge N_0} ||\mathcal{A}_N^{-1}|| < \infty$. *For* $(v, u) \in V_1$ *the solution of* $\mathcal{A}(v, u) = \mathcal{F}_0$ *with* $\mathcal{F}_0$ *as in (6.3) and for* $(v_N, u_N) \in V_1$ *the solution of* $\mathcal{A}_N(v, u) = \mathcal{F}_0$ *with the same* $\mathcal{F}_0$, *we obtain the estimate* $||(v, u) - (v_N, u_N)||_{V_1} \le c||(v, u)||_{V_1} q^N$ *for any N. Here, c and q are constants independent of N and* (v, u) *s.t.* $c > 0$ *and* $0 < q < 1$.

Remark 6.2. To simplify the formulas in part ii) of the Lemma, we have assumed $\eta_n \ne 0$. The at most finite number of terms with $\eta_n = 0$ does not affect the asymptotics. Note that, for $\eta_n = 0$, the modes $x \mapsto \exp(i\alpha_n \cdot \tilde{x} \pm \gamma_n x_3)$ are to be replaced by $x \mapsto \exp(i\alpha_n \cdot \tilde{x})(1 \pm x_3)$. Moreover, for the simplicity of the formulas, we assume $\gamma_n \ne 0$ and $\beta_n \ne 0$. Again, the at most finite number of exceptional terms does not affect the asymptotics.

Proof. i. Fixing a small ϵ s.t. $\Gamma^+ + 2\varepsilon < b$, we conclude

$$|v^+_n \exp(i\eta_n (\Gamma^+ + 2\epsilon - b))| \le \sum_n |v^+_n \exp(i\eta_n (\Gamma^+ + 2\varepsilon - b))|^2$$

$$\le \sqrt{\sum_n |v^+_n \exp(i\eta_n (\Gamma^+ + \varepsilon - b))|^2 (1 + |n|^2)^{1/2}}$$

$$\times \sqrt{\sum_n |\exp(i\eta_n\varepsilon)|^2 (1 + |n|^2)^{-1/2}}$$

$$\le c||v||_{H^{1/2}_\alpha(\{x:x_3=\Gamma^+ + \varepsilon\})} \le c||v||_{H^1_\alpha(\Omega^+_b)}, \tag{6.9}$$

where $\eta_n \sim i|n|$ was used in the last step. Consequently, we obtain $|v_n^+| \le cq^{|n|}$ with the constant $q := \exp((\Gamma^+ + 2\varepsilon) - b) < 1$. The assertions for u follow analogously.

ii. According to the integral $\int_{\Gamma_b^+} \mathcal{T}_N^+ v\,\varphi$ in the variational form (6.2), the solution v_N satisfies the boundary condition $\partial_3 v_N|_{\Gamma_b^+} = \mathcal{T}_N^+(v_N|_{\Gamma_b^+})$, that is, by (6.6) we conclude

$$\eta_n v_{N,n}^+ - \eta_n v_{N,n}^- = \begin{cases} \eta_n(v_{N,n}^+ + v_{N,n}^-) & \text{if } |n| \le N \\ 0 & \text{if } |n| > N. \end{cases} \tag{6.10}$$

Hence, $v_{N,n}^- = 0$ for $|n| \le N$ and $v_{N,n}^+ = v_{N,n}^-$ if $|n| > N$. The proof of the remaining assertions for v_N is analogous to that of part i). Suppose, e.g., that $|n| > N$. Then, similarly to (6.9), we obtain

$$|v_{N,n}^+[\exp(i\eta_n\left(\Gamma^+ + 2\epsilon - b)\right) + \exp(-i\eta_n\left(\Gamma^+ + 2\varepsilon - b)\right)]| \le c||v_N||_{H_\alpha^1(\Omega_b^+)}. \tag{6.11}$$

Again $\eta_n \sim i|n|$ implies $|v_{N,n}^+| \le cq^{|n|}$ with $q := \exp((\Gamma^+ + 2\varepsilon) - b) < 1$. The assertions for u_N follow analogously.

iii. In accordance with Lemma 4.1 the operator $\mathcal{A} : V_1 \to V_1'$ is strongly elliptic. Due to the proof of this lemma, $\mathcal{A}_N : V_1 \to V_1'$ is strongly elliptic too. Indeed, the only N dependent parts of $\mathcal{A}_N$ are the integrals over $\Gamma_b^\pm$. The truncated operator $-\mathrm{Re}\,\mathcal{T}_N$ is positive semidefinite [cf. (3.8)] and its quadratic form can be estimated from below by zero too. Now suppose the $\mathcal{T}_j$ are the operators of the splitting $\mathcal{T}^- = \mathcal{T}_1 + \mathcal{T}_2$ in the proof of Lemma 4.1 and the truncation $\mathcal{T}_{j,N}$ the restriction of $\mathcal{T}_j$ to the subspace of all functions with Fourier coefficients vanishing for $|n| > N$. Then together with $-\mathrm{Re}\,\mathcal{T}_N$ the truncation $-\mathrm{Re}\,\mathcal{T}_{1,N}$ is positive semidefinite. The truncation $\mathcal{T}_{2,N}$ of the compact operator $\mathcal{T}_2$, however, tends to zero in operator norm as $N \to \infty$. Thus $\mathcal{A}_N : V_1 \to V_1'$ is strongly elliptic at least for sufficiently large N.

Moreover, the above mentioned proof of Lemma 4.1 implies the uniform strong ellipticity estimate $\mathrm{Re}\,\langle \mathcal{A}_N(v,u),(v,u)\rangle \ge c||(v,u)||_{V_1}^2 - \mathrm{Re}\,\langle \mathcal{U}(v,u),(v,u)\rangle$ with a constant c and a compact operator $\mathcal{U}$ independent of N. We define $\mathcal{B}_N := \mathcal{A}_N + \mathrm{Re}\,\mathcal{U}$ and $\mathcal{B} := \mathcal{A} + \mathrm{Re}\,\mathcal{U}$. Then the uniform strong ellipticity of the $\mathcal{A}_N$ and $\mathcal{A}$ implies that $\mathrm{Re}\,\mathcal{B}_N$ and $\mathrm{Re}\,\mathcal{B}$ are coercive, that is, the $\mathcal{B}_N^{-1}$ are uniformly bounded and $\mathcal{B}_N^{-1}$ converges to $\mathcal{B}^{-1}$ strongly. From

$$\begin{aligned}
\mathcal{A}_N &= \mathcal{B}_N(I - \mathcal{B}_N^{-1}\mathrm{Re}\,\mathcal{U}) = \mathcal{B}_N(I - \mathcal{B}^{-1}\mathrm{Re}\,\mathcal{U}) - \mathcal{B}_N(\mathcal{B}_N^{-1} - \mathcal{B}^{-1})\mathrm{Re}\,\mathcal{U} \\
&= \mathcal{B}_N\mathcal{B}^{-1}(\mathcal{B} - \mathrm{Re}\,\mathcal{U}) - \mathcal{B}_N(\mathcal{B}_N^{-1} - \mathcal{B}^{-1})\mathrm{Re}\,\mathcal{U} \\
&= \left(\mathcal{A}^{-1}\mathcal{B}\mathcal{B}_N^{-1}\right)^{-1} - \mathcal{B}_N(\mathcal{B}_N^{-1} - \mathcal{B}^{-1})\mathrm{Re}\,\mathcal{U}, \quad ||(\mathcal{B}_N^{-1} - \mathcal{B}^{-1})\mathrm{Re}\,\mathcal{U}|| \to 0,
\end{aligned}$$

we conclude that $\mathcal{A}_N^{-1}$ is uniformly bounded. Using this fact and the exponential decay of the Rayleigh coefficients in the parts i) and ii) of the lemma, the estimate in the third assertion is a simple consequence of

$$\begin{aligned}
(v,u) - (v_N,u_N) &= \mathcal{A}^{-1}\mathcal{F}_0 - \mathcal{A}_N^{-1}\mathcal{F}_0 = \mathcal{A}_N^{-1}(\mathcal{A}_N - \mathcal{A})\mathcal{A}^{-1}\mathcal{F}_0, \\
||(v,u) - (v_N,u_N)||_{V_1} &\le c||(\mathcal{A}_N - \mathcal{A})(v,u)||_{V_1}.
\end{aligned}$$

■

B. FEM

Now we consider the classical FEM. We introduce FE meshes over the domains $\Omega_b^{\pm}$ and denote the mesh size, that is, the maximal diameter of the simplex subdomains by h. Using this h, we denote the space of piecewise linear functions in V_1, which are linear over each subdomain of the mesh, by V_h. Note that, for the sake of simplicity, we restrict ourselves to the linear case. Higher order elements can be treated analogously and are useful especially for large wavenumbers. For a given truncation number N and a given mesh of mesh size h, we compute the approximate solution $(v_{N,h}, u_{N,h}) \in V_h$ as the solution of the finite-element system

$$A_N((v_{N,h}, u_{N,h}), (\varphi_h, \psi_h)) = \mathcal{F}_0(\varphi_h), \quad \text{for all } (\varphi_h, \psi_h) \in V_h. \tag{6.12}$$

To get convergence estimates for this FEM, we need the following two assumptions on the regularity of the solution. Suppose the Sobolev space indices s_1, s_2 are fixed in the intervals $(1, 2]$ and $[0, 1)$, respectively.

(RA1) For given v_0 and u_0, consider the boundary value problem of quasiperiodic functions $(v, u) \in H^1_\alpha(\Omega_a^+) \times H^1_\alpha(\Omega_a^-)^3$ defined by

$$\begin{cases} (\Delta + k^2)\, v = 0 & \text{in } \Omega_a^+ := \{x \in \Omega_b^+ : x_3 < \frac{1}{2}(b + \Gamma^+)\}, \\ (\Delta^* + \omega^2 \rho) u = 0 & \text{in } \Omega_a^- := \{x \in \Omega_b^- : \frac{1}{2}(-b + \Gamma^-) < x_3\}, \\ \eta\, u \cdot \nu = \partial_\nu v & \text{on } \Gamma, \\ Tu = -v\, \nu & \text{on } \Gamma, \\ v = v_0 & \text{on } \Gamma_a^+ := \{x : 0 < x_1, x_2 < 2\pi, x_3 = \frac{1}{2}(b + \Gamma^+)\}, \\ u = u_0 & \text{on } \Gamma_a^- := \{x : 0 < x_1, x_2 < 2\pi, x_3 = \frac{1}{2}(-b + \Gamma^-)\}. \end{cases} \tag{6.13}$$

Suppose that any solution (v, u) of the variational formulation corresponding to (6.13) with $v_0 \in H^{s_1 - 1/2}(\Gamma_a^+)$ and $u_0 \in H^{s_1 - 1/2}(\Gamma_a^-)^3$ has the regularity $v \in H^{s_1}(\Omega_a^+)$ and $u \in H^{s_1}(\Omega_a^-)^3$.

(RA2) Consider the sesquilinear form corresponding to (6.13)

$$\begin{aligned} C((v, u), (\varphi, \psi)) := & \int_{\Omega_a^+} \left[\operatorname{grad} v \cdot \operatorname{grad} \overline{\varphi} - k^2 v \overline{\varphi} \right] dx - \eta \int_\Gamma u \cdot \nu\, \overline{\varphi}\, ds \\ & + \eta \left[\int_{\Omega_a^-} [\mathcal{E}(u, \overline{\psi}) - \omega^2 \rho u \cdot \overline{\psi}\,]\, dx - \int_\Gamma v\, \nu \cdot \overline{\psi} ds \right]. \end{aligned} \tag{6.14}$$

Clearly, for any functional $\mathcal{F} \in V_1'$, the solution (φ, ψ) of the adjoint variational equation $C((v, u), (\varphi, \psi)) = \mathcal{F}(v, u), \forall (v, u) \in V_1$ is in V_1. We suppose that, for $\mathcal{F}(v, u) := \langle v, f_v \rangle + \eta \langle u, f_u \rangle$ with functions $f_v \in H^{-s_2}(\Omega_a^+)$ and $f_u \in H^{-s_2}(\Omega_a^-)^3$, the solution (φ, ψ) is in $H^{2-s_2}(\Omega_a^+) \times H^{2-s_2}(\Omega_a^-)^3$ and satisfies the estimate

$$||\varphi||_{H^{2-s_2}(\Omega_a^+)} + ||\psi||_{H^{2-s_2}(\Omega_a^-)^3} \le c\{||f_v||_{H^{-s_2}(\Omega_a^+)} + ||f_u||_{H^{-s_2}(\Omega_a^-)^3}\}, \tag{6.15}$$

where c is independent of f_v and f_u.

Remark 6.3. The assumptions (RA1) and (RA2) are fulfilled for smooth boundaries Γ with $s_1 = 2$ and $s_2 = 0$. If Γ is piecewise linear, then the assumptions hold with s_1 and s_2 depending on the singularities at the vertices and edges (compare the asymptotic analysis of, e.g., [38]). For Lipschitz boundaries Γ, (RA1) and (RA2) are fulfilled at least with $s_1 = 1.5$ and $s_2 = 0.5$.

These regularity results have been observed by Estécahandy [39, Remark I.1.3.1] and are based on a deep result by Jerison and Kenig [40]. For instance in the case of (RA1) for a smooth interface Γ, we can argue as follows. The regularity results contained in (RA1) and (RA2) are of local nature s.t. the regularity of the solution is to be proved in the neighbourhood of Γ only. If $(v, u) \in H^1_\alpha(\Omega_a^+) \times H^1_\alpha(\Omega_a^-)^3$ is the solution of (6.13), then $\partial_\nu v = \eta u \cdot \nu \in H^{1/2}_\alpha(\Gamma)$ and $Tu = -v\nu \in H^{1/2}_\alpha(\Gamma)^3$. Well-known regularity theorems for the Neumann boundary value problem over smooth domains imply $(v, u) \in H^2_\alpha(\Omega_a^+) \times H^2_\alpha(\Omega_a^-)^3$. Repeating this trick, even higher order regularity can be derived.

Theorem 6.4. *Suppose the operator $\mathcal{A} : V_1 \to V_1'$ is invertible, that is, the variational equation (3.11) is uniquely solvable for any right-hand side from V_1'.*

i. There exist $N_0 > 0$ and $h_0 > 0$ s.t., for any $N > N_0$ and $h < h_0$, the FEM system (6.12) has a unique solution $(v_{N,h}, u_{N,h}) \in V_h$. For $N \to \infty$ and $h \to 0$, the FEM solutions $(v_{N,h}, u_{N,h})$ converge in the norm of V_1 to the solution $(u, v) \in V_1$ of (3.11).
ii. Suppose the right-hand side $\mathcal{F}_0$ is defined as in (6.3), that is, in accordance to the plane wave incidence in the scattering problem BVP. Furthermore, suppose regularity assumption (RA1) is satisfied with $1 < s_1 \le 2$. Then there exist constants c and q with $c > 0$ and $0 < q < 1$ s.t., for any $N > N_0$ and $h < h_0$,

$$||(v_{N,h}, u_{N,h}) - (v, u)||_{V_1} \le c||(v, u)||_{H^{s_1}(\Omega_b^+)\times H^{s_1}(\Omega_b^-)^3}\{h^{s_1-1} + q^N\}. \tag{6.16}$$

iii. Suppose the right-hand side $\mathcal{F}_0$ is defined as in (6.3) . Furthermore, suppose the regularity assumptions (RA1) with $1 < s_1 \le 2$ and (RA2) with $0 \le s_2 < 1$ are satisfied. Then there exist constants c and q with $c > 0$ and $0 < q < 1$ s.t., for any $N > N_0$ and $h < h_0$,

$$||(v_{N,h}, u_{N,h}) - (v, u)||_{H^{s_2}(\Omega_b^+)\times H^{s_2}(\Omega_b^-)^3} \le c||(v, u)||_{H^{s_1}(\Omega_b^+)\times H^{s_1}(\Omega_b^-)^3}\{h^{s_1-s_2} + q^N\}. \tag{6.17}$$

Proof. i. Clearly, $(v_{N,h}, u_{N,h}) - (v, u) = [(v_{N,h}, u_{N,h}) - (v_N, u_N)] + [(v_N, u_N) - (v, u)]$. In view of Lemma 6.1, it remains to analyze the convergence $[(v_{N,h}, u_{N,h}) - (v_N, u_N)] \to 0$. However, all estimates for this FEM must be shown uniformly w.r.t. N. We denote the L^2 orthogonal projection of V_1 onto the spline space V_h by P_h. From the proof of Lemma 6.1, we recall $\mathcal{A}_N = \mathcal{B}_N - \mathrm{Re}\,\mathcal{U}$ with a compact operator $\mathcal{U}$, the uniform coercivity $\mathrm{Re}\,\langle \mathcal{B}_N(v, u), (v, u)\rangle \ge c||(v, u)||^2$ and the strong convergence $\mathcal{A}_N^{-1} \to \mathcal{A}^{-1}$. In accordance with the proof of [20, Lemma 5.5], the uniform stability follows if we can show that the operator norm of $(P_h - I)\mathcal{A}_N^{-1}\mathrm{Re}\,\mathcal{U} : V_1 \to V_1$ is smaller than any prescribed threshold for h sufficiently small (compare the operator$(P_h - I)B^{-1}T$ in [20, Lemma 5.5]). However, this is true as

$$(P_h - I)\mathcal{A}_N^{-1}\mathrm{Re}\,\mathcal{U} = (P_h - I)[\mathcal{A}^{-1}\mathrm{Re}\,\mathcal{U}] + (P_h - I)[\mathcal{A}_N^{-1} - \mathcal{A}^{-1}]\mathrm{Re}\,\mathcal{U},$$

as $[\mathcal{A}^{-1}\mathrm{Re}\,\mathcal{U}]$ and $\mathcal{U}$ are compact, and as $P_h \to I$ as well as $\mathcal{A}_N^{-1} \to \mathcal{A}^{-1}$. Now the uniform stability implies

$$||(v_N, u_N) - (v_{N,h}, u_{N,h})||_{V_1} \le c \inf_{(\varphi_h,\psi_h)\in V_h} ||(v_N, u_N) - (\varphi_h, \psi_h)||_{V_1}. \tag{6.18}$$

The uniform convergence of the FEM in the norm of V_1 follows as the discrete set $\{(v_N, u_N) : N = 0, 1, \dots\}$ is precompact due to $(v_N, u_N) \to (v, u)$.

ii) This part follows from (6.18) and the approximation property of finite-element functions if we can prove $||(v_N, u_N)||_{H^{s_1}(\Omega_b^+)\times H^{s_1}(\Omega_b^-)^3} < c$. However, $||(v_N, u_N)||_{V_1} < c$ and the proof to Lemma 6.1 i) and ii) implies that the H^{s_1} norms over $\Omega_b^\pm \backslash \Omega_a^\pm$ are uniformly bounded. Consequently, we conclude $v|_{\Gamma_a^+} \in H^{s_1-1/2}(\Gamma_a^+)$ and $u|_{\Gamma_a^-} \in H^{s_1-1/2}(\Gamma_a^-)^3$ s.t. assumption (RA1) yields $||(v_N, u_N)||_{H^{s_1}(\Omega_b^+)\times H^{s_1}(\Omega_b^-)^3} < c$.

iii) The estimate in Sobolev norms of order less than 1 follows from Nitsche's trick, from part ii) of the Lemma and from the approximation property. It remains only to show that the operators $\mathcal{A}_N^* : H^{2-s_2}(\Omega_b^+) \times H^{2-s_2}(\Omega_b^-)^3 \to H^{-s_2}(\Omega_b^+) \times H^{-s_2}(\Omega_b^-)^3$ are invertible with uniformly bounded inverse operators. More precisely, for given $g_v \in H^{-s_2}(\Omega_b^+)$ and $g_u \in H^{-s_2}(\Omega_b^+)^3$, we have to show that the solution $(\varphi, \psi) = [\mathcal{A}_N^*]^{-1}(g_u, g_v) \in V_1$ satisfies

$$\begin{aligned} ||\varphi||_{H^{2-s_2}(\Omega_b^+)} &\leq c\left\{||g_u||_{H^{-s_2}(\Omega_b^+)} + ||g_v||_{H^{-s_2}(\Omega_b^+)^3}\right\} \\ ||\psi||_{H^{2-s_2}(\Omega_b^+)^3} &\leq c\left\{||g_u||_{H^{-s_2}(\Omega_b^+)} + ||g_v||_{H^{-s_2}(\Omega_b^+)^3}\right\}. \end{aligned}$$

We choose a partition of unity $1 = \sum_{j=1}^3 \chi_j(x_3)$ with smooth functions χ_j s.t.

$$\left[\frac{1}{4}\Gamma^+ + \frac{3}{4}b, b\right] \subseteq \{x_3 : \chi_1(x_3) = 1)\} \subseteq \operatorname{supp}\chi_1 \subseteq \left[\frac{1}{2}(\Gamma^+ + b), b\right],$$

$$\left[\frac{3}{4}\Gamma^- - \frac{1}{4}b, \frac{3}{4}\Gamma^+ + \frac{1}{4}b\right] \subseteq \{x_3 : \chi_2(x_3) = 1)\} \subseteq \operatorname{supp}\chi_2 \subseteq \left[\frac{1}{2}(\Gamma^- - b), \frac{1}{2}(\Gamma^+ + b)\right],$$

$$\left[-b, \frac{1}{4}\Gamma^- - \frac{3}{4}b\right] \subseteq \{x_3 : \chi_3(x_3) = 1)\} \subseteq \operatorname{supp}\chi_3 \subseteq \left[-b, \frac{1}{2}(\Gamma^- - b)\right].$$

So it is sufficient to prove the regularity estimates for the functions $[\chi_1\varphi]$, $[\chi_2\varphi]$, $[\chi_2\psi]$ and $[\chi_3\psi]$ instead of φ and ψ.

The functions $[\chi_2\varphi]$ and $[\chi_2\varphi]$, however, are solutions of the boundary value problem appearing in the assumption (RA2) with H^{-s_2} bounded right-hand side. Thus $[\chi_2\psi]$ and $[\chi_3\psi]$ have bounded H^{2-s_2} norms according to assumption (RA2). The function $[\chi_1\varphi]$ is a solution of the Helmholtz equation with inhomogeneous H^{-s_2} bounded right-hand side and the boundary condition $\partial_3[\chi_1\varphi]|_{\Gamma_b^+} = \mathcal{T}_N^*([\chi_1\varphi]|_{\Gamma_b^+})$. Now we take a quasiperiodic H^{-s_2} extension of the right-hand side of the Helmholtz equation, which has a bounded support in x_3-direction. Using a volume potential based on a quasiperiodic Green's function satisfying the radiation condition for the lower half plane, we can construct a quasiperiodic solution φ_0 of the inhomogeneous Helmholtz equation with the just extended right-hand side. As this is H^{2-s_2} bounded, it remains to estimate the H^{2-s_2} norm of $\varphi_{00} := [\chi_1\varphi] - \varphi_o$. This function, however, is a solution of the homogeneous Helmholtz equation in $\{x : 0 < x_1, x_2 < 2\pi, x_3 < b\}$ satisfying the radiation condition and the inhomogeneous boundary condition $\partial_3\varphi_{00}|_{\Gamma_b^+} - \mathcal{T}_N^*(\varphi_{00}|_{\Gamma_b^+}) = \partial_3\varphi_0|_{\Gamma_b^+} - \mathcal{T}_N^*(\varphi_0|_{\Gamma_b^+})$. The uniform H^{2-s_2} bound of the solution of the latter problem can be derived easily by Rayleigh expansions. Finally, the estimate for $[\chi_3\psi]$ is analogous to that for $[\chi_1\varphi]$. ■

Now we discuss how to choose the truncation of the DtN map in an optimal way. Optimal means that the error estimates in (6.16) and (6.17) are majorized by the typical FEM bound ch^κ with $\kappa := s_1 - 1$ and $\kappa := s_1 - s_2$, respectively, and that the number of nontruncated modes in the DtN map is as small as possible. We shall use the following notation. Again c stands for a generic positive constant, the value of which varies from instance to instance. By d we denote the period of the periodic grating structure. Thus, till the end of this section, we do not suppose that the period is normalized to 2π. However, for simplicity, we assume the same period in the x_1- and the x_2-direction. Moreover, we assume that the thickness of the grating is also in the size of d. Note that the x_1- and x_2-components of the wave vector for the nth Rayleigh mode is given by $\alpha_n := \alpha + \frac{2\pi}{d} n$ [cf. (2.6)]. We introduce the thickness Θ of the additional strips between the artificial boundaries $\Gamma_{\pm b}$ and the interface Γ. More precisely, for a sufficiently small $\epsilon > 0$, we suppose $\Theta := \min\{b - \Gamma^+, \Gamma^- - (-b)\} - \epsilon$, which is supposed to be fixed. Finally, we slightly modify the truncation algorithm. We replace $\mathcal{T}_N^{\pm}$ [cf. (6.1)] by $\mathcal{T}_\ell^{\pm}$, where

$$(\mathcal{T}_\ell^+ w)(\tilde{x}) := \sum_{\substack{cn\in\mathbb{Z}^2:\\ |\alpha_n|^2\le k^2+\ell^2}} i\eta_n\, \hat{w}_n \exp(i\alpha_n\cdot\tilde{x}),\ (\mathcal{T}_\ell^- w)(\tilde{x}) := \sum_{\substack{n\in\mathbb{Z}^2:\\ |\alpha_n|^2\le k_s^2+\ell^2}} iW_n\, \hat{w}_n \exp(i\alpha_n\cdot\tilde{x}).$$

For the truncated modes, we conclude $|\eta_n| > \ell$ and $|\gamma_n| > \ell$. Due to $k_s > k_p$, we even get $|\beta_n| > \ell$. Consequently, we arrive at $|e^{i\eta_n[b-\Gamma^+]}| < e^{-\ell\Theta}$ as well as $|e^{i\beta_n[\Gamma^- -(-b)]}| < e^{-\ell\Theta}$ and $|e^{i\gamma_n[\Gamma^- -(-b)]}| < e^{-\ell\Theta}$. Following the estimates in the proofs of this section, we get the error estimates (6.16) and (6.17) with q^N replaced by $e^{-\ell\Theta}$. In other words, for an optimal truncation we should have $e^{-\ell\Theta} = ch^\kappa$ or equivalently $\ell = (c + \kappa\log h^{-1})/\Theta$. Using the truncation operators $\mathcal{T}_N^{\pm}$, this corresponds to choosing $N = k_m + \frac{c+\kappa\log h^{-1}}{\Theta}$ with $k_m := \max\{k, k_s\}$.

Now suppose h is small and that k_m is a fixed number, which is small in comparison to $\ell = (c + \kappa\log h^{-1})/\Theta$. Then the complexity of the proposed algorithm, that is, the number of arithmetic operations necessary to compute an approximate solution with an error less than a prescribed threshold, is comparable with that for a classical FEM applied to an elliptic boundary value problem with local boundary condition. Indeed, in this case the number of degrees of freedom included into the DtN map is of order $\mathcal{O}(1 + [\log h^{-1}]^2)$ and much less than the degrees of freedom of the finite elements. Conversely, if the mesh size h is chosen as large as possible, for example, by a rule like $h = \lambda/8 = 2\pi/k_m 8$, then the complexity might be slightly higher. Using an iterative solver with a good preconditioner and applying one of the fast numerical algorithms for integral equations to the DtN, the complexity of conventional FEM can be reached again. However, we note that a rule like $h = \lambda/8$ is usually not sufficient due to a numerical effect, which is called numerical pollution or numerical dispersion (cf. [41]).

VII. VARIATIONAL FORMULATION IN TWO DIMENSIONS

In this section, we change the notation. For a reduction to two dimensions, the last component should be dropped. Therefore, we suppose that the geometry is periodic in the x_1-component, that the component normal to the interface is the x_2-component (formerly the x_3-component), and that the geometry is invariant in the x_3 direction (formerly the x_2 direction of the second period). The cross-section of Γ in the (x_1, x_2)-plane will be represented by a curve Λ, which is 2π-periodic in x_1. All elastic waves are assumed to be propagating perpendicular to the x_3-axis, so that the problem can be treated as a problem of plane elasticity. This implies that the incident plane wave is of the form

$$v^{\text{in}}(x_1, x_2) = \exp(i\alpha x_1 - i\eta_0 x_2), \quad \alpha := k\sin\theta, \eta_0 := k\cos\theta, \tag{7.1}$$

where $\theta \in (-\pi/2, \pi/2)$ denotes the angle of incidence.

The boundary value problem for finding α-quasiperiodic solutions $v = v(x_1, x_2)$ and $u = (u_1(x_1, x_2), u_2(x_1, x_2))^\top$ can be formulated analogously to (2.2) with the two-dimensional *traction* operator having the form

$$Tu = 2\mu\, \partial_{\mathbf{n}} u + \lambda \operatorname{div} u\, \mathbf{n} + \mu \begin{pmatrix} n_2\,(\partial_1 u_2 - \partial_2 u_1) \\ n_1\,(\partial_2 u_1 - \partial_1 u_2) \end{pmatrix} \quad \text{on } \Lambda\,, \tag{7.2}$$

where $\mathbf{n} = (n_1, n_2)^\top$ denotes the exterior unit normal on Λ. As done in 3D, we will confine ourselves to a single periodic cell by setting

$$\Lambda_b^{\pm} := \left\{(x_1, \pm b)^\top : 0 \le x_1 \le 2\pi\right\}, \Omega_b^{\pm} := \left\{(x_1, x_2)^\top : \exists x \in \Omega^{\pm} \text{ s.t. } 0 < x_1 < 2\pi,\ x_2 \lessgtr \pm b\right\}.$$

The upward and downward Rayleigh expansions for v^{sc} and u can be expressed as

$$\begin{aligned} v^{sc}(x) &= \sum_{n\in\mathbb{Z}} v_n \exp(i\alpha_n x_1 + i\eta_n x_2), \quad x_2 > \Lambda^+, \\ u(x) &= \sum_{n\in\mathbb{Z}} \left\{ A_{p,n} \begin{pmatrix} \alpha_n \\ -\beta_n \end{pmatrix} \exp(i\alpha_n x_1 - i\beta_n x_2) - A_{s,n} \begin{pmatrix} \gamma_n \\ \alpha_n \end{pmatrix} \exp(i\alpha_n x_1 - i\gamma_n x_2) \right\}, x_2 < \Lambda^- \end{aligned} \tag{7.3}$$

with α_n, η_n, β_n, and γ_n defined analogously to the 3D case. The DtN maps $\mathcal{T}^{\pm}$ can be represented as

$$(\mathcal{T}^+ w)(x) := \sum_{n\in\mathbb{Z}} i\eta_n\, \hat{w}_n \exp(i\alpha_n x_1) \quad \text{for } w = \sum_{n\in\mathbb{Z}} \hat{w}_n \exp(i\alpha_n x_1) \in H_\alpha^s(\Lambda_b^+), s \ge 1/2, \tag{7.4}$$

$$(\mathcal{T}^- w)(x) := \sum_{n\in\mathbb{Z}} i\, W_n\, \hat{w}_n \exp(i\alpha_n x_1) \ \text{ for } w = \sum_{n\in\mathbb{Z}} \hat{w}_n \exp(i\alpha_n x_1) \in H_\alpha^s(\Lambda_b^-)^2, s \ge 1/2, \tag{7.5}$$

where W_n is the 2×2 matrix

$$W_n := \begin{pmatrix} \omega^2\beta_n/d_n & -2\mu\alpha_n + \omega^2\alpha_n/d_n \\ 2\mu\alpha_n - \omega^2\alpha_n/d_n & \omega^2\gamma_n/d_n \end{pmatrix},\ d_n := \alpha_n^2 + \beta_n\gamma_n. \tag{7.6}$$

The expression (7.6) follows from the arguments of [18] and differs from the matrix corresponding to upward propagating elastic waves only in the signs of the off-diagonal terms. We state the variational formulation for the FSI problem in the two-dimensional setting as follows: Find $(v, u) \in V_1 := H_\alpha^1(\Omega_b^+) \times H_\alpha^1(\Omega_b^-)^2$ s.t.

$$A((v, u), (\varphi, \psi)) = \int_{\Lambda_b^+} f_0 \overline{\varphi}\, ds \quad \text{for all } (\varphi, \psi) \in V_1, \tag{7.7}$$

where the sesquilinear form $A : V_1 \times V_1 \to \mathbb{C}$ is defined analogously to (3.12) with Λ, $\Lambda_b^{\pm}$, $\mathbf{n}$ in place of Γ, $\Gamma_b^{\pm}$ and ν, and

$$\begin{aligned} \mathcal{E}(u, \overline{\varphi}) = (2\mu + \lambda)\,(\partial_1 u_1\, \partial_1 \overline{\varphi}_1 + \partial_2 u_2\, \partial_2 \overline{\varphi}_2) + \mu\,(\partial_2 u_1\, \partial_2 \overline{\varphi}_1 + \partial_1 u_2\, \partial_1 \overline{\varphi}_2) \\ +\ \lambda\,(\partial_1 u_1\, \partial_2 \overline{\varphi}_2 + \partial_2 u_2\, \partial_1 \overline{\varphi}_1) + \mu\,(\partial_2 u_1\, \partial_1 \overline{\varphi}_2 + \partial_1 u_2\, \partial_2 \overline{\varphi}_1)\,. \end{aligned}$$

The function $f_0 \in H_\alpha^{-1/2}(\Lambda_b^+)$ on the right-hand side of (7.7) is given by $f_0(x_1) := -2i\eta_0 \exp(i\alpha x_1 - i\eta_0 b)$. All the uniqueness, existence and nonuniqueness results in Section IV carry over to the 2D case. Moreover, there holds the energy balance formula

$$1 = \sum_{n\in\mathbb{Z}:\eta_n>0} \frac{\eta_n}{\eta_0} |v_n|^2 + \omega^2\rho\,\eta \left(\sum_{n\in\mathbb{Z}:\beta_n>0} \frac{\beta_n}{\eta_0} |A_{p,n}|^2 + \sum_{n\in\mathbb{Z}:\gamma_n>0} \frac{\gamma_n}{\eta_0} |A_{s,n}|^2 \right). \tag{7.8}$$

When the scattering interface Γ coincides with the straight line $\Gamma_0 := \{(x_1, x_2) : x_2 = 0\}$, the energy balance formula (7.8) can be verified straightforwardly. Indeed, for the incident plane wave (7.1), the unique solution of (2.2) takes the form

$$v^{sc}(x) = a_1 \exp(i\alpha x_1 + i\eta_0 x_2), \quad x \in \Omega^+, \tag{7.9}$$

$$u(x) = a_2 \begin{pmatrix} \alpha \\ -\beta_0 \end{pmatrix} \exp(i\alpha x_1 - i\beta_0 x_2) + a_3 \begin{pmatrix} \gamma_0 \\ \alpha \end{pmatrix} \exp(i\alpha x_1 - i\gamma_0 x_2), x \in \Omega^-, \tag{7.10}$$

where the coefficients $a_j \in \mathbb{C}$, $j = 1, 2, 3$ can be obtained by solving the linear system

$$\begin{pmatrix} i\eta_0 & \rho_f\omega^2\beta_0 & -\rho_f\omega^2\alpha \\ 0 & 2i\mu\alpha\beta_0 & 2i\mu\gamma_0^2 - i\mu k_s^2 \\ 1 & 2i\mu\beta_0^2 + i\lambda k_p^2 & -2i\mu\alpha\gamma_0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} i\eta_0 \\ 0 \\ -1 \end{pmatrix}. \tag{7.11}$$

We suppose further that $k_p < k_s \le \alpha = k\sin\theta < k$. This physically implies that the acoustic wave velocity in the fluid is smaller than the shear and compressional wave velocities in the solid. Straightforward calculations show that a_1 is of the form

$$a_1 = \frac{c_1 + ic_2}{c_1 - ic_2},$$

where $c_1, c_2 \in \mathbb{R}$ are some constants. This verifies the energy balance $|a_1|^2 = 1$. Note that the terms in the bracket of (7.8) vanish under our assumptions.

The variational formulations (7.7) and (3.11) are convenient for theoretical justifications. However, in numerical implementations we prefer the following formulation equivalent to (7.7):

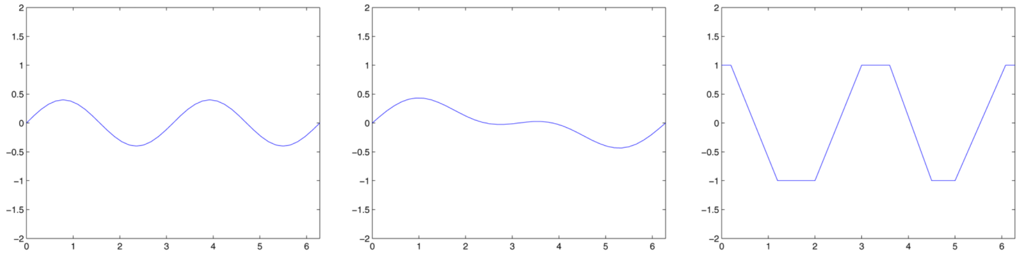
$$A((v^{sc}, u), (\varphi, \psi)) = \int_\Gamma \left(\partial_{\mathbf{n}} v^{in}\overline{\varphi} - \eta \mathbf{n} v^{in} \cdot \overline{\psi} \right) ds \quad \text{for all } (\varphi, \psi) \in V_1. \tag{7.12}$$

In other words, we compute the scattered field $v^{sc} = v - v^{in}$ instead of the total field v over the domain Ω_b^+.

The truncation of the DtN mappings and the FEM can be defined analogously to the 3D case. With a straightforward modification of the conditions (RA1) and (RA2), Theorem 6.4 remains true.

VIII. NUMERICAL EXAMPLES

In this section, we present several numerical tests to confirm our theoretical results in 2D. The computational domains $\Omega_b^\pm$ are discretized by quasiuniform triangular elements. A direct solver



(a) Grating 1 (b) Grating 2 (c) Grating 3

FIG. 3. One-dimensional periodic interfaces. [Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com.]

for sparse systems is employed for computing solutions of the resulting linear system. In our numerical tests, the energy function is defined by

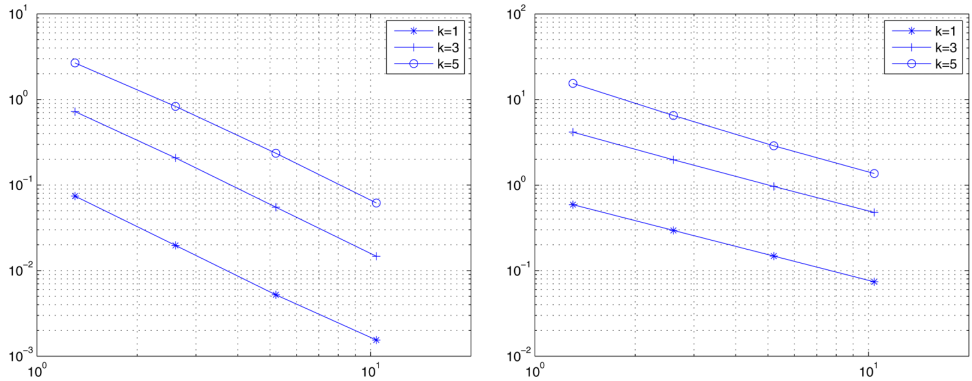
$$
\begin{aligned}
E_{N,h} &:= \sum_{n=-N:\eta_n>0}^{N} \frac{\eta_n}{\eta_0} |v_n^{N,h}|^2 + \omega^2\rho\eta \left(\sum_{n=-N:\beta_n>0}^{N} \frac{\beta_n}{\eta_0} |A_{p,n}^{N,h}|^2 + \sum_{n=-N:\gamma_n>0}^{N} \frac{\gamma_n}{\eta_0} |A_{s,n}^{N,h}|^2 \right), \\
v_n^{N,h} &:= \frac{1}{2\pi} \int_{\Gamma_b^+} v_{N,h}(x_1, b) \exp(-i(\alpha_n x_1 + \eta_n b)) dx_1, \\
A_{p,n}^{N,h} &:= \frac{1}{2\pi} \frac{1}{\alpha^2 + \beta_n\gamma_n} \int_{\Gamma_b^+} u_{N,h}(x_1, b) \cdot (\alpha_n, -\gamma_n)^\top \exp(-i(\alpha_n x_1 + \beta_n b)) dx_1, \\
A_{s,n}^{N,h} &:= \frac{1}{2\pi} \frac{1}{\alpha^2 + \beta_n\gamma_n} \int_{\Gamma_b^+} u_{N,h}(x_1, b) \cdot (-\beta_n, -\alpha_n)^\top \exp(-i(\alpha_n x_1 + \gamma_n b)) dx_1,
\end{aligned}
$$

where $N = 20$ is the truncation number of the Rayleigh series. Note that the exact value of the energy function is $E_{\infty,0} = 1$ [cf. (7.8)].

Example 1. In this example, we check whether our code provides the correct solution. We consider two grating profiles, one is smooth (Grating 1) and another one is piecewise linear (Grating 3), shown in Fig. 3. We take the parameters $\omega = 1, \mu = 1, \lambda = 1, \rho_f = 2, \rho = 1, b = 3, \theta = \pi/6$. To the system (2.2) we add inhomogeneous right-hand sides $g \in H^{-1/2}(\Gamma)$ and $h \in H^{-1/2}(\Gamma)^2$ over the interface, that is,

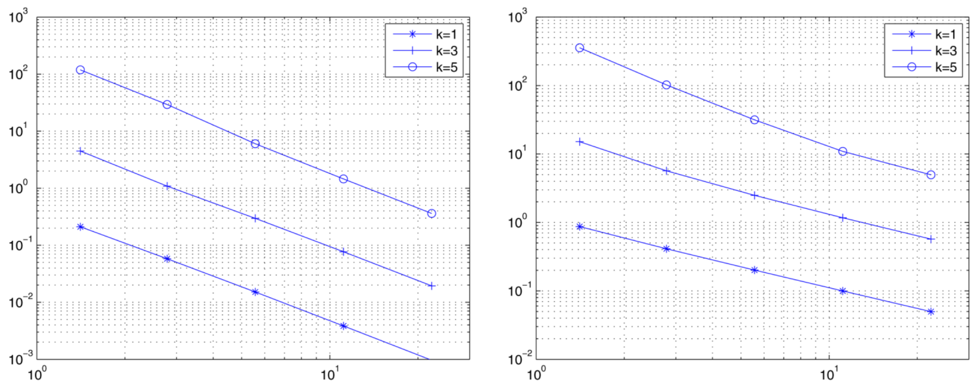
$$
\begin{cases}
(\Delta + k^2)\, v = 0 & \text{in } \Omega^+, \\
(\Delta^* + \omega^2\rho)\, u = 0 & \text{in } \Omega^-, \\
\eta\, u \cdot \nu - \partial_\nu v = g & \text{on } \Gamma, \\
T u + v\, \nu = h & \text{on } \Gamma.
\end{cases}
\tag{8.1}
$$

The results presented in Sections II–VII are still true for (8.1). We choose g and h s.t. the exact solutions of (8.1) take the forms (7.9) and (7.10) with $a_1 = 1, a_2 = 2$ and $a_3 = -1$. In Figs. 4 and 5, we present the numerical error $||(v^{sc}, u) - (v^{sc}_{N,h}, u_{N,h})||$ in the spaces $V_0 := L^2_\alpha(\Omega_b^+) \times L^2_\alpha(\Omega_b^-)^2$ and $V_1 := H^1_\alpha(\Omega_b^+) \times H^1_\alpha(\Omega_b^-)^2$ with respect to $1/h$ for $k = 1, 3$ and 5. We can obviously observe that $||(v^{sc}, u) - (v^{sc}_{N,h}, u_{N,h})||_{V_0} = O(h^2)$ and $||(v^{sc}, u) - (v^{sc}_{N,h}, u_{N,h})||_{V_1} = O(h)$.



(a) $||(v^{sc}, u) - (v^{sc}_{N,h}, u_{N,h})||_{V_0}$ (b) $||(v^{sc}, u) - (v^{sc}_{N,h}, u_{N,h})||_{V_1}$

FIG. 4. Log-log plot of errors vs. $1/h$. Errors in V_0-norm (left) and V_1-norm (right) of FEM solution for (8.1) with Grating 1. [Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com.]



(a) $||(v^{sc}, u) - (v^{sc}_{N,h}, u_{N,h})||_{V_0}$ (b) $||(v^{sc}, u) - (v^{sc}_{N,h}, u_{N,h})||_{V_1}$

FIG. 5. Log–log plot of errors vs. $1/h$. Errors in V_0-norm (left) and V_1-norm (right) of FEM solution for (8.1) with Grating 3. [Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com.]

Example 2. In the second example, we consider the system (2.2) with homogeneous boundary conditions and the smooth grating profiles 1 and 2 illustrated in Fig. 3. We plot the numerical energy functions with respect to $1/h$ in Fig. 6. In these cases the wave number in the fluid is taken as $k = 9$ and the other coefficients are set as in Example 1. The numerical solutions are consistent with the proposed energy balance formula and thus support our theoretical results. Next we consider the problem (2.2) with Grating 2 for a fixed mesh. We define $k_0 := \max\{k, k_p, k_s\}$ and set $N_0 := \max\{|n| : |\alpha_n| \le k_0 \text{ or } |\alpha_{-n}| \le k_0\}$ and

$$N_\tau := \min\left\{N : \frac{||(v^{sc}_{N,h}, u_{N,h}) - (v^{sc}_{20,h}, u_{20,h})||_{V_0}}{||(v^{sc}_{20,h}, u_{20,h})||_{V_0}} \le 0.01\right\}.$$

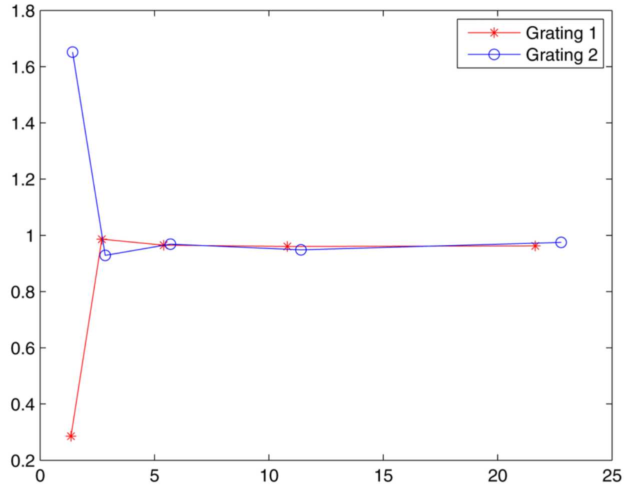


FIG. 6. Numerical energy function $E_{N,h}$ vs. $1/h$ for the one-dimensional periodic interfaces.

TABLE I. The values of N_τ (cols. 2–5) compared with the N_0 (col. 6) depending on k and b.

	$b = 0.7$ ($h = 0.3181$)	$b = 1$ ($h = 0.3385$)	$b = 2$ ($h = 0.3083$)	$b = 3$ ($h = 0.3621$)	N_0
$k = 1$	**4**	**3**	**2**	**2**	**2**
$k = 3$	**5**	**4**	**4**	**4**	**5**
$k = 5$	**7**	**7**	**7**	**9**	**8**

Table I exhibits the numbers N_τ and N_0 depending on the wave number k and the x_2-coordinates $\pm b$ of the truncation boundaries $\Gamma_b^\pm$. The truncation number N can be chosen relatively small. In our example, we even do not need to choose N_τ larger than N_0.

Example 3. In this example, we consider a water–brass interaction problem with the physical parameters taken from [7]. We set the period of the interface to $\Lambda = 2.2$ mm and the height of the corrugation $h = 0.05$ mm (cf. Fig. 7, left). The frequency and speed of sound are $\omega = 2\pi\ 1.5 \cdot 10^6$ Hz and $c_0 = 1\,480$ m/s, and the density of water is $\rho_f = 1\,000$ kg/m^3. The density of brass is $\rho = 8\,100$ kg/m^3. The velocities for shear and pressure waves are $c_s = 2\,270$ m/s and $c_p = 4\,840$ m/s. Figure 8 shows the numerical solutions of the elastic displacement in the brass and the scattered acoustic field in the water, where we have taken the incident angle $\theta = 0$, $b = 3$ mm and the mesh size $h = 0.0462$ mm. We compute the relative V_0-error and V_1-error of the scattered field in Ω_b^+ via $RE_1 = ||(v^{sc}, u) - (v^{sc}_{N_0,h_0}, u_{N_0,h_0})||_{V_0}$ and $RE_2 = ||(v^{sc}, u) - (v^{sc}_{N_0,h_0}, u_{N_0,h_0})||_{V_1}$, where $(v^{sc}_{N_0,h_0}, u_{N_0,h_0})$ is the numerical solution with $N_0 = 20$ and $h_0 = 0.0209$ mm. In Table II, we present the relative errors and the order of accuracy, and one can observe the expected second order accuracy for RE_1 and first order accuracy for RE_2. Furthermore, we plot the frequency spectrum of the normally reflected signals in Fig. 9 (left), showing the reflection efficiency R_0 (in dB) with respect to $f := \omega/(2\pi)$. Here, $R_0 := |v_0^{N,h}|^2$ with $N = 20$ and $h = 0.0836$ mm, and $R_0(\text{ in dB }) := 20\log_{10} R_0$. In Fig. 9 (left), the clearly identified frequency dip shows the existence of a Wood anomaly. This implies the possibility of generating interface waves by the mechanical coupling of acoustic waves incident upon a fluid–solid interface and mode conversion (cf.

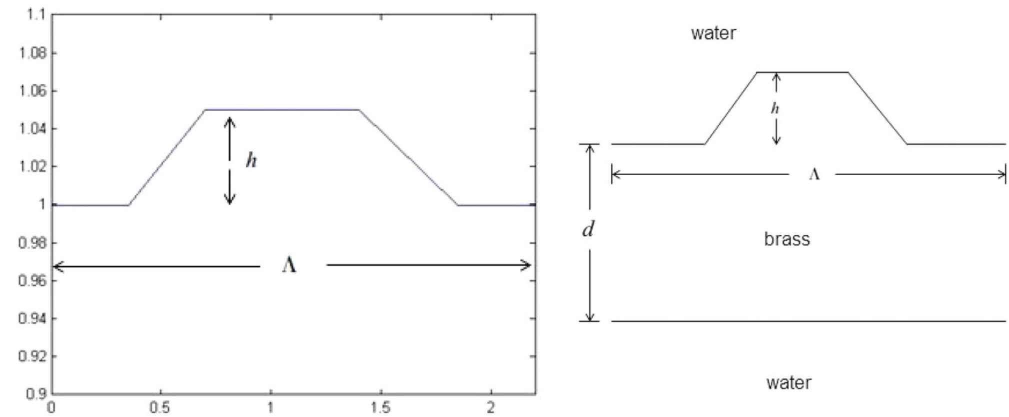
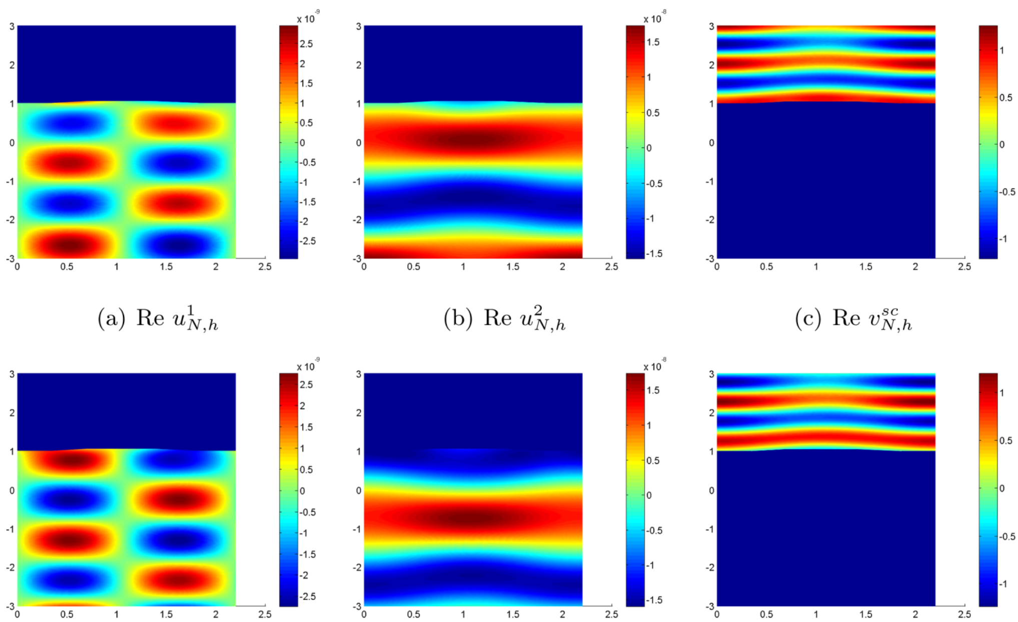


FIG. 7. The geometry settings of the interface for Examples 3 and 4. [Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com.]



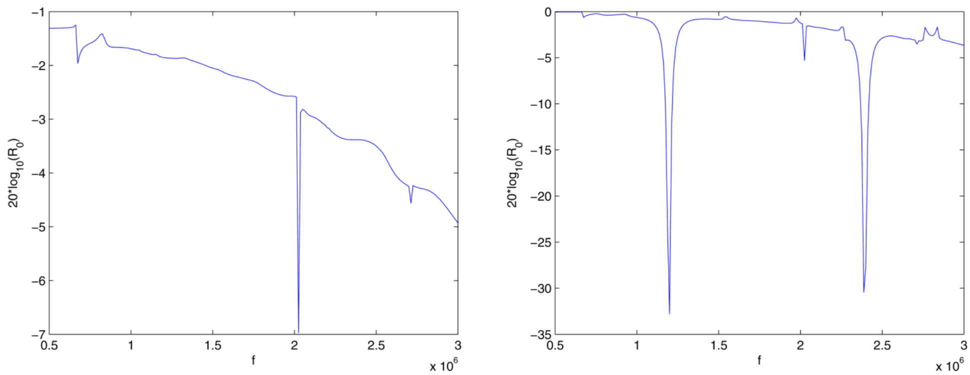
(a) Re $u^1_{N,h}$ (b) Re $u^2_{N,h}$ (c) Re $v^{sc}_{N,h}$

(d) Im $u^1_{N,h}$ (e) Im $u^2_{N,h}$ (f) Im $v^{sc}_{N,h}$

FIG. 8. Numerical solutions with $b = 3$ mm and mesh size $h = 0.0209$ mm for example 3. [Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com.]

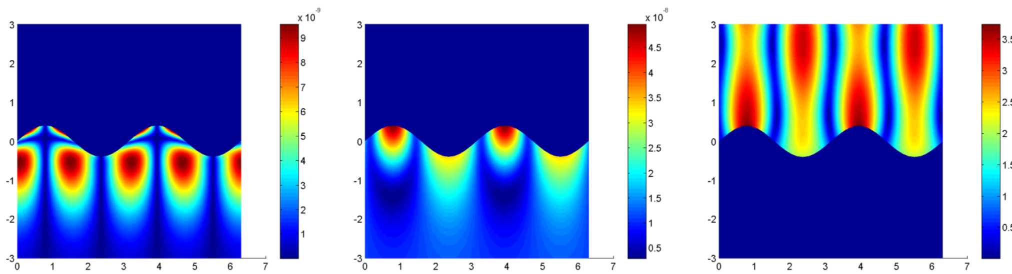
TABLE II. Relative L^2-error and order of accuracy for Example 3.

h	RE_1	Order	RE_2	Order
6.69 E– 1	2.21E0	–	1.45E1	–
1.51	7.76 E– 1		6.54E0	1.15
1.67 E– 1	2.30 E– 1	1.75	2.74E0	1.26
8.36 E– 2	5.88 E– 2	1.97	1.21E0	1.18
4.18 E– 2	1.21 E– 2	2.28	5.24 E– 1	1.21



(a) R_0(in dB) for Example 3 (b) R_0(in dB) for Example 4

FIG. 9. The reflection efficiency R_0 (in dB) vs. f for Examples 3 and 4. [Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com.]



(a) $|u^1_{N,h}|$ (b) $|u^2_{N,h}|$ (c) $|v^{sc}_{N,h}|$

FIG. 10. Numerical solutions with $b = 3$ mm and mesh size $h = 0.0462$ mm for the plane wave normally incident onto Grating 1. [Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com.]

[2, 7]), even if the amplitude of the interface is rather small compared to the period. Note that surface waves cannot be incited by a plane wave normally incident onto a flat surface, due to the unique solvability of the linear system (7.11) when $\alpha = 0$. The numerical solutions for the above water–brass interaction problem with Grating 1 are illustrated in Fig. 10, where the amplitude of the interface is comparable with the period.

Example 4. In the last example, we consider an interesting water–brass–water interaction problem in periodic structures. The geometry of the two-dimensional diffraction problem is shown in Fig. 7 (right). The upper boundary of the layer is the same as Fig. 7 (left), while the lower boundary is a flat surface. The thickness of the layer is taken as $d = 2$ mm, and the other parameters are set to be the same as in Example 3. We plot the reflection efficiency R_0 (in dB) with respect to f in Fig. 9 (right). It can be seen that there are two major dips occurring in R_0, which differ drastically from those for the water–brass interaction problem shown in Fig. 9 (left). The extra dip implies that additional interface waves may exist in a layer structure. Our computational results are in a good agreement with those of [7]. Of course, an exact comparison was not possible as we do not know the exact geometry used for the calculations in [7].

References

1. N. F. Declercq, J. Degrieck, R. Briers, and O. Leroy, Diffraction of homogeneous and inhomogeneous plane waves on a doubly corrugated liquid/solid interface, Ultrasonics 43 (2005), 605–618.
2. J. M. Claeys, O. Leroy, A. Jungman, and L. Adler, Diffraction of ultrasonic waves from periodically rough liquid-solid surface, J Appl Phys 54 (1983), 5657.
3. S. W. Herbison, Ultrasonic diffraction effects on periodic surfaces, George W. Woodruff School of Mechanical Engineering, PhD Thesis, Georgia Institute of Technology, 2011.
4. M. Sanna, Numerical simulation of fluid-structure interaction between acoustic and elastic waves, University of Jyväskylä, PhD Thesis, Jyväskylä Studies in Computing 133, 2011.
5. N. Favretto-Anrès and G. Rabau, Excitation of the Stoneley-Scholte wave at the boundary between an ideal fluid and a viscoelastic solid, J Sound Vibration 203 (1997), 193–208.
6. J. W. S. Lord Rayleigh, On the dynamical theory of gratings, Proc R Soc Lond A 79 (1907), 399–416.
7. K. E. A. Van Den Abeele, R. Briers, and O. Leroy, Inhomogeneous plane-wave scattering and mode stimulation on periodic rough surfaces, J Acoust Soc Am 99 (1996), 2883–2897.
8. T. Arens, The scattering of plane elastic waves by a one-dimensional periodic surface, Math Methods Appl Sci 22 (1999), 55–72.
9. A. Meier, T. Arens, S. N. Chandler-Wilde, and A. Kirsch, A Nyström method for a class of integral equations on the real line with applications to scattering by diffraction gratings and rough surfaces, J Integral Equations Appl 12 (2000), 281–321.
10. J. C. Nédélec and F. Starling, Integral equation method in a quasi-periodic diffraction problem for the time-harmonic Maxwell equations, SIAM J Math Anal 22 (1991), 1679–1701.
11. A. Rathsfeld, G. Schmidt and B. H. Kleemann, On a fast integral equation method for diffraction gratings, Commun Comput Phys 1 (2006), 984–1009.
12. G. Schmidt, Electromagnetic scattering by periodic structures, J Math Sci 124 (2004), 5390–5405.
13. T. Abboud, Formulation variationnelle des équations de Maxwell dans un réseau bipériodique de $\mathbb{R}^3$, C R Acad Sci Paris 317 (1993), 245–248.
14. G. Bao, Variational approximation of Maxwell's equation in biperiodic structures, SIAM J Appl Math 57, (1997), 364–381.
15. A. S. Bonnet-Bendhia and P. Starling, Guided waves by electromagnetic gratings and non-uniqueness examples for the diffraction problem, Math Methods Appl Sci 17 (1994), 305–338.
16. D. Dobson and A. Friedman, The time-harmonic Maxwell equations in a doubly periodic structure, J Math Anal Appl 166 (1992), 507–528.
17. J. Elschner and M. Yamamoto, An inverse problem in periodic diffractive optics: reconstruction of Lipschitz grating profiles, Appl Anal 81, (2002) 1307–1328.
18. J. Elschner and G. Hu, Variational approach to scattering of plane elastic waves by diffraction gratings, Math Methods Appl Sci 33 (2010), 1924–1941.
19. G. Hu and A. Rathsfeld, Scattering of time-harmonic electromagnetic plane waves by perfectly conducting diffraction gratings, IMA J Appl Math 80 (2015), 508–532.
20. G. Hu and A. Rathsfeld, Convergence analysis of the FEM coupled with Fourier-mode expansion for the electromagnetic scattering by biperiodic structures, Electron Trans Numer Anal 41 (2014), 350–375.
21. A. Kirsch, Diffraction by periodic structures, L. Päivärinta and E. Somersalo, Proc. Lapland Conf. Inverse Problems, Springer, Berlin, 1993, pp. 87–102.
22. H. Ammari and J. C. Nédélec, Couplage éléments finis/équations intégrales pour la résolution des équations de Maxwell en milieu hétérogéne, Équations aux Dérivées Partielles et Applications, Articles dédiés à Jacques-Louis Lions. Gauthier-Villars, Paris, 1998, pp. 19–33.

23. H. Ammari and G. Bao, Coupling of finite element and boundary element methods for the scattering by periodic chiral structures, J Comput Math 3 (2008), 261–283.
24. G. Bao, Z. Chen, and H. Wu, Adaptive finite element method for diffraction gratings, J Opt Soc Am. A 22 (2005), 1106–1114.
25. M. Huber, J. Schoeberl, A. Sinwel, and S. Zaglmayr, Simulation of diffraction in periodic media with a coupled finite element and plane wave approach, SIAM J Sci Comput 31 (2009), 1500–1517.
26. J. Elschner and G. Hu, Scattering of plane elastic waves by three-dimensional diffraction gratings, Math Models Methods Appl Sci 22 (2012), 1150019.
27. J. J. Faran, Sound scattering by solid cylinders and spheres, J Acoust Soc Am 23 (1951), 405–418.
28. G. C. Hsiao, R. E. Kleinman, and G. F. Roach, Weak solution of fluid-solid interaction problem, Math Nachr 218 (2000), 139–163.
29. C. J. Luke and P. A. Martin, Fluid-solid interaction: Acoustic scattering by a smooth elastic obstacle, SIAM J Appl Math 55 (1995), 904–922.
30. T. Huttunen, J. P. Kaipio, and P. Monk, An ultra-weak method for acoustic fluid-solid interaction, Journal of Computational and Applied Math 213, 166–185.
31. V. D. Kupradze, T.G. Gegelia, M. O. Basheleishvili and T.V. Burchuladze, Three-dimensional problems of the mathematical theory of elasticity and thermoelasticity, Amsterdam, North-Holland, 1979.
32. S. N. Chandler-Wilde and J. Elscner, Variational approach in weighted Sobolev spaces to scattering by unbounded rough surfaces, SIAM J Math Anal 42 (2010), 2554–2580.
33. J. Elschner and G. Hu, Elastic scattering by unbounded rough surfaces: Solvability in weighted Sobolev spaces, Appl.Anal 94 (2015), 251–278.
34. T. Arens, Scattering by biperiodic layered media: The integral equation approach, Habilitation Thesis, Universität Karlsruhe, Karlsruhe, Germany, 2010.
35. J. Elschner and G. Schmidt, Diffraction in periodic structures and optimal design of binary gratings I. Direct problems and gradient formulas, Math Methods Appl Sci 21 (1998), 1297–1342.
36. D. Colton, and R. Kress, Inverse acoustic and electromagnetic scattering theory, Springer, Berlin, 1998.
37. I. C. Gohberg and M. G. Krein, Introduction to the theory of linear nonselfadjoint operators in Hilbert space, Translations of Mathematical Monographs Vol. 18, Providence, AMS, 1969.
38. M. Dauge, Elliptic boundary value problems in corner domains, Lecture Notes in Mathematics 1341, Springer-Verlag, Berlin, Heidelberg, 1988.
39. É. Estécahandy, Contribution à l'analyse mathématique et à la résolution numérique d'un problème inverse de scattering élasto-acoustique, PhD Thesis, L'Université de Pau et des Pays de l'Adour, 2013.
40. D. S. Jerison and C. E. Kenig, The Neumann problem on Lipschitz domains, Bull Am Soc 4 (1981), 203–207.
41. F. Ihlenburg, Finite element analysis of acoustic scattering, Applied mathematical methods, 132, Springer-Verlag, Inc., New York, 1998.