

## INVERSE SOURCE PROBLEMS IN AN INHOMOGENEOUS MEDIUM WITH A SINGLE FAR-FIELD PATTERN\*

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**Abstract.** This paper concerns time-harmonic inverse source problems with a single far-field pattern in two dimensions, where the source term is compactly supported in an a priori given inhomogeneous background medium. For convex-polygonal source terms, we prove that the source support together with the zeroth and first-order derivatives of the source function at corner points can be uniquely determined. Further, we prove that an admissible set of source functions (including harmonic functions) having a convex-polygonal support can be uniquely identified by a single far-field pattern. A class of radiating sources is characterized, and the extension of the radiated field across a corner point is proven impossible. Finally, the corner scattering theory motivates us to design a data-driven inversion scheme for imaging a convex polygonal source support.

**Key words.** uniqueness, inverse source problem, Helmholtz equation, single measurement

**AMS subject classifications.** Primary, 35R30, 78A46; Secondary, 51M20, 35P25

**DOI.** 10.1137/20M1325289

**1. Introduction and main results.** Consider the radiating of a time-harmonic acoustic source in an inhomogeneous background medium. In two dimensions, this can be modeled by the inhomogeneous Helmholtz equation

$$(1.1a) \quad \begin{cases} \Delta u + k^2 n(x)u = f & \text{in } \mathbb{R}^2, \\ \lim_{r \rightarrow +\infty} \sqrt{r} \left( \frac{\partial u}{\partial r} - iku \right) = 0, & r := |x|. \end{cases}$$

In this paper, the refractive index function is supposed to satisfy  $n(x) \in C^{0,\alpha'}(\mathbb{R}^2)$  ( $0 < \alpha' < 1$ ) and  $n(x) = 1$  in  $|x| > R$  for some  $R > 0$ . The number  $k > 0$  represents the wavenumber of the homogeneous medium in  $|x| > R$ , and  $f \in L^2(\mathbb{R}^2)$  is a source term compactly supported in  $D \subset B_R := \{x : |x| < R\}$  (see Figure 1). The Sommerfeld Radiation Condition (1.1b) (which was introduced by Sommerfeld [36] in 1912) excludes inwardly propagated waves to characterize the outgoing nature of the radiated wave [6] and thus guarantees the uniqueness of solutions to the system (1.1). This radiation condition gives arise to the asymptotic expansion of  $u$  at infinity

$$(1.2) \quad u(x) = \frac{e^{ik|x|}}{\sqrt{|x|}} u^\infty(\hat{x}) + \mathcal{O}(|x|^{-3/2}) \quad \text{as } |x| \rightarrow \infty,$$

which is uniform in all directions  $\hat{x} := x/|x| \in \mathbb{S} := \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\}$ . The function  $u^\infty(\hat{x})$  is referred to as the far-field pattern (or scattering amplitude). It is an analytic function with respect to the observation direction  $\hat{x} \in \mathbb{S}$  (see [6]).

\*Received by the editors March 13, 2020; accepted for publication (in revised form) July 23, 2020; published electronically October 22, 2020.

<https://doi.org/10.1137/20M1325289>

**Funding:** The work of the second author was partially supported by the National Natural Science Foundation of China grants 11971221 and 11731006 and by the Shenzhen Sci-Tech Fund grant JCYJ20190809150413261.

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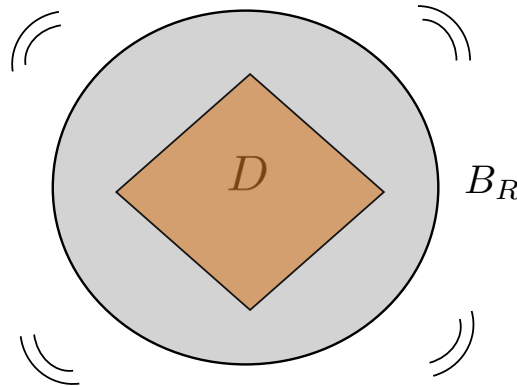


FIG. 1. Illustration of the acoustic source problem in an inhomogeneous background medium  $B_R$ . We assume that the source support  $D \subset B_R$  is a convex polygon and that the medium in the exterior of  $B_R$  is homogeneous.

Using the variational approach, it is easy to prove that the system (1.1) admits a unique solution in  $H_{loc}^2(\mathbb{R}^2)$ ; see [6, Chapter 5] or [5, Chapter 5]. Since the far-field pattern encodes information on the source, we are interested in the inverse problem of recovering the source support  $\partial D$  and/or the source term  $f$  from the far-field pattern over all observation directions at a fixed frequency. Throughout the paper we suppose that the refractive index function  $n(x)$  is known and  $k > 0$  is an arbitrarily fixed wave number. Note that the relation between  $\partial D$  and  $u^\infty$  is nonlinear, but the operator mapping  $f$  to  $u^\infty$  is linear.

It is well known that a single far-field pattern cannot uniquely determine a source function (even its support) in general due to the existence of nonradiating sources, for instance,  $f_0 := (\Delta + k^2 n(x))\varphi$ , where  $\varphi \in C_0^\infty(\mathbb{R}^2)$ . In fact, by linear superposition, it is easy to conclude that  $f$  and  $f + f_0$  could radiate identical far-field data at infinity. It was proved in [1, 11] that the far-field data over a range of frequencies can be used to uniquely determine a spatially dependent source term compactly supported in a homogeneous medium. Similar uniqueness results can be proved if the background medium is inhomogeneous but known in advance. We refer the reader to [17] for recent uniqueness results for various inverse source problems in the time domain.

In [22], Ikehata proposed the enclosure method to recover the convex hull of a polygonal source in a homogeneous medium using a single Cauchy pair of near-field data. Ikehata's arguments were based on the asymptotic behavior of an oscillatory integral obtained by using special complex geometrical optic (CGO) solutions to the homogeneous Helmholtz equation  $(\Delta + k^2)u = 0$  and led to a reconstruction formula for calculating the support function with a regular direction in two dimensions. Uniqueness was also verified as a by-product of the enclosure method if the source function is Hölder continuous and nonvanishing near corners. For general source functions, Kusiak and Sylvester [26] introduced the concept "scattering support" to define the minimal set that supports the scattered field; see also [37]. In a recent paper [2], Blåsten revealed that, in both two and three dimensions, the far-field data determine not only the convex-polygonal/polyhedral support but also the source values at corner/edge points. Moreover, it was proved that nonradiating sources must vanish at corner/edge points. Blåsten's approach relies on the construction of new

type of CGO solutions which decay exponentially in all directions except one and provides a new uniqueness proof to Ikehata’s result in two dimensions. Other related studies on corner scattering are devoted to the absence of real nonscattering energies [3, 7, 8, 19, 34] and uniqueness with a single measurement datum in shape identification arising from inverse conductivity [12, 21, 35] and inverse medium scattering problems [7, 8, 18, 19, 20, 22, 27].

This paper will generalize the existing results in the literature in several aspects, providing new insights into inverse source problems with a single measurement datum. The major novelties are summarized as follows. First, we consider the inverse source problem with a single far-field pattern in an inhomogeneous background medium rather than a homogeneous one. Second, we prove that the gradient of  $C^{1,\alpha}$ -smooth source terms at corner points can be uniquely identified, in addition to the information on source values at corners and the convex-polygonal support, which were already discussed in [2, 22, 26]. Moreover, we show that a convex-polygonal source support can be uniquely identified under the more general assumption that the lowest-order expansion of the source function is harmonic near corner points (see Corollary 1.6 and the remarks thereafter); Third, we define an admissible set of source functions (including harmonic functions) which can be uniquely identified by a single far-field pattern. Our approach relies heavily on the singularity analysis of the inhomogeneous Laplace equation in a corner domain, which was already used in [7] for treating inverse medium scattering problems. Our arguments are presented in a two-dimensional setting for simplicity; however, they can be carried over to curvilinear polyhedral sources in three dimensions following the lines of [7, section 5].

Below we state the main uniqueness results of this paper. Denote by  $B_a(z) = \{x \in \mathbb{R}^2 : |x - z| < a\}$  the disk centered at  $z \in \mathbb{R}^2$  with radius  $a > 0$ .

**THEOREM 1.1** (determination of source support). *Suppose the following:*

- (i)  $D := \text{supp}(f) \subset \mathbb{R}^2$  is a convex polygon.
- (ii) For each corner point  $O \in \partial D$ , it holds that  $f \in C^{1,\alpha}(\overline{D \cap B_\epsilon(O)})$  for some  $\epsilon > 0$ ,  $0 < \alpha < 1$  and that  $|f(O)| + |\nabla f(O)| > 0$ .

*Then the source support  $\partial D$  together with the values of  $f$  and  $\nabla f$  at each corner point can be uniquely determined by  $u^\infty(\hat{x})$  for all  $\hat{x} \in \mathbb{S}$ .*

The proof of Theorem 1.1 implies the following weaker result, which was proved in [2, 22, 26].

**COROLLARY 1.2.** *If the second condition (ii) in Theorem 1.1 is replaced by*

- (ii)  $f \in C^{0,\alpha}(\overline{D \cap B_\epsilon(O)})$  for some  $\epsilon > 0$ ,  $0 < \alpha < 1$  and  $|f(O)| > 0$  at each corner point  $O \in \partial D$ .

*Then the source support  $\partial D$  and the value of  $f(O)$  at each corner point  $O \in \partial D$  can be uniquely determined by  $u^\infty$ .*

Theorem 1.1, Corollary 1.2 together with Corollary 3.1 (see section 3) state that a source support of convex-polygonal type and partial information (derivatives of the zeroth and first orders) of the source function at corner points can be uniquely identified. In addition to these information at corner points, we prove that the entire source function from an admissible class can be also uniquely determined by a single far-field pattern.

Given  $A(x) = (a_1(x), a_2(x)) \in (L^\infty(B_R))^2$ , and  $b \in L^\infty(B_R)$ , introduce the admissible set

$$(1.3) \quad S(A, b) := \{v(x) : \Delta v(x) + A(x) \cdot \nabla v(x) + b(x)v(x) = 0 \text{ in } B_R\}.$$

In the special case that  $A \equiv 0$  and  $b \equiv 0$ , the set  $S(A, b)$  consists of all harmonic functions in  $B_R$ . Theorem 1.3 below states that a single far-field pattern can be used to determine a source term given by the restriction to a convex polygon of a special function from the admissible set  $S(A, b)$ .

**THEOREM 1.3** (determination of source terms). *Assume that  $D = \text{supp}(f)$  is a (nonempty) convex polygon and that  $f = v|_{\overline{D}}$  for some  $v \in S(A, b)$ . Suppose further that  $A$  and  $b$  are known functions which are analytic near each corner of  $\partial D$ . Then the source term  $f$  (and also its support  $D$ ) can be uniquely determined by  $u^\infty(\hat{x})$  for all  $\hat{x} \in \mathbb{S}$ .*

The source term  $f$  is called nonradiating if the resulting far-field pattern vanishes identically. Below we describe a class of radiating sources in a piecewise homogeneous background medium. Here the source support is not necessarily a convex polygon. For  $j \in \mathbb{N}_0$ , we introduce  $\nabla^j$  as the vector of all partial derivatives of order  $j$  with respect to the spatial variable, that is,

$$\nabla^j \varphi(x) := \{ \partial_{x_1}^{j_1} \partial_{x_2}^{j_2} \varphi(x); j_1, j_2 \in \mathbb{N}_0, j_1 + j_2 = j \}.$$

**COROLLARY 1.4** (characterization of radiating sources). *Let  $n(x) \equiv n_0 \neq 1$  in  $B_R$ . Suppose that the support  $D = \text{supp}(f) \neq \emptyset$  contains at least one corner point  $O \in \partial D$  and its exterior  $\mathbb{R}^2 \setminus \overline{D}$  is connected. Then  $f$  is a radiating source under one of the following conditions:*

- (i) *There exists some  $l \in \mathbb{N}_0$  and  $\alpha \in (0, 1)$  such that  $f \in C^{l+1, \alpha}(\overline{D \cap B_\epsilon(O)}) \cap W^{l, \infty}(B_\epsilon(O))$  and*

$$(1.4) \quad |\nabla^l f(O)| + |\nabla^{l+1} f(O)| > 0.$$

- (ii)  *$f = v|_{\overline{D}}$  for some  $v \in S(A, b)$ , where  $A$  and  $b$  are analytic functions near  $O$ .*

The condition (1.4) means that there exists some multi-index  $\beta = (\beta_1, \beta_2)$  ( $\beta_j \in \mathbb{N}_0$ ) with  $|\beta| := \beta_1 + \beta_2 = l$  such that

$$(1.5) \quad |\partial^\beta f(O)| + |\partial^{\beta'} f(O)| > 0$$

with  $\beta' = (\beta_1 + 1, \beta_2)$  or  $\beta' = (\beta_1, \beta_2 + 1)$ . A condition analogously to (1.4) has been used in [7] to describe medium discontinuity in a low-contrast case. In the special case of  $l = 0$ , the first condition in Corollary 1.4 is equivalent to the condition (ii) of Theorem 1.1 at the corner  $O$ ; see also Remark 3.1. Corollary 1.4(i) in the special case  $l = 0$  implies that the zeroth and first-order derivatives of a nonradiating source must vanish at the corner point (see also [2]).

As a corollary of the proofs of Theorems 1.1 and 1.3, we claim that the radiated field must be "singular" (that is, nonanalytic) at corner points; see Corollary 1.5 below. This excludes the possibility of analytical extension in a corner domain for solutions of forward source problems, which is important in designing inversion algorithms with a single measurement datum; see, e.g., the enclosure method [22], the range test approach [25, 26], and [32, Chapter 5] and [9].

**COROLLARY 1.5** (singularity at corner points). *Let  $n(x) \equiv n_0 \neq 1$  in  $B_R$ . Suppose that  $O \in \partial D$  is a corner point lying on the boundary of  $D = \text{supp}(f) \neq \emptyset$ , and assume that one of the conditions in Corollary 1.4 holds at  $O$ . Then  $u$  cannot be analytically extended from  $\mathbb{R}^2 \setminus \overline{D}$  to the interior of  $D$  across the corner point  $O$ . In other words,  $u$  cannot be analytic at  $O$ .*

In an inhomogeneous background medium,  $u$  may be not analytic in  $\mathbb{R}^2 \setminus \overline{D}$  if  $n(x) \in C^{0,\alpha}(\mathbb{R}^2)$ . However, Corollary 1.5 still holds if we properly define the *extension* across a corner point. Corollary 1.5 motivates us to design a data-driven and domain-defined sampling scheme for imaging a convex polygonal source support (see section 6 for more details).

From the proofs of Theorems 1.1 and 1.3, we can easily show the following results.

**COROLLARY 1.6.**

- (i) Assume that  $D = \text{supp}(f)$  is a (nonempty) convex polygon. For each corner point  $O_j \in \partial D$ , suppose that  $f \in C^{0,\alpha_j}(D \cap B_\epsilon(O_j))$  ( $\alpha_j \in (0, 1)$ ) has the asymptotic behavior

$$(1.6) \quad f(x) = r^{N_j}(A_j \cos N_j\theta + B_j \sin N_j\theta) + o(r^{N_j}), \quad |x| \rightarrow 0,$$

for some  $N_j \in \mathbb{N}_0$  and  $A_j, B_j \in \mathbb{C}$  with  $|A_j| + |B_j| > 0$ , uniformly in all directions  $\hat{x} = x/|x|$  such that  $x \in D \cap B_\epsilon(O)$ . Then  $\partial D$  can be uniquely determined by a single far-field pattern  $u^\infty$ .

- (ii) The results in Corollaries 1.4 and 1.5 hold true if the lowest-order expansion of  $f$  around the corner takes the form (1.6).

We remark that the opening angle of the corner domain in Corollaries 1.4, 1.5, and 1.6(ii) is allowed to be any number in  $(0, 2\pi) \setminus \{\pi\}$ . The condition (1.6) in Corollary (1.6) covers those assumptions made in Theorem 1.1, Corollary 1.2, and Theorem 1.3 for identifying the source support.

The rest of the paper is organized as follows. In section 2, we give some preliminaries of weighted Hölder spaces in an infinite sector and prove an important Lemma 2.3, which plays the key role to all of our uniqueness proofs. Section 3 presents uniqueness results for recovering a convex-polygonal source support. Section 4 is devoted to the uniqueness in recovering source terms. In section 5, we provide the proofs of Corollaries 1.4, 1.5, and 1.6. The reconstruction scheme will be described and compared with other approaches in section 6.

**2. Preliminaries.** In this section, we present some notations and auxiliary lemmas for weighted Hölder spaces in an infinite sector. Denote by  $O = (0, 0)$  the origin in  $\mathbb{R}^2$ . Let  $(r, \theta)$  be the polar coordinates of  $x = (x_1, x_2) \in \mathbb{R}^2$ . Define  $\Sigma = \Sigma_{\theta_0} := \{(r, \theta); r > 0, |\theta| < \theta_0/2\}$ , an infinite sector in  $\mathbb{R}^2$  with the opening angle  $\theta_0 \in (0, \pi)$  and with the vertex  $O$  located at the origin. For  $0 < \alpha < 1$ ,  $l \in \mathbb{N}_0$ , and  $\beta \in \mathbb{R}$ , we define the weighted Hölder spaces  $\Lambda_\beta^{l,\alpha}(\Sigma)$  endowed with the norm (see [23, 24, 29, 33])

$$(2.1) \quad \|\varphi\|_{\Lambda_\beta^{l,\alpha}(\Sigma)} := \sup_{x \in \Sigma} \left\{ \sum_{j=0}^l |x|^{\beta-\alpha-l+j} |\nabla^j \varphi(x)| \right\} + \sup_{x,y \in \Sigma} \left\{ \frac{||x|^\beta \nabla^l \varphi(x) - |y|^\beta \nabla^l \varphi(y)|}{|x-y|^\alpha} \right\}.$$

We may denote  $\Lambda_\beta^{l,\alpha}(\Sigma)$  as  $\Lambda_\beta^{l,\alpha}$  if it is clear in the context. It can be verified that if  $u \in \Lambda_\beta^{l,\alpha}$ , then  $\nabla^j u \in \Lambda_\beta^{l-j,\alpha}$  for all  $j = 0, 1, \dots, l$ . The weight  $\beta$  characterizes the asymptotic behavior at the origin and at infinity. In our applications we shall only care about the asymptotic behavior near the origin, where the corner point of a planar sector is supposed to be located. Obviously, any function from  $\Lambda_\beta^{l,\alpha}(\Sigma)$  lies in  $C^{l,\alpha}(\overline{\Sigma^{a,b}})$  for all  $0 < a < b$ , where  $\Sigma^{a,b} := \Sigma \cap \{(r, \theta) : a < r < b\}$ . For any function  $u \in \Lambda_\beta^{l,\alpha}(\Sigma)$  with a compact support in  $\Sigma \cap B_R(O)$  ( $R > 0$ ), we have

$$\nabla^j u(x) = O(r^{l-j+\alpha-\beta}) \quad \text{as } r \rightarrow 0^+ \quad \text{in } \Sigma$$

for  $j = 0, 1, \dots, l$ . In particular, we have the decay rate  $u(x) = O(r^{l+\alpha-\beta})$  if  $\beta < l+\alpha$  and the inclusion relation

$$(2.2) \quad \Lambda_\beta^{l,\alpha}(\Sigma) \subset \Lambda_{\beta+1}^{l,\alpha}(\Sigma).$$

In the following two lemmas we prove elementary properties of  $\Lambda_\beta^{l,\alpha}(\Sigma)$ .

LEMMA 2.1. *Let  $\beta \in \mathbb{R}$  and  $\alpha, \alpha' \in (0, 1)$  with  $\alpha' \geq \alpha$ , and let  $\Sigma$  be an infinite sector. Assume that  $g \in \Lambda_\beta^{0,\alpha}(\Sigma)$  with compact support and that  $f \in C^{0,\alpha'}(\overline{\Sigma})$ . Then the product  $fg$  belongs to the space  $\Lambda_\beta^{0,\alpha}(\Sigma)$ .*

*Proof.* We need to prove the boundedness of the norm

$$(2.3) \quad \begin{aligned} \|fg\|_{\Lambda_\beta^{0,\alpha}(\Sigma)} &= \sup_{x \in \Sigma} |x|^{\beta-\alpha} |f(x)g(x)| + \sup_{x,y \in \Sigma} \frac{||x|^\beta f(x)g(x) - |y|^\beta f(y)g(y)|}{|x-y|^\alpha} \\ &:= \mathcal{I}_1 + \mathcal{I}_2. \end{aligned}$$

The first term  $\mathcal{I}_1$  in (2.3) can be bounded by

$$(2.4) \quad \mathcal{I}_1 \leq \sup_{x \in \overline{\Sigma}} |f(x)| \cdot \sup_{x \in \Sigma} |x|^{\beta-\alpha} |g(x)| \leq \|f\|_{C^{0,\alpha'}(\overline{\Sigma})} \cdot \|g\|_{\Lambda_\beta^{0,\alpha}(\Sigma)} < +\infty.$$

Using the compactness of  $\text{supp}(g)$  and the triangle inequality, we can estimate the second term  $\mathcal{I}_2$  by

$$(2.5) \quad \begin{aligned} \mathcal{I}_2 &\leq \sup_{x,y \in \Sigma} \frac{||x|^\beta f(x)g(x) - |y|^\beta f(x)g(y)|}{|x-y|^\alpha} + \sup_{x,y \in \Sigma} \frac{||y|^\beta f(x)g(y) - |y|^\beta f(y)g(y)|}{|x-y|^\alpha} \\ &\leq \sup_{x \in \Sigma} |f(x)| \cdot \sup_{x,y \in \Sigma} \frac{||x|^\beta g(x) - |y|^\beta g(y)|}{|x-y|^\alpha} \\ &\quad + \sup_{y \in \text{supp } g} |y|^\alpha \cdot \sup_{y \in \Sigma} |y|^{\beta-\alpha} |g(y)| \cdot \sup_{x,y \in \Sigma} \frac{|f(x) - f(y)|}{|x-y|^\alpha} \\ &\leq \|f\|_{C^{0,\alpha'}(\overline{\Sigma})} \cdot \|g\|_{\Lambda_\beta^{0,\alpha}(\Sigma)} + C \|g\|_{\Lambda_\beta^{0,\alpha}(\Sigma)} \cdot \|f\|_{C^{0,\alpha'}(\overline{\Sigma})} \\ (2.6) \quad &< +\infty. \end{aligned}$$

Combining (2.3)–(2.6), we arrive at our conclusion.  $\square$

LEMMA 2.2. *Any compactly supported function in  $\Lambda_{-N+1}^{2,\alpha'}(\Sigma)$  ( $N \in \mathbb{N}_0$ ) belongs to  $\Lambda_{-N}^{0,\alpha'}(\Sigma)$ .*

*Proof.* Suppose that  $\varphi \in \Lambda_{-N+1}^{2,\alpha'}(\Sigma)$  has a compact support in  $\Sigma$ . By the definition of the norm (2.1), we obtain

$$(2.7) \quad \mathcal{M}_0 := \sup_{x \in \Sigma} |x|^{-N-1-\alpha'} |\varphi(x)| < +\infty \quad \text{and} \quad \mathcal{M}_1 := \sup_{x \in \Sigma} |x|^{-N-\alpha'} |\nabla \varphi(x)| < +\infty.$$

To prove  $\varphi \in \Lambda_{-N}^{0,\alpha'}(\Sigma)$ , we only need to prove the boundedness of

$$(2.8) \quad \mathcal{M}_2 := \sup_{x \in \Sigma} |x|^{-N-\alpha'} |\varphi(x)|, \quad \mathcal{M}_3 := \sup_{x,y \in \Sigma} \frac{||x|^{-N}\varphi(x) - |y|^{-N}\varphi(y)|}{|x-y|^{\alpha'}}.$$

We first prove  $\mathcal{M}_3 < \infty$ . Write  $\psi(x) := |x|^{-N}\varphi(x)$ . The derivative of  $\psi$  can be estimated by

$$\begin{aligned} |\partial_{x_j}\psi(x)| &= \left| -N|x|^{-N-1}\frac{x_j}{|x|}\varphi(x) + |x|^{-N}\partial_{x_j}\varphi(x) \right| \\ &\leq |N||x|^{-N-1}|\varphi(x)| + |x|^{-N}|\partial_{x_j}\varphi(x)| \\ &\leq |N||x|^{\alpha'}\mathcal{M}_0 + |x|^{\alpha'}\mathcal{M}_1 \end{aligned}$$

for all  $x \in \text{supp}(\varphi)$  and  $j = 1, 2$ , implying that

$$(2.9) \quad |\nabla\psi(x)| \leq C(\mathcal{M}_0 + \mathcal{M}_1) < +\infty \quad \text{for all } x \in \Sigma \cap \text{supp } \varphi.$$

Now we can estimate  $\mathcal{M}_3$  by applying the mean-value theorem

$$(2.10) \quad \mathcal{M}_3 = \sup_{x,y \in \Sigma} \frac{|\psi(x) - \psi(y)|}{|x - y|^{\alpha'}} \leq \|\nabla\psi\|_{L^\infty(\Sigma \cap \text{supp } \varphi)} \sup_{x,y \in \Sigma \cap \text{supp } \varphi} |x - y|^{1-\alpha'} < \infty.$$

Writing  $\mathcal{M}_2$  as  $\mathcal{M}_2 = \sup_{x \in \Sigma} (|x| \cdot |x|^{-N-1-\alpha'}|\varphi(x)|)$  and using the compactness of  $\text{supp } \varphi$ , we get

$$(2.11) \quad \mathcal{M}_2 \leq \left( \sup_{x \in \Sigma \cap \text{supp } \varphi} |x| \right) \cdot \sup_{x \in \Sigma} (|x|^{-N-1-\alpha'}|\varphi(x)|) \leq C\mathcal{M}_0 < +\infty.$$

Combining (2.10) and (2.11) we conclude that  $\varphi \in \Lambda_{-N}^{0,\alpha'}(\Sigma)$ . The proof is complete.  $\square$

The proofs of our main results depend on Lemma 2.3 below, which is motivated by Propositions 10, 12, and 13 of [7]. Introduce the finite sector

$$\Sigma_\epsilon := \{(r, \theta) \in \mathbb{R}^2 : 0 < r < \epsilon, 0 < \theta < \theta_0\}, \quad \theta_0 \in (0, \pi), \quad 0 < \epsilon < 1$$

and its partial boundary

$$\Gamma_\epsilon := \{(r, \theta) \in \mathbb{R}^2 : 0 < r < \epsilon, \theta = 0, \theta_0\}.$$

LEMMA 2.3. *Suppose that  $q, h \in C^{0,\alpha}(\overline{\Sigma_\epsilon})$  for some  $\alpha \in (0, 1)$  and that the lowest-order expansion of  $h$  as  $|x| \rightarrow 0$  is harmonic; that is, the asymptotic expansion*

$$(2.12) \quad h(x) = r^N (A \cos(N\theta) + B \sin(N\theta)) + \mathcal{O}(r^{N+\alpha}), \quad |x| \rightarrow 0, x \in \Sigma_\epsilon$$

*holds uniformly in all  $\theta \in (0, \theta_0)$  for some  $N \in \mathbb{N}_0$  and  $A, B \in \mathbb{C}$ . Let  $u \in H^2(\Sigma_\epsilon)$  be a solution to the Cauchy problem*

$$(2.13) \quad (BVP) : \begin{cases} \Delta u + q(x)u = h(x) & \text{in } \Sigma_\epsilon, \\ u = \partial_\nu u = 0 & \text{on } \Gamma_\epsilon, \end{cases}$$

*where  $\partial_\nu u$  denotes the normal derivative of  $u$ . Then it holds that  $A = B = 0$ .*

The proof of Lemma 2.3 is based on a precise characterization of the singularity of solutions to the inhomogeneous Laplacian equation in a corner domain (see, e.g., [33, Chapters 2 and 3]). Our argument is a refinement of the proof of [7, Lemma 2] under the assumption (2.12). Below we sketch the proof for the reader's convenience and refer the reader to [7, Lemma 2] for more details.

*Proof of Lemma 2.3.* Introduce the cutoff function  $\chi \in C_0^\infty(\mathbb{R})$  satisfying  $\chi(r) \equiv 1$  when  $r < \frac{\epsilon}{2}$  and  $\chi(r) \equiv 0$  when  $r > \epsilon$ . From (2.13) we deduce the inhomogeneous Laplacian equation defined over the infinite sector  $\Sigma$ ,

$$(2.14) \quad \Delta(\chi u) = \chi h - (\chi u)q + [\Delta, \chi]u =: f \quad \text{in } \Sigma,$$

with the commutator

$$[\Delta, \chi]u := \Delta(\chi u) - \chi \Delta u = 2\nabla \chi \cdot \nabla u + (\Delta \chi)u.$$

We are going to analyze the regularity of  $\chi u$  (and thus  $u$  itself) in a neighborhood of the origin by using the vanishing of the Cauchy data of  $\chi u$  on  $\theta = 0, \theta_0$  together with the decaying rate of  $f$  near  $O$ . For clarity we shall divide the rest of the proof into four steps.

**Step 1:** Show that the right-hand side of (2.14) satisfies  $f \in \Lambda_1^{0,\alpha}(\Sigma)$ .

By the assumption of  $h$  and the inclusion relation (2.2) for compactly supported functions, we obtain

$$(2.15) \quad \chi h \in \Lambda_{-j}^{0,\alpha}(\Sigma) \quad \text{for every } j = -1, 0, 1, \dots, N-1.$$

The assumption  $u \in H^2(\Sigma_\epsilon)$  together with the choice of the cutoff function  $\chi$  implies

$$(2.16) \quad \chi u \in \Lambda_1^{0,\alpha}(\Sigma) \quad \text{and} \quad (\chi u)q \in C^{0,\alpha}(\overline{\Sigma}) \subseteq \Lambda_1^{0,\alpha}(\Sigma),$$

where the first relation follows from the Sobolev embedding theorem and the second one from Lemma 2.1. We now consider the regularity of  $[\Delta, \chi]u$  appearing in (2.14). Since the supports of  $\nabla \chi$  and  $\Delta \chi$  are contained in  $\{x \in \mathbb{R}^2 : \frac{\epsilon}{2} \leq |x| \leq \epsilon\}$ , we consider the equation of  $u$

$$(2.17) \quad \Delta u = h - q(x)u$$

over the truncated domain  $\Sigma'_\epsilon := \Sigma_{2\epsilon} \cap \{\epsilon/4 < |x| < 2\epsilon\}$ . Since  $h \in C^{0,\alpha}(\overline{\Sigma'_\epsilon})$ ,  $qu \in C^{0,\alpha}(\Sigma'_\epsilon)$  and there exists a  $C^\infty$ -smooth subdomain of  $\Sigma'_\epsilon$  which contains  $\Sigma''_\epsilon := \Sigma_\epsilon \cap \{\epsilon/2 < |x| < \epsilon\}$ , by standard interior elliptic regularity theory (see [13, Theorem 4.11]) it holds that

$$(2.18) \quad u \in C^{2,\alpha}(\overline{\Sigma''_\epsilon}),$$

implying the relation

$$(2.19) \quad [\Delta, \chi]u \in \Lambda_{-j}^{0,\alpha}(\Sigma) \quad \text{for any integer } j.$$

**Step 2:** Prove  $\chi u \in \Lambda_{-N}^{0,\alpha}(\Sigma)$ .

We can summarize from (2.15), (2.16), and (2.19) that

$$(2.20) \quad \begin{cases} \chi h \in \Lambda_j^{0,\alpha}(\Sigma) & \text{for every } j = 1, 0, -1, \dots, -N+1, \\ (\chi u)q \in \Lambda_\ell^{0,\alpha}(\Sigma), & \text{where } \ell = 1, \\ [\Delta, \chi]u \in \Lambda_j^{0,\alpha}(\Sigma) & \text{for any integer } j. \end{cases}$$

This implies that the right-hand side of (2.14) belongs to  $f \in \Lambda_1^{0,\alpha}(\Sigma)$ . By the solvability of the Laplace equation in an infinite sector (see, e.g., [33, Theorem 6.11, Chapter 3]), we obtain

$$(2.21) \quad \chi u \in \Lambda_1^{2,\alpha}(\Sigma).$$



Now, applying Lemma 2.2 gives  $\chi u \in \Lambda_0^{0,\alpha}(\Sigma)$ , which improves the weight  $\beta$  in the first step (cf. (2.16)) from  $\beta = 1$  to  $\beta = 0$ . This means that  $u$  at the corner point is getting less singular. Moreover, combining the fact  $\chi u \in \Lambda_0^{0,\alpha}(\Sigma)$  with Lemma 2.1 leads to those relations in (2.20) where the subscript  $\ell$  is replaced by 0, which in turn gives  $\chi u \in \Lambda_0^{2,\alpha}(\Sigma)$ . Repeating this process, we finally arrive at

$$(2.22) \quad f \in \Lambda_{-N+1}^{0,\alpha}(\Sigma), \quad \chi u \in \Lambda_{-N+1}^{2,\alpha}(\Sigma) \subset \Lambda_{-N}^{0,\alpha}(\Sigma').$$

**Step 3:** Singularity analysis of (2.14).

Write  $h_1(x) := f - r^N(A \cos N\theta + B \sin N\theta)\chi$ . From (2.13) and the definition of  $\chi$ , it follows that the function  $\chi u$  solves the following Dirichlet and Neumann boundary value problems:

$$(2.23) \quad (\text{Dirichlet BVP}) \quad \begin{cases} \Delta(\chi u) = r^N(A \cos N\theta + B \sin N\theta)\chi + h_1(x) & \text{in } \Sigma \\ (\chi u) = 0 & \text{on } \partial\Sigma, \end{cases}$$

$$(2.24) \quad (\text{Neumann BVP}) \quad \begin{cases} \Delta(\chi u) = r^N(A \cos N\theta + B \sin N\theta) + h_1(x) & \text{in } \Sigma \\ \partial_\nu(\chi u) = 0 & \text{on } \partial\Sigma. \end{cases}$$

Note that  $r^N(A \cos N\theta + B \sin N\theta)\chi \in \Lambda_{-N+1}^{0,\alpha}(\Sigma)$ . From (2.12), we see

$$\chi [h - r^N(A \cos N\theta + B \sin N\theta)] \in \Lambda_{-N}^{0,\alpha}(\Sigma).$$

Together with (2.19) and (2.22), this gives  $h_1 \in \Lambda_{-N}^{0,\alpha'}(\Sigma) \subset \Lambda_{-N+1}^{0,\alpha'}(\Sigma)$  for all  $0 < \alpha' < \alpha$ . Therefore, from [33, Theorem 6.11, Chapter 3], the Dirichlet BVP (2.23) admits a unique solution in  $\Lambda_{-N+1}^{2,\alpha'}(\Sigma)$  if  $\alpha' \neq \frac{j\pi}{\theta_0} - 1 - N$  ( $j \in \mathbb{Z}$ ), where  $\theta_0$  is the opening angle of the sector  $\Sigma$ . Further, using [7, Proposition 5], [33, Proposition 2.12, Chapter 2], and the arbitrariness of  $\alpha' \in (0, \alpha)$ , the function  $\chi u \in \Lambda_{-N}^{0,\alpha'}(\Sigma)$  around the corner can be decomposed into two parts,  $\chi u = u_{\mathcal{D}}^{(1)} + u_{\mathcal{D}}^{(2)}$ , where

$$\begin{aligned} u_{\mathcal{D}}^{(1)} &= q_{\mathcal{D},N+2} + C_{\mathcal{D}} r^{N+2}[\ln r \sin(N+2)\theta + \theta \cos(N+2)\theta], \quad C_{\mathcal{D}} \in \mathbb{C}, \\ u_{\mathcal{D}}^{(2)} &= d_{\mathcal{D}} r^{N+2} \sin(N+2)\theta + \sum_{j \in I(\theta_0, N)} d_{\mathcal{D},j} r^{\frac{j\pi}{\theta_0}} \sin\left(\frac{j\pi}{\theta_0}\theta\right) + \mathcal{O}(r^{N+2+\alpha'}), \quad d_{\mathcal{D}}, d_{\mathcal{D},j} \in \mathbb{C}, \end{aligned}$$

where

$$I(\theta_0, N) := \left\{ j : \frac{j\pi}{\theta_0} \in (N+1, N+2) \right\}, \quad d_{\mathcal{D}} = 0 \quad \text{if } (N+2)\theta_0/\pi \notin \mathbb{N},$$

and  $q_{\mathcal{D},N+2}$  is a polynomial of order  $N+2$  satisfying

$$(2.25) \quad \Delta q_{\mathcal{D},N+2} = r^N(A \cos N\theta + B \sin N\theta) \quad \text{in } \Sigma, \quad q_{\mathcal{D},N+2} = 0 \quad \text{on } \partial\Sigma.$$

Similarly, by [33, Chapter 2, Proposition 2.12], the Neumann BVP (2.24) admits a unique solution  $\chi u = u_{\mathcal{N}}^{(1)} + u_{\mathcal{N}}^{(2)} \in \Lambda_{-N+1}^{2,\alpha'}(\Sigma) \subset \Lambda_{-N}^{0,\alpha'}(\Sigma)$ , where

$$\begin{aligned} u_{\mathcal{N}}^{(1)}(x) &= q_{\mathcal{N},N+2}(x) + C_{\mathcal{N}} r^{N+2}[\ln r \cos(N+2)\theta - \theta \sin(N+2)\theta], \quad C_{\mathcal{N}} \in \mathbb{C}, \\ u_{\mathcal{N}}^{(2)}(x) &= d_{\mathcal{N}} r^{N+2} \cos(N+2)\theta + \sum_{j \in I(\theta_0, N)} d_{\mathcal{N},j} r^{\frac{j\pi}{\theta_0}} \cos\left(\frac{j\pi}{\theta_0}\theta\right) + \mathcal{O}(r^{N+2+\alpha'}) \end{aligned}$$

with  $d_{\mathcal{N}}, d_{\mathcal{N},j} \in \mathbb{C}$ . Here,  $d_{\mathcal{N}} = 0$  if  $(N+2)\theta_0/\pi \notin \mathbb{N}$ , and  $q_{\mathcal{N},N+2}$  is a polynomial of order  $N+2$  satisfying

$$(2.26) \quad \Delta q_{\mathcal{N},N+2} = r^N (A \cos N\theta + B \sin N\theta) \quad \text{in } \Sigma, \quad \partial_\nu q_{\mathcal{N},N+2} = 0 \quad \text{on } \partial\Sigma.$$

**Step 4:** Show that  $A = B = 0$ .

We observe that  $r^{N+2} \ln r = o(r^{N+1+\tau})$  ( $r \rightarrow 0^+$ ) for any  $\tau \in (0, 1)$  and

$$(2.27) \quad r^{\frac{j\pi}{\theta_0}} > r^{N+2} \ln r > r^{N+2} > r^{N+2+\alpha'} \quad \text{as } r \rightarrow 0^+ \quad \text{for all } j \in I(\theta_0, N).$$

Thus, we can conclude from  $\chi u = u_{\mathcal{D}}^{(1)} + u_{\mathcal{D}}^{(2)} = u_{\mathcal{N}}^{(1)} + u_{\mathcal{N}}^{(2)}$  in  $\Sigma$  that

$$(2.28) \quad C_{\mathcal{D}} = C_{\mathcal{N}} = 0 \quad \text{and} \quad d_{\mathcal{D},j} = d_{\mathcal{N},j} = 0 \quad \text{for all } j \in I(\theta_0, N)$$

and

$$q_{\mathcal{D},N+2}(x) + d_{\mathcal{D}} r^{N+2} \sin(N+2)\theta = q_{\mathcal{N},N+2}(x) + d_{\mathcal{N}} r^{N+2} \cos(N+2)\theta =: q_{N+2}.$$

Combining (2.25) and (2.26) yields

$$(2.29) \quad \Delta q_{N+2} = r^N (A \cos N\theta + B \sin N\theta)$$

and further

$$(2.30) \quad \Delta^2 q_{N+2} = 0 \quad \text{in } \Sigma, \quad q_{N+2} = \partial_\nu q_{N+2} = 0 \quad \text{on } \partial\Sigma.$$

Applying [7, Proposition 12] to (2.30), we obtain  $q_{N+2} = 0$  in  $\Sigma$ . Finally, it can be concluded from (2.29) that  $r^N (A \cos N\theta + B \sin N\theta) \equiv 0$  in  $\Sigma$ , which implies  $A = B = 0$ . The proof is complete.  $\square$

In the subsequent two corollaries we present examples of  $h$  fulfilling the condition (2.12) in Lemma 2.3.

**COROLLARY 2.1.** *Assume in Lemma 2.3 that  $h \in C^{1,\alpha}(\overline{\Sigma}_\epsilon)$  for some  $0 < \alpha < 1$ . Then  $h(O) = 0$  and  $|\nabla h(O)| = 0$ .*

*Proof.* Since  $h \in C^{1,\alpha}(\overline{\Sigma}_\epsilon)$ , the function  $h$  admits the asymptotic behavior

$$(2.31) \quad h(x) = h(O) + \nabla h(O) \cdot x + \mathcal{O}(|x|^{1+\alpha}), \quad |x| \rightarrow 0$$

in  $\Sigma_\epsilon$ . Note that the expression of  $h$  in (2.31) takes the form (2.12) with  $N = 0$ . Applying Lemma 2.3 gives  $h(O) = 0$ . Then we have

$$h(x) = \nabla h(O) \cdot x + \mathcal{O}(|x|^{1+\alpha}), \quad |x| \rightarrow 0,$$

which is of the form (2.12) with  $N = 1$ . Again using Lemma 2.3, we arrive at  $|\nabla h(O)| = 0$ .  $\square$

**COROLLARY 2.2.** *Suppose in Lemma 2.3 that  $h \in S(A, b)$ , where  $A = (a_1, a_2)$  and  $b$  are analytic near  $O$ . Then  $h \equiv 0$  in  $B_\epsilon(O)$ .*

*Proof.* We first prove that the lowest-order nonvanishing term in the Taylor expansion of  $h$  at the corner point  $O$  is harmonic if  $h$  does not vanish identically. This extends the proof of [7, Proposition 10] in the special case  $a_1 = a_2 \equiv 0$  and  $b = b_0 \in \mathbb{C}$  to a more general setting.

Since  $A$  and  $b$  are both analytic functions, by elliptic regularity theory,  $h$  is also analytic in  $B_\epsilon(O)$ . By [10, Lemma 2.1], there exists a positive integer  $M \in \mathbb{N}_0$  such that  $h$  can be expanded in the polar coordinates  $(r, \theta)$  into the convergent series

$$h(x) = \sum_{j \geq M} r^j F_j(\theta), \quad \text{where } F_j(\theta) = \sum_{n+2m=j} (C_{n,m}^+ \cos n\theta + C_{n,m}^- \sin n\theta)$$

with  $C_{n,m}^\pm \in \mathbb{C}$ . Here, we have  $C_{n,m}^\pm = 0$  if  $n+2m < M$  because  $r^M F_M(\theta)$  is supposed to be the lowest-order term. In two dimensions, it is easy to check that

$$\begin{aligned} \Delta h &= \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) h \\ &= \sum_{j \geq M} (j(j-1)r^{j-2} F_j + jr^{j-2} F_j + r^{j-2} F_j'') \\ (2.32) \quad &= \sum_{j \geq M-2} r^j [(j+2)^2 F_{j+2} + F_{j+2}'']. \end{aligned}$$

On the other hand, assuming

$$A(x) = \sum_{j \geq 0} r^j A_j(\theta), \quad b(x) = \sum_{j \geq 0} r^j b_j(\theta),$$

we find

$$(2.33) \quad A(x) \cdot \nabla h(x) = \sum_{j \geq M-1} r^j \tilde{A}_j(\theta)$$

and

$$b(x) h(x) = \left( \sum_{j \geq 0} r^j b_j(\theta) \right) \left( \sum_{\ell \geq M} r^\ell F_\ell(\theta) \right) = \sum_{j \geq M} r^j \tilde{b}_j(\theta),$$

where

$$(2.34) \quad \tilde{b}_j(\theta) = \sum_{k+\ell \geq j; k \geq 0, \ell \geq M} b_k(\theta) F_\ell(\theta).$$

Inserting (2.32)–(2.34) into the equation

$$\Delta h(x) + A(x) \cdot \nabla h(x) + b(x) h(x) = 0$$

and comparing the coefficients of  $r^{M-1}$ , we obtain  $M^2 F_M + F_M'' = 0$ ,  $\theta \in [0, 2\pi]$ . By the definition of  $F_j(\theta)$ , it follows that

$$\begin{aligned} F_M''(\theta) &= (-M^2) \sum_{n+2m=M} (C_{n,m}^+ \cos n\theta + C_{n,m}^- \sin n\theta) \\ (2.35) \quad &= -M^2 (C_{M,0}^+ \cos M\theta + C_{M,0}^- \sin M\theta) \\ &\quad - n^2 \sum_{n+2m=M, m \neq 0} (C_{n,m}^+ \cos n\theta + C_{n,m}^- \sin n\theta) \end{aligned}$$

and

$$\begin{aligned} M^2 F_M(\theta) &= M^2 (C_{M,0}^+ \cos M\theta + C_{M,0}^- \sin M\theta) \\ (2.36) \quad &+ M^2 \sum_{n+2m=M, m \neq 0} (C_{n,m}^+ \cos n\theta + C_{n,m}^- \sin n\theta). \end{aligned}$$

Combining the previous two identities we get

$$(2.37) \quad \sum_{\substack{n+2m=M \\ m \neq 0}} (C_{n,m}^+ \cos n\theta + C_{n,m}^- \sin n\theta)(M^2 - n^2) = 0 \quad \text{for all } \theta \in [0, 2\pi].$$

Since  $M^2 - n^2 \neq 0$  for all  $(n, m) \in \{(n, m) \in \mathbb{N}^2; n + 2m = M, m \neq 0\}$ , the relation (2.37) implies

$$C_{n,m}^\pm = 0 \quad \text{if } n + 2m = M, m \neq 0.$$

Hence,

$$h(x) = r^M [C_{M,0}^+ \cos M\theta + C_{M,0}^- \sin M\theta] + \sum_{j \geq M+1} r^j F_j(\theta);$$

that is,  $h$  is of the form (2.12) near the corner point  $O$ . Applying Lemma 2.3 we obtain  $C_{M,0}^\pm = 0$ . This is a contradiction to the fact that  $F_M(\theta)$  does not vanish. Hence,  $h \equiv 0$  in  $B_\epsilon(O)$ .  $\square$

**3. Determination of convex-polygonal source support.** In this section, we first prove Theorem 1.1 for  $C^{1,\alpha}$ -continuous source functions around a corner point and then generalize the uniqueness result to a larger class of smooth source functions with a low contrast to the background medium.

*Proof of Theorem 1.1.* Let  $\tilde{u}$  be the solution of  $\Delta \tilde{u} + k^2 n \tilde{u} = \tilde{f}$  in  $\mathbb{R}^2$ , where  $\tilde{D} := \text{supp } \tilde{f}$  is also a convex polygon and  $\tilde{f} \in C^{1,\alpha}(\overline{\tilde{D} \cap B_\epsilon(\tilde{O})})$  near each corner  $\tilde{O}$  of  $\partial \tilde{D}$ . Suppose further that  $|\tilde{f}| + |\nabla \tilde{f}| > 0$  at each corner point of  $\partial \tilde{D}$ . Assume that

$$(3.1) \quad u^\infty(\hat{x}) = \tilde{u}^\infty(\hat{x}) \quad \text{for all } \hat{x} \in \mathbb{S}.$$

Applying Rellich's lemma and the unique continuation for Helmholtz equations (see [16, Theorem 17.2.6, Chapter XVII], where the refractive index  $n$  is allowed to be  $L^\infty$ ), we see

$$(3.2) \quad u = \tilde{u} \quad \text{in } \mathbb{R}^2 \setminus \overline{D \cup \tilde{D}}.$$

Note that here we have used the assumption that the refractive index function  $n(x)$  is given.

The rest of the proof of Theorem 1.1 is divided into two steps. In the first step we prove the unique determination of the source support, and in the second step the determination of the zeroth and first-order derivatives of the source term at corner points.

**Step 1:** Prove  $D = \tilde{D}$ . If the geometric shapes of  $D$  and  $\tilde{D}$  are not identical, without loss of generality we may suppose there exists some corner  $O \in \partial D$  and a neighborhood  $B_\epsilon(O)$  of  $O$  such that  $B_\epsilon(O) \cap \tilde{D} = \emptyset$ . Set  $\Sigma_\epsilon := B_\epsilon(O) \cap D$  with the opening angle  $\theta_0 \in (0, \pi)$ , and write  $\Gamma_\epsilon := \partial D \cap B_\epsilon(O)$ . This scenario is illustrated in Figure 2.

By coordinate translation, we may suppose without loss of generality that the corner point  $O$  is located at the origin. Then we have

$$(3.3) \quad \begin{cases} \Delta u + k^2 n(x)u = f & \text{in } \Sigma_\epsilon, \\ \Delta \tilde{u} + k^2 n(x)\tilde{u} = 0 & \text{in } \Sigma_\epsilon. \end{cases}$$

Since  $f \in L^2(B_\epsilon(O)) \cap C^{1,\alpha}(\overline{D \cap B_\epsilon(O)})$ , by standard elliptic regularity theory we have  $u, \tilde{u} \in H^2(B_\epsilon(O))$ . Recall the transmission conditions for  $u$ :

$$u^+ = u^-, \quad \partial_\nu u^+ = \partial_\nu u^- \quad \text{on } \Gamma_\epsilon,$$

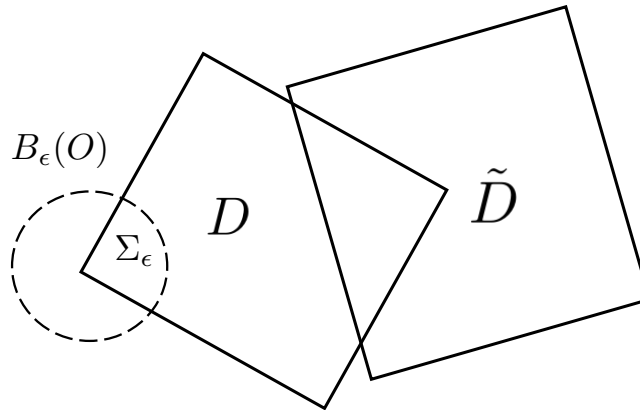


FIG. 2. Illustration of two convex-polygonal source supports  $D$  and  $\tilde{D}$ .

where  $(\cdot)^\pm$  denote the traces of  $u \in H^2(B_\epsilon(O))$  taking from  $D(+)$  and  $\mathbb{R}^2 \setminus \bar{D}(-)$ , respectively. We deduce from (3.2) and the previous transmission conditions that

$$(3.4) \quad u = \tilde{u}, \quad \partial_\nu u = \partial_\nu \tilde{u} \quad \text{on } \Gamma_\epsilon.$$

Set  $w = u - \tilde{u}$ . From (3.3) and (3.4) it follows that  $w \in H^2(B_\epsilon(O))$  solves the Cauchy problem

$$(3.5) \quad \begin{cases} \Delta w + k^2 n(x)w = f_0 + f_1 & \text{in } \Sigma_\epsilon, \\ w = \partial_\nu w = 0 & \text{on } \Gamma_\epsilon, \end{cases}$$

where

$$f_0(x) := \begin{cases} f(O) & \text{when } f(O) \neq 0, \\ r[(\partial_1 f)(O) \cos \theta + (\partial_2 f)(O) \sin \theta] & \text{when } f(O) = 0, \end{cases}$$

and

$$f_1(x) := f(x) - f_0(x), \quad x \in \Sigma_\epsilon.$$

The assumption  $|f(O)| + |\nabla f(O)| > 0$  gives  $f_0 \not\equiv 0$ . However, noting that  $f \in C^{1,\alpha}(\overline{B_\epsilon(O) \cap D})$  and  $f_0$  is harmonic, we deduce from Lemma 2.3 that  $f_0 \equiv 0$ . This contradiction implies  $\partial D = \partial \tilde{D}$ .

**Step 2:** Prove  $f(O) = \tilde{f}(O)$ ,  $\nabla f(O) = \nabla \tilde{f}(O)$ , where  $O$  is an arbitrary corner point of the source support.

We still use the notations  $\Sigma_\epsilon$  and  $\Gamma_\epsilon$  defined in Step 1 with  $D = \tilde{D}$ . Repeating the arguments in Step 1, we obtain

$$\begin{cases} \Delta u + k^2 n(x)u = f & \text{in } \Sigma_\epsilon, \\ \Delta \tilde{u} + k^2 n(x)\tilde{u} = \tilde{f} & \text{in } \Sigma_\epsilon, \\ u = \tilde{u}, \partial_\nu u = \partial_\nu \tilde{u} & \text{on } \Gamma_\epsilon. \end{cases}$$

This implies that  $w := u - \tilde{u} \in H^2(B_\epsilon(O))$  is a weak solution of

$$\begin{cases} \Delta w + k^2 n(x)w = f - \tilde{f} & \text{in } \Sigma_\epsilon, \\ w = \partial_\nu w = 0 & \text{on } \Gamma_\epsilon. \end{cases}$$

Analogously to Step 1, we may define  $f_0(x) := f(O) - \tilde{f}(O)$  when  $f(O) \neq \tilde{f}(O)$  and define

$$f_0(x) := r\{[(\partial_1 f)(O) - (\partial_1 \tilde{f})(O)] \cos \theta + [(\partial_2 f)(O) - (\partial_2 \tilde{f})(O)] \sin \theta\}$$

when  $f(O) = \tilde{f}(O)$  and

$$f_1(x) := f(x) - \tilde{f}(x) - f_0(x).$$

Applying Lemma 2.3 again, we obtain  $f_0(x) \equiv 0$ , which implies  $f(O) = \tilde{f}(O)$  and  $\nabla f(O) = \nabla \tilde{f}(O)$ . Since the corner  $O$  is taken arbitrarily, we finish the proof of Theorem 1.1.  $\square$

If the smoothness of  $f$  at corner points can be weakened to be  $f \in C^{0,\alpha}(\overline{D \cap B_\epsilon(O)})$ , it follows from the proof of Theorem 1.1 that the source support  $\partial D$  and  $f(O)$  can be uniquely determined by  $u^\infty(\hat{x})$  for all  $\hat{x} \in \mathbb{S}$  if  $f(O) \neq 0$ . This proves Corollary 1.2. Combining the arguments in proving Theorem 1.1 and Lemma 2.3, we immediately get the shape identification result of Corollary 1.6(i).

Theorem 1.1 can be generalized to more smooth source terms with a compact support under an extra condition concerning the source discontinuity at corner points (see (3.6) below). This leads to Corollary 3.1.

**COROLLARY 3.1.** *Suppose that  $D := \text{supp}(f) \subset \mathbb{R}^2$  is a convex polygon and  $n(x) \equiv 1$  in  $\mathbb{R}^2$ . Assume that there exists an  $l \in \mathbb{N}_0$  such that*

$$f \in C^{l+1,\alpha}(\overline{D \cap B_\epsilon(O)}) \cap W^{l,\infty}(B_\epsilon(O)), \quad 0 < \alpha < 1$$

and

$$(3.6) \quad |\nabla^l f(O)| + |\nabla^{l+1} f(O)| > 0$$

for each corner  $O$  of  $\partial D$ . Then  $\partial D$ ,  $\partial^\beta f(O)$ , and  $\partial^{\beta'} f(O)$  can be uniquely determined by  $u^\infty(\hat{x})$  for all  $\hat{x} \in \mathbb{S}$ .

*Remark 3.1.* By Sobolev embedding theorems, the condition  $f \in W^{l,\infty}(B_\epsilon(O))$  implies that  $f \in C^{l-1,\alpha}(B_\epsilon(O))$ . Hence,  $\nabla^j f(O) = 0$  for all  $|j| \leq l-1$ . If  $l = 0$ , Corollary 3.1 is equivalent to the result of Theorem 1.1 when  $n(x) \equiv 1$ .

*Proof of Corollary 3.1.* Obviously, we have  $f \in H^l(B_\epsilon(O))$ , and by the interior regularity of elliptic equations (see, e.g., [13]) we get  $u \in H^{l+2}(B_{\epsilon'}(O))$  for some  $0 < \epsilon' < \epsilon$ . By the trace lemma,

$$\partial_\nu^j u^+ = \partial_\nu^j u^- \quad \text{on } \Gamma_{\epsilon'} := \partial D \cap B_{\epsilon'}(O)$$

for  $j = 0, 1, \dots, l+1$ . Proceeding as in the proof of Theorem 1.1, we suppose there exist two sources  $f$  and  $\tilde{f}$  ( $\tilde{D} := \text{supp } \tilde{f}$  is a convex polygon), which generate identical far-field patterns over all observation directions. If  $\partial D \neq \partial \tilde{D}$ , we suppose there exists a corner point  $O$  of  $D$  and a neighborhood  $B_{\epsilon'}(O)$  of  $O$  such that  $B_{\epsilon'}(O) \cap \tilde{D} = \emptyset$ . Set  $\Sigma'_{\epsilon'} = D \cap B_{\epsilon'}(O)$ . For any fixed multi-index  $\beta = (\beta_1, \beta_2)$  with  $|\beta| = l$ , we define  $v := \partial^\beta(u - \tilde{u})$ . Then it holds that

$$(3.7) \quad \begin{cases} \Delta v + k^2 v = \partial^\beta f & \text{in } \Sigma'_{\epsilon'}, \\ v = \partial_\nu v = 0 & \text{on } \Gamma_{\epsilon'}. \end{cases}$$

By the assumption of  $f$ , we have  $\partial^\beta f \in C^{1,\alpha}(\overline{\Sigma_{\epsilon'}})$  and thus

$$\partial^\beta f(x) = \partial^\beta f(O) + x \cdot [\nabla \partial^\beta f(O)] + o(|x|^\alpha) \quad \text{as } |x| \rightarrow 0,$$

uniformly in all  $x/|x|$  such that  $x \in \overline{\Sigma_{\epsilon'}}$ . Applying Lemma 2.3 to (3.7), we conclude  $\partial^\beta f(O) = \partial^{\beta'} f(O) = 0$  for all  $\beta' \geq \beta$ ,  $|\beta' - \beta| = 1$ , which contradicts our assumption (3.6) due to the arbitrariness of  $\beta$ . Thus,  $D = \tilde{D}$ . In the same manner, one can prove

$$\partial^\beta (f - \tilde{f})(O) = \partial^{\beta'} (f - \tilde{f})(O) = 0.$$

The proof is complete. □

**4. Determination of source terms.** In this section we prove Theorem 1.3.

*Proof of Theorem 1.3.* Assume that the two convex polygons  $D$  and  $\tilde{D} := \text{supp}(\tilde{f})$  produce identical far-field patterns. Here we suppose that  $\tilde{f} = \tilde{v}|_{\tilde{D}}$  for some  $\tilde{v} \in S(A, b)$ , which is analytic near corner points of  $\tilde{D}$ . Denote by  $\tilde{u}$  the unique solution corresponding to  $\tilde{D}$  and  $\tilde{f}$  that satisfies the Sommerfeld radiation condition.

If  $D \neq \tilde{D}$ , we may choose at least one corner point  $O \in \partial D$  as done in the proof of Theorem 1.1. Since  $f = v|_{\overline{D}}$  for some  $v \in S(A, b)$ , the source function  $f$  must be analytic on  $\overline{B_\epsilon(O)} \cap \overline{D}$ . By the proof of Corollary 2.2, the lowest-order term in the Taylor expansion of  $f$  in  $B_\epsilon(O) \cap D$  is harmonic. Arguing analogously to the proofs of Corollary 2.2 and Theorem 1.1, we may conclude from  $u^\infty = \tilde{u}^\infty$  that  $f$  vanishes identically on  $\overline{B_\epsilon(O)} \cap \overline{D}$ . Hence,  $v \equiv 0$  on  $\overline{B_\epsilon(O)} \cap \overline{D}$ , and by unique continuation we get  $v \equiv 0$  on  $\overline{D}$ . This implies that  $f$  vanishes identically on  $\overline{D}$  and thus  $\text{supp}(f) = \emptyset$ , which contradicts our assumption. Hence, we obtain the uniqueness in determining the source support.

To determine the source term, we set  $w = u - \tilde{u}$  and consider the Cauchy problem (cf. (3.5))

$$\begin{cases} \Delta w + k^2 n(x)w = f - \tilde{f} & \text{in } \Sigma_\epsilon, \\ w = \partial_\nu w = 0 & \text{on } \Gamma_\epsilon, \end{cases}$$

where  $\Sigma_\epsilon$  and  $\Gamma_\epsilon$  are defined as the same ones in the proof of Theorem 1.1. Note that  $f - \tilde{f} = (v - \tilde{v})|_{\overline{D}}$ , where  $v - \tilde{v} \in S(A, b)$  is analytic in  $B_\epsilon(O)$ . By the proof of Lemma 2.2, the function  $f - \tilde{f}$  takes the form

$$f(x) - \tilde{f}(x) = r^N (A \cos N\theta + B \sin N\theta) + \mathcal{O}(r^{N+\alpha}), \quad |x| \rightarrow 0, x \in \Sigma_\epsilon$$

for some  $N \in \mathbb{N}_0$ . Recalling Lemma 2.3 and Corollary 2.2, we arrive at  $f = \tilde{f}$  and thus  $v = \tilde{v}$  on  $\Sigma_\epsilon$ . Since  $A$  and  $b$  are a priori given, the relation  $f = \tilde{f}$  in  $D$  follows from the unique continuation property of elliptic equations (see [16, Theorem 17.2.6, Chapter XVII]). □

**5. Characterization of radiating sources.**

*Proof of Corollary 1.4.* Assume  $u^\infty \equiv 0$ . Then we obtain  $u \equiv 0$  in  $\mathbb{R}^3 \setminus \overline{D}$  by Rellich’s lemma, and in particular the traces of  $u$  vanish on  $\Gamma_\epsilon = \partial D \cap B_\epsilon(O)$  for some  $\epsilon > 0$ . Under the condition (i) in Corollary 1.4, we can deduce from the proof of Corollary 3.1 that  $v = \partial^\beta u$  with  $|\beta| = l$  satisfies

$$(5.1) \quad \begin{cases} \Delta v + k^2 n_0 v = \partial^\beta f & \text{in } \Sigma_\epsilon = D \cap B_\epsilon(O), \\ v = \partial_\nu v = 0 & \text{on } \Gamma_\epsilon. \end{cases}$$

It then follows that  $|\partial^\beta f(O)| = |\nabla \partial^\beta f(O)| = 0$ , which contradicts the condition (1.4).

If  $f$  fulfills the condition (ii), by the proof of Theorem 1.3,  $f$  must vanish identically, leading to a contradiction to the assumption that  $D = \text{supp}(f) \neq \emptyset$ .  $\square$

*Proof of Corollary 1.5.* Since  $n(x) \equiv n_0$  in  $B_R \setminus \overline{D}$ ,  $u$  is analytic in  $B_R \setminus \overline{D}$ . Suppose on the contrary that  $u$  can be analytically extended from  $(\mathbb{R}^2 \setminus \overline{D}) \cap B_\epsilon(O)$  to  $D \cap B_\epsilon(O)$  for some  $\epsilon > 0$ . Denote by  $w$  the extended solution in  $B_\epsilon(O)$ , which solves the Helmholtz equation  $\Delta w + k^2 n_0 w = 0$  in  $B_\epsilon(O)$  and coincides with  $u$  in  $(\mathbb{R}^2 \setminus \overline{D}) \cap B_\epsilon(O)$ . Setting  $v = u - w$ , we may arrive at the same Cauchy problem (5.1) with  $l = 0$  and then the same contradictions as in the proof of Corollary 1.4.  $\square$

*Proof of Corollary 1.6(ii).* If the lowest-order expansion of  $f$  around the corner is harmonic, the results of Corollaries 1.4 and 1.5 can be proved analogously by applying Corollary 2.2.  $\square$

**6. A data-driven inversion scheme.** The aim of this section is to propose a noniterative numerical scheme to reconstruct the shape  $\partial D$  of a convex-polygonal source embedded in  $B_R$  from a single far-field pattern. For simplicity we shall assume  $n(x) = n_0 > 0$  in  $|x| < R$  and consider a slightly different model as follows:

$$\begin{aligned} \Delta u + k^2 u &= 0 & \text{in } |x| > R, \\ \Delta u + k^2 n_0 u &= f & \text{in } |x| < R, \\ u^+ &= u^-, \quad \partial_\nu u^+ = \lambda \partial_\nu^- u & \text{on } |x| = R, \end{aligned}$$

where  $u$  is required to fulfill the Sommerfeld radiation condition at infinity. Here, the medium inside  $B_R$  is supposed to be homogeneous and isotropic, and the parameter  $\lambda > 0$  ( $\lambda \neq 1$ ) models the ratio of the constant densities in  $|x| > R$  and  $|x| < R$ . This implies that the background medium is piecewise constant, which often occurs in the real world. In this section, the source function  $f$  is supposed to satisfy the following:

- (i)  $D := \text{supp}(f) \subset B_R$  is a convex polygon.
- (ii) For each corner point  $O \in \partial D$ , it holds that  $f \in C^{0,\alpha}(\overline{D \cap B_\epsilon(O)})$  for some  $\epsilon > 0$ ,  $0 < \alpha < 1$  and that  $|f(O)| > 0$ .

By Theorem 1.1 and Corollary 1.2, the source support  $\partial D$  can be uniquely determined by the radiated far-field pattern  $u^\infty(\hat{x})$  for all  $\hat{x} \in \mathbb{S}$ . Moreover, we obtain the following result (see the proof of Theorem 1.1 and Corollary 1.5).

**PROPOSITION 6.1.** *The solution  $u$  cannot be (analytically) extended from  $B_R \setminus \overline{D}$  into a neighboring area of any corner point of  $\partial D$  as a solution of the Helmholtz equation  $\Delta u + k^2 n_0 u = 0$ .*

To determine the position and shape of  $\partial D$ , we introduce a test domain  $\Omega$  which represents an acoustically sound-soft obstacle inside  $B_R$ . Here  $\Omega \subset B_R$  is called a test domain if it is a connected convex domain and  $k^2 n_0$  is not a Dirichlet eigenvalue of  $-\Delta$  over  $\Omega$ . Consider the time-harmonic scattering of a plane wave from the obstacle  $\Omega$  modeled by

$$\begin{aligned} \Delta v + k^2 v &= 0 & \text{in } \mathbb{R}^2 \setminus \overline{B_R}, & \quad v = e^{ikx \cdot d} + v^{sc}, \\ \Delta v + k^2 n_0 v &= 0 & \text{in } B_R \setminus \overline{\Omega}, \\ v &= 0 & \text{on } \partial\Omega, \\ v^+ &= v^-, \quad \partial_\nu v^+ = \lambda \partial_\nu^- v & \text{on } \partial B_R, \end{aligned}$$

where  $v^{sc}$  is a Sommerfeld radiating solution in  $|x| > R$ . Note that  $d \in \mathbb{S}$  denotes the incident direction of the plane wave  $v^{in}(x) = e^{ikx \cdot d}$  in  $\mathbb{R}^2$ . The far-field patterns



$v^\infty(\hat{x}, d)$  of  $v^{sc}$  for all incident directions define the far-field operator  $F^{(\Omega)} : L^2(\mathbb{S}) \rightarrow L^2(\mathbb{S})$ :

$$(F^{(\Omega)}g)(\hat{x}) = \int_{\mathbb{S}} v^\infty(\hat{x}, d)g(d) ds(d), \quad \hat{x} \in \mathbb{S}.$$

Analogously, denote by  $F_0 : L^2(\mathbb{S}) \rightarrow L^2(\mathbb{S})$  the far-field operator corresponding to the penetrable scatterer  $B_R$  without the embedded sound-soft obstacle  $\Omega$  and by  $\mathcal{S}_0$  the scattering operator corresponding to  $F_0$  defined by

$$\mathcal{S}_0 := I + 2ik\gamma F_0, \quad \gamma := \frac{e^{i\pi/4}}{\sqrt{8k\pi}}.$$

It was proved in [4] via the factorization method that the spectral system  $(\lambda_j^{(\Omega)}, \psi_j^{(\Omega)})$  of the operator

$$F_{\#}^{(\Omega)} := |\operatorname{Re}((F_0 - F^{(\Omega)})\mathcal{S}_0)| + |\operatorname{Im}((F_0 - F^{(\Omega)})\mathcal{S}_0)|$$

can be used to characterize the embedded obstacle  $\Omega$ . Note that the spectra are uniquely determined by  $\Omega, B_R, k, n_0$ , and  $\lambda$ , all of which are known. To proceed, we recall that the data-to-pattern operator  $G^{(\Omega)} : H^{1/2}(\partial\Omega) \rightarrow L^2(\mathbb{S})$  is defined as  $G^{(\Omega)}(h) := v^\infty$ , where  $v^\infty$  is the far-field pattern of the unique radiating solution  $v$  to the boundary value problem

$$\begin{aligned} \Delta v + k^2 v &= 0 \quad \text{in } \mathbb{R}^2 \setminus \overline{B}_R, & \Delta v + k^2 n_0 v &= 0 \quad \text{in } B_R \setminus \overline{\Omega}, \\ v &= h \quad \text{on } \partial\Omega, & v^+ &= v^-, \partial_\nu v^+ = \lambda \partial_\nu^- v \quad \text{on } \partial B_R. \end{aligned}$$

LEMMA 6.1. *Let  $u^\infty$  be the far-field pattern corresponding to the source term  $f$  with the convex polygonal support  $D \subset B_R$ . Then  $u^\infty$  belongs to the range of the data-to-pattern operator  $G^{(\Omega)}$  if and only if  $D \subset \overline{\Omega}$ .*

*Proof.* Suppose  $D \subset \overline{\Omega} \subset B_R$ . By the definition of  $G^{(\Omega)}$ , we have  $G^{(\Omega)}(h) = u^\infty$  with  $h := u|_{\partial\Omega}$ , implying that  $u^\infty$  is indeed in the range of  $G^{(\Omega)}$ . Now suppose that  $u^\infty = G^{(\Omega)}(h)$  for some  $h \in H^{1/2}(\Omega)$  but  $D \subset \overline{\Omega}$  does not hold. Since  $D$  is a convex polygon and  $\Omega$  is a connected convex domain, one can always find a corner point of  $\partial D$  which lies in the exterior of  $\Omega$ . Using Rellich’s lemma and the unique continuation for solutions of the Helmholtz equation,  $u$  can be analytically extended into a full neighborhood of this corner point. However, this is a contradiction to Proposition 6.1, implying that  $D \subset \overline{\Omega}$ .  $\square$

By [4], the range of  $G^{(\Omega)}$  coincides with the range of  $F_{\#}^{1/2}$ . Hence,  $u^\infty$  belongs to the range of  $F_{\#}^{1/2}$  if and only if  $D \subset \overline{\Omega}$ . This together with Picard’s criterion implies a noniterative inversion scheme stated as below.

THEOREM 6.1. *Introduce the domain-defined indicator*

$$(6.1) \quad W(\Omega) := \sum_{j=1}^{\infty} \frac{|\langle u^\infty, \psi_j^{(\Omega)} \rangle_{L^2(\mathbb{S})}|^2}{|\lambda_j^{(\Omega)}|}.$$

*We have  $W(\Omega) < \infty$  if and only if  $D \subset \overline{\Omega}$ . Hence, the source support  $D$  can be regarded as the intersection of all test domains  $\Omega$  such that  $W(\Omega) < \infty$ .*

By Theorem 6.1, the source support  $D$  can be reconstructed by first selecting different test domains  $\Omega \subset B_R$  and then computing the indicator  $W(\Omega)$  to get the inclusion relation between  $D$  and  $\Omega$ . In particular, these test domains can be taken as sound-soft disks with different centers and radii. We refer to [30] for numerical examples of recovering convex polygonal sound-soft obstacles and source support in a homogeneous background medium. If  $\partial D$  contains no corners, only partial information on  $D$  can be recovered; see [28], where the linear sampling method with a single far-field pattern was discussed. In the case of a variable inhomogeneous background medium, Theorem 6.1 can be similarly justified by taking inspiration from the corresponding factorization schemes proved in [15, 31].

In Theorem 6.1 we established a domain-defined factorization method in an inhomogeneous background medium using only a single far-field pattern. A one-wave factorization method was discussed earlier in [9] for inverse elastic scattering problems. The point-defined factorization scheme [4] makes use of far-field data over all incident directions at a fixed energy. In comparison to [4], we use in Lemma 6.1 a sampling domain  $\Omega \subset \mathbb{R}^2$  in place of a sampling point  $z \in \mathbb{R}^2$ . Correspondingly, the far-field data of the background Green's tensor  $G_0(\cdot, z)$  (or its equivalence function used in [4]) have been substituted by the measurement far-field pattern  $u^\infty$  in Lemma 6.1, and the role of the singularity of  $G_0$  was replaced by the absence of analytical extension in a corner domain (see Proposition 6.1). We refer to [25] and [32, Chapter 15] for other domain-defined sampling methods.

One can interpret Theorem 6.1 as a data-driven reconstruction scheme because far-field patterns of a priori given obstacles (test domains) are involved in the inversion process. Theorem 6.1 explains how these a priori data encode the information on our target. The test domains  $\Omega$  can also be replaced by sound-hard, impedance, and penetrable scatterers embedded in  $B_R$ .

**Acknowledgments.** The authors are grateful to the two anonymous referees for their valuable suggestions and comments which helped improve this paper.

#### REFERENCES

- [1] G. BAO, J. LIN, AND F. TRIKI, *A multi-frequency inverse source problem*, J. Differential Equations, 249 (2010), pp. 3443–3465.
- [2] E. BLÅSTEN, *Nonradiating sources and transmission eigenfunctions vanish at corners and edges*, SIAM J. Math. Anal., 6 (2018), pp. 6255–6270.
- [3] E. BLÅSTEN, L. PÄIVÄRINTA, AND J. SYLVESTER, *Corners always scatter*, Commun. Math. Phys., 331 (2014), pp. 725–753.
- [4] O. BONDARENKO, A. KIRSCH, AND X. LIU, *The factorization method for inverse acoustic scattering in a layered medium*, Inverse Problems, 29 (2013), 045010.
- [5] F. CAKONI AND D. COLTON, *A Qualitative Approach to Inverse Scattering Theory*, Springer, New York, 2014.
- [6] D. COLTON AND R. KRESS, *Inverse Acoustic and Electromagnetic Scattering Theory*, Springer, New York, 1998.
- [7] J. ELSCHNER AND G. HU, *Acoustic scattering from corners, edges and circular cones*, Arch. Ration. Mech. Anal., 228 (2018), pp. 653–690.
- [8] J. ELSCHNER AND G. HU, *Corners and edges always scatter*, Inverse Problems, 31 (2015), 015003.
- [9] J. ELSCHNER AND G. HU, *Uniqueness and factorization method for inverse elastic scattering with a single incoming wave*, Inverse Problems, 35 (2019), 094002.
- [10] J. ELSCHNER, G. HU, AND M. YAMAMOTO, *Uniqueness in inverse elastic scattering from unbounded rigid surfaces of rectangular type*, Inverse Probl. Imaging, 9 (2015), pp. 127–141.
- [11] M. ELLER AND N. VALDIVIA, *Acoustic source identification using multiple frequency information*, Inverse Problems, 25 (2009), 115005.

- [12] A. FRIEDMAN AND V. ISAKOV, *On the uniqueness in the inverse conductivity problem with one measurement*, Indiana Univ. Math. J., 38 (1989), pp. 563–579.
- [13] D. GILBARG AND N. S. TRUDINGER, *Elliptic Partial Differential Equations of Second Order*, Springer, Berlin, 2001.
- [14] P. GRISVARD, *Singularities in Boundary Value Problems*, Masson, Paris, 1992.
- [15] Y. GRISEL, V. MOUYSET, P. A. MAZET, AND J. P. RAYMOND, *Determining the shape of defects in non-absorbing inhomogeneous media from far-field measurements*, Inverse Problems, 28 (2012), 055003.
- [16] L. HÖRMANDER, *The Analysis of Linear Partial Differential Operators III*, Springer, Berlin, 1985.
- [17] G. HU, Y. KIAN, AND Y. ZHAO, *Uniqueness to some inverse source problems for the wave equation in unbounded domains*, Acta Math. Appl. Sin. Engl. Ser., 36 (2020), pp. 134–150.
- [18] G. HU, L. LI, AND J. ZOU, *Unique determination of a penetrable scatterer of rectangular type for inverse Maxwell equations by a single incoming wave*, Inverse Problems, 35 (2019), 035006.
- [19] G. HU, M. SALO, AND E. V. VESALAINEN, *Shape identification in inverse medium scattering problems with a single far-field pattern*, SIAM J. Math. Anal., 48 (2016), pp. 152–165.
- [20] M. IKEHATA, *An inverse transmission scattering problem and the enclosure method*, Computing, 75 (2005), pp. 133–156.
- [21] M. IKEHATA, *On reconstruction in the inverse conductivity problem with one measurement*, Inverse Problems, 16 (2000), pp. 785–793.
- [22] M. IKEHATA, *Reconstruction of a source domain from the Cauchy data*, Inverse Problems, 15 (1999), pp. 637–645.
- [23] V. A. KONDRATIEV, *Boundary value problems for elliptic equations in domains with conical or angular points*, Trans. Moscow Math. Soc., 16 (1967), pp. 227–313.
- [24] V. A. KOZLOV, V. G. MAZ’YA, AND J. ROSSMANN, *Elliptic Boundary Value Problems in Domains with Point Singularities*, American Mathematical Society, Providence, RI, 1997.
- [25] S. KUSIAK, R. POTTHAST, AND J. SYLVESTER, *A “range test” for determining scatterers with unknown physical properties*, Inverse Problems, 19 (2003), pp. 533–547.
- [26] S. KUSIAK AND J. SYLVESTER, *The scattering support*, Comm. Pure Appl. Math., 56 (2003), pp. 1525–1548.
- [27] L. LI, G. HU, AND J. YANG, *Interface with weakly singular points always scatter*, Inverse Problems, 34 (2018), 075003.
- [28] J. LIU AND J. SUN, *Extended sampling method in inverse scattering*, Inverse Problems, 34 (2018), 085007.
- [29] V. G. MAZ’YA, S. A. NAZAROV, AND B. A. PLAMENEVSKII, *Asymptotic Theory of Elliptic Boundary Value Problems in Singularly Perturbed Domains I*, Birkhäuser-Verlag, Basel, 2000.
- [30] G. MA AND G. HU, *A data-driven approach to inverse acoustic scattering*, in preparation.
- [31] A. NACHMAN, L. PÄIVÄRINTA, AND A. TEIRLILÄ, *On imaging obstacle inside inhomogeneous media*, J. Funct. Anal., 252 (2007), pp. 490–516.
- [32] G. NAKAMURA AND R. POTTHAST, *Inverse Modeling—An Introduction to the Theory and Methods of Inverse Problems and Data Assimilation*, Institute of Physics Publishing, Bristol, UK, 2015.
- [33] S. A. NAZAROV AND B. A. PLAMENEVSKY, *Elliptic Problems in Domains with Piecewise Smooth Boundaries*, Walter de Gruyter, Berlin, 1994.
- [34] L. PÄIVÄRINTA, M. SALO, AND E. V. VESALAINEN, *Strictly convex corners scatter*, Rev. Mat. Iberoam., 33 (2017), pp. 1369–1396.
- [35] J. SEO, *On the uniqueness in the inverse conductivity problem*, J. Fourier Anal. Appl., 2 (1996), pp. 227–235.
- [36] A. SOMMERFELD, *Die Greensche Funktion der Schwingungsgleichung*, Jber. Deutsch. Math. Verein., 21 (1912), pp. 309–353.
- [37] J. SYLVESTER, *Notions of support for far fields*, Inverse Problems, 22 (2006), pp. 1273–1288.