Singularities of the analytical continuation of time-harmonic scattering by sound-soft ellipses

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Abstract

This paper is concered with the analytical continuation of time-harmonic wave fields from the exterior of a sound-soft circle or ellipse to inside. The incoming wave is supposed to be a point source wave emitting from the exterior. The Schwart reflection principle across the Dirichlet boundary is used to locate all possible singularities. The first part of the singularities is caused by the imaging point of the point source by the Schwartz reflection, which truns out to be of logarithimic type, while the second part consists of algebraic singularities of the Schwartz function of the boundary.

Keywords: inverse scattering, Schwartz reflection principle, Helmholtz equation, singularities.

1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be a sound-soft disc or ellipse embedded in a homogeneous and isotropic medium. Suppose that Ω is illuminated by a time-harmonic point source wave u^{in} defined by

$$u^{in}(x;y) = \frac{i}{4} H_0^{(1)}(k|x-y|), \quad x \neq y, \tag{1.1}$$

with $x = (x_1, x_2) \in \mathbb{R}^2$, $y = (y_1, y_2) \in \Omega^c := \mathbb{R}^2 \backslash \overline{\Omega}$. Here, $H_0^{(1)}$ denotes the Hankel function of the first kind of order zero. The proragation of scattered (perturbed) field u^{sc} can be modeled by the exterior boundary value problem of the Helmholtz equation

$$\begin{cases} \Delta u^{sc} + k^2 u^{sc} = 0 & \text{in} \quad \Omega^c \backslash \{y\}, \\ u^{sc} = -u^{in}, & \text{on } \partial\Omega, \\ \lim_{r\to\infty} \sqrt{r}\,(\partial_r u^{sc} - iku^{sc}) = 0, & r = |x|,\ x \in \Omega^c, \end{cases} \tag{1.2}$$

where $k > 0$ is the wave number and the boundary condition at infinity is known as the Sommerfeld radiation condition. Since $\partial\Omega$ is C^∞-smooth, It is well known (see [1]) that there exists

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a unique solution $u^{sc} \in C^\infty(\overline{\Omega}^c)$. Moreover, the solution is even analytic up to $\partial\Omega$ by standard elliptic regularity. The Sommerfeld radiation condition of u^{sc} leads to the asymptotic expansion

$$u^{sc}(x) = \frac{e^{ikr}}{\sqrt{r}} \left(u^\infty(\hat{x}) + O\left(\frac{1}{r}\right) \right), \qquad \hat{x} := x/|x|,$$

uniformly in all directions $\hat{x} \in \mathbb{S} := \{x \in \mathbb{R}^2 : |x| = 1\}$, where the function u^∞ defined on the unit circle is known as the far-field pattern of u^{sc}. In this paper, we discuss the extension of the total wave $u(x;y) = u^{in}(x;y) + u^{sc}(x;y)$ from Ω^c into Ω and find the location and type of possible singularities.

2 Schwartz function for circles and ellipses

Consider a non-singular real-analytic curve $\Gamma \subset \mathbb{R}^2$ given by

$$\Gamma := \{x = (x_1, x_2) \in \mathbb{R}^2 : \ f(x) = 0, \ \nabla f(x) \neq 0\}$$

where f is an algebraic function defined on $\mathbb{R}^2$, that is, f is polynomial in x_1 and x_2 with real coefficients. Below we define the Schwartz reflection $\mathcal{R}_\Gamma$ with respect to Γ by the Schwartz function of Γ. Let $U \subset \mathbb{R}^2$ be an open connected set separated into two parts U^+ and U^- by Γ. Now consider a complex domain $V \subset \mathbb{C}^2$ such that $V \cap \mathbb{R}^2 = U$. Using the the bicharacteristic coordinates $z = x_1 + ix_2$, $\omega = x_1 - ix_2$, we can identify the real plane $x \in \mathbb{R}^2$ with the complex plane $z \in \mathbb{C}$. The curve Γ can be complexified to a curve $\Gamma_\mathbb{C} \subset \mathbb{C}^2$ defined by

$$\Gamma_\mathbb{C} := \left\{ (z, \omega) \in \mathbb{C}^2 : f\left(\frac{z+\omega}{2}, \frac{z-\omega}{2i}\right) = 0 \right\}. \tag{2.3}$$

Note that $\Gamma_\mathbb{C}$ coincides with Γ on those points $(z, w) \in \mathbb{C}^2$ such that $z = \overline{\omega}$, where the bar denotes the complex conjugate, that is, $\Gamma = \left\{x \in \mathbb{R}^2 : (x_1 + ix_2, x_1 - ix_2) \in \Gamma_\mathbb{C}\right\}$. Since $\nabla f \neq 0$ on Γ, the above Equation (2.3) is unique solvable for z and w in a neighborhood of $\Gamma_\mathbb{C}$ in $\mathbb{C}^2$. We denote the corresponding solutions by $\omega = S(z)$ and $z = \widetilde{S}(\omega)$. Here the function $S(z)$ is called the *Schwarz function* of the curve Γ and $z = \widetilde{S}(\omega)$ represents the inverse of $S(z)$. The Schwartz reflection $\mathcal{R}_\Gamma$ is defined as $\mathcal{R}_\Gamma(z) := \overline{S(z)}$ for $z \in \mathbb{C}$ ([7]). Equivalently, we have on the real plane that

$$\mathcal{R}_\Gamma(x) := \{y = (y_1, y_2) \in \mathbb{R}^2 : y_1 + iy_2 = \overline{S(z)}\Big|_{z = x_1 + ix_2}\}.$$

It is well known that $\mathcal{R}_\Gamma : U \to U$ is a conformal mapping permuting U^+ and U^- (that is, if $x \in U^\pm$, then $\mathcal{R}_\Gamma(x) \in U^\mp$). For a geometric interpretation of $\mathcal{R}_\Gamma$ we refer to [15, 14]. Below we list the Schwartz functions for straight lines, circles and ellipses (see e.g., [7, Chapter 5]).

- The straight line passing through $z_1 = x_1 + ix_2$ and $z_2 = y_1 + iy_2$:

$$S(z) = \frac{\overline{z}_1 - \overline{z}_2}{z_1 - z_2}(z - z_2) + \overline{z}_2.$$

- The circle centered at the $z_1 = x_1 + ix_2$ with radius $a > 0$:

$$S(z) = \frac{a^2}{z - z_1} + \overline{z}_1. \tag{2.4}$$

- The ellipse $(x_1/a)^2 + (x_2/b)^2 = 1$ with $a > b$:

$$S(z) = \frac{a^2 + b^2}{c^2} z - \frac{2ab}{c^2}\sqrt{z^2 - c^2}, \quad c^2 := a^2 - b^2. \tag{2.5}$$

Obviously, the Schwartz function of a straight line is an entire function over $\mathbb{C}$, while that has an algebraic pole of order one at the origin for circles and has ordinary algebraic singularities of square-root type at the two focal points of an ellipse. In general, only straight lines admit the entire Schwartz function, and rational Schwartz functions are possible only for straight lines and circles; see [7, Chapter 10]. In the remaining part of this paper, we always assume that the disc B_R denotes the disc centered at the origin with radius R, that is

$$B_R = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < R^2\right\} \tag{2.6}$$

and the ellipse D denotes

$$D = \left\{(x_1, x_2) \in \mathbb{R}^2 : \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} < 1\right\} \quad \text{for some } a > b > 0, \tag{2.7}$$

with the two focal points $(\pm c, 0)$ on the x_1-axis. Let $L \subset \mathbb{R}^2$ be the line segment connecting $(-c, 0)$ and $(c, 0)$. Then we take L as a branch cut of the square root $\sqrt{z^2 - c^2}$ in (2.5) and define $\sqrt{\cdot}$ to be positive on the positive x_1-axis.

Remark 2.1. *The Schwartz function of a closed curve is usually a meromorphic function. In the elliptic case, we give an explicit representation of the Schwartz reflection on the real plane. It follows from* (2.5) *for* $z = |z|e^{i\theta} \in \mathbb{C}\backslash L$ *that*

$$S(z) = \begin{cases} \frac{a^2+b^2}{c^2}(x_1 + ix_2) - \frac{2ab}{c^2}\sqrt{x_1^2 - x_2^2 - c^2 + i2x_1x_2} & \text{if} \quad \theta \in (-\pi/2, \pi/2], \\ \frac{a^2+b^2}{c^2}(x_1 + ix_2) + \frac{2ab}{c^2}\sqrt{x_1^2 - x_2^2 - c^2 + i2x_1x_2} & \text{if} \quad \theta \in (\pi/2, 3\pi/2], \end{cases}$$

where $z = x_1 + ix_2 \in \mathbb{C}$*. Write*

$$\eta(x) := \left(\left(x_1^2 - x_2^2 - c^2\right)^2 + 4x_1^2x_2^2\right)^{1/2} > 0,$$
$$\alpha(x) := \arg(x_1^2 - x_2^2 - c^2 + 2ix_1x_2) \in (-\pi, \pi].$$

We get from the definition of $\mathcal{R}_\Gamma$ *that* $\mathcal{R}_\Gamma(x) := \overline{S(z)}\Big|_{z=x_1+x_2} = (y_1, y_2) \in \mathbb{R}^2$*, where*

$$y_1 = \mathrm{Re}[\overline{\mathrm{S(z)}}] = \begin{cases} \frac{a^2+b^2}{c^2}x_1 - \frac{2ab}{c^2}\sqrt{\eta(x)}\cos\frac{\alpha(x)}{2}, & \text{if} \quad \theta \in (-\pi/2, \pi/2], \\ \frac{a^2+b^2}{c^2}x_1 + \frac{2ab}{c^2}\sqrt{\eta(x)}\cos\frac{\alpha(x)}{2}, & \text{if} \quad \theta \in (\pi/2, 3\pi/2], \end{cases}$$

and

$$y_2 = \mathrm{Im}[\overline{S(z)}] = \begin{cases} -\frac{a^2+b^2}{c^2}x_2 + \frac{2ab}{c^2}\sqrt{\eta(x)}\sin\frac{\alpha(x)}{2}, & \text{if} \quad \theta \in (-\pi/2, \pi/2], \\ -\frac{a^2+b^2}{c^2}x_2 - \frac{2ab}{c^2}\sqrt{\eta(x)}\sin\frac{\alpha(x)}{2}, & \text{if} \quad \theta \in (\pi/2, 3\pi/2]. \end{cases}$$

Denote by D_j $(j = 1, 2, \cdots)$ the ellipse confocal to D with the semi-major and semi-minor axis a_j and b_j along the x_1- and x_2-axes, respectively. Set $\Gamma_j = \partial D_j$ and define D^* with the boundary $\Gamma^* := \partial D^*$ as the unique ellipse confocal to D with the half-axes $a^* > b^* > 0$ such that

$$\frac{a^*}{b^*} = \frac{a^2 + b^2}{2ab}, \quad (a^*)^2 - (b^*)^2 = c^2. \tag{2.8}$$

Simple calculations show that

$$a^* = \frac{a^2 + b^2}{c} > a, \quad b^* = \frac{2ab}{c} > b. \tag{2.9}$$

Below we collect reflecting properties of $\mathcal{R}_\Gamma$. For simplicity we write $R_\Gamma = R$, but we must always keep in mind that it is the Schwartz reflection with respect to $\Gamma = \partial D$.

Lemma 2.1 ([2]). *(i) For $a_1 > c$, we have $R(\Gamma_1) = \Gamma_2$ with the half-axis $a_2 > b_2$ satisfying*

$$a_2 = \frac{a^2 + b^2}{c^2} a_1 - \frac{2ab}{c^2}\sqrt{a_1^2 - c^2}, \quad b_2 = -\frac{a^2 + b^2}{c^2} b_1 + \frac{2ab}{c^2}\sqrt{b_1^2 + c^2}.$$

(ii) If $a_1 = a$ and $b_1 = b$, it holds that $a_2 = a$, $b_2 = b$, that is, $\mathcal{R}(\Gamma) = \Gamma$; If $a_1 = a^*$ and $b_1 = b^*$, we have $a_2 = c, b_2 = 0$, implying that $\mathcal{R}(\Gamma^*) = L$. If the semi-major axis a_1 of Γ_1 increases from a to a^*, then the semi-axis a_2 of its image $\Gamma_2 = \mathcal{R}(\Gamma_1)$ decreases from a to c and vice versa. Hence, the domain $D \backslash L$ and $D^* \backslash \overline{D}$ are symmetric with respect to Γ in the sense of the Schwartz reflection $\mathcal{R}$.

(iii) If the semi-major axis a_1 of Γ_1 increases from a^* to $+\infty$, then the semi-major axis a_2 of $\Gamma_2 = \mathcal{R}(\Gamma_1)$ also increases from c to ∞.

In Figure 1, we plot four images of R_Γ when the reflecting ellipse is given by $(x_1/5)^2 + (x_2/3)^2 = 1$. In this case, we have $c = 4$ and $a^* = 8.5$. When the major-axis a_1 along the x_1-axis of the outmost ellipse increases from $a = 5$ to $a^* = 8.5$, the half-axis a_2 of its image (i.e., the innermost ellipse) decreases from $a = 5$ to $c = 4$. In the special case of $a_1 = a^*$, the image $R(\Gamma^*)$ degenerates to the line segment L connecting the two foci $(\pm 4, 0)$. The four points A, B, C and D are reflected to A^*, B^*, C^* and D^*, respectively. Clearly, when a point moves in the positive direction along the outmost/innermost ellipse, its image moves in the positive direction along the innermost/outmost ellipse. It is shown in Figure 2 that, when the half-axis of the outmost ellipse increases from $a^* = 8.5$, the half-axis of its image increases from $c = 4$. However, the points lying on Γ_1 with $a_1 > a^*$ and those on $R_\Gamma(\Gamma_1)$ move in opposite directions.

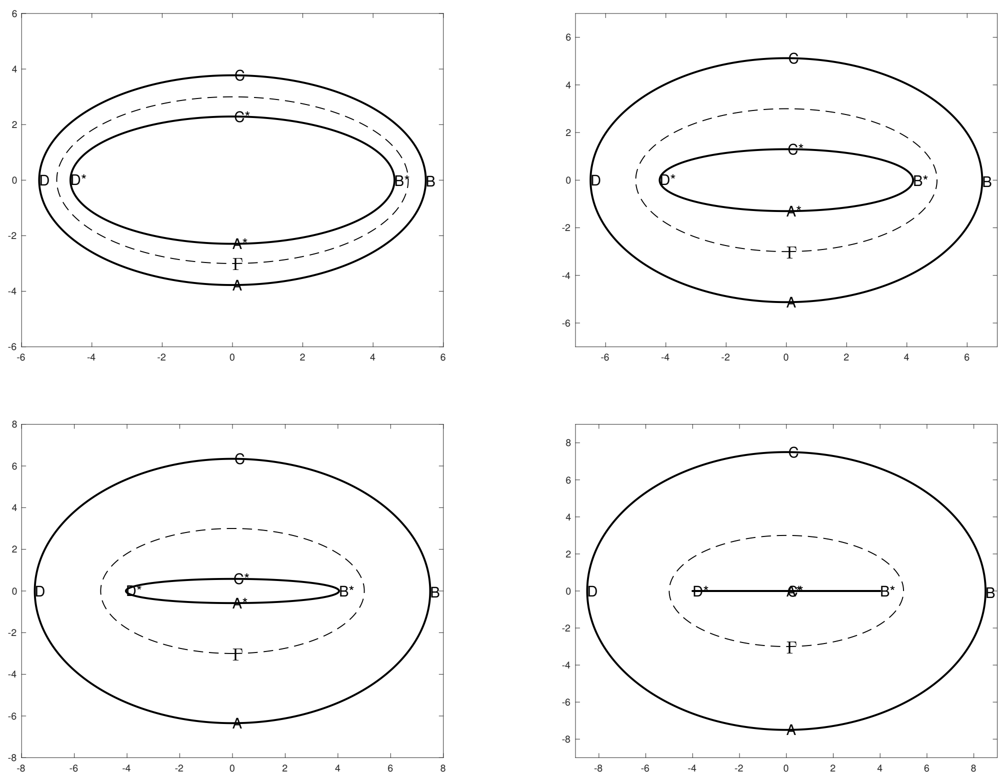

Figure 1: The dashed curve denotes the ellipse given by $(x_1/5)^2 + (x_2/3)^2 = 1$. The two solid ellipses are symmetric w.r.t in the sense of the Schwartz reflection $\mathcal{R}_\Gamma$. The half-axis of the outmost ellipses along the x_1-axis are 5.5 (left-top), 6.5 (right-top), 7.5 (left-bottom) and 8.5 (right-bottom), respectively.

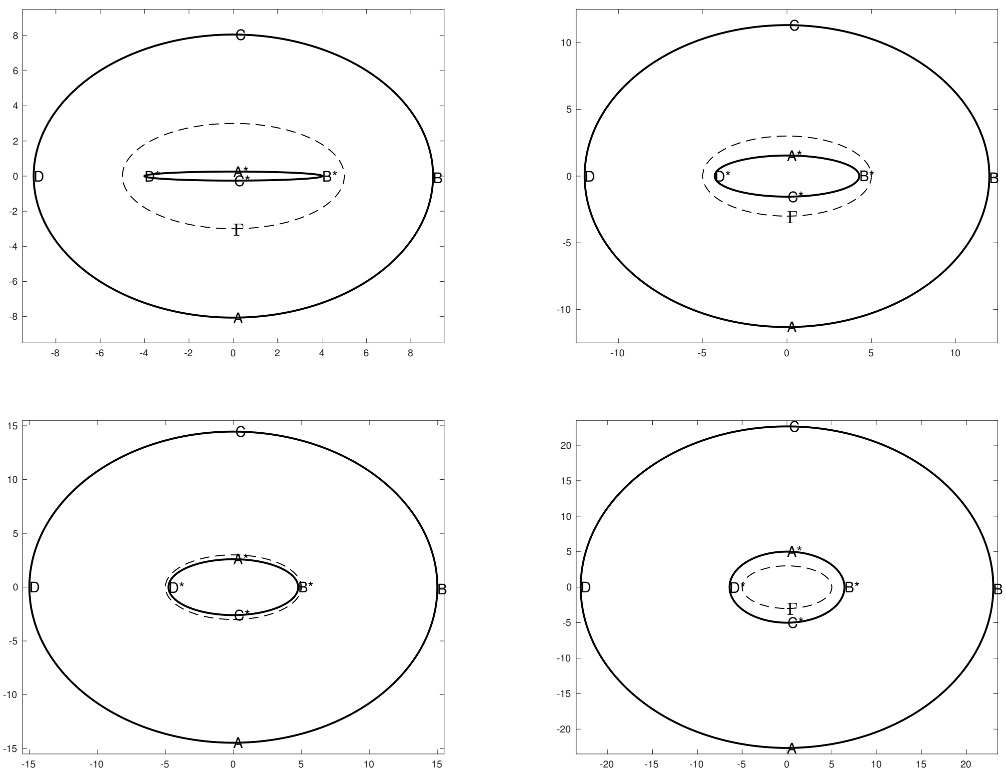

Figure 2: The outmost solid ellipse is reflected to another solid ellipse inside by the Schwartz reflection of $\Gamma := \{x : (x_1/5)^2 + (x_2/3)^2 = 1\}$. The half-axis of the outmost ellipses along the x_1-axis are 9 (left-top), 12 (right-top), 15 (left-bottom) and 23 (right-bottom), respectively, all of them are greater than the critical value $a^* = 8.5$.

3 Schwartz reflection principle for Helmholtz equation

It is well known that the Schwartz function can be used to continue harmonic functions with analytic Cauchy data on a non-singular algebraic curve; see e.g. [2, 20, 8] and [7, Chapter 13]. In this paper, we need the Schwartz reflection principle for the Helmholtz equation with respect to a Dirichlet boundary, which was proved in [13] for general non-singular analytic curves; see also [21, 14]. To state the reflection principle, we first introduce a Cauchy-Gousat problem.

Let $\Gamma_{\mathbb{C}}$ be the complexificated curve of Γ and let $S(z)$ be the Schwartz function of $\Gamma \subset \mathbb{R}^2$. Consider the Cauchy-Gousat problem in $\mathbb{C}^2$.

$$\begin{cases} \left(\frac{\partial^2}{\partial z \partial w} + \frac{k^2}{4}\right) G_j(z, w; z_0, w_0) = 0 \quad \text{near} \quad \Gamma_{\mathbb{C}}, \\ G_j|_{\Gamma_{\mathbb{C}}} = J_0(k\sqrt{(z - z_0)(w - w_0)})|_{\Gamma_{\mathbb{C}}}, \\ G_j = 1 \quad \text{on the characteristic rays } \tilde{l}_j \text{ staring from } (\widetilde{S}(w_0), S(z_0)) \in \mathbb{C}^2,\ j = 1, 2. \end{cases} \tag{3.10}$$

Here, J_0 is the zero order Bessel function (a Riemann function for the Helmholtz operator) and

$$\tilde{l}_1 = \{(z, w) \in \mathbb{C}^2 : \widetilde{S}(w) - z_0 = 0\}, \quad \tilde{l}_2 = \{(z, w) \in \mathbb{C}^2 : S(z) - w_0 = 0\}, \tag{3.11}$$

For the circle centered at the $z = x_1 + ix_2$ with radius $a > 0$, we have

$$\widetilde{S}(w) = \frac{a^2}{w - \overline{z}} + z, \quad w \in \mathbb{C}. \tag{3.12}$$

For the ellipse in (2.7),

$$\widetilde{S}(w) = \frac{a^2 + b^2}{c^2} w - \frac{2ab}{c^2}\sqrt{w^2 - c^2}, \quad c^2 := a^2 - b^2. \tag{3.13}$$

Denote the function $G_j(x; y)$ the restriction to $\mathbb{R}^2 \times \mathbb{R}^2$ of the multi-value analytic function $G_j(z, w; z_0, w_0)$ in $\mathbb{C}^4$ (that is, $z = \overline{w} = x_1 + ix_2, z_0 = \overline{w}_0 = y_1 + iy_2$). The singularities of the function $z \mapsto G_j(z, w; z_0, w_0)$ and $w \mapsto G_j(z, w; z_0, w_0)$ coincide with those of the Schwartz function S, and the singularities of the function $z_0 \mapsto G_j(z, w; z_0, w_0)$ and $w_0 \mapsto G_j(z, w; z_0, w_0)$ coincide with those of $\widetilde{S}$.

Denote $G(x; y)$ as the difference

$$G(x; y) = G_1(x; y) - G_2(x; y), \tag{3.14}$$

where $G_1(z, w; z_0, w_0)$ and $G_2(z, w; z_0, w_0)$ are solutions to the Cauchy-Gousat problems (3.10), which will be used later.

Below we state the reflection principle across a non-singular algebraic curve.

Lemma 3.1 ([13]). *Let $\gamma \subset \Gamma$ be a subset of Γ and let $U = U^+ \cup \gamma \cup U^- \subset \mathbb{R}^2$ be a symmetric domain in the sense of the Schwartz reflection of Γ. Let u be a solution to the Helmholz equation with inhomogeneous Dirichlet data on γ:*

$$\Delta u + k^2 u = 0 \quad in \quad U^+, \qquad u = g \quad on \quad \gamma, \tag{3.15}$$

where g extends to a holomorphic function of two variables at least in a neighborhood of Γ. Then u can be analytically extended from U^+ to U^- by the Schwarz reflection principle

$$u(x) := -u(\mathcal{R}_\Gamma(x)) + \frac{1}{2i}\int_{\mathcal{C}_\gamma^{(x)}} G(x;y)\frac{\partial u(y)}{\partial\nu(y)}\mathrm{d}y - \frac{1}{2i}\int_{\mathcal{C}_\gamma^{(x)}} u(y)\frac{\partial G(x;y)}{\partial\nu(y)}\mathrm{d}y + \mathcal{F}_g(x) \qquad (3.16)$$

for $x \in U^-$. Here, $\mathcal{C}_\gamma^{(x)}$ denotes a path starting at an arbitrary point in γ and ending at $\mathcal{R}_\Gamma(x)$; the unit normal vector ν points at the left side of the integral path; $\mathcal{R}_\Gamma$ denotes the Schwartz reflection with respect to Γ; the function $G(x;y)$ is defined in (3.14) *The operator $\mathcal{F}[g]$ in* (3.16) *is an integral operator over two contours lying in the complexificated curve $\Gamma_\mathbb{C}$ which surround respectively the two intersection points of the characteristic rays* (3.11) *starting from $(z,\overline{z}) \in \mathbb{C}^2$ ($z = x_1 + ix_2 \in \mathbb{C}$) with $\Gamma_\mathbb{C}$.*

Remark 3.1. *The reflection formula* (3.16) *is independent on the path of integration joining γ and $\mathcal{R}_\Gamma(x)$, because both u and G vanish on $\Gamma_\mathbb{C}$ and the integrand is a closed form. The function $\mathcal{F}_g(x)$ vanishes identically if $g \equiv 0$ on Γ, and the singularities of $\mathcal{F}_g(x)$ coincide with those of the function $\mathcal{R}_\Gamma(x)$.*

The right hand side of (3.16) only relies on the boundary data g and $u|_{U^-}$. It is supposed in Lemma 3.1 that the domain U^- does not contain singularities of S. Although the mapping $\mathcal{R}_\Gamma$ is originally constructed in a small neighborhood of $\Gamma_\mathbb{C}$, by the uniqueness theorem for analytic functions the above extension formula remains valid in the large. In the special case that Γ is a straight line and $g \equiv 0$, both G and $\mathcal{F}_g$ vanish identically and thus one can obviously get a global (odd) extension formula in the real space $\mathbb{R}^2$. For harmonic functions (that is, $k = 0$), it hold that $G \equiv 0$ and

$$\mathcal{F}_g(x) = \mathcal{F}_g(z,\overline{z}) = g(\widetilde{S}(\overline{z}), z) + g(z, S(z)), \quad z = x_1 + ix_2 \in \mathbb{C}, \quad x = (x_1, x_2) \in \mathbb{R}^2.$$

Note that $(\widetilde{S}(\overline{z}), z), (z, S(z)) \in \Gamma_\mathbb{C}$. In our applications of Lemma 3.1, we will take Γ as a circle $\Gamma = \partial B_R$ or an ellipse $\Gamma = \partial D$ centered at the origin.

4 Analytical extension and singularities

In this section, we consider time-harmonic scattering of a point source wave by circular and elliptic obstacles of sound-soft type. Before studying these two cases, we define different kinds of singularities. Our aim is to find the possible singularities in the continuation and investigate their singular type.

Definition 4.1. *The (multi-valued) analytic function $u : \mathbb{C} \to \mathbb{C}$ has an ordinary algebraic singularity at a point $\omega \in \mathbb{C}$ if it has a convergent series expansion*

$$u(z) = \sum_{m\geq 0} a_m\,(z-\omega)^{m/l},$$

in some punctured disk centered at ω for some $l \in \mathbb{N}^+$. In the special case that $u(z) = (z-\omega)^{1/2}$, we say that u has a square-root singularity at ω. If $u(z) = \ln(z-\omega)$, we say that ω is a logrithmic singularity of u. A real-analytic function $x = (x_1, x_2) \to u(x)$ has a singularity of a certain type if its complex analytic extension $z \to f(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i})$ with $z = x_1 + ix_2 \in \mathbb{C}$ admits such kind of singularity.

4.1 Circular case: $\Omega = B_R$

Consider the exterior boundary value problem (1.2) in $|x| > R$, where the sound-soft obstacle B_R is represented by (2.6) and the incoming wave is a point source wave emitting from the source position $y \in \mathbb{R}^2 \setminus \overline{B_R}$.

From the expression of u^{in} in (1.1), we have

$$u^{in}(x; y) = -\frac{1}{2\pi} \ln|x-y| + O(1) \quad \text{as} \quad x \to y.$$

Since $u^{sc}(x; y)$ is real-analytic at $x = y$, it follows that

$$u(x; y) = u^{in}(x; y) + u^{sc}(x; y) = -\frac{1}{2\pi} \ln|x-y| + O(1) \quad \text{as} \quad x \to y. \tag{4.17}$$

According to (2.4), the Schwartz function of $\Gamma = \partial B_R$ is $S(z) = \frac{R^2}{z}$, that is

$$\mathcal{R}_\Gamma(x) = \left(\frac{R^2 x_1}{x_1^2 + x_2^2}, \frac{R^2 x_2}{x_1^2 + x_2^2} \right), \quad (x_1, x_2) \in \mathbb{R}^2, \tag{4.18}$$

where $R > 0$ is the radius of the disc B_R.

We state the main result of this subsection:

Theorem 4.1. *Let $\Omega = B_R$ be a sound-soft disc. For $y = (y_1, y_2) \in \mathbb{R}^2$ with $|y| = \sqrt{y_1^2 + y_2^2} > R$, the total wave can be analytically extended from $\mathbb{R}^2 \setminus \big(\overline{B_R} \cup \{y\}\big)$ to $\mathbb{R}^2 \setminus \{O, R_\Gamma(y), y\}$, while $\mathcal{R}_\Gamma(y)$ and y are logrithmic singularities of u in $\mathbb{R}^2 \setminus \{O\}$.*

Proof. Since the disc B_R is symmetric, without loss of generality, we set $y_2 = 0$. By (4.17), $y = (y_1, 0)$ is a logrithmic singularity of u in $\mathbb{R}^2 \setminus \overline{B_R}$.

We decompose the total field into the sum $u(x; y) = -\frac{1}{2\pi} \ln|x-y| + f(x; y)$, where f is smooth near the incoming point source y. Taking the derivatives with respect to x_1 and x_2 gives

$$\frac{\partial u}{\partial x_1}(x; y) = -\frac{1}{2\pi} \frac{x_1 - y_1}{(x_1 - y_1)^2 + x_2^2} + \frac{\partial f}{\partial x_1}(x; y), \tag{4.19}$$

$$\frac{\partial u}{\partial x_2}(x; y) = -\frac{1}{2\pi} \frac{x_2}{(x_1 - y_1)^2 + x_2^2} + \frac{\partial f}{\partial x_2}(x; y). \tag{4.20}$$

Set $\eta = (\eta_1, \eta_2) \in \mathbb{R}^2$ with $0 < |\eta| = \sqrt{\eta_1^2 + \eta_2^2} < R$. Suppose that η is close to $\mathcal{R}_\Gamma(y) = \left(\frac{R^2}{y_1}, 0\right)$; see Figure 3. By the reflection principle, $u(\eta; y)$ is defined by the right hand side

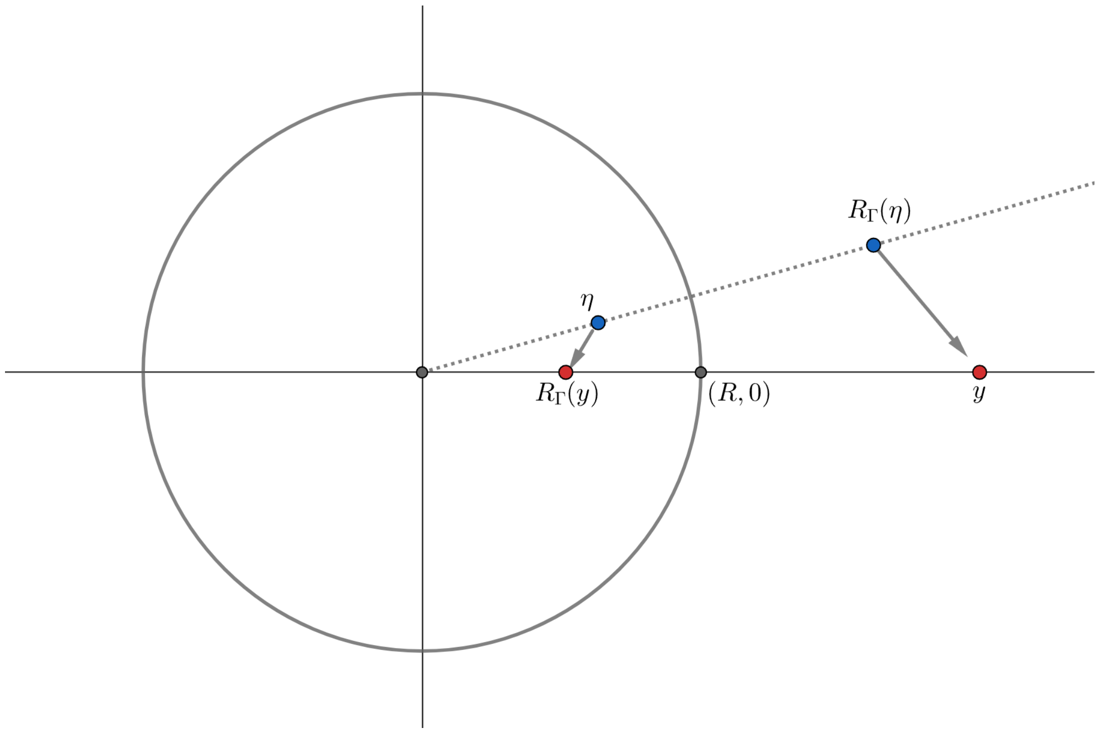

Figure 3: Schematic diagram of Theorem 4.1

of (3.16), in which the integral path $C_\gamma^{(\eta)}$ is chosen as the line segment starting from $\frac{R}{|\eta|}\eta = \left(\frac{R\eta_1}{\sqrt{\eta_1^2+\eta_2^2}}, \frac{R\eta_2}{\sqrt{\eta_1^2+\eta_2^2}}\right) \in \partial B_R$ to $\mathcal{R}_\Gamma(\eta) = \frac{R^2}{|\eta|^2}\eta$. To compute the integral in (3.16), we parameterize $C_\gamma^{(\eta)}$ by

$$C_\gamma^{(\eta)} = \left\{ \left(\xi_1\left(t\right), \xi_2\left(t\right)\right) : t \in \left[1, \frac{R}{|\eta|}\right] \right\}, \tag{4.21}$$

where $\xi_1(t) = \frac{R\eta_1}{\sqrt{\eta_1^2+\eta_2^2}}t$, $\xi_2(t) = \frac{R\eta_2}{\sqrt{\eta_1^2+\eta_2^2}}t$. By (3.16), we obtain

$$\begin{aligned} u(\eta; y) &= -u\left(\mathcal{R}_\Gamma(\eta); y\right) + \frac{1}{2i} \int_{\mathcal{C}_\gamma^{(\eta)}} G(\eta; \xi) \left[\frac{\partial u(\xi; y)}{\partial \xi_2} \mathrm{d}\xi_1 - \frac{\partial u(\xi; y)}{\partial \xi_1} \mathrm{d}\xi_2\right] \\ &\quad - \frac{1}{2i} \int_{\mathcal{C}_\gamma^{(\eta)}} u(\xi; y) \left[\frac{\partial G(\eta; \xi)}{\partial \xi_2} \mathrm{d}\xi_1 - \frac{\partial G(\eta; \xi)}{\partial \xi_1} \mathrm{d}\xi_2\right] \\ &= I_1 + I_2 + I_3, \end{aligned}$$

where the smooth function G is given in (3.14) and

$$
\begin{aligned}
I_1(\eta) &:= -u\left(\mathcal{R}_\Gamma(\eta); y\right), \\
I_2(\eta) &:= \frac{1}{2i} \int_{\mathcal{C}_\gamma^{(\eta)}} G(\eta;\xi) \left[\frac{\partial u(\xi;y)}{\partial \xi_2} \mathrm{d}\xi_1 - \frac{\partial u(\xi;y)}{\partial \xi_1} \mathrm{d}\xi_2\right] \\
&= \frac{1}{2i} \int_1^{\frac{R}{|\eta|}} G\left(\eta; \xi_1(t), \xi_2(t)\right) \left[\frac{\partial u\left(\xi_1(t), \xi_2(t); y\right)}{\partial \xi_2} \xi_1'(t) - \frac{\partial u\left(\xi_1(t), \xi_2(t), ; y\right)}{\partial \xi_1} \xi_2'(t)\right] \mathrm{d}t, \\
I_3(\eta) &:= -\frac{1}{2i} \int_1^{\frac{R}{|\eta|}} u\left(\xi_1(t), \xi_2(t); y\right) \left[\frac{\partial G\left(\eta; \xi_1(t), \xi_2(t)\right)}{\partial \xi_2} \xi_1'(t) - \frac{\partial G(\eta; \xi_1(t), \xi_2(t))}{\partial \xi_1} \xi_2'(t)\right] \mathrm{d}t,
\end{aligned}
$$

Next we analyze the singularity of $u(\eta; y)$ as $\eta \to R_\Gamma(y)$. For the first term, we have

$$
\begin{aligned}
I_1(\eta) = -u\left(\mathcal{R}_\Gamma(\eta); y\right) &= \frac{1}{2\pi} \ln\left|y - \mathcal{R}_\Gamma(\eta)\right| + O(1) \\
&= \frac{1}{2\pi} \ln\left|\mathcal{R}_\Gamma\left(\mathcal{R}_\Gamma(y)\right) - \mathcal{R}_\Gamma(\eta)\right| + O(1) \\
&= \frac{1}{2\pi} \ln\left|\left(\triangledown \mathcal{R}_\Gamma\right)(\xi) \cdot \left(\mathcal{R}_\Gamma(y) - \eta\right)\right| + O(1) \\
&= \frac{1}{2\pi} \ln\left|\mathcal{R}_\Gamma(y) - \eta\right| + O(1)
\end{aligned}
$$

as $\eta \to \mathcal{R}_\Gamma(y)$, so the singularity of $I_1(\eta)$ is a logrithmic singularity at $\eta = \mathcal{R}_\Gamma(y)$.

Applying (4.19) and (4.20), we know $I_2(\eta)$ and $I_3(\eta)$ are convergent as $\eta \to \mathcal{R}_\Gamma(y)$. This imlies that the function $\eta \mapsto u(\eta; y)$ has a logrithmic singularity at $\eta = \mathcal{R}_\Gamma(y)$. □

4.2 The case of ellipse: $\Omega = \left\{x \in \mathbb{R}^2 : \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1\right\}$

Consider the exterior boundary value problem (1.2) in D^c, where the sound-soft obstacle D is represented by (2.7) and the incoming wave is a point source wave emitting from the source position $y \in D^c \setminus D^*$, where D^* is defined as in (2.8).

We first investigate the singularities of G at the two focuses of the ellipse.

Lemma 4.1. *Let Ω be an ellipse and let G be defined as in* (3.14)*. Then the functions $z \mapsto G(z, w; z_0, w_0)$ and $z_0 \mapsto G(z, w; z_0, w_0)$ both have square-root-type singularities at $z = \pm c$, $z_0 = \pm c$.*

Proof. It is known from [21] that $G(z, w; z_0, w_0)$ can be represented as

$$
G(z, w; z_0, w_0) = \sum_{j=0}^{\infty} a_j^{(1)} \frac{(\psi_1)^j}{j!} - \sum_{j=0}^{\infty} a_j^{(2)} \frac{(\psi_2)^j}{j!}, \tag{4.22}
$$

where $a_j^{(1)}(z, w; z_0, w_0)$ and $a_j^{(2)}(z, w; z_0, w_0)$ satisfy

$$
\begin{cases}
\frac{\partial a_{j+1}^{(1)}}{\partial w} \frac{\partial S(z)}{\partial z} = -\frac{\partial^2 a_j^{(1)}}{\partial z \partial w} - \frac{k^2}{4} a_j^{(1)}, \\
a_j^{(1)}\big|_{z = \widetilde{S}(w)} = \frac{(-k^2)^j (w - w_0)^j}{4^j j!}, j \geq 0.
\end{cases} \tag{4.23}
$$

$$\begin{cases} \frac{\partial a_{j+1}^{(2)}}{\partial w}\frac{\partial S(z)}{\partial z} = -\frac{\partial^2 a_j^{(2)}}{\partial z \partial w} - \frac{k^2}{4}a_j^{(2)}, \\ a_j^{(2)}\big|_{w=S(z)} = \frac{(-k^2)^j (z-z_0)^j}{4^j j!}, j \geq 0, \end{cases} \tag{4.24}$$

By the definition of S and $\widetilde{S}$ for the ellipse D,

$$\psi_1(z, w; z_0, w_0) = O(\sqrt{w \pm c}), \quad \text{as} \quad w \to \pm c, \tag{4.25}$$

$$\psi_2(z, w; z_0, w_0) = O(\sqrt{z \pm c}), \quad \text{as} \quad z \to \pm c. \tag{4.26}$$

Integrating (4.23) with respect to w from $w_0 = S(z_0)$ to w, we get

$$a_{j+1}^{(2)}(z, w; z_0, w_0)\frac{\partial S(z)}{\partial z} = -\left(\frac{\partial a_j^{(2)}}{\partial z} - \frac{a_j^{(2)}}{\partial z}\Bigg|_{w=w_0, z=z_0} \right) - \frac{k^2}{4}\int_{w_0}^{S(z)} a_j^{(2)} \mathrm{d}w \tag{4.27}$$

where we have used the fact that $a_{j+1}^{(2)}(z, w; z_0, w_0)\big|_{w=w_0, z=z_0} = 0$ if $j \geq 0$. By (4.27), the types of singularities of $a_j^{(2)}$ at $z = \pm c$ are square-root singularities. Similarly, the types of singularities of $a_j^{(1)}$ at $z = \pm c$ are also square-root singularities. By (4.22), the types of singularities of G at $z = \pm c$ is square-root singularity. □

Remark 4.1. *Setting $z = x_1 + ix_2$, $w = x_1 - ix_2$, $z_0 = y_1 + iy_2$, $w_0 = y_1 - iy_2$, we get the equations for $a_j^{(1)}(x_1, x_2; y_1, y_2)$ in $\mathbb{R}^2 \times \mathbb{R}^2$:*

$$\begin{cases} \frac{1}{4}\left(\frac{\partial a_{j+1}^{(1)}}{\partial x_1} - \frac{1}{i}\frac{a_{j+1}^{(1)}}{\partial x_2}\right)\left(\frac{\partial S(x_1+ix_2)}{\partial x_1} + \frac{1}{i}\frac{S(x_1+ix_2)}{\partial x_2}\right) = -(\triangle + k^2)a_j^{(1)} \\ a_0^{(1)}(x_1, x_2; y_1, y_2)\Big|_{(x_1,x_2)=R_\Gamma(y_1,y_2)} = 1, \\ a_j^{(1)}(x_1, x_2; y_1, y_2)\Big|_{(x_1,x_2)=R_\Gamma(y_1,y_2)} = 0, \quad \text{if } j \geq 1 \end{cases} \tag{4.28}$$

Then we state the main result.

Theorem 4.2. *Let D be given by (2.7) with the half-axes a, b along the x_1- and x_2-axis such that $a > b > 0$, and let $D^* \supset D$ be the unique ellipse confocal to D with the half-axes a^* and b^* given by (2.8). If $y \in D^*\backslash\overline{D}$, then the total field can be extended into $D\backslash\{\mathcal{R}_\Gamma(y), (\pm c, 0)\}$ with $c = \sqrt{a^2 - b^2}$. Moreover, the extended solution is of logarithmic singularity at $\mathcal{R}_\Gamma(y) \in D$ and develops at most square root singularities at $(\pm c, 0)$. If $y \in \mathbb{R}^2\backslash\overline{D^*}$, the total field can be extended into $D\backslash\{(\pm c, 0)\}$.*

Proof. By Lemma 2.1, $\mathcal{R}_\Gamma$ is an involution mapping in $D^*\backslash\{L\}$ and is identical on Γ. Here L denotes again the interfocal segment. In the exterior of D^*, $\mathcal{R}_\Gamma$ cannot be regarded as the Schwartz reflection, because it is no longer a involution mapping, although it is still well defined.

Suppose that $y \in D^*\backslash\overline{D}$. In this case it is clear that $\mathcal{R}_\Gamma(y) \in D\backslash\{L\}$, Given $x \in D\backslash\{\mathcal{R}_\Gamma(y), (c, 0), (-c, 0)\}$, we have $\mathcal{R}_\Gamma(x) \in D^*\backslash\{\overline{D} \cup \{y\}\}$ and the Schwartz reflection $\mathcal{R}_\Gamma$

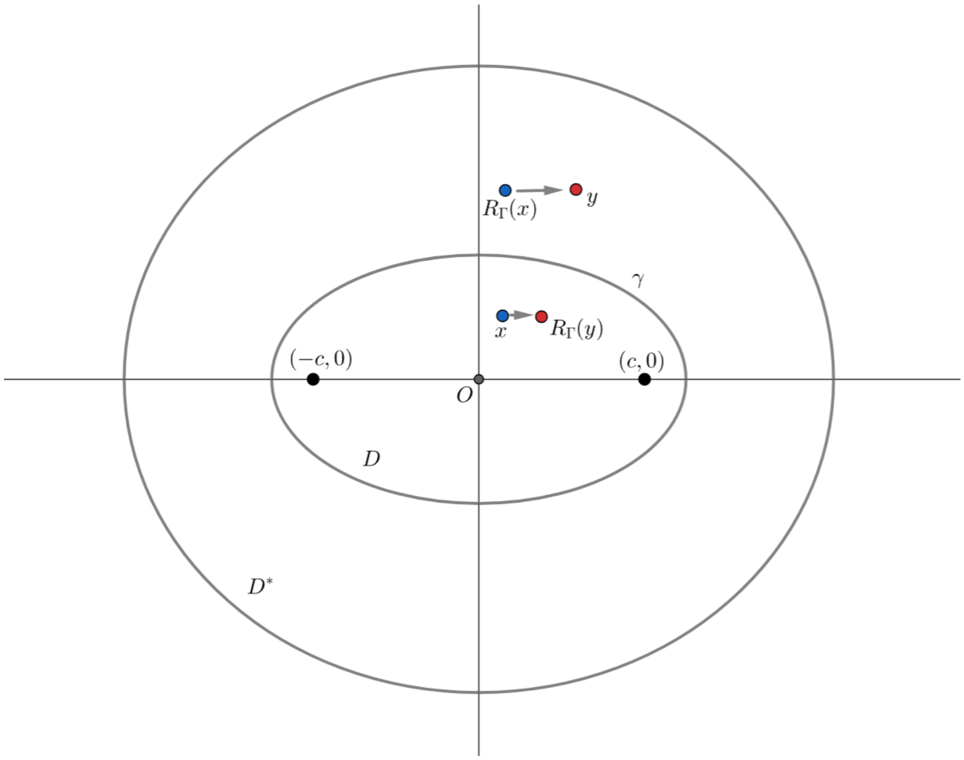

Figure 4: Schematic diagram of Theorem 4.2

is analytic near x. Hence, the total field u can be extended into $D\backslash\{\mathcal{R}_\Gamma(y),(c,0),(-c,0)\}$ by the reflection formula (3.16) with $\mathcal{F}[g]=0$. To check the limiting behavior of $u(x)$ as $x\to\mathcal{R}_\Gamma(y)$, we note that by construction the "reflected" fundamental solution $G(x;\tilde{x})$ is analytic in the variable $\tilde{x}$ in a neighborhood of $\mathcal{R}_\Gamma(x)$, if $z=x_1+ix_2$ is not a singularity of both S and $\widetilde{S}$. Hence, the function $\tilde{x}\to\nabla_{\tilde{x}}G(x;\tilde{x})$ is sufficiently smooth at $\tilde{x}=\mathcal{R}_\Gamma(y)$ and for x lying in a neighborhood of $\mathcal{R}_\Gamma(y)$. Further, it follows from $G_1(\mathcal{R}_\Gamma(y);y)=G_2(\mathcal{R}_\Gamma(y);y)=J_0(0)=1$ (see (3.10)) that

$$G(\mathcal{R}_\Gamma(y);\tilde{x})=G_1(\mathcal{R}_\Gamma(y);\tilde{x})-G_2(\mathcal{R}_\Gamma(y);\tilde{x})=O(|y-\tilde{x}|)\quad\text{as}\quad\tilde{x}\to y.$$

Together with the fact that

$$u(x;y)=u^{in}(x;y)+u^{sc}(x;y)=-\frac{1}{2\pi}\ln|x-y|+O(1),\quad x\to y,$$

we can conclude

$$\begin{aligned}
&\lim_{x\to\mathcal{R}_\Gamma(y)}\int_{\mathcal{C}_\gamma^{(x)}} G(x;\tilde{x})\left[\frac{\partial u(\tilde{x};y)}{\partial\tilde{x}_2}\mathrm{d}\tilde{x}_1-\frac{\partial u(\tilde{x};y)}{\partial\tilde{x}_1}\mathrm{d}\tilde{x}_2\right]\\
&-u(\tilde{x};y)\left[\frac{\partial G(x;\tilde{x})}{\partial\tilde{x}_2}\mathrm{d}\tilde{x}_1-\frac{\partial G(x;\tilde{x})}{\partial\tilde{x}_1}\mathrm{d}\tilde{x}_2\right]\\
&=\int_{\mathcal{C}_\gamma^{(\mathcal{R}_\Gamma(y))}} G(\mathcal{R}_\Gamma(y);\tilde{x})\left[\frac{\partial u(\tilde{x};y)}{\partial\tilde{x}_2}\mathrm{d}\tilde{x}_1-\frac{\partial u(\tilde{x};y)}{\partial\tilde{x}_1}\mathrm{d}\tilde{x}_2\right]\\
&-u(\tilde{x};y)\left[\frac{\partial G(\mathcal{R}_\Gamma(y);\tilde{x})}{\partial\tilde{x}_2}\mathrm{d}\tilde{x}_1-\frac{\partial G(\mathcal{R}_\Gamma(y);\tilde{x})}{\partial\tilde{x}_1}\mathrm{d}\tilde{x}_2\right]\\
&<\infty
\end{aligned}$$

where the integral is understood as the same as in Lemma 3.1. Hence, the leading singularity on the right hand side of (3.16) is governed by the first term, given by

$$-u(R_\Gamma(x);y)=-u^{in}(\mathcal{R}_\Gamma(x);y)+O(1)=\frac{1}{2\pi}\ln|\mathcal{R}_\Gamma(x)-y|+O(1),\quad x\to\mathcal{R}_\Gamma(y).$$

Applying the mean value theorem and using the smoothness of R_Γ at $R_\Gamma(y)$, we obtain

$$u(x;y)=\frac{1}{2\pi}\ln|x-\mathcal{R}_\Gamma(y)|+O(1),$$

as $x\to\mathcal{R}_\Gamma(y)$, where $\xi=\xi(x)$ lies on the line segment connecting x and $\mathcal{R}_\Gamma(y)$. This proves the logarithmic singularity of u at $x=\mathcal{R}_\Gamma(y)\in D\backslash\{L\}$.

By lemma 4.1 and (3.16), the function $u(x;y)$ admits at most square-root singularities at the two focal points $x=(\pm c,0)$.

If the source position y lies in the exterior of $\overline{D^*}$, the total field u is analytic in $D^*\backslash\overline{D}$. Since $\mathcal{R}_\Gamma(D\backslash\ L)=D^*\backslash\overline{D}$ by Lemma 2.1 (ii), the function $u(\mathcal{R}_\Gamma(x))$ is analytic for all $x\in D\backslash\{(c,0),(-c,0)\}$. This implies that the incoming point source wave for $y\in\mathbb{R}^2\backslash\overline{D^*}$ cannot induce new singularities in $D\backslash\{L\}$. □

5 Conclusion

In this paper, we use the Schwartz reflection principle to extend the wave fields for sound-soft obstacles into the interior. For circular and elliptic obstacles, we analyze the singular points in the continuation and investigate the singular behavior at these points.

In future, we shall consider applications of these singularities to inverse scattering problems. For example, determine the shape and location of a sound-soft ellipse.

References

[1] D. Colton and R. Kress. *Inverse acoustic and electromagnetic scattering theory*, volume 93 of *Applied Mathematical Sciences*. Springer, Switzerland AG, fourth edition, 2019.

[2] P. Ebenfelt, Singularities encountered by the analytic continuation of solutions to Dirichlet's problem, *Complex Variables* **20** (1992): 75-91.

[3] G. Alessandrini and L. Rondi, Determining a sound-soft polyhedral scatterer by a single far-field measurement, *Proc. Amer. Math. Soc.* **133** (2005): 1685-1691 (Corrigendum: arXiv: math/0601406v1, 2006).

[4] J. Cheng and M. Yamamoto, Uniqueness in an inverse scattering problem with non-trapping polygonal obstacles with at most two incoming waves, *Inverse Problems* **19** (2003): 1361-1384 (Corrigendum: *Inverse Problems* **21** (2005): 1193).

[5] D. Colton, A reflection principle for solutions to the Helmholtz equation and an application to the inverse scattering problem, Glasgow Math. J. 18 (1977): 125-130.

[6] D. Colton and B. D. Sleeman, Uniqueness theorems for the inverse problem of acoustic scattering, *IMA J. Appl. Math.* **31** (1983): 253-259.

[7] Ph. Davis, *The Schwarz Function and Its Applications*, Carus Math. Monogr., Math. Assoc. Amer., 1979.

[8] P. Garabedian, Partial differential equations with more than two independent variables in the complex domain, *J. Math. Mech.* **9** (1960): 241-271.

[9] D. Gintides, Local uniqueness for the inverse scattering problem in acoustics via the Faber-Krahn inequality, *Inverse Problems* **21** (2005): 1195–1205.

[10] G. Hu and X. Liu, Unique determination of balls and polyhedral scatterers with a single point source wave, *Inverse Problems* **30** (2014): 065010.

[11] P.D. Lax and R.S. Phillips, *Scattering Theory*, Academic Press, New York, 1967.

[12] C. Liu, An inverse obstacle problem: a uniqueness theorem for balls, *Inverse Problems in Wave Propagation*, Springer-Verlag, Berlin, 1997, 347-355.

[13] T.V. Savina, V.E. Shatalov and B.Yu. Sternin, On a reflection law for the Helmholtz equation, *Soviet Math. Dokl.,* **45** (1992): 42-45.

[14] V.E. Shatalov and B.Yu. Sternin, Reflection formulas and continuation of solutions of boundary value problems, *Computational and Mathematical Modeling* **7** (1996): 117-125.

[15] E. Study, Einige elementare Bemerkungen uber den Prozess der analytischen Fortsetzung, *Math. Ann.* **63** (1907): 239-245.

[16] R. F. Millar, Singularities and the Rayleigh Hypothesis for solutions to the Helmholtz equation, *IMA Journal of Applied Mathematics* **37** (1986): 155-171.

[17] S. Kusiak and J. Sylvester, The scattering support, *Commnications on Pure and Applied Mathematics* **56**: 1525-1548.

[18] Lord Rayleigh (J. W. Strutt), Proceedings of the Royal Society of London. Series A, 79, 399-416 (1907).

[19] Rother, T. (2009). The Rayleigh Hypothesis. In book: Electromagnetic Wave Scattering on Nonspherical Particles, Springer Series in Optical Sciences, 165-195.

[20] D. Khavinson and H. S. Shapiro, Dirichlet's problem when the data is an entire function, *Bulletin of the London Mathematical Society* **24** (1992): 456-468.

[21] B. Sternin B. and V. Shatalov V., Continuation of Solutions to Elliptic Equations. In: *Differential Equations on Complex Manifolds,* Mathematics and Its Applications, vol. 276. Springer, Dordrecht, 1994.