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ABSTRACT
We prove a conditional stability estimate of log-type for determining unknown boundaries from a single Cauchy data taken on an accessible sub-boundary. Our approach relies on new interior and boundary estimates for elliptic equations which are derived from the Carleman estimate. Stability results for target identification of an acoustic sound-soft scatterer from one or several far-field patterns are also obtained.

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1. Introduction and main results

1.1. Shape identification problems

Let $\Omega \subset \mathbb{R}^n, n = 2, 3$ be a bounded domain with smooth boundary $\partial\Omega$. Consider the elliptic differential operator

$$(Au)(x) := -\sum_{i,j=1}^{n} \partial_i(a_{ij}(x)\partial_j u) + \sum_{i=1}^{n} b_i(x)\partial_i u + c(x)u, \quad x \in \Omega, \tag{1}$$

where $a_{ij} = a_{ji} \in C^3(\overline{\Omega})$, $b_i, c \in W^{2,\infty}(\Omega)$. We assume

$$c \geq 0 \quad \text{in } \Omega, \tag{2}$$

and there exists a positive constant σ such that

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \sigma \sum_{i=1}^{n} \xi_i^2, \quad \xi_1, \ldots, \xi_n \in \mathbb{R},\ x \in \Omega. \tag{3}$$

Let $D \subset \Omega$ be a star-shaped subdomain such that $\overline{D} \subset \Omega$. Throughout the paper, we define the complement of D in Ω as $D^c := \Omega \backslash \overline{D}$. It is supposed that the boundaries ∂D and $\partial\Omega$ are both of C^4-class.

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Let $u = u(D)$ be a solution to the Dirichlet boundary value problem

$$Au = 0 \quad \text{in } D^c, \quad u|_{\partial D} = 0.$$

Denote by $\nu = (\nu_1, \ldots, \nu_n)$ the unit outward normal vector at $\partial\Omega$. For simplicity, we write $\partial_A u = \sum_{i,j=1}^n a_{ij}(\partial_j u)\nu_i$, which will be referred to as the Neumann data of u at ∂D. The first part of this paper concerns a stability estimate of the following inverse problem with a single Cauchy data: *Inverse Problem 1 (IP1)*: Determine the shape ∂D from knowledge of the Cauchy data $(u, \partial_A u)|_\Gamma$ where $\Gamma \subset \partial\Omega$ is an arbitrarily chosen sub-boundary.

The above inverse problem arises from, for example, the detection of the inaccessible interior corroded boundary ∂D by the measurement data taken on an accessible outer sub-boundary Γ. There have been many papers on this inverse boundary problem. For reconstruction methods related to non-destructive testing, we refer to [1–5]. It is widely acknowledged that the stability of the Cauchy problem for elliptic equations is closely connected to the quantitative unique continuation theory. In fact, both the stability estimate and the unique continuation property can be derived from either Carleman estimates or three-spheres inequalities. We refer to [6–8] for the stable determination of unknown boundaries in the case of the scalar elliptic equation and the Lamé system, which relies essentially on three-sphere inequalities in combination with doubling inequalities on the boundary and lower estimates of gradients of solutions. As for the Laplace operator, we refer to [9–11] where double logarithmic conditional stability estimates were given in two and three dimensions.

The purpose of this paper is to propose an alternative method for proving conditional stability estimates of logarithmic type. The arguments of using three-sphere and doubling inequalities are not involved in the present paper. Our approach relies essentially on new interior and boundary stability estimates (see Lemmas 2.1 and 2.3 in Section 2) in combination with the quantitative unique continuation (see Lemma 3.1 in Section 3), all of which are verified using Carleman estimates for elliptic equations (see Lemma 2.2). For completeness, we will provide in the appendix a proof of the elliptic Carleman estimate based on the integration by parts only. This paper provides a new insight into the stability of determining unknown boundaries with a single Cauchy data. Since Carleman estimates apply to vectorial elliptic equations such as the Lamé system and the Navier-Stokes equations, in an analogous manner, we could also establish the single logarithmic conditional stability for these equations within the framework of this paper.

We state the stability result as follows. Let $D_1, D_2 \subset \Omega$ be two star-shaped domains centered at the origin, with the boundaries parameterized in polar coordinates by

$$\partial D_j = \{(r, \hat{x}) : r = f_j(\hat{x})\}, \quad \hat{x} \in \mathbb{S}^{n-1} := \{x \in \mathbb{R}^n : |x| = 1\}. \tag{4}$$

Due to technical reasons, we suppose that ∂D_j are of C^4-class, i.e. $f_j \in C^4(\mathbb{S}^2)$. Let $u_j = u(D_j)$ satisfy

$$\begin{aligned} Au_j &= 0 \quad \text{in } D_j^c, \\ u_j &= 0 \quad \text{on } \partial D_j, \\ u_j &= g_j, \quad \partial_A u_j = h_j \quad \text{on } \Gamma \end{aligned} \tag{5}$$

for $j = 1,2$, where $g_j \in H^3(\Gamma)$ and $h_j \in H^2(\Gamma)$. Since $c \geq 0$ (see (2)), it is well-known that the above boundary value problems admit unique solutions $u_j \in H^4(D_j^c)$.

We make the following assumptions for (IP1):

Condition A: There exist $M, \delta > 0$ such that

$$1/M \leq \|f_j\|_{C^4(\mathbb{S}^2)} \leq M, \quad \operatorname{dist}(\partial D_j, \partial\Omega) \geq \delta > 0 \tag{6}$$

for $j = 1,2$. *Condition B:*

$$\inf_{x \in \Gamma} |g_j(x)| > C_0 > 0, \quad j = 1, 2. \tag{7}$$

It is seen from Condition A and the elliptic regularity that the norm $\|u_j\|_{H^4(D_j^c)}$ is uniformly bounded from above. Without loss of generality, we suppose that

$$\|u_j\|_{H^4(D_j^c)} \leq M, \quad j = 1, 2, \tag{8}$$

with the same constant M as in (6). The Condition B implies that u_j does not vanish identically on Γ. Below we state the first result of this paper.

Theorem 1.1: *Under the conditions (A) and (B) there exist constants $\theta \in (0, 1)$ and $C > 0$ only depending on M, δ and C_0 such that*

$$d(\partial D_1, \partial D_2) \leq C \left(\frac{1}{\log 1/(\|u_1 - u_2\|_{H^3(\Gamma)} + \|\partial_A(u_1 - u_2)\|_{H^2(\Gamma)})} \right)^{\theta}$$

provided $\|u_1 - u_2\|_{H^3(\Gamma)} + \|\partial_A(u_1 - u_2)\|_{H^2(\Gamma)}$ is sufficiently small. Here $d(\partial D_1, \partial D_2)$ is the Hausdorff distance defined by

$$d(\partial D_1, \partial D_2) := \max \left(\sup_{x \in \partial D_1} \mathrm{d}\,(x, \partial D_2), \sup_{x \in \partial D_2} \mathrm{d}\,(x, \partial D_1) \right).$$

If the condition (2) is not fulfilled, additional assumptions on the geometry of D are needed in order to get the same stability estimate. In the special case of $a_{ij}(x) \equiv \delta_{ij}$, $b_i = 0$ and $c(x) = -k^2$ for some $k > 0$, the equation $-Au = 0$ reduces to the Helmholtz equation $(\Delta + k^2)u = 0$ which models the time-harmonic acoustic wave propagation in an isotropic homogeneous medium. Hence, our inverse problem (IP1) in this case is closely related to the shape identification problem arising from inverse obstacle scattering with a single incoming wave; see subsection 1.2 below.

1.2. Inverse scattering problems

In this section, we present a local stability result for target identification of a sound-soft obstacle from a single far-field pattern with a priori assumptions on the underlying scatterer. Let $D_1, D_2 \in \mathbb{R}^n$ be two distinct sound-soft obstacles embedded in an isotropic homogeneous medium. Assume an incoming plane wave of the form $u^{in}(x) = \exp(ik\alpha \cdot x)$ with the direction $\alpha \in \mathbb{S}^{n-1} := \{x \in \mathbb{R}^n : |x| = 1\}$ is incident onto D_j, where $k > 0$ is the wavenumber. Denote by $u_j = u_j(D_j)$ the total field corresponding to D_j. Then the scattered field $u_j^{sc} := u_j - u^{in}$ satisfies the boundary value problem

$$(\Delta + k^2)u_j^{sc} = 0 \quad \text{in } \mathbb{R}^n \backslash \bar{D}_j, \quad u_j^{sc} = -u^{in} \quad \text{on } \partial D_j, \tag{9}$$

and the Sommerfeld radiation condition

$$\lim_{|x| \to \infty} |x|^{n-1/2} \left\{ \frac{\partial u_j^{sc}}{\partial |x|} - iku_j^{sc} \right\} = 0, \quad j = 1, 2. \tag{10}$$

In particular, the Sommerfeld radiation condition (10) leads to the asymptotic expansion

$$u^{sc}(x) = \frac{e^{ik|x|}}{|x|^{(n-1)/2}} u^{\infty}(\hat{x}) + \mathcal{O}\left(\frac{1}{|x|^{n/2}} \right), \quad |x| \to +\infty, \tag{11}$$

uniformly in all directions $\hat{x} := x/|x| \in \mathbb{S}^{n-1}$. The function $u^{\infty}(\hat{x})$ is an analytic function defined on $\mathbb{S}^{n-1}$ and is referred to as the *far-field pattern* or the *scattering amplitude*. The vector $\hat{x} \in \mathbb{S}^{n-1}$ is called the observation direction of the far field. The inverse obstacle scattering problem with a single far-field pattern can be stated as

Inverse Problem 2 (IP2): Determine the boundary ∂D from the far-field pattern $u^\infty(\hat{x})$ for all $\hat{x} \in \mathbb{S}^{n-1}$ with fixed $k > 0$ and $\alpha \in \mathbb{S}^{n-1}$.

It remains a long-standing open problem whether a single Cauchy data (or equivalently, a single far-field pattern) can uniquely determine the boundary of a general sound-soft scatterer; see e.g. Colton and Kress [12, Chapter 5.1]. Local uniqueness results were obtained in [13,14] under the *smallness* and *closeness* assumptions. Correspondingly, local stability estimates of the double logarithmic type were verified in [15,16] under these *a priori* assumptions. Note that the arguments of [16] are closest to those of [6] using three-spheres inequalities, and that in [16] a sharper upper bound of the *closeness* of two sound-soft obstacles was derived from the Faber-Krahn inequality. As a by-product of the proof of Theorem 1.1, we present a novel approach to the stable determination of the boundary of a soft obstacle from a single far-field pattern.

Let $B_R(z) = \{x \in \mathbb{R}^n : |x - z| \leq R\}$ and $B_R = B_R(O)$. Clearly, B_1 is the unit ball in $\mathbb{R}^n$. Denote by $\mathrm{Vol}(D)$ the volume of D in $\mathbb{R}^n$. We assume one of the following a priori conditions holds:

Condition C:

$$D_j \subset B_R \quad \text{with } kR < \eta_n, \quad n = 1, 2, \tag{12}$$

where η_n denotes the first root of the spherical Bessel function $(n=3)$ or Bessel function $(n=2)$ of the first order. *Condition D*: There exist two bounded domains $D^\pm \subset \mathbb{R}^n$ such that

$$D^- \subset D_j \subset D^+, \quad \mathrm{Vol}\,(D^+ \backslash D^-) \leq \left(\frac{\eta_n}{k}\right)^n \mathrm{Vol}(B_1), \tag{13}$$

where η_n is defined as in condition C.

The stability of the inverse problem (IP2) is stated as follows.

Theorem 1.2: *Suppose that $D_j (j = 1, 2)$ are sound-soft obstacles with C^4-smooth star-shaped boundaries centered at the origin* (4) *which satisfy the uniform smoothness assumption* $1/M \leq \|f_j\|_{C^4(\mathbb{S}^2)} \leq M$ *for some $M > 0$. Assume further that D_j fulfill either the smallness condition C or the closeness type condition D. Then the Hausdorff distance of ∂D_1 and ∂D_2 can be estimated by*

$$d\,(\partial D_1, \partial D_2) \leq C \left| \frac{\log \varrho}{1 + \log(e + \log 1/\varrho)} \right|^{-\theta}, \quad \varrho = \|u_1^\infty - u_2^\infty\|_{L^2(\mathbb{S}^{n-1})},$$

where $e := \lim_{n\to+\infty}(1 + 1/n)^n$ and the constants $\theta \in (0, 1)$, $C > 0$ depend on the wavenumber k, the a priori data M and the regions $D^\pm$ in (13) *or the radius R in* (12).

Remark 1.1: (i) The upper bounds in (12) and (13) are derived from the Faber-Krahn inequality which provides a lower bound for the first Dirichlet eigenvalue $\lambda_1(\Omega)$ of the Laplace equation over a bounded domain $\Omega \subset \mathbb{R}^n$, i.e.

$$\lambda_1(\Omega) \geq \lambda_1(B_1) \left[\frac{\mathrm{Vol}\,(B_1)}{\mathrm{Vol}\,(\Omega)}\right]^{2/n}. \tag{14}$$

The inequality (14) has been used in [17] to improve the local uniqueness results of [13,14].

(ii) The rate in Theorem 1.2 is stronger than the double logarithmic rate of [15], but weaker than a single logarithmic estimate. The same stability result was derived in [15] for sound-soft obstacles with analytic boundaries. In [8], a single log stability estimate was proved with different a priori assumptions on unknown domains.

The proofs of Theorems 1.1 and 1.2 will be carried out in Section 4. Stability estimates for inverse scattering with several incoming plane waves will be addressed at the end of Section 4.

2. Interior and boundary estimates

2.1. Interior stability estimate and elliptic Carleman estimate

We introduce some notation before stating our interior estimate. Given $y = (y_1, \ldots, y_n) \in \Omega$, $\lambda > 0$ and a unit vector $\nu \in \mathbb{S}^{n-1}$, we denote by $\Lambda(y, \lambda, \nu)$ a paraboloidal domain with the vertex located at y and the axis parallel to ν which is congruent to $y_n < -\lambda \sum_{j=1}^{n-1} y_j^2$. For $\delta > 0$, set

$$\Lambda(y, \lambda, \nu) + \delta\nu := \{x : x - \delta\nu \in \Lambda(y, \lambda, \nu)\} = \bigcup_{x \in \Lambda(y,\lambda,\nu)} \{x + \delta\nu\}, \tag{15}$$

that is, the translation of $\Lambda(y, \lambda, \nu)$ along the direction ν. Note that there are exactly two paraboloidal domains $\Lambda(y, \lambda, \nu)$ determined by y, λ and ν. In this paper, $\Lambda(y, \lambda, \nu)$ is always chosen such that $\Lambda(y, \lambda, \nu) + \delta\nu \subset \Lambda(y, \lambda, \nu)$ for any $\delta > 0$. Since $\Lambda(y, \lambda, \nu) \cap \Omega$ may have several connected components if Ω is not convex, we make the convention that the paraboloidal domain $\Lambda(y, \lambda, \nu)$ always means the connected component of $\Lambda(y, \lambda, \nu) \cap \Omega$ whose boundary contains y. Analogously, the notation $\Lambda(y, \lambda, \nu) \cap \partial\Omega$ always means the intersection of the boundary of this connected domain with $\partial\Omega$. This convention also applies to the paraboloidal domain $\Lambda(y, \lambda, \nu) + \delta\nu$ for $\delta > 0$. An essential ingredient in our analysis is the following solution estimate in the level sets $\Lambda(y, \lambda, \nu) + \delta\nu$.

Lemma 2.1 (interior estimate): *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with the boundary $\partial\Omega$ of C^2-class. Let $y \in \Omega$, $\gamma = \partial\Omega \cap \Lambda(y, \lambda, \nu)$ and $\ell = \min\{t : y + t\nu \in \partial\Omega, t > 0\}$. For $0 < \delta < \ell$, set $\Lambda_\delta := (\Lambda(y, \lambda, \nu) + \delta\nu) \cap \Omega$ (see Figure 1). Suppose that $u \in H^2(\Omega)$ is a solution to the elliptic Equation (1). Then there exist constants $C > 0$ and $\kappa \in (0, 1)$, which depend on $\ell, \delta, \lambda, a_{ij}, b_i$ and c, such that*

$$\begin{aligned} &\|u\|_{H^1(\Lambda_\delta)} \\ &\leq C\left(\|u\|_{H^1(\gamma)} + \|\partial_\nu u\|_{L^2(\gamma)}\right) + C\left(\|u\|_{H^1(\gamma)} + \|\partial_\nu u\|_{L^2(\gamma)}\right)^\kappa \|u\|_{H^1(\Omega)}^{1-\kappa}. \end{aligned}$$

Here C and κ do not depend on γ.

Lemma 2.1 yields a stability estimate for u provided that $\|u\|_{H^1(\Omega)}$ is bounded which is called a conditional stability estimate. Further, it implies that a solution to the elliptic Equation (1) with vanishing Cauchy data on an arbitrary non-empty open sub-boundary of $\partial\Omega$ must vanish identically. Lemma 2.1 was proved in [18] by applying the following elliptic Carleman estimate.

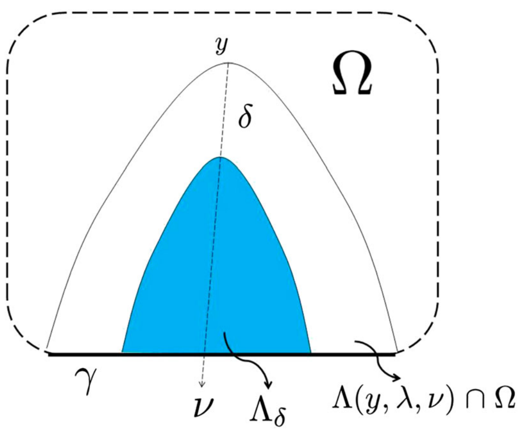

Figure 1. Configurations of $\Lambda(y, \lambda, \nu) \cap \Omega$ and $(\Lambda(y, \lambda, \nu) + \delta\nu) \cap \Omega =: \Lambda_\delta$ with $y \in \Omega$ and $\gamma := \partial\Omega \cap \Lambda(y, \lambda, \nu)$.

Lemma 2.2 (Carleman estimate): *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with the boundary $\partial\Omega$ of C^2-class, and let $D \subset \Omega$ be a domain such that $\overline{D} \subset \Omega$ and ∂D is of C^2-class. Suppose that $d \in C^2(\overline{\Omega})$ satisfies $|\nabla d| \neq 0$ on $\overline{\Omega}$ and set*

$$\varphi(x) := e^{\lambda \mathrm{d}(x)}, \qquad x \in \Omega,$$

with a positive parameter $\lambda > 0$. Then there exists positive constants $\lambda_0, s_0(\lambda)$ and $C(s_0, \lambda)$ such that

$$\int_D \{s\lambda^2\varphi|\nabla u|^2 + s^3\lambda^4\varphi^3u^2\}e^{2s\varphi}\,\mathrm{d}x \leq C\int_D |Au|^2e^{2s\varphi}\,\mathrm{d}x + Ce^{C(\lambda)s}\int_{\partial D}(|\nabla u|^2 + |u|^2)\,\mathrm{d}s$$

for all $s > s_0$, $\lambda \geq \lambda_0$ and for all $u \in H^2(D)$. Here the constants s_0, C are dependent on λ, but independent of s and the geometry of D, and they are bounded provided that $\max_{1\leq i,j\leq n}\|a_{ij}\|_{C^3(\overline{\Omega})}$, $\max_{1\leq i\leq n}\|b_i\|_{W^{2,\infty}(\Omega)}$, $\|c\|_{W^{2,\infty}(\Omega)}$, $\|d\|_{C^2(\overline{\Omega})}$ are bounded.

In particular, fixing $\lambda > 0$ sufficiently large, we can rewrite the above estimate as

$$\int_D \{s|\nabla u|^2 + s^3u^2\}e^{2s\varphi}\,\mathrm{d}x \leq C\int_D |Au|^2e^{2s\varphi}\,\mathrm{d}x + Ce^{Cs}\int_{\partial D}(|\nabla u|^2 + |u|^2)\,\mathrm{d}s \tag{16}$$

for all $s > s_0$ and all $u \in H^2(D)$.

For clarity, we shall present the proof of Lemma 2.2 in the Appendix. We emphasize that the proofs of our interior estimate (see Lemma 2.1) and the estimate at a boundary point (see Lemma 2.3) both rely heavily on the Carleman estimate (16).

2.2. Stability at a boundary point

For a boundary point $x_0 \in \partial\Omega$, let $\nu = \nu(x_0)$ be the unit normal vector pointing into the interior of Ω. Given $\lambda > 0$ sufficiently large, we denote by $\Lambda(x_0, \lambda, \nu)$ the paraboloidal domain with the vertex located at x_0 and the axis parallel to ν which is congruent to $x_n < -\lambda\sum_{i=1}^{n-1}x_i^2$. Further, one can observe that $\partial\Omega$ intersects with $\Lambda(x_0, \lambda, \nu)$ tangentially at x_0. Moreover, we assume that the surface $\Gamma := \{\Lambda(x_0, \lambda, \nu) \cap \partial\Omega\} \setminus \{x_0\}$ is a non-empty connected relatively open subset of $\partial\Omega$ and there exists $\widetilde{x} \in \Gamma$ such that $\overline{x_0\widetilde{x}}$ is parallel to ν (Figure 2). We set $\ell = |\overline{x_0\widetilde{x}}|$. Assume that $\partial\Omega$ is of C^4-class and $u \in H^4(\Omega)$ is a solution to (1). Next, we discuss a conditional stability estimate of u at the boundary point x_0.

Lemma 2.3: (i) *There exist constants $C_2 > 0$ and $\kappa_1 \in (0, 1)$, which depend on $\ell, \lambda, \max_{1\leq i,j\leq n}\|a_{ij}\|_{C^3(\overline{\Omega})}, \max_{1\leq i\leq n}\|b_i\|_{W^{2,\infty}(\Omega)}, \|c\|_{W^{2,\infty}(\Omega)}$, such that*

$$|u(x_0)| \leq C_2\max\{1, \|u\|_{H^3(\Omega)}\}\left\{\left(\frac{1}{|\log 1/\varrho|}\right)^{\frac{1}{2}} + \varrho^{\kappa_1}\right\}, \tag{17}$$

with $\varrho := \|u_1 - u_2\|_{H^3(\Gamma)} + \|\partial_A(u_1 - u_2)\|_{H^2(\Gamma)}$. Here, the constants C_2 and κ_1 are independent of the choice of x_0, and can be chosen uniformly in $\ell \in [\ell_0, \ell_1]$, where $\ell_0, \ell_1 > 0$ are arbitrarily fixed such that $\ell_0 < \ell_1$.

(ii) *If $\varrho \leq 1/e$, then the estimate in the first assertion can be rewritten as*

$$|u(x_0)| \leq C_2\max\{1, \|u\|_{H^3(\Omega)}\}\left(\frac{1}{\log 1/\varrho}\right)^{\min\{\frac{1}{2},\kappa_1\}}.$$

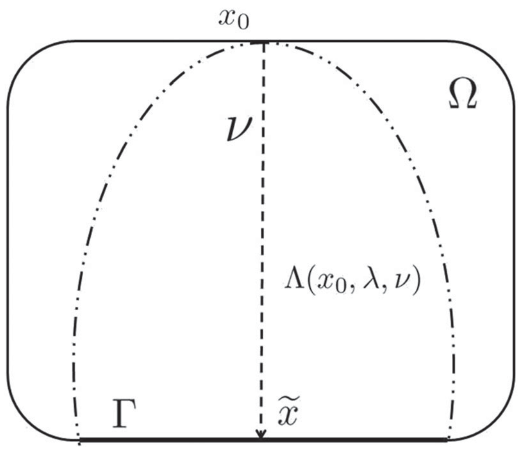

Figure 2. Configurations of $\Lambda(x_0, \lambda, \nu)$ with $x_0 \in \partial\Omega$ and $\Gamma := \partial\Omega \cap \Lambda(x_0, \lambda, \nu)$.

Proof: (i) By the Sobolev embedding we have $|u(x_0)| \le C_2 \|u\|_{H^3(\Omega)}$, whence the first assertion follows if $\varrho \ge 1$. Hence, it remains to prove the lemma under the assumption that $\varrho \le 1$.

Without loss of generality, after translation and rotation, we can define the paraboloidal domain $\Lambda(x_0, \lambda, \nu)$ as

$$\Lambda(x_0, \lambda, \nu) = \{(x', x_n) : x_n < -\lambda \sum_{i=1}^{n-1} x_i^2 + \ell\}, \quad \lambda, \ell > 0$$

with $\nu = (0, \dots, 0, -1), x_0 = (0, \dots, 0, \ell)$. Further, we may assume that the line segment x_0O is parallel to ν where the origin O is located at Γ. Set

$$\mathrm{d}(x) = -x_n - \lambda \sum_{i=1}^{n-1} x_i^2 + \ell, \quad D_t := \{x \in \Lambda(x_0, \lambda, \nu) \cap \Omega : \mathrm{d}(x) > t\} \quad \text{for } 0 \le t < l/2.$$

We note that $D_{t_2} \subset D_{t_1}$ if $t_1 < t_2$ and $D_t = (\Lambda(x_0, \lambda, \nu) + t\nu) \cap \Omega$. In particular, $D_0 = \Lambda(x_0, \lambda, \nu) \cap \Omega$. We can always choose a cut-off function $\chi_t \in C^\infty(\mathbb{R}^n)$ such that $0 \le \chi_t \le 1$ and

$$\chi_t(x) = \begin{cases} 1, & x \in D_t, \\ 0, & x \in D_0 \setminus D_{t/2}, \end{cases} \qquad \|\chi_t\|_{C^2(\mathbb{R}^n)} \le C_3/t^2, \quad 0 < t < l/2. \tag{18}$$

In fact, we may choose $\tilde{\chi} \in C^\infty(\mathbb{R}^n)$ such that $0 \le \tilde{\chi} \le 1$ and

$$\tilde{\chi}(\eta) = \begin{cases} 1, & \eta \ge 1, \\ 0, & \eta \le 0. \end{cases}$$

Then the function $\chi_t(x) = \widetilde{\chi}((2d(x) - t)/t)$ satisfies (18). Set $v := \chi_t u$. Using the fact that $D_{2t} \subset D_0$ and applying the Carleman estimate (16) to v in D_0, we obtain

$$\int_{D_{2t}} (s|\nabla v|^2 + s^3 v^2) e^{2s\varphi}\, \mathrm{d}x$$
$$\le \int_{D_0} (s|\nabla v|^2 + s^3 v^2) e^{2s\varphi}\, \mathrm{d}x$$

$$\leq C\int_{D_0}\left|\sum_{i,j=1}^{n} a_{ij}((\partial_i\chi_t)\partial_j u + (\partial_j\chi_t)\partial_i u + (\partial_i\partial_j\chi_t)u) + \sum_{i=1}^{n} b_i(\partial_i\chi_t)u\right|^2 e^{2s\varphi}\,\mathrm{d}x$$
$$+ Ce^{Cs}\int_{\Gamma}(|\nabla v|^2 + v^2)\,\mathrm{d}s$$
$$\leq C\int_{D_{t/2}\backslash\overline{D_t}}\left|\sum_{i,j=1}^{n} a_{ij}((\partial_i\chi_t)\partial_j u + (\partial_j\chi_t)\partial_i u + (\partial_i\partial_j\chi_t)u) + \sum_{i=1}^{n} b_i(\partial_i\chi_t)u\right|^2 e^{2s\varphi}\,\mathrm{d}x$$
$$+ Ce^{Cs}\int_{\Gamma}(|\nabla u|^2 + u^2)\,\mathrm{d}s$$

where $\varphi(x) = \exp(\lambda\,\mathrm{d}(x))$, $\lambda > 0$ is sufficiently large and $s > s_0$ for some $s_0 > 0$. Since $\varphi(x) \geq \exp(2\lambda t)$ in D_{2t} and $\varphi(x) \leq \exp(\lambda t)$ in $D_{t/2}\backslash D_t$, it can be derived from the previous relation that

$$\|u\|^2_{H^1(D_{2t})} \leq \frac{C_4}{t^4}e^{-2sr(t)}\|u\|^2_{H^1(\Omega)} + C_5e^{C_0s}(\|u\|^2_{H^1(\Gamma)} + \|\partial_A u\|^2_{L^2(\Gamma)}) \tag{19}$$

for all $s \geq s_0$, with $r(t) := e^{2\lambda t} - e^{\lambda t}$. Analogously, applying the Carleman estimate to $v_i = \chi_t\partial_i u$ and $v_{ij} = \chi_t\partial_i\partial_j u$, $1 \leq i,j \leq n$ we can obtain

$$\|\nabla u\|^2_{H^1(D_{2t})} + \|\nabla^2 u\|^2_{H^1(D_{2t})} \leq \frac{C_4}{t^4}e^{-2sr(t)}M^2 + C_5e^{C_0s}\,\varrho^2, \quad s \geq s_0, \tag{20}$$

where $\|u\|_{H^3(\Omega)} \leq M$. Combining (19) and (20) gives

$$\|u\|^2_{H^3(D_{2t})} \leq \frac{C_4}{t^4}e^{-2sr(t)}M^2 + C_5e^{C_0s}\varrho^2, \quad s \geq s_0, \tag{21}$$

Choose $t_0 = \min(1, \ell_0/4)$. By the Sobolev embedding theorem, there exists a constant $C_6 = C_6(t) > 0$ such that

$$\|u\|_{C^1(\overline{D_{2t}})} \leq C_6(t)\,\|u\|_{H^3(D_{2t})}, \quad 0 \leq t \leq t_0.$$

Recall that D_t is defined by a translation of D_0 and that $D_{2t_0} \neq \emptyset$, $D_{2t_0} \subset D_{2t} \subset D_0$. Since $\lambda > 0$ is sufficiently large, we may suppose that D_{2t} are Lipschitz domains with uniformly bounded Lipschitz constants in all $t \in [0, t_0]$. This allows us to choose a constant $C_7 > 0$ such that

$$\|u\|_{C^1(\overline{D_{2t}})} \leq C_7\,\|u\|_{H^3(D_{2t})}, \quad \text{for all} \quad 0 \leq t \leq t_0.$$

It then follows from (21) that

$$\|u\|_{C^1(D_{2t})} \leq \frac{C_8}{t^2}e^{-sr(t)}M + C_8e^{C_0s}\varrho \tag{22}$$

for all $s \geq 0$ and all $0 < t \leq t_0$. We find a value s minimizing the right-hand side of (22), that is, we choose $s \geq s_0$ such that

$$e^{-sr(t)}M = e^{C_0s}\,\varrho.$$

Consequently, we have

$$\|u\|_{C^1(\overline{D_{2t}})} \leq \frac{C_9}{t^2}M^{C_0/(C_0+r(t))}\varrho^{r(t)/(C_0+r(t))} \leq \frac{C_9}{t^2}M_1\varrho^{r(t)/(C_0+r(t))} \tag{23}$$

for all $0 < t \leq t_0$, where we set $M_1 := \max\{M, 1\}$.

For simplicity we write $\partial_n = \partial/\partial_{x_n}$. Since $(0,\dots,0,\ell-2t) \in \overline{D_{2t}}$, we observe from (23) that

$$|\partial_n u(0,\dots,0,\ell-2t)| \le \frac{C_9}{t^2} M_1\, \varrho^{r(t)/(C_0+r(t))}, \quad 0 < t \le t_0. \tag{24}$$

Using the inequalities

$$e^{2\lambda t} - 2e^{\lambda t} + 1 \ge 0, \quad e^{\lambda t} - \lambda t - 1 \ge 0 \quad \text{for all} \quad t > 0,$$

it is easy to check that

$$\frac{r(t)}{C_0 + r(t)} \le \frac{e^{\lambda t} - 1}{C_0 + e^{\lambda \ell_0} - e^{\lambda \ell_0/2}} \le \frac{\lambda}{C_0 + e^{\lambda \ell_0} - e^{\lambda \ell_0/2}} t \equiv C_{10} t \tag{25}$$

for some $C_{10} > 0$. Since $\varrho \le 1$, we have by (24) and (25) that

$$|\partial_n u(0,\dots,0,\ell-2t)| \le \frac{C_9}{t^2} M_1 \varrho^{C_{10} t}, \quad 0 < t \le t_0.$$

Hence,

$$\begin{aligned}
|\partial_n u(0,\dots,0,\ell-2t)| &= |\partial_n u(0,\dots,0,\ell-2t)|^{3/4}\, |\partial_n u(0,\dots,0,\ell-2t)|^{1/4} \\
&\le \|u\|^{3/4}_{C^1(\overline{\Omega})} \left(C_9\, t^{-2}\, M_1 \varrho^{C_{10} t}\right)^{1/4} \\
&\le M^{3/4} M_1^{1/4}\, C_9^{1/4} t^{-1/2} \varrho^{C_{10} t/4} \\
&\le C_{11} M_1\, t^{-1/2}\, \varrho^{C_{12} t},
\end{aligned}$$

where we have used again the Sobolev embedding $\|u\|_{C^1(\overline{\Omega})} \le C\,\|u\|_{H^3(\Omega)}$. Therefore, by (23) we obtain

$$\begin{aligned}
|u(x_0)| = |u(0,\dots,0,\ell)| &= \left| u(0,\dots,0,\ell-2t_0) + \int_{t_0}^{0} \frac{\partial}{\partial t}(u(0,\dots,0,\ell-2t))\,\mathrm{d}t \right| \\
&\le \|u\|_{C(\overline{D_{2t_0}})} + \int_0^{t_0} 2C_{11} M_1\, t^{-1/2}\, \varrho^{C_{12} t}\,\mathrm{d}t \\
&\le C_{13} \|u\|_{H^2(D_{2t_0})} + \int_0^{t_0} C_{13} M_1\, t^{-1/2} \exp\left(-\left(C_{12}\log\frac{1}{\varrho}\right) t\right) \mathrm{d}t \\
&\le \frac{C_{14}}{t_0^2} M_1 \varrho^{\frac{r(t_0)}{C_0 + r(t_0)}} + C_{13} M_1 \int_0^{\infty} t^{-1/2} \exp\left(-\left(C_{12}\log\frac{1}{\varrho}\right) t\right) \mathrm{d}t \\
&= C_{15} M_1 \varrho^{\kappa_1} + C_{15} M_1 \frac{\Gamma\left(\frac{1}{2}\right)}{\left(C_{12}\log\frac{1}{\varrho}\right)^{\frac{1}{2}}} \\
&\le C_{16} M_1 \left\{ \left(\frac{1}{\log 1/\varrho}\right)^{1/2} + \varrho^{\kappa_1} \right\}
\end{aligned}$$

from which the stability estimate (17) follows.

(ii) The second assertion follows straightforwardly from (17) in combination with the inequality

$$\varrho \le \frac{1/e}{\log\frac{1}{\varrho}} < \frac{1}{\log\frac{1}{\varrho}} \quad \text{for all} \quad 0 \le \varrho \le \frac{1}{e}.$$

■

3. Quantitative unique continuation

The aim of this section is to verify the quantitative unique continuation for solutions of the elliptic equation $Au = 0$ (see (1)). Set $m = [n/2] + 2$, where the notation $[a]$ denotes the largest natural number not exceeding $a > 0$. Lemma 3.1 will be used in the subsequent section for the proofs of Theorems 1.1 and 1.2.

Lemma 3.1: *(Quantitative unique continuation) Let $Au = 0$ in Ω and $\|u\|_{H^m(\Omega)} \le M$, where $M > 0$ is an a priori bound. We assume there exists $z \in \Omega$ such that $|u(z)| > C_0$. Suppose further that*

$$|u(x)| < \delta \quad \textit{for all} \quad x \in B_r(y) \subset \Omega, \tag{26}$$

for some $y \in \Omega$ and $\delta, r > 0$. Then an upper bound of the radius r can be estimated by

$$r \le C/C_0^{\kappa}\, \delta^{\theta},$$

where κ, θ and C are positive constants depending only on the space dimension, the region Ω and the distance between z and $\partial\Omega$.

The unique continuation follows directly from Lemma 3.1.

Corollary 3.1: *Let $Au = 0$ in Ω and $u \equiv 0$ in $B_r(y) \subset \Omega$ for some $r > 0$, $y \in \Omega$. Then $u \equiv 0$.*

Proof: Assume on the contrary that $|u(z)| > C_0 > 0$ for some $z \in \Omega$. Since $u \equiv 0$ in $B_r(y)$, we have $|u(x)| < \delta$ for any $\delta > 0$ and for all $x \in B_r(y)$. Applying Lemma 3.1 we see $r \le C/C_0^{\kappa}\, \delta^{\theta}$ for all $\delta > 0$. Now, letting $\delta \to 0$ yields the relation $r = 0$, which contradicts the fact that $r > 0$. Hence $u \equiv 0$ in Ω. ■

Below we carry out the proof of Lemma 3.1, relying on the interior estimate in Lemma 2.1.

Proof of Lemma 3.1.: For notational convenience, we write $x' = (x_2, \ldots, x_n)$ so that $x = (x_1, x')$, $z = (z_1, z') \in \mathbb{R}^n$. Without loss of generality, we suppose that y coincides with the origin O, $|z'| = 0, z_1 > 0$ and $0 < r < 1$. Using the interior estimate (see [19]), it follows from (26) that

$$\|\nabla u\|_{L^\infty(B_{r/2})} \le (C_1/r)\|u\|_{L^\infty(B_r)} \le C_1\delta/r, \tag{27}$$

where the constant $C_1 > 0$ is independent of r. Hence,

$$\|u\|_{W^{1,\infty}(B_{r/2})} \le C_1\delta(1 + 1/r). \tag{28}$$

We may always choose a paraboloidal domain $\Lambda(x_0, \lambda, \nu)$ with $x_0 \in \Omega \cap \mathbb{R}^n_+$, $\nu = (-1, 0, \ldots, 0)$ such that for some $r_0, \delta_0 > 0$

$$B_{rr_0}(z) \subset \{\Lambda(x_0, \lambda, \nu) + \delta_0\nu\} \cap \{\Omega \cap \mathbb{R}^n_+\} =: \Lambda_{\delta_0}.$$

Note that the point x_0 and the parameters λ, r_0 and δ involved are dependent only on the geometry of Ω and the distance between z and $\partial\Omega$. Applying Lemma 2.1 to Λ_{δ_0} yields

$$\|u\|_{H^1(B_{rr_0}(z))} \le \|u\|_{H^1(\Omega_{\delta_0})} \le C_2\left(\|u\|_{H^1(\gamma)} + \|\partial_\nu u\|_{L^2(\gamma)}\right)^{\kappa} \tag{29}$$

for some $\kappa \in (0, 1]$ and $C_2 > 0$ independent of $\gamma = \{\Lambda(y, \lambda, \nu) + \delta_0\nu\} \cap \{\Omega \cap \{(0, x')\}\}$. Further, without loss of generality we may suppose that $\gamma \subset \{(0, x') : |x'| < r/2\}$. Otherwise, this can be achieved by constructing a finite number of paraboloidal domains $\Lambda(y_j,\lambda_j,\nu_j)$ with $y_j\in\Omega$ and uniformly bounded parameters λ_j and ν_j, and then our arguments should be applied successively to each paraboloidal domain.

Combining the estimates in (28) and (29), we obtain

$$\|u\|_{H^1(B_{rr_0}(z))}\le C_2(C_1\delta(1+1/r)\,r^{(n-1)/2})^{\kappa}\le C_3\,\delta^{\kappa}\,(1+r^{n-3})^{\kappa/2},\tag{30}$$

where $C_3>0$ does not depend on δ. Moreover, recalling the inequality $(r^{n-3})^{\kappa/2}\le Cr^{-\kappa}$ for all $r\in(0,1]$, it holds that

$$\|u\|_{H^1(B_{rr_0}(z))}\le C_4\delta^{\kappa}\,r^{-\kappa},\quad C_4>0.$$

Now, applying Lemma 3.2 below we obtain for $m=[n/2]+1$ and $\theta=1/m\in(0,1)$ that

$$\begin{aligned}\|u\|_{L^\infty(B_{rr_0}(z))}&\le C\,(rr_0)^{-m-n/2}\|u\|^{\theta}_{H^1(B_{rr_0}(z))}\\&\le C\,r_0^{-m-\frac{n}{2}}r^{-m-\frac{n}{2}}r^{-\kappa\theta}\delta^{\kappa\theta}\\&=C\,r^{-\mu_1}\delta^{\mu_2}\end{aligned}$$

where $\mu_1=m+\frac{n}{2}+\kappa\theta>0$ and $\mu_2=\kappa\theta\in(0,1)$. Since $|u(z)|>C_0>0$, we have

$$(rr_0)^n\,C_nC_0\le\|u_1\|_{L^\infty(B_{rr_0}(z))}<Cr^{-\mu_1}\delta^{\mu_2},$$

leading to the relation

$$r^{n+\mu_1}\le C\,C_0^{-1}\delta^{\mu_2}.$$

Finally, an upper bound of r can be estimated by

$$r\le C\,\delta^{\mu_2/(n+\mu_1)}\,C_0^{-1/(n+\mu_1)}.$$

The proof of the lemma is complete. ■

In proving the quantitative unique continuation we have used the following result.

Lemma 3.2: *Let $B_r=B_r(O)\subset\mathbb{R}^n$ for some $r\in(0,1)$. Suppose that*

$$\|u\|_{H^{m+1}(B_r)}\le M,\quad m:=[n/2]+1.$$

Then there exists a constant $C=C(M,n)>0$ such that

$$\|u\|_{L^\infty(B_r)}\le C\,r^{-m-n/2}\,\|u\|^{1/m}_{H^1(B_r)}.\tag{31}$$

Proof: By change of variables $y=x/r$ and $\widetilde{u}(y):=u(ry)$, we have

$$\int_{B_r}\sum_{|\alpha|\le m}|\partial_x^{\alpha}u|^2\,\mathrm{d}x=\int_{B_1}\sum_{|\alpha|\le m}|\partial_y^{\alpha}\widetilde{u}|^2r^{n-2\alpha}\,\mathrm{d}y.$$

Hence there exist $C_0, C_1 > 0$ independent of $r \in (0,1)$ such that

$$C_0 r^{\frac{n}{2}} \|\widetilde{u}\|_{H^m(B_1)} \leq \|u\|_{H^m(B_r)} \leq C_1\, r^{\frac{n}{2}-m} \|\widetilde{u}\|_{H^m(B_1)}. \tag{32}$$

Let $m = [\frac{n}{2}] + 1$ and $m' = [\frac{n}{2}] + 2$. In B_1 we have the interpolation inequality

$$\|\widetilde{u}\|_{H^m(B_1)} \leq C \|\widetilde{u}\|_{H^1(B_1)}^{\frac{m'-m}{m'-1}} \|\widetilde{u}\|_{H^{m'}(B_1)}^{\frac{m-1}{m'-1}} = C \|\widetilde{u}\|_{H^1(B_1)}^{1/m} \|\widetilde{u}\|_{H^{m'}(B_1)}^{1-1/m}.$$

Using (32), we get

$$\|u\|_{H^m(B_r)} \leq C_2\, r^{-m} \|u\|_{H^1(B_r)}^{1/m} \|u\|_{H^{m'}(B_r)}^{1-1/m}. \tag{33}$$

Moreover, applying the Sobolev embedding theorem yields

$$\|\widetilde{u}\|_{L^\infty(B_1)} \leq C_3 \|\widetilde{u}\|_{H^m(B_1)}.$$

Together with the definition of $\tilde{u}$ and the first inequality in (32), this implies that

$$\|u\|_{L^\infty(B_r)} \leq C_3\, C_0^{-1} r^{-\frac{n}{2}} \|u\|_{H^m(B_r)}.$$

We use (33) to estimate the right-hand side of the previous inequality to obtain

$$\|u\|_{L^\infty(B_r)} \leq C_4\, r^{-m-\frac{n}{2}} \|u\|_{H^1(B_r)}^{1/m} \|u\|_{H^{m'}(B_r)}^{1-1/m} \leq C_5\, r^{-m-\frac{n}{2}} \|u\|_{H^1(B_r)}^{1/m},$$

which proves (31). ∎

4. Proof of Theorems 1.1 and 1.2

Proof of Theorem 1.1: Set

$$u = u_1 - u_2 \quad \text{in } D_1^c \cap D_2^c$$

and

$$\varrho := \|u_1 - u_2\|_{H^3(\Gamma)} + \|\partial_A (u_1 - u_2)\|_{H^2(\Gamma)}.$$

Let $\Omega_0 = \{x : \operatorname{dist}(x, \partial\Omega) < \delta/2\}$ where δ is the a priori data given in (6); see Figure 3. Since the parameter $\lambda > 0$ of the parabolic domain $\Lambda(y, \lambda, \nu)$ in Lemma 2.1 can be chosen arbitrarily large, we can always construct a family of paraboloidal domains to prove that

$$\|u\|_{H^1(\Omega_0)} \leq C_1 \varrho^{\kappa_1},$$

where the constants $\kappa_1 \in (0,1]$ and $C_1 > 0$ depend on $\partial\Omega$ and the data M, δ involved in Condition A. We set $\Gamma_0 = \partial\Omega_0 \setminus \partial\Omega$. By the interpolation inequality and Condition A, we find

$$\|u\|_{H^{7/2}(\Omega_0)} \leq C \|u\|_{H^1(\Omega_0)}^{1/6} \|u\|_{H^4(\Omega_0)}^{5/6} \leq C\, M^{5/6} \|u\|_{H^1(\Omega_0)}^{1/6} \leq C_2\, \varrho^{\kappa_2}.$$

Applying the trace theorem gives

$$\|u\|_{H^3(\Gamma_0)} + \|\partial_A u\|_{H^2(\Gamma_0)} \leq C_3\, \varrho^{\kappa_2},$$

where $C_3 > 0$ depends on $\partial\Omega$, δ and M. Let E be any connected component of $D_1^c \backslash \overline{D_2^c}$; see the shadow area in Figure 3. Since D_1 and D_2 are star-shaped centered at the origin, the boundary $\partial E \cap \partial D_2$ can

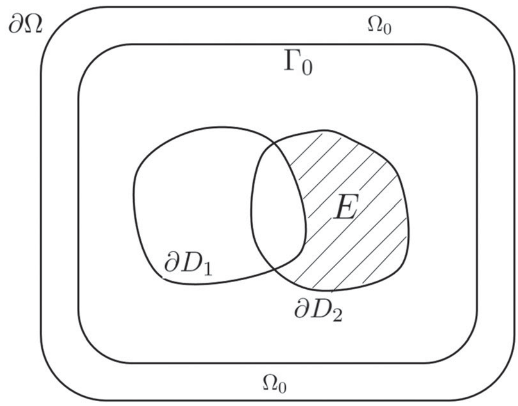

Figure 3. Illustration of two sub-boundaries ∂D_1, ∂D_2 and the domain $E := D_1^c \backslash \overline{D_2^c}$.

be connected to Γ_0 in $\Omega \setminus (\overline{D_1 \cup D_2})$. We apply Lemma 2.3 (ii) to the region $\Omega \setminus (\overline{D_1 \cup D_2})$ to obtain an estimate of u on $\partial E \cap \partial D_2$:

$$\|u\|_{L^\infty(\partial E \cap \partial D_2)} \le C_4 \left(\frac{1}{\log 1/(\|u\|_{H^3(\Gamma_0)} + \|\partial_A u\|_{H^2(\Gamma_0)})} \right)^{\kappa_3} \le C_5 \left(\frac{1}{\log 1/\varrho} \right)^{\kappa_3},$$

for some $\kappa_3 \in (0, 1/2]$, where $\varrho > 0$ is supposed to be sufficiently small. Since $u_2 = 0$ on ∂D_2, we have

$$\|u_1\|_{L^\infty(\partial E \cap \partial D_2)} \le C_5 \left(\frac{1}{\log\ 1/\varrho} \right)^{\kappa_3}.$$

Using the fact that $u_1 = 0$ on ∂D_1, the previous inequality can be written as

$$\|u_1\|_{L^\infty(\partial E)} \le C_5 \left(\frac{1}{\log\ 1/\varrho} \right)^{\kappa_3}. \tag{34}$$

We set $B_r(z) := \{x \in \mathbb{R}^n;\ |x - z| < r\}$. Let

$$r_0 = \sup\{r :\ B_r(z) \subset E \text{ with some } z \in E\}.$$

That is, r_0 is the radius of the inscribed ball in E. Suppose that $B_{r_0}(z_0) \subset E$ for some $z_0 \in E$. The maximum principle in E yields

$$\|u_1\|_{L^\infty(B_{r_0}(z_0))} \le \|u_1\|_{L^\infty(E)} \le C_5 \left(\frac{1}{\log\ 1/\varrho} \right)^{\kappa_3} := \delta_0. \tag{35}$$

On the other hand, it is seen form Condition B that there exist $C_0 > 0$ and $z \in \Omega_\epsilon$ such that $|u_1(z)| \ge C_0/2$. Now applying the quantitative unique continuation, we see that

$$r_0 \le C\delta_0^\kappa \le C \left(\frac{1}{\log\ 1/\varrho} \right)^{\theta} \tag{36}$$

for some $\kappa, \theta \in (0, 1)$. Note that the constant C depends on the a priori bounds involved in Conditions A and B, the region Ω and the upper bounds of the coefficients in equation (1). Note that the estimate (36) applies to the radius of the inscribed ball in any connected component of $D_1^c\backslash\overline{D_2^c}$ and $D_1^c\backslash\overline{D_2^c}$. Without loss of generality we suppose that

$$\mathrm{d}(\partial D_1, \partial D_2) = |z_1 - z_2| =: \rho, \quad z_1 \in \partial D_1, z_2 \in \partial D_2, \tag{37}$$

and that

$$\rho = \sup_{x\in\partial D_2} \mathrm{d}(x, \partial D_1). \tag{38}$$

Then the line segment connecting z_1 and z_2 is contained in E and is orthogonal to the tangent plane of ∂D_1 at z_1. Hence, we can always find a finite cone contained in E with the vertex at z_1 and the axis parallel to z_1z_2. Moreover, the opening angle and the height of this cone both depend on ρ and the a priori bound $M > 0$. This implies that the ratio of ρ and r_0 can be bounded by some constant depending on M only. Hence, the Hausdorff distance can also be bounded by the right-hand side of (36). This finishes the proof of Theorem 1.1. ■

Proof of Theorem 1.2: Let $D \subset \mathbb{R}^n$ be the unbounded connected component of $(\mathbb{R}^n\backslash\overline{D}_1) \cap (\mathbb{R}^n\backslash\overline{D}_2)$. Analogously to the proof of Theorem 1.1, we set

$$u := u_1 - u_2 \quad \text{in} \quad D, \quad \varrho := \|u_1^\infty - u_2^\infty\|_{L^2(\mathbb{S}^{n-1})}.$$

We first estimate the near field data in D from the far-field pattern. By [15], there exist a radius $R_1 > R$ and a constant $C > 0$ such that

$$\|u\|_{L^2(B_{R_1+1}\backslash B_{R_1})} \le C\,\varrho^{\alpha(\varrho)},$$

where the function $\alpha : \mathbb{R}^+ \to \mathbb{R}$ is defined as

$$\alpha(\varrho) := (1 + \log(-\log\varrho + e))^{-1}.$$

Setting $\Omega := B_{R_1+1/2}$ and $\Gamma = \partial\Omega = \{|x| = R_1 + 1/2\}$, it follows from the interior elliptic estimate that

$$\|u\|_{H^3(\Gamma)} + \|\partial_\nu u\|_{H^2(\Gamma)} \le C\,\varrho^{\alpha(\varrho)}.$$

Now, we may restrict our discussions to the bounded domain Ω, following the lines in the proof of Theorem 1.1. For this purpose, it is necessary to check the conditions A and B for the inverse problem (IP1). By well-posedness of the forward scattering and the uniform C^4-smoothness assumption of ∂D_j, there exist $M, \delta > 0$ such that the relations in (6) hold. On the other hand, since $|u^{in}(x)| = 1$ and u_j^{sc} decays at infinity, the boundary Γ can be chosen depending on the a priori data only such that (see e.g. [20, Corollary 3.3])

$$|u_j(x)| > 1/2 \quad \text{for all} \quad x \in \Gamma, \quad j = 1, 2,$$

which implies Condition B in (7). Arguing as in the proof of Theorem 1.1, we get (cf. (34))

$$\|u_1\|_{L^\infty(\partial E)} \le C\,|\alpha(\varrho)\log\varrho|^{-\theta} := \delta_0, \quad \theta \in (0, 1), \tag{39}$$

where the region $E \subset \Omega$ is defined as in the proof of Theorem 1.1. Under Conditions C and D, k^2 is not a Dirichlet eigenvalue of $-\Delta$ in E. Hence the estimate (35) still holds with δ_0 given by (39). Consequently,

$$\mathrm{d}(\partial D_1, \partial D_2) \le C\,|\alpha(\varrho)\log\varrho|^{-\theta}, \tag{40}$$

for some $\theta \in (0, 1)$. ■

We conclude this section by a remark on the stability estimate of the inverse scattering problem with several incoming waves. Condition C or D ensures uniqueness to the inverse scattering problem with a single incoming wave. Without these two conditions, one can get the same estimate from the far-field data of a finite number of incident directions $\alpha_j \in \mathbb{S}^{n-1}$ at a fixed frequency or a finite number of frequencies $k_j \in \mathbb{R}_+$ with fixed incident direction. More precisely, the smallness and closeness type assumptions in Theorem 1.2 can be removed in the following cases:

$$\text{Case (a):} \quad \varrho = \max\{\|u_1^\infty(\hat{x};\alpha_j,k) - u_2^\infty(\hat{x};\alpha_j,k)\|_{L^2(\mathbb{S}^{n-1})} : j = 1,2,\dots N_1+1\}$$

where $N_1 := \sum_{t_{ml}<k^*R}(2m+1)$. Here, for the dimension $n=3$ and $m = 0,1,\dots$, we denote the positive zeros of the spherical Bessel functions j_m by $t_{ml}, l = 0,1,\dots$; for $n = 2$, t_{ml} are the positive zeros of the Bessel functions J_m. The number $R>0$ is the radius of a ball centered at the origin which contains D_j inside.

$$\text{Case (b):} \quad \varrho = \max\{\|u_1^\infty(\hat{x};\alpha,k_j) - u_2^\infty(\hat{x};\alpha,k_j)\|_{L^2(\mathbb{S}^{n-1})} : j = 1,2,\dots N_2+1\}$$

where $k_j := k_* + (j-1)(k^* - k_*)/N_2$ with $k_* < k^*$ and $N_2 := \sum_{t_{ml}<k^*R}(2m+1)$.

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Appendix. Proof of Carleman estimate

In this section, we give a direct derivation of the Carleman estimate for the elliptic operator A, i.e. Lemma 2.2. There is an approach based on the general theory (e.g. [21–23]), but we present a direct proof which is based on integration by parts. One can refer to [24,25] for similar direct derivation of a parabolic Carleman estimate and to [26] for a hyperbolic Carleman estimate.

Thanks to the large parameter s, it is sufficient to prove the Carleman estimate in the case of $b_i = c = 0, 1 \le i \le n$, i.e. to verify Lemma 2.2 for the principal part of the elliptic operator A, given by

$$(A_0 u)(x) \equiv - \sum_{i,j=1}^{n} a_{ij}(x)\partial_i\partial_j u = f, \quad x \in \Omega.$$

In fact, regarding the lower-order part $\sum_{i=1}^{n} b_i\partial_i u + cu$ as the right-hand side, we can absorb the weighted L^2-norms of the lower-order part into the left-hand side by applying the Carleman estimate for A_0 and taking the parameter $s > 0$ sufficiently large.

Let $D \subset \Omega$ and $\varphi(x) = e^{\lambda d(x)}$ be given as in Lemma 2.2. For notational simplicity we set

$$\sigma(x) = \sum_{i,j=1}^{n} a_{ij}(x)(\partial_i d)(x)(\partial_j d)(x), \quad x \in \overline{D}.$$

Define

$$w(x) := e^{s\varphi(x)}u(x)$$

and

$$Pw(x,t) := e^{s\varphi}A_0(e^{-s\varphi}w) = e^{s\varphi}A_0 u = e^{s\varphi}f.$$

Below we give some technical remarks on the proof of the Carleman estimate. The derivation argument consists of three steps:

Step 1: Decomposition of the differential operator P into the sum of P_1 and P_2, where P_1 is composed of the second-order and zeroth-order terms in x, whereas P_2 is composed of first-order terms in x. Here the terms in Pw are classified by the highest order of s, λ and φ.

Step 2: Estimation of $\int_D 2(P_1w)(P_2w)\,dx$ from below.

Step 3: Derivation of an estimate for the term

$$\int_D s\lambda^2\varphi\sigma \sum_{i,j=1}^{n} a_{ij}(\partial_i w)(\partial_j w)\,dx,$$

which appears in the lower bound of $\int_D 2(P_1w)(P_2w)\,dx$ in Step 2.

Moreover the estimate in the second step produces the estimate of u with desirable order of s, λ, φ but not the term of ∇u. This is caused by the different orders of the derivatives of terms under consideration. Therefore, another estimate in the third step is necessary. Such kind of double estimates have been used in proving the observability inequality of the time-dependent wave equation by the multiplier method. As for the multiplier method, the two estimates are obtained from (see e.g. Komornik [27, p. 36–39]):

$$\int_0^T \int_\Omega (\partial_t^2 v - \Delta v)(h(x) \cdot \nabla v)\, \mathrm{d}x\, \mathrm{d}t$$

and

$$\int_0^T \int_\Omega (\partial_t^2 v - \Delta v) v\, \mathrm{d}x\, \mathrm{d}t$$

respectively, with a suitable vector-valued function $h(x)$, and then the estimates are summed up to obtain an L^2-estimate of v. The second estimate for the wave equation via the multiplier method is similar to the third step in our case.

Proof of Lemma 2.2: *Step 1*. Let $\nu = \nu(x)$ be the outward unit normal vector to ∂D. Simple calculations show that

$$Pw = -\sum_{i,j=1}^n a_{ij}\, \partial_i \partial_j w + 2s\lambda\varphi \sum_{i,j=1}^n a_{ij}\, \partial_i d\, \partial_j w$$
$$- s^2\lambda^2\varphi^2\sigma w + s\lambda^2\varphi\sigma w + s\lambda\varphi w \sum_{i,j=1}^n a_{ij}\partial_i\partial_j \mathrm{d}$$

in D. Note that in the previous identity we have specified all the dependency of coefficients on s, λ and φ. The last two terms in Pw can be rewritten as $A_1 w$, where $A_1 = A_1(x; s, \lambda, \varphi, \sigma)$ is defined as

$$A_1(x; s, \lambda, \varphi, \sigma) := s\lambda^2\varphi\sigma + s\lambda\varphi \sum_{i,j=1}^n a_{ij}\partial_i\partial_j d =: s\lambda^2\varphi\, a_1(x; s, \lambda),$$

$$a_1(x; s, \lambda) := \sigma + (1/\lambda) \sum_{i,j=1}^n a_{ij}\partial_i\partial_j d.$$

Hence,

$$Pw = -\sum_{i,j=1}^n a_{ij}(x)\partial_i\partial_j w + 2s\lambda\varphi \sum_{i,j=1}^n a_{ij}(x)(\partial_i d)\partial_j w - s^2\lambda^2\varphi^2\sigma w + A_1 w.$$

We note that a_1 depends on s and λ, and

$$|a_1(x; s, \lambda)| \le C \quad \text{for } x \in \overline{D} \text{ and all sufficiently large } \lambda > 0 \text{ and } s > 0.$$

Here and henceforth by C, C_1, etc., we denote generic constants which are dependent on λ, but independent of s and the geometry of D, and are bounded provided that $\max_{1\le i,j\le n} \|a_{ij}\|_{C^3(\overline{\Omega})}$, $\max_{1\le i\le n} \|b_i\|_{W^{2,\infty}(\Omega)}$, $\|c\|_{W^{2,\infty}(\Omega)}$, $\|d\|_{C^2(\overline{\Omega})}$ are bounded.

Taking into account the orders of (s, λ, φ), we split P into the sum of P_1 and P_2, where P_1 is composed of second-order and zeroth-order terms in x, whereas P_2 is composed of first-order terms in x. That is,

$$P_1 w := -\sum_{i,j=1}^n a_{ij}(x)\partial_i\partial_j w - s^2\lambda^2\varphi^2 w\sigma(x) + A_1 w,$$

$$P_2 w := 2s\lambda\varphi \sum_{i,j=1}^n a_{ij}(x)(\partial_i d)\partial_j w.$$

By $\|fe^{s\varphi}\|^2_{L^2(D)} = \|P_1 w + P_2 w\|^2_{L^2(D)}$, we have

$$2\int_D (P_1 w)(P_2 w)\, \mathrm{d}x \le \int_D f^2 e^{2s\varphi}\, \mathrm{d}x. \tag{A1}$$

Step 2: We need to derive a lower bound of the left-hand side of (A1). Clearly, we have

$$\int_D (P_1 w)(P_2 w)\, \mathrm{d}x = \sum_{k=1}^3 J_k,$$

where

$$J_1 := -\sum_{i,j=1}^n \int_D a_{ij}(\partial_i\partial_j w) 2s\lambda\varphi \sum_{k,\ell=1}^n a_{k\ell}(\partial_k d)(\partial_\ell w)\, \mathrm{d}x,$$

$$J_2 := -\int_D 2s^3\lambda^3\varphi^3\sigma w \sum_{i,j=1}^{n} a_{ij}(\partial_i d)(\partial_j w)\,\mathrm{d}x,$$

$$J_3 := \int_D (A_1 w) 2s\lambda\varphi \sum_{i,j=1}^{n} a_{ij}(\partial_i d)(\partial_j w)\,\mathrm{d}x. \tag{A2}$$

Now, applying integration by parts, $a_{ij} = a_{ji}$ and $u \in H^2(D)$ and assuming that $\lambda > 1$ and $s > 1$ are sufficiently large, we reduce all the derivatives of w to $w, \partial_i w$. We continue the estimation of $J_k, k = 1, 2, 3$ as follows. First,

$$\begin{aligned}
J_1 &= -\sum_{i,j=1}^{n}\sum_{k,\ell=1}^{n}\int_D 2s\lambda\varphi a_{ij}a_{k\ell}(\partial_k d)(\partial_\ell w)(\partial_i\partial_j w)\,\mathrm{d}x \\
&= 2s\lambda \int_D \sum_{i,j=1}^{n}\sum_{k,\ell=1}^{n} \lambda(\partial_i d)\varphi a_{ij}a_{k\ell}(\partial_k d)(\partial_\ell w)(\partial_j w)\,\mathrm{d}x \\
&\quad + 2s\lambda \int_D \sum_{i,j=1}^{n}\sum_{k,\ell=1}^{n} \varphi\partial_i(a_{ij}a_{k\ell}\partial_k d)(\partial_\ell w)(\partial_i w)\,\mathrm{d}x \\
&\quad + 2s\lambda \int_D \sum_{i,j=1}^{n}\sum_{k,\ell=1}^{n} \varphi a_{ij}a_{k\ell}(\partial_k d)(\partial_i\partial_\ell w)(\partial_j w)\,\mathrm{d}x \\
&:= J_1^{(1)} + J_1^{(2)} + J_1^{(3)}.
\end{aligned}$$

The first and third terms in J_1 can be estimated by

$$J_1^{(1)} = 2s\lambda^2 \int_D \varphi \left| \sum_{i,j=1}^{n} a_{ij}(\partial_i d)(\partial_j w)\right|^2 \mathrm{d}x \ge 0,$$

and

$$\begin{aligned}
J_1^{(3)} &= \int_D 2s\lambda \sum_{k,\ell=1}^{n} \left(\sum_{i>j} \varphi a_{ij}a_{k\ell}(\partial_k d)\{(\partial_i\partial_\ell w)(\partial_j w) + (\partial_j\partial_\ell w)(\partial_i w)\}\,\mathrm{d}x \right. \\
&\quad \left. + \sum_{k,\ell=1}^{n}\sum_{i=1}^{n} \varphi a_{ii}a_{k\ell}(\partial_k d)(\partial_i\partial_\ell w)(\partial_i w) \right) \mathrm{d}x \\
&= s\lambda \sum_{i,j=1}^{n}\sum_{k,\ell=1}^{n} \int_D \varphi a_{ij}a_{k\ell}(\partial_k d)\partial_\ell((\partial_i w)(\partial_j w))\,\mathrm{d}x \\
&= s\lambda \int_{\partial D} \sum_{i,j=1}^{n}\sum_{k,\ell=1}^{n} \varphi a_{ij}a_{k\ell}(\partial_k d)(\partial_i w)(\partial_j w)\nu_\ell\,\mathrm{d}s \\
&\quad - s\lambda^2 \int_D \varphi\sigma \sum_{i,j=1}^{n} a_{ij}(\partial_i w)(\partial_j w)\,\mathrm{d}x \\
&\quad - s\lambda \int_D \varphi \sum_{i,j=1}^{n}\sum_{k,\ell=1}^{n} \partial_\ell(a_{ij}a_{k\ell}\partial_k d)(\partial_i w)(\partial_j w)\,\mathrm{d}x.
\end{aligned}$$

Hence, we can estimate J_1 from below by

$$\begin{aligned}
J_1 &\ge -\int_D s\lambda^2\varphi\sigma \sum_{i,j=1}^{n} a_{ij}(\partial_i w)(\partial_j w)\,\mathrm{d}x \\
&\quad - C\int_D s\lambda\varphi|\nabla w|^2\,\mathrm{d}x + 2s\lambda^2 \int_D \varphi \left|\sum_{i,j=1}^{n} a_{ij}(\partial_i d)(\partial_j w)\right|^2 \mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&+ s\lambda \int_{\partial D} \sum_{i,j=1}^{n} \sum_{k,\ell=1}^{n} \varphi a_{ij} a_{k\ell} (\partial_k d)(\partial_i w)(\partial_j w) \nu_\ell \, dS \\
&\geq - \int_D s\lambda^2 \varphi \sigma \sum_{i,j=1}^{n} a_{ij} (\partial_i w)(\partial_j w) \, \mathrm{d}x - C \int_D s\lambda\varphi |\nabla w|^2 \, \mathrm{d}x \\
&- Cs\lambda \int_{\partial D} \varphi |\nabla w|^2 \, \mathrm{d}s. \qquad \text{(A3)}
\end{aligned}
$$

On the other hand, the other two terms J_2 and J_3 in the integral $\int_D 2(P_1 w)(P_2 w)\,\mathrm{d}x$ can be estimated by

$$
\begin{aligned}
J_2 &= - \int_D 2s^3\lambda^3\varphi^3 \sigma w \sum_{i,j=1}^{n} a_{ij} (\partial_i d)(\partial_j w) \, \mathrm{d}x \\
&= - \int_D s^3\lambda^3\varphi^3 \sum_{i,j=1}^{n} \sigma a_{ij} (\partial_i d) \partial_j (w^2) \, \mathrm{d}x \\
&= \int_D s^3\lambda^3 \sum_{i,j=1}^{n} 3\varphi^2 \{\lambda(\partial_j d)\varphi\} \sigma a_{ij} (\partial_i d) w^2 \, \mathrm{d}x \\
&+ \int_D s^3\lambda^3\varphi^3 \sum_{i,j=1}^{n} \partial_j (\sigma a_{ij} \partial_i d) w^2 \, \mathrm{d}x - \int_{\partial D} \sum_{i,j=1}^{n} s^3\lambda^3\varphi^3 \sigma a_{ij} (\partial_i d) w^2 \nu_j dS \\
&\geq \int_D 3s^3\lambda^4\varphi^3\sigma^2 w^2 \, \mathrm{d}x - C \int_D s^3\lambda^3\varphi^3 w^2 \, \mathrm{d}x - C \int_{\partial D} s^3\lambda^3\varphi^3 w^2 \, \mathrm{d}s \qquad \text{(A4)}
\end{aligned}
$$

and

$$
\begin{aligned}
|J_3| &= \left| \int_D (s\lambda^2\varphi a_1)\, (2s\lambda\varphi w) \sum_{i,j=1}^{n} a_{ij} (\partial_i d)(\partial_j w) \, \mathrm{d}x \right| \\
&= \left| \int_D 2a_1 s^2\lambda^3\varphi^2 \sum_{i,j=1}^{n} a_{ij} (\partial_i d) w (\partial_j w) \, \mathrm{d}x \right| \\
&= \left| \int_D a_1 s^2\lambda^3\varphi^2 \sum_{i,j=1}^{n} a_{ij} (\partial_i d) \partial_j (w^2) \, \mathrm{d}x \right| \\
&= \left| - \int_D \sum_{i,j=1}^{n} \partial_j (a_1 s^2\lambda^3\varphi^2 a_{ij} (\partial_i d)) w^2 \, \mathrm{d}x + \int_{\partial D} \sum_{i,j=1}^{n} a_1 s^2\lambda^3\varphi^2 a_{ij} (\partial_i d) w^2 \nu_j dS \right| \\
&\leq C \int_D s^2\lambda^4\varphi^2 w^2 \, \mathrm{d}x + C \int_{\partial D} s^2\lambda^3\varphi^2 w^2 ds. \qquad \text{(A5)}
\end{aligned}
$$

Hence, combining (A2)–(A5) we obtain

$$
\begin{aligned}
\int_D (P_1 w)(P_2 w) \, \mathrm{d}x \geq 3 \int_D s^3\lambda^4\varphi^3\sigma^2 w^2 \, \mathrm{d}x - \int_D s\lambda^2\varphi\sigma \sum_{i,j=1}^{n} a_{ij} (\partial_i w)(\partial_j w) dx \\
- C \int_D s\lambda\varphi |\nabla w|^2 \, \mathrm{d}x - C \int_D (s^3\lambda^3\varphi^3 + s^2\lambda^4\varphi^2) w^2 \, \mathrm{d}x \\
- C \int_{\partial D} s\lambda\varphi |\nabla w|^2 dS - C \int_{\partial D} (s^3\lambda^3\varphi^3 + s^2\lambda^3\varphi^2) w^2 \, \mathrm{d}s.
\end{aligned}
$$

Rearranging the terms in the previous inequality yields

$$
3 \int_D s^3\lambda^4\varphi^3\sigma^2 w^2 \, \mathrm{d}x - \int_D s\lambda^2\varphi\sigma \sum_{i,j=1}^{n} a_{ij} (\partial_i w)(\partial_j w) \, \mathrm{d}x
$$

$$
\begin{aligned}
&\leq \frac{1}{2}\int_D f^2 e^{2s\varphi}\,\mathrm{d}x + C\int_D s\lambda\varphi|\nabla w|^2\,\mathrm{d}x \\
&+ C\int_D (s^3\lambda^3\varphi^3 + s^2\lambda^4\varphi^2)w^2\,\mathrm{d}x \\
&+ C\int_{\partial D} (s\lambda\varphi|\nabla w|^2 + (s^3\lambda^3\varphi^3 + s^2\lambda^3\varphi^2)w^2)\,\mathrm{d}s. \qquad \text{(A6)}
\end{aligned}
$$

Step 3. The first and the second terms on the left-hand side of (A6) have different signs, so we need another estimate. In this step will obtain another estimation of

$$\int_D s\lambda^2\varphi\sigma\sum_{i,j=1}^n a_{ij}(\partial_i w)(\partial_j w)\,\mathrm{d}x$$

by means of

$$\int_D (P_1 w + P_2 w)(s\lambda^2\varphi\sigma w)\,\mathrm{d}x.$$

Here the factor $s\lambda^2\varphi\sigma w$ is necessary for obtaining the term of $|\nabla w|^2$ with the desirable (s,λ,φ)-factor $s\lambda^2\varphi$. That is, multiplying $s\lambda^2\varphi\sigma w$ to both sides of the equation

$$2s\lambda\varphi\sum_{i,j=1}^n a_{ij}(\partial_i d)(\partial_j w) - \sum_{i,j=1}^n a_{ij}\partial_i\partial_j w - s^2\lambda^2\varphi^2\sigma w + A_1 w = fe^{s\varphi},$$

we obtain

$$\int_D fe^{s\varphi}s\lambda^2\varphi\sigma w dx = \sum_{k=1}^4 I_k, \qquad \text{(A7)}$$

where

$$
\begin{aligned}
I_1 &:= \int_D 2s\lambda\varphi\sum_{i,j=1}^n a_{ij}(\partial_i d)(\partial_j w)s\lambda^2\varphi\sigma w\,\mathrm{d}x, \\
I_2 &:= -\int_D \left(\sum_{i,j=1}^n a_{ij}\partial_i\partial_j w\right) s\lambda^2\varphi\sigma w\,\mathrm{d}x, \\
I_3 &:= -\int_D s^3\lambda^4\varphi^3\sigma^2 w^2\,\mathrm{d}x, \\
I_4 &:= \int_D (A_1 w)(s\lambda^2\varphi\sigma w)\,\mathrm{d}x.
\end{aligned}
$$

Now, using integration by parts and the relation $\partial_i\varphi = \lambda(\partial_i d)\varphi$, we estimate the terms I_j $(j = 1,2,3,4)$ as follows.

$$
\begin{aligned}
|I_1| &= \left|\int_D s^2\lambda^3\varphi^2\sigma\sum_{i,j=1}^n a_{ij}(\partial_i d)\partial_j(w^2)\,\mathrm{d}x\right| \\
&= \Bigg|-\int_D \sum_{i,j=1}^n s^2\lambda^3\{2\lambda(\partial_j d)\varphi^2\}\sigma a_{ij}(\partial_i d)w^2\,\mathrm{d}x \\
&\quad - \sum_{i,j=1}^n s^2\lambda^3\varphi^2\partial_j(\sigma a_{ij}(\partial_i d))w^2\,\mathrm{d}x + \int_{\partial D}\sum_{i,j=1}^n s^2\lambda^3\varphi^2\sigma a_{ij}(\partial_i d)w^2\nu_j dS\Bigg| \\
&\leq C\int_D s^2\lambda^4\varphi^2 w^2\,\mathrm{d}x + C\int_{\partial D} s^2\lambda^3\varphi^2 w^2\,\mathrm{d}s; \qquad \text{(A8)}
\end{aligned}
$$

$$
\begin{aligned}
I_2 &= -\int_D s\lambda^2\sum_{i,j=1}^n \varphi\sigma a_{ij}w(\partial_i\partial_j w)\,\mathrm{d}x \\
&= \int_D s\lambda^2\sum_{i,j=1}^n \varphi\sigma a_{ij}(\partial_i w)(\partial_j w)\,\mathrm{d}x + \int_D s\lambda^2\sum_{i,j=1}^n \partial_i(\varphi\sigma a_{ij})w(\partial_j w)\,\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
& - \int_{\partial D} s\lambda^2 \sum_{i,j=1}^{n} \varphi\sigma\, a_{ij} w(\partial_j w)\nu_i \,\mathrm{d}s \\
& \geq \int_D s\lambda^2 \varphi\sigma \sum_{i,j=1}^{n} a_{ij}(\partial_i w)(\partial_j w)\,\mathrm{d}x - C\int_D s\lambda^3\varphi|\nabla w||w|\,\mathrm{d}x \\
& - C\int_{\partial D} s\lambda^2\varphi|w||\nabla w|\,\mathrm{d}s; \qquad \text{(A9)}
\end{aligned}
$$

$$
I_3 = -\int_D s^3\lambda^4\varphi^3\sigma^2 w^2\,\mathrm{d}x; \qquad \text{(A10)}
$$

$$
|I_4| \leq C\left|\int_D (s\lambda^2\varphi)\,(s\lambda^2\varphi\sigma w^2)\,\mathrm{d}x\right| \leq C\int_D s^2\lambda^4\varphi^2 w^2\,\mathrm{d}x. \qquad \text{(A11)}
$$

Hence, by (A7)–(A11) we obtain

$$
\begin{aligned}
& \int_D s\lambda^2\varphi\sigma \sum_{i,j=1}^{n} a_{ij}(\partial_i w)(\partial_j w)\,\mathrm{d}x - \int_D s^3\lambda^4\varphi^3\sigma^2 w^2\,\mathrm{d}x \\
& \leq C\int_D |fe^{s\varphi} s\lambda^2\varphi\sigma w|\,\mathrm{d}x + C\int_D s^2\lambda^4\varphi^2 w^2\,\mathrm{d}x + C\int_D s\lambda^3\varphi|\nabla w||w|\,\mathrm{d}x \\
& + C\int_{\partial D} (s^2\lambda^3\varphi^2 w^2 + s\lambda^2\varphi|w||\nabla w|)\,\mathrm{d}s. \qquad \text{(A12)}
\end{aligned}
$$

Since

$$
s\lambda^3\varphi|\nabla w||w| = (s\lambda^2\varphi|w|)(\lambda|\nabla w|) \leq (1/2)s^2\lambda^4\varphi^2 w^2 + (1/2)\lambda^2|\nabla w|^2,
$$

we have

$$
\int_D s\lambda^3\varphi|\nabla w||w|\,\mathrm{d}x \leq (1/2)\int_D (s^2\lambda^4\varphi^2 w^2 + \lambda^2|\nabla w|^2)\,\mathrm{d}x. \qquad \text{(A13)}
$$

Furthermore, using the inequalities

$$
\begin{aligned}
s\lambda^2\varphi|w||\nabla w| &= (s^{1/2}\lambda^{1/2}\varphi^{1/2}|\nabla w|)(s^{1/2}\lambda^{\frac{3}{2}}\varphi^{1/2}w) \\
&\leq (1/2)s\lambda\varphi|\nabla w|^2 + (1/2)s\lambda^3\varphi w^2, \\
|fe^{s\varphi} s\lambda^2\varphi\sigma w| &\leq (1/2)f^2e^{2s\varphi} + (1/2)s^2\lambda^4\varphi^2\sigma^2 w^2 \\
&\leq (1/2)f^2e^{2s\varphi} + Cs^2\lambda^4\varphi^2 w^2,
\end{aligned}
$$

it follows from (A12) and (A13) that

$$
\begin{aligned}
& \int_D s\lambda^2\varphi\sigma \sum_{i,j=1}^{n} a_{ij}(\partial_i w)(\partial_j w)\,\mathrm{d}x - \int_D s^3\lambda^4\varphi^3\sigma^2 w^2\,\mathrm{d}x \\
& \leq C\int_D f^2e^{2s\varphi}\,dx + C\int_D s^2\lambda^4\varphi^2 w^2\,\mathrm{d}x + C\int_D \lambda^2|\nabla w|^2\,\mathrm{d}x \\
& + C\int_{\partial D} (s\lambda\varphi|\nabla w|^2 + (s\lambda^3\varphi + s^2\lambda^3\varphi^2)w^2)\,\mathrm{d}s. \qquad \text{(A14)}
\end{aligned}
$$

End of the proof. Multiplying (A14) by two, adding the resulting expression to (A6), and making use of (3) and the relation $\sigma_0 \equiv \inf_{(x,t)\in Q}\sigma(x,t) > 0$, we obtain

$$
\begin{aligned}
& \int_D s^3\lambda^4\varphi^3\sigma_0^2 w^2\,\mathrm{d}x + \int_D s\lambda^2\varphi|\nabla w|^2\,\mathrm{d}x \\
& \leq C\int_D f^2e^{2s\varphi}\,\mathrm{d}x + C\int_D (s\lambda\varphi + \lambda^2)|\nabla w|^2\,\mathrm{d}x \\
& + C\int_D (s^3\lambda^3\varphi^3 + s^2\lambda^4\varphi^2)w^2\,\mathrm{d}x \\
& + C\int_{\partial D} (s\lambda\varphi|\nabla w|^2 + (s^3\lambda^3\varphi^3 + s^2\lambda^3\varphi^2 + s\lambda^3\varphi)w^2)\,\mathrm{d}s. \qquad \text{(A15)}
\end{aligned}
$$

Therefore, taking $\lambda > 0$ and $s > 0$ sufficiently large, we can absorb the second and the third terms on the right-hand side of (A15) into the left-hand side. Consequently, it follows that

$$\begin{aligned}&\int_D s^3\lambda^4\varphi^3 w^2\,\mathrm{d}x + \int_D s\lambda^2\varphi|\nabla w|^2\,\mathrm{d}x\\&\quad\le C\int_D f^2 e^{2s\varphi}\,\mathrm{d}x + C\int_{\partial D}(s\lambda\varphi|\nabla w|^2 + s^3\lambda^3\varphi^3 w^2)\,\mathrm{d}s.\end{aligned}$$

Noting $w = ue^{s\varphi}$, we have

$$\begin{aligned}&\int_D (s\lambda^2\varphi|\nabla u|^2 + s^3\lambda^4\varphi^3 u^2)e^{2s\varphi}\,\mathrm{d}x\\&\quad\le C\int_D f^2 e^{2s\varphi}\,\mathrm{d}x\mathrm{d} + Ce^{C(\lambda)s}\int_{\partial D}(|\nabla u|^2 + u^2)\,\mathrm{d}s,\end{aligned}$$

which finishes the proof of the Carleman estimate. ■