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INVERSE TIME-HARMONIC ELECTROMAGNETIC SCATTERING FROM COATED POLYHEDRAL SCATTERERS WITH A SINGLE FAR-FIELD PATTERN*

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Abstract. It is proved that a convex polyhedral scatterer of impedance type can be uniquely determined by the electric far-field pattern of a nonvanishing incident field. The incoming wave is allowed to be an electromagnetic plane wave, a vector Herglotz wave function, or a point source wave incited by some magnetic dipole. Our proof relies on the reflection principle for Maxwell's equations with the impedance (or Leontovich) boundary condition enforcing on a hyperplane. We prove that it is impossible to analytically extend the total field across any vertex of the scatterer. This leads to a data-driven inversion scheme for imaging an arbitrary convex polyhedron.

Key words. uniqueness, inverse electromagnetic scattering, polyhedral scatterer, reflection principle, impedance boundary condition, single incident wave

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1. Introduction and main result.

The propagation of time-harmonic electromagnetic waves in a homogeneous isotropic medium in $\mathbb{R}^3$ is modeled by the Maxwell's equations

$$\nabla \times E(x) - ikH(x) = 0,\ \nabla \times H(x) + ikE(x) = 0 \qquad \text{for } x \in \mathbb{R}^3, \tag{1.1}$$

where E and H represent the electric and magnetic fields, respectively, and $k > 0$ is known as the wave number. Let E^{in} and H^{in} satisfying (1.1) denote the incident electric and magnetic fields, respectively. Consider the scattering of given incoming waves E^{in} and H^{in} from a convex polyhedral scatterer $D \subset \mathbb{R}^3$ coated by a thin dielectric layer, which can be modeled by the impedance (or Leontovich) boundary value problem of the Maxwell's equations (1.1) in $\mathbb{R}^3\backslash\overline{D}$. Then the total fields $E = E^{in} + E^{sc}$, $H = H^{in} + H^{sc}$, where E^{sc} and H^{sc} denote the scattered fields, are governed by the following set of equations:

$$\nabla \times E - ikH = 0, \quad \nabla \times H + ikE = 0 \ \text{ in } \mathbb{R}^3\backslash\overline{D}, \tag{1.2}$$

$$E = E^{in} + E^{sc},\ H = H^{in} + H^{sc} \text{ in } \mathbb{R}^3\backslash\overline{D}, \tag{1.3}$$

$$\lim_{|x|\to\infty} (H^{sc} \times x - |x|E^{sc}) = 0, \tag{1.4}$$

$$\nu \times (\nabla \times E) + i\lambda\nu \times (\nu \times E) = 0 \text{ on } \partial D, \tag{1.5}$$

where ν denotes the outward unit normal to ∂D and the impedance coefficient $\lambda > 0$ is supposed to be a constant. Equation (1.4) is known as the Silver–Müller radiation

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condition and is uniform in all directions $\widehat{x} := x/|x|$. For the existence and uniqueness of the solution (E, H) to the forward system (1.2)–(1.5), we refer the reader to [10] when ∂D is C^2-smooth and to [4, 6] when ∂D is Lipschitz with connected exterior. Moreover, the Silver–Müller radiation condition (1.4) ensures that scattered fields E^{sc} and H^{sc} satisfy the following asymptotic behavior (see [10]):

$$E^{sc}(x) = \frac{e^{ik|x|}}{|x|}\left(E^{\infty}(\widehat{x}) + O\left(\frac{1}{|x|}\right)\right) \text{ as } |x| \to \infty, \tag{1.6}$$

$$H^{sc}(x) = \frac{e^{ik|x|}}{|x|}\left(H^{\infty}(\widehat{x}) + O\left(\frac{1}{|x|}\right)\right) \text{ as } |x| \to \infty, \tag{1.7}$$

where the vector fields E^{∞} and H^{∞} defined on the unit sphere $\mathbb{S}^2$ are called the electric and magnetic far-field patterns of the scattered waves E^{sc} and H^{sc}, respectively. It is well known that E^{∞} and H^{∞} are analytic functions with respect to the observation direction $\widehat{x} \in \mathbb{S}^2$ and satisfy the following relations:

$$H^{\infty} = \nu \times E^{\infty}, \ \ \nu \cdot E^{\infty} = \nu \cdot H^{\infty} = 0, \tag{1.8}$$

where ν denotes the unit normal vector to the unit sphere $\mathbb{S}^2$.

Given the incoming wave (E^{in}, H^{in}) and the scatterer $D \subset \mathbb{R}^3$, the direct problem arising from electromagnetic scattering is to find the scattered fields (E^{sc}, H^{sc}) and their far-field patterns. The inverse problem to be considered in this paper consists of determining the location and shape of D from knowledge of the far-field patterns (E^{∞}, H^{∞}). We assume that the incident fields E^{in} and H^{in} are nonvanishing vector fields which are solutions to the Maxwell's equations (1.1) in a neighboring area of the obstacle D. For instance, one can take the incident fields E^{in} and H^{in} to be one among the following:

1. **Plane waves:**

$$E^{in}(x, d, p) = pe^{ikx\cdot d}, \qquad H^{in}(x, d, p) = (d \times p)e^{ikx\cdot d}, \tag{1.9}$$

where $d \in \mathbb{S}^2$ is known as the incident direction and $p \in \mathbb{S}^2$ with $p \perp d$ is known as the polarization direction.

2. **Point source waves:**

$$E^{in}(x) = \nabla_x \times (\Phi(x, y)\vec{a}), \ \ H^{in}(x) = \frac{1}{ik}\nabla \times E^{in}(x), \ \ x \neq y, \tag{1.10}$$

where $\vec{a}$ is a constant vector and $\Phi(x, y) := \frac{1}{4\pi}\frac{e^{ik|x-y|}}{|x-y|}$, $x \neq y$ is the fundamental solution to $\Delta + k^2$ in $\mathbb{R}^3$. $\left(E^{in}, H^{in}\right)$ given by (1.10) represent the electromagnetic field generated by a magnetic dipole located at y, and they solve the Maxwell's equations (1.1) for $x \neq y$.

3. **Electromagnetic Herglotz pairs:**

$$E^{in}(x) = \int_{\mathbb{S}^2} e^{ikx\cdot d}a(d)ds(d), \ \ H^{in}(x) = \frac{1}{ik}\nabla \times E^{in}(x), \ x \in \mathbb{R}^3, \tag{1.11}$$

where the square integrable tangential field $a \in L_t^2(\mathbb{S}^2)^3$ is known as the Herglotz kernel of E^{in} and H^{in}.

The present article is concerned with a uniqueness result of determining the convex polyhedral scatterer D appearing in the system of (1.2)–(1.5) from the knowledge of a single electric far-field pattern $E^\infty(\widehat{x})$ measured over all observation directions $\widehat{x} \in \mathbb{S}^2$. We mention that in case of plane wave incidence, the incident direction $d \in \mathbb{S}^2$, the polarization direction $p \in \mathbb{S}^2$, and the wave number $k > 0$ are all fixed.

We state our main result as follows.

THEOREM 1.1. *Let D_1 and D_2 be two convex polyhedral scatterers of impedance type. Given an incident field (E^{in}, H^{in}) as mentioned above, we denote by E_j^∞ ($j = 1, 2$) the electric far-field patterns of the scattering problem* (1.2)–(1.5) *when $D = D_j$. Then the relation*

$$E_1^\infty(\widehat{x}) = E_2^\infty(\widehat{x}) \quad \text{for all } \widehat{x} \in \mathbb{S}^2 \tag{1.12}$$

implies that $D_1 = D_2$.

It is widely open how to uniquely determine the shape of a general impenetrable/penetrable scatterer using a single far-field pattern. As in the acoustic case [1, 2, 7, 27], quite limited progress has also been made in inverse time-harmonic electromagnetic scattering. To the best of our knowledge, global uniqueness with a single measurement datum is proved only for perfectly conducting obstacles with restrictive geometric shapes, such as balls [26] and convex polyhedrons [10, Chapter 7.1]. Without the convexity condition, it was shown in [29] that a *general* perfect polyhedral conductor (the closure of which may contain screens) can be uniquely determined by the far-field pattern for plane wave incidence with one direction and two polarizations. We shall prove Theorem 1.1 by using the reflection principle for Maxwell's equations with the impedance boundary condition enforcing on a hyperplane. It seems that such a reflection principle has not been studied in prior works, although the corresponding principle under the perfectly conducting boundary condition is well known in optics (see, e.g., [29]). Theorem 1.1 carries over to perfectly conducting polyhedrons with a single far-field pattern (see Corollary 4.5) and thus improves the acoustic uniqueness result for impedance scatterers [8] where two incident directions were used. It is also worth mentioning other works in the literature related to reflection principles for the Helmholtz and Navier equations together with their applications to uniqueness in inverse acoustic and elastic scattering [1, 7, 9, 13, 14, 15, 28]. We believe that the reflection principle, as a special case of unique continuation, provides a powerful tool for gaining new insights into inverse scattering problems. More remarks concerning our uniqueness proof will be concluded in section 4.2.

In the second part of this paper, we shall propose a novel noniterative scheme for imaging an arbitrary convex polyhedron from a single electric far-field pattern. The linear sampling method in inverse electromagnetic scattering [3, 4, 5, 6] was earlier studied with an infinite number of plane waves at a fixed energy. We are mostly motivated by the uniqueness proof of Theorem 1.1 (see also Corollary 4.4) and the one-wave factorization method in inverse elastic and acoustic scattering [15, 18]. The proposed scheme is essentially a domain-defined sampling approach, requiring no forward solvers. Promising features of our imaging scheme are summarized as follows.

(i) It requires lower computational cost and only a single measurement datum. The proposed domain-defined indicator function involves only inner product calculations. Since the number of sampling variables is comparable with the original linear sampling method and factorization method [3, 10, 22], the computational cost is not heavier than the aforementioned pointwise-defined sampling methods. (ii) It can be interpreted as a data-driven approach because it relies on measurement data corresponding to a priori given scatterers (which are also called test domains in the literature or samples in the terminology of learning theory and data science). There is a variety of choices on the shape and physical properties of these samples, giving rise to quite "rich" a priori sample data in addition to the measurement data of the unknown target. In this paper, we choose perfectly conducting balls (or impedance balls) as test domains because the spectra of the resulting far-field operator admit explicit representations. However, these test domains can also be chosen as any other convex penetrable and impenetrable scatterers, provided the classical factorization scheme for imaging this test domain can be verified using all incident and polarization directions. We refer the reader to [23] for the factorization method applied to inverse electromagnetic medium scattering problems. (iii) It provides a necessary and sufficient criterion for imaging convex polyhedrons (see Theorem 5.2) with a single incoming wave. We prove that the wave fields cannot be analytic around any vertex of D (see Corollary 4.4), excluding the possibility of analytical extension across a vertex. Some other domain-defined sampling approaches, such as the range test approach [24, 25] and the one-wave no-response test [31, 32], usually preassume such extensions, leading to a sufficient condition for imaging general targets. Our approach is closest to the no-response test for reconstructing perfectly conducting polyhedral scatterers with a few incident plane waves [33] and is comparable with the one-wave enclosure method by Ikehata [20, 21] for capturing singular points of ∂D. Detailed discussions on the issue of analytic continuation tests can be found in the monograph [36, Chapter 15] (see also [17]). If ∂D contains no singular points, only partial information of D can be numerically recovered; see [30], where the linear sampling method with a single far-field pattern was tested.

We organize the article as follows. In section 2, we prove Theorem 1.1 when the incident fields are given by (1.9). In section 3, we state and prove the reflection principle for Maxwell's equations with the impedance boundary condition on a hyperplane in $\mathbb{R}^3$ (see Theorem 3.1). Using this reflection principle, we prove in section 4.1 the main uniqueness results for electromagnetic Herglotz waves and point source waves. The data-driven reconstruction scheme will be described in section 5.

2. Uniqueness with a single plane wave. In this section, we prove Theorem 1.1, when the incident fields (E^{in}, H^{in}) are given by

$$E^{in}(x,d,p) = pe^{ikx\cdot d}, \qquad H^{in}(x,d,p) = (d\times p)e^{ikx\cdot d},$$

where $d, p \in \mathbb{S}^2$ satisfying $p \perp d$ and $k > 0$ are all fixed. Now recall from (1.12) that $E_1^\infty(\widehat{x}) = E_2^\infty(\widehat{x})$ for all $\widehat{x} \in \mathbb{S}^2$. Using Rellich's lemma (see [10]), we get

$$E_1 = E_2 \text{ and } H_1 = H_2 \text{ in } \mathbb{R}^3 \setminus \left(\overline{D_1 \cup D_2}\right).$$

Assuming that $D_1 \neq D_2$, we shall prove the uniqueness by deriving a contradiction. By the convexity of D_1 and D_2, we may assume that there exists a vertex O of ∂D_1 and a neighborhood V_O of O such that $V_O \cap \overline{D_2} = \emptyset$. Next, using the impedance boundary condition of E_1 on ∂D_1 and $E_1 = E_2$ in $\mathbb{R}^3 \setminus \left(\overline{D_1 \cup D_2}\right)$, we have that $\nu \times (\nabla \times E_2) + i\lambda\nu \times (\nu \times E_2) = 0$ on $V_O \cap \partial D_1$. Since D_1 is a convex polyhedron, there exists m ($m \geq 3$) convex polygonal faces Λ_j ($j = 1, 2, \ldots, m$) of ∂D_1, whose closures meet at O and which can be analytically extended to infinity in $\mathbb{R}^3 \backslash \overline{D}_2$; for example, see Figure 1, where $m = 4$ (left) and $m = 3$ (right). Denote by $\widetilde{\Pi} \supseteq \Lambda_j$ the maximum extension of Λ_j in $\mathbb{R}^3 \backslash \overline{D}_2$. Using the real analyticity of E_2 in $\mathbb{R}^3 \backslash \overline{D}_2$ together with the fact that $\lambda > 0$ is a constant, we conclude that E_2 satisfies the impedance boundary condition on $\widetilde{\Pi}_j$. Recalling (1.4) and (1.6), we have $\lim_{|x|\to\infty} |\nabla \times E_2^{sc}| = 0, \lim_{|x|\to\infty} |E_2^{sc}| = 0.$

Hence,

$$\nu \times \left(\nabla \times E^{in}\right) + i\lambda \nu \times \left(\nu \times E^{in}\right) = 0 \text{ on } \widetilde{\Pi}_j, \quad j = 1, 2, \ldots, m. \tag{2.1}$$

By (1.9), it follows that

$$ik\nu \times (d \times p) + i\lambda \nu \times (\nu \times p) = 0 \tag{2.2}$$

holds for any outward unit normal ν to $\widetilde{\Pi}_j$. Without loss of generality, we suppose that $p = \mathbf{e}_1$, $d \times p = \mathbf{e}_2$, $\nu = c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3$ with $c_1^2 + c_2^2 + c_3^2 = 1$, where $\mathbf{e}_j \in \mathbb{S}^2$ ($j = 1, 2, 3$) denotes the Cartesian coordinates in $\mathbb{R}^3$. By (2.2), simple calculations show that

$$-\left(kc_3 + \lambda(c_2^2 + c_3^2)\right)\mathbf{e}_1 + \lambda c_1 c_2 \mathbf{e}_2 + (kc_1 + \lambda c_1 c_3)\,\mathbf{e}_3 = 0.$$

This gives us the equations of c_j,

$$kc_3 = -\lambda\left(c_2^2 + c_3^2\right), \quad \lambda c_1 c_2 = 0, \quad kc_1 = -\lambda c_1 c_3,$$

which have the following solutions for $\nu = (c_1, c_2, c_3) \in \mathbb{S}^2$:

$$\begin{aligned} \nu &= (0, 0, -1) && \text{if} \quad k = \lambda, \\ \nu &= \left(\pm\sqrt{1 - (k/\lambda)^2}, 0, -k/\lambda\right) && \text{if} \quad k < \lambda, \\ \nu &= \left(0, \pm\sqrt{1 - (\lambda/k)^2}, -\lambda/k\right) && \text{if} \quad k > \lambda. \end{aligned}$$

Hence, the relation (2.2) cannot hold for three linearly independent unit normal vectors ν. This contradiction implies that $D_1 = D_2$.

Remark 2.1. The above uniqueness proof with a single plane wave cannot be applied to the Helmholtz equation in two dimensions under the impedance boundary condition $\partial_\nu u + i\lambda u = 0$. In the two-dimensional case, we deduce a corresponding relation $k\nu \cdot d + \lambda = 0$, which holds for only two linearly independent unit normal vectors if $D_1 \neq D_2$. However, this cannot lead to a contradiction when $k > \lambda$. It was proved in [8] that the far-field patterns of two incident directions uniquely determine a convex polygonal obstacle of impedance type.

The above proof relies on the form of electromagnetic plane waves and therefore it is not applicable to the incident fields given by electromagnetic point source waves and vector Herglotz wave functions. For these kinds of incident fields, we shall apply the reflection principle for Maxwell's equations to prove Theorem 1.1; see section 3 and subsection 4.1 below.

3. Reflection principle for Maxwell's equations. Let $\Omega \subseteq \mathbb{R}^3$ be an open connected set which is symmetric with respect to a plane Π in $\mathbb{R}^3$, and we define by $\gamma := \Omega \cap \Pi$. Denote by Ω^+ and Ω^- the two symmetric parts of Ω which are divided by Π and by R_Π the reflection operator about Π; that is, if $x \in \Omega^\pm$, then $R_\Pi x \in \Omega^\mp$ for $x = (x_1, x_2, x_3) \in \Omega$. Throughout this article, Ω will be assumed in such a way that any line segment with end points in Ω and intersected with Π by the angle $\pi/2$ lies completely in Ω. In other words, the projection of any line segment in Ω onto the hyperplane Π is a subset of γ. This geometrical condition was also used in [12], where the reflection principle for the Helmholtz equation with the impedance boundary condition was verified in $\mathbb{R}^n$ $(n \geq 2)$. Now consider the time-harmonic Maxwell's equations with the impedance boundary condition by

$$\nabla \times E - ikH = 0, \quad \nabla \times H + ikE = 0 \qquad \text{in} \quad \Omega^+, \tag{3.1}$$

$$\nu \times (\nabla \times E) + i\lambda \nu \times (\nu \times E) = 0 \qquad \text{on} \quad \gamma \subset \Pi. \tag{3.2}$$

It is well known from Theorem 6.4 in [10] that a solution (E, H) of (3.1) satisfies the vectorial Helmholtz equations with the divergence-free condition:

$$\Delta E + k^2 E = 0, \ \Delta H + k^2 H = 0, \quad \nabla \cdot E = \nabla \cdot H = 0. \tag{3.3}$$

Since (3.3) and (3.1) are rotational invariant, without loss of generality we can assume that the plane Π mentioned above coincides with the ox_1x_2-plane, i.e., $\Pi = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : \ x_3 = 0\}$. Consequently, we have $\nu = \mathbf{e}_3 := (0, 0, 1)^T$ and $R_\Pi x = (x_1, x_2, -x_3)$. In this section, we study the reflection principle for solutions to the Maxwell's equation satisfying the impedance boundary condition on γ. Our aim is to extend the solution (E, H) of (3.1) from Ω^+ to Ω^- by an analytical formula. The reflection principle is stated as follows.

THEOREM 3.1. *Let* $\Pi := \{(x', x_3) \in \mathbb{R}^3 : x_3 = 0\}$ *with* $x' := (x_1, x_2)$, $\gamma \subset \Omega \cap \Pi$, *and* $\Omega^\pm := \{(x_1, x_2, x_3) \in \Omega : \ \pm x_3 > 0\}$. *Assume that* (E, H) *satisfies* (3.1) *with the boundary condition* (3.2). *Then* (E, H) *can be analytically extended to* Ω^- *as a solution to* (3.1). *Moreover, the extended electric field* $\widetilde{E} := (\widetilde{E}_1, \widetilde{E}_2, \widetilde{E}_3)^T$ *is given explicitly by*

$$\widetilde{E}(x', x_3) = \begin{cases} E(x) & \text{if } x \in \Omega^+ \cup \gamma, \\ \mathcal{D}E(x', -x_3) & \text{if } x \in \Omega^-, \end{cases} \tag{3.4}$$

where the operator $\mathcal{D}E := ((\mathcal{D}E)_1, (\mathcal{D}E)_2, (\mathcal{D}E)_3)^T$ *is defined by*

$$(\mathcal{D}E)_3(x) := E_3(x', x_3) - \frac{2k^2}{i\lambda} \int_0^{x_3} e^{\frac{k^2}{i\lambda}(s - x_3)} E_3(x', s) \mathrm{d}s \quad \text{for } x \in \Omega^+, \tag{3.5}$$

$$\begin{aligned} (\mathcal{D}E)_j(x', x_3) := {} & E_j(x', x_3) + 2i\lambda \int_0^{x_3} e^{-i\lambda(s - x_3)} E_j(x', s) \mathrm{d}s \\ & + \frac{2\lambda^2}{k^2 - \lambda^2} \int_0^{x_3} e^{-i\lambda(s - x_3)} \partial_j E_3(x', s) \mathrm{d}s - \frac{2k^2}{k^2 - \lambda^2} \int_0^{x_3} e^{\frac{k^2}{i\lambda}(s - x_3)} \partial_j E_3(x', s) \mathrm{d}s \end{aligned} \tag{3.6}$$

for $j = 1, 2$ *and* $x \in \Omega^+$.

Obviously, the extension operator $(\mathcal{D}E)_3$ only relies on E_3 in Ω^+, whereas $(\mathcal{D}E)_j$ $(j = 1, 2)$ depends on both E_j and E_3. The formula given by (3.4) is a "non-point-to-point" reflection formula which is in contrast with the "point-to-point" reflection formula for the Maxwell's equations with the Dirichlet boundary condition (see [29]). As $\lambda \to \infty$, the impedance boundary condition will be reduced to the Dirichlet boundary condition $\nu \times E = 0$ on Π. It is easy to observe that $(\mathcal{D}E)_3 \to E_3$ and that, by applying integration by parts, $(\mathcal{D}E)_j \to -E_j$ for $j = 1, 2$ as λ tends to infinity. Hence, the reflection formula (3.4) becomes $\widetilde{E}(x) = -R_\Pi E(R_\Pi x)$ for $x \in \Omega$, which is valid for any symmetric domain with respect to $\Pi = \{x : x_3 = 0\}$.

Before going to the proof of Theorem 3.1, we first state the reflection principle for the Helmholtz equation with an impedance boundary condition. The result in the following Lemma 3.2 has already been proved in [12].

LEMMA 3.2 (see [12]). *Let Ω, Π, γ, and $\Omega^{\pm}$ be defined as in Theorem 3.1. If u is a solution to the boundary value problem of the Helmholtz equation*

$$\Delta u + k^2 u = 0 \quad in \quad \Omega^+, \qquad \partial_\nu u + i\lambda u = 0 \quad on \quad \gamma, \tag{3.7}$$

then u can be extended from Ω^+ to Ω as a solution to the Helmholtz equation, with the extended solution $\widetilde{u}$ given by the formula $\widetilde{u} := u$ in $\Omega^+ \cup \gamma$ and

$$\widetilde{u}(x) := u(x_1, x_2, -x_3) + 2i\lambda e^{-i\lambda x_3} \int_0^{-x_3} e^{-i\lambda s} u(x_1, x_2, s) \mathrm{d}s \quad in \quad \Omega^-. \tag{3.8}$$

Our proof of Theorem 3.1 is essentially motivated by the proof of Lemma 3.2.

3.1. Proof of Theorem 3.1. From (3.1), we deduce that E is a solution to

$$\nabla \times (\nabla \times E) - k^2 E = 0 \ \text{ in } \Omega^+, \quad \mathbf{e}_3 \times (\nabla \times E) + i\lambda \mathbf{e}_3 \times (\mathbf{e}_3 \times E) = 0 \ \text{ on } \gamma. \tag{3.9}$$

Define the vector function F and the scalar function V by

$$\begin{aligned} F &:= \mathbf{e}_3 \times (\nabla \times E) + i\lambda \mathbf{e}_3 \times (\mathbf{e}_3 \times E) =: (F_1, F_2, 0)^T \quad \text{in} \quad \Omega^+, \\ F_1 &= \partial_1 E_3 - \partial_3 E_1 - i\lambda E_1, \quad F_2 = \partial_2 E_3 - \partial_3 E_2 - i\lambda E_2, \\ V &:= \frac{\partial_1 F_1 + \partial_2 F_2}{i\lambda} = \frac{\partial_1^2 E_3 - \partial_{31}^2 E_1 - i\lambda \partial_1 E_1 + \partial_2^2 E_3 - \partial_{32}^2 E_2 - i\lambda \partial_2 E_2}{i\lambda}. \end{aligned} \tag{3.10}$$

Now using $\Delta E + k^2 E = 0$ and $\nabla \cdot E = 0$ in Ω^+, we have

$$V = \frac{-\partial_3^2 E_3 - k^2 E_3 - \partial_3 \left(\partial_1 E_1 + \partial_2 E_2\right) - i\lambda \left(\partial_1 E_1 + \partial_2 E_2\right)}{i\lambda} = \partial_3 E_3 - \frac{k^2}{i\lambda} E_3.$$

Since $F_1 = F_2 = 0$ on $\gamma \subseteq \{x \in \mathbb{R}^3 : \ x_3 = 0\}$, we get $\partial_1 F_1 = \partial_2 F_2 = 0$ on γ and thus

$$\Delta E_3 + k^2 E_3 = 0 \quad \text{in} \quad \Omega^+, \quad \partial_3 E_3 - k^2/(i\lambda)\, E_3 = 0 \quad \text{on} \quad \gamma. \tag{3.11}$$

Applying Lemma 3.2, we can extend E_3 from Ω^+ to Ω by $\widetilde{E}_3 := E_3$ in $\Omega^+ \cup \gamma$ and

$$\widetilde{E}_3(x) := E_3(x', -x_3) - \frac{2k^2}{i\lambda} e^{\frac{k^2}{i\lambda} x_3} \int_0^{-x_3} e^{\frac{k^2}{i\lambda} s} E_3(x', s) \mathrm{d}s \text{ in } \Omega^-,$$

which gives the extension formula for E_3.

To find the extension formula for E_j $(j = 1, 2)$, we observe that F_j $(j = 1, 2)$ given by (3.10) satisfy the Helmholtz equation with the Dirichlet boundary condition,

$$\Delta F_j + k^2 F_j = 0 \text{ in } \Omega^+, \qquad F_j = 0 \text{ on } \gamma.$$

Applying the reflection principle with the Dirichlet boundary condition (see [12]), we can extend F_j through $\widetilde{F}_j := F_j$ in $\Omega^+ \cup \gamma$ and $\widetilde{F}_j(x) := -F_j(x', -x_3)$ in Ω^-. As done for the Helmholtz equation, we will derive the extension formula for E_j for $j = 1, 2$ by considering the boundary value problem of the ODE (cf. (3.10))

$$\partial_3 \widetilde{E}_j + i\lambda \widetilde{E}_j - \partial_j \widetilde{E}_3 = -\widetilde{F}_j \quad \text{in} \quad \Omega, \qquad \widetilde{E}_j = E_j \quad \text{on} \quad \gamma,$$

where $\widetilde{E}_j$ $(j = 1, 2)$ denote the extended functions. Multiplying the above equation by $e^{i\lambda x_3}$ and integrating between 0 to x_3, we have

$$\int_0^{x_3} e^{i\lambda s} \left(\partial_s \widetilde{E}_j(x', s) + i\lambda \widetilde{E}_j(x', s)\right) \mathrm{d}s - \int_0^{x_3} e^{i\lambda s} \partial_j \widetilde{E}_3(x', s) \mathrm{d}s = -\int_0^{x_3} e^{i\lambda s} \widetilde{F}_j(x', s) \mathrm{d}s,$$

which gives us

$$\widetilde{E}_j(x) = e^{-i\lambda x_3}E_j(x',0) + \int_0^{x_3} e^{i\lambda(s-x_3)}\partial_j \widetilde{E}_3(x',s)\mathrm{d}s - \int_0^{x_3} e^{i\lambda(s-x_3)}\widetilde{F}_j(x',s)\mathrm{d}s.$$

Since $\widetilde{F}_j = F_j$ and $\widetilde{E}_3 = E_3$ in $\in \Omega^+$, the above equation can be rewritten as

$$\begin{aligned}\widetilde{E}_j(x) = e^{-i\lambda x_3}E_j(x',0) + e^{-i\lambda x_3}\int_0^{x_3} e^{i\lambda s}\partial_j E_3(x',s)\mathrm{d}s \\ - e^{-i\lambda x_3}\int_0^{x_3} e^{i\lambda s}\left(\partial_j E_3(x',s) - \partial_s E_j(x',s) - i\lambda E_j(x',s)\right)\mathrm{d}s,\end{aligned}$$

which can be proved to be identical with E_j in Ω^+ by applying integration by parts. Next, we want to simplify the expression of $\widetilde{E}_j(x)$ in Ω^-. Using the expression for $\widetilde{F}_j$ and $\widetilde{E}_3$ (see (3.5)), we obtain

$$\begin{aligned}&\widetilde{E}_j(x) \\ &= e^{-i\lambda x_3}E_j(x',0) + \int_0^{x_3} e^{i\lambda(s-x_3)}\left(\partial_j E_3(x',-s) + \partial_s E_j(x',-s) - i\lambda E_j(x',-s)\right)\mathrm{d}s \\ &\quad + \int_0^{x_3} e^{i\lambda(s-x_3)}\partial_j E_3(x',-s)\mathrm{d}s - \frac{2k^2}{i\lambda}\int_0^{x_3} e^{i\lambda(s-x_3)}\left(\int_0^{-s} e^{\frac{k^2}{i\lambda}(t+s)}\partial_j E_3(x',t)\mathrm{d}t\right)\mathrm{d}s.\end{aligned}$$

This gives

$$\begin{aligned}&\widetilde{E}_j(x) \\ &= E_j(x',-x_3) + 2i\lambda\int_0^{-x_3} e^{-i\lambda(s+x_3)}E_j(x',s)\mathrm{d}s - 2\int_0^{-x_3} e^{-i\lambda(s+x_3)}\partial_j E_3(x',s)\mathrm{d}s \\ &\quad + \frac{2k^2}{i\lambda}\int_0^{x_3} e^{i\lambda(s-x_3)}\int_0^{s} e^{-\frac{k^2}{i\lambda}(t-s)}\partial_j E_3(x',-t)\mathrm{d}t\mathrm{d}s.\end{aligned}$$

Changing the order of integration in the last term of the previous equation, we obtain

$$\begin{aligned}\widetilde{E}_j(x) &= E_j(x',-x_3) + 2i\lambda\int_0^{-x_3} e^{-i\lambda(s+x_3)}E_j(x',s)\mathrm{d}s - 2\int_0^{-x_3} e^{-i\lambda(s+x_3)}\partial_j E_3(x',s)\mathrm{d}s \\ &\quad + \frac{2k^2}{i\lambda}e^{-i\lambda x_3}\int_{t=0}^{t=x_3} e^{-\frac{k^2}{i\lambda}t}\partial_j E_3(x',-t)\int_{s=t}^{s=x_3} e^{\frac{k^2-\lambda^2}{i\lambda}s}\mathrm{d}s\mathrm{d}t \\ &= E_j(x',-x_3) + 2i\lambda\int_0^{-x_3} e^{-i\lambda(s+x_3)}E_j(x',s)\mathrm{d}s - 2\int_0^{-x_3} e^{-i\lambda(s+x_3)}\partial_j E_3(x',s)\mathrm{d}s \\ &\quad + \frac{2k^2}{k^2-\lambda^2}\int_0^{x_3} e^{-\frac{k^2}{i\lambda}(s-x_3)}\partial_j E_3(x',-s)\mathrm{d}s - \frac{2k^2}{k^2-\lambda^2}\int_0^{x_3} e^{i\lambda(s-x_3)}\partial_j E_3(x',-s)\mathrm{d}s.\end{aligned}$$

After combining similar terms, we get

$$\begin{aligned}\widetilde{E}_j(x) &= E_j(x',-x_3) + 2i\lambda\int_0^{-x_3} e^{-i\lambda(s+x_3)}E_j(x',s)\mathrm{d}s \\ &\quad + \frac{2\lambda^2}{k^2-\lambda^2}\int_0^{-x_3} e^{-i\lambda(s+x_3)}\partial_j E_3(x',s)\mathrm{d}s - \frac{2k^2}{k^2-\lambda^2}\int_0^{-x_3} e^{\frac{k^2}{i\lambda}(s+x_3)}\partial_j E_3(x',s)\mathrm{d}s.\end{aligned}$$

This proves (3.6). Notice that the right-hand side of $\widetilde{E}_j$ $(j = 1, 2)$ depends on both E_j and E_3 in Ω^+.

In order to show that $\widetilde{E}_j$ given by (3.6) and (3.5) is indeed the required extension formula for E_j, we need to verify that $\Delta\widetilde{E}_j + k^2\widetilde{E}_j = 0$ and $\nabla\cdot\widetilde{E} = 0$ in Ω. For this purpose, we shall proceed with the following three steps.

Step 1. Prove that the Cauchy data of $\widetilde{E}_j$ taking from $\Omega^\pm$ are identical on γ. By Lemma 3.2, this is true for the third component $\widetilde{E}_3$. On the other hand, it is clear from (3.6) that $\widetilde{E}_j$ $(j=1,2)$ are continuous functions in Ω. Therefore, we only need to show that $\partial_3^+\widetilde{E}_j = \partial_3^-\widetilde{E}_j$ on γ, $j=1,2$. Simple calculations show that

$$\partial_3\widetilde{E}_j(x) = \begin{cases} \partial_3 E_j(x',x_3) \quad \text{in } \Omega^+, \\ -\partial_3 E_j(x',-x_3) + 2\lambda^2 e^{-i\lambda x_3}\int_0^{-x_3} e^{-i\lambda s}E_j(x',s)\mathrm{d}s - 2i\lambda E_j(x',-x_3) \\ \quad - \frac{2i\lambda^3}{k^2-\lambda^2}e^{-i\lambda x_3}\int_0^{-x_3} e^{-i\lambda s}\partial_j E_3(x',s)\mathrm{d}s - \frac{2\lambda^2}{k^2-\lambda^2}\partial_j E_3(x',-x_3) \\ - \frac{2k^4}{i\lambda(k^2-\lambda^2)}e^{\frac{k^2}{i\lambda}x_3}\int_0^{-x_3} e^{\frac{k^2}{i\lambda}s}\partial_j E_3(x',s)\mathrm{d}s + \frac{2k^2}{k^2-\lambda^2}\partial_j E_3(x',-x_3) \text{ in } \Omega^-, \end{cases}$$

from which it follows that

$$\partial_3^-\widetilde{E}_j(x',0) = -\partial_3^+ E_j(x',0) - 2i\lambda E_j(x',0) + \frac{2(k^2-\lambda^2)}{k^2-\lambda^2}\partial_j E_3(x',0).$$

Recalling the relation $\partial_j E_3(x',0) - \partial_3^+ E_j(x',0) - i\lambda E_j(x',0) = 0$ for $j=1,2$, we get from the previous equation that $\partial_3^+ E_j = \partial_3^-\widetilde{E}_j$ on γ.

Step 2. Prove that $\Delta\widetilde{E}_j + k^2\widetilde{E}_j = 0$ in Ω for $j=1,2,3$. In view of Step 1, it suffices to verify that $\Delta\widetilde{E}_j + k^2\widetilde{E}_j = 0$ in Ω^- for $j=1,2$. From (3.6), we have

$$\Delta\widetilde{E}_j(x) = \Delta E_j(x',-x_3) + I_1 + I_2 + I_3, \quad x\in\Omega^-, \tag{3.12}$$

where $j=1$ or $j=2$ is fixed and I_k's for $k=1,2,3$ are given by

$$I_1 := 2i\lambda\Delta\left(e^{-i\lambda x_3}\int_0^{-x_3} e^{-i\lambda s}E_j(x',s)\mathrm{d}s\right),$$

$$I_2 := \frac{2\lambda^2}{k^2-\lambda^2}\Delta\left(e^{-i\lambda x_3}\int_0^{-x_3} e^{-i\lambda s}\partial_j E_3(x',s)\mathrm{d}s\right),$$

$$I_3 := -\frac{2k^2}{k^2-\lambda^2}\Delta\left(e^{\frac{k^2}{i\lambda}x_3}\int_0^{-x_3} e^{\frac{k^2}{i\lambda}s}\partial_j E_3(x',s)\mathrm{d}s\right).$$

Using $\Delta E_j + k^2 E_j = 0$ for $j=1,2,3$ in Ω^+ and applying integration by parts, the three terms I_j $(j=1,2,3)$ can be calculated as follows:

$$\begin{aligned} I_1 = &-2i\lambda\partial_3 E_j(x',-x_3) + 2i\lambda e^{-i\lambda x_3}\partial_3 E_j(x',0) + 2\lambda^2 E_j(x',-x_3) - 2\lambda^2 e^{-i\lambda x_3}E_j(x',0) \\ &+ 2i\lambda^3 e^{-i\lambda x_3}\int_0^{-x_3} e^{-i\lambda s}E_j(x',s)\mathrm{d}s - 2i\lambda k^2 e^{-i\lambda x_3}\int_0^{-x_3} e^{-i\lambda s}E_j(x',s)\mathrm{d}s \\ &- 2i\lambda^3 e^{-i\lambda x_3}\int_0^{-x_3} e^{-i\lambda s}E_j(x',s)\mathrm{d}s - 2\lambda^2 E_j(x',-x_3) + 2i\lambda\partial_3 E_j(x',-x_3), \end{aligned}$$

$$
\begin{aligned}
I_2 \;=& -\frac{2\lambda^2}{k^2-\lambda^2}\partial_3\partial_j E_3(x',-x_3) + \frac{2\lambda^2}{k^2-\lambda^2}e^{-i\lambda x_3}\partial_3\partial_j E_3(x',0) - \frac{2i\lambda^3}{k^2-\lambda^2}\partial_j E_3(x',-x_3) \\
&+ \frac{2i\lambda^3}{k^2-\lambda^2}e^{-i\lambda x_3}\partial_j E_3(x',0) + \frac{2\lambda^4}{k^2-\lambda^2}e^{-i\lambda x_3}\int_0^{-x_3} e^{-i\lambda s}\partial_j E_3(x',s)\mathrm{d}s \\
&- \frac{2\lambda^2 k^2}{k^2-\lambda^2}e^{-i\lambda x_3}\int_0^{-x_3} e^{-i\lambda s}\partial_j E_3(x',s)\mathrm{d}s - \frac{2\lambda^4}{k^2-\lambda^2}e^{-i\lambda x_3}\int_0^{-x_3} e^{-i\lambda s}\partial_j E_3(x',s)\mathrm{d}s \\
&+ \frac{2i\lambda^3}{k^2-\lambda^2}\partial_j E_3(x',-x_3) + \frac{2\lambda^2}{k^2-\lambda^2}\partial_3\partial_j E_3(x',-x_3),
\end{aligned}
$$

$$
\begin{aligned}
I_3 =& \frac{2k^2}{k^2-\lambda^2}\partial_3\partial_j E_3(x',-x_3) - \frac{2k^2}{k^2-\lambda^2}e^{\frac{k^2}{i\lambda}x_3}\partial_3\partial_j E_3(x',0) + \frac{2ik^4}{\lambda\,(k^2-\lambda^2)}\partial_j E_3(x',-x_3) \\
&- \frac{2ik^4}{\lambda\,(k^2-\lambda^2)}e^{\frac{k^2}{i\lambda}x_3}\partial_j E_3(x',0) - \frac{2k^6}{\lambda^2\,(k^2-\lambda^2)}e^{\frac{k^2}{i\lambda}x_3}\int_0^{-x_3} e^{\frac{k^2}{i\lambda}s}\partial_j E_3(x',s)\mathrm{d}s \\
&+ \frac{2k^4}{k^2-\lambda^2}e^{\frac{k^2}{i\lambda}x_3}\int_0^{-x_3} e^{\frac{k^2}{i\lambda}s}\partial_j E_3(x',s)\mathrm{d}s + \frac{2k^6}{\lambda^2\,(k^2-\lambda^2)}e^{\frac{k^2}{i\lambda}x_3}\int_0^{-x_3} e^{\frac{k^2}{i\lambda}s}\partial_j E_3(x',s)\mathrm{d}s \\
&- \frac{2ik^4}{\lambda\,(k^2-\lambda^2)}\partial_j E_3(x',-x_3) - \frac{2k^2}{k^2-\lambda^2}\partial_3\partial_j E_3(x',-x_3).
\end{aligned}
$$

Using again the Helmholtz equation $\Delta E_j + k^2 E_j = 0$ in Ω^+ and inserting expressions of I_1, I_2, and I_3 into (3.12), we get

$$
\begin{aligned}
\Delta\widetilde{E}_j(x) = -k^2\Bigg(& E_j(x',-x_3) + 2i\lambda\int_0^{-x_3} e^{-i\lambda(s+x_3)}E_j(x',s)\mathrm{d}s \\
&+ \frac{2\lambda^2}{k^2-\lambda^2}\int_0^{-x_3} e^{-i\lambda(s+x_3)}\partial_j E_3(x',s)\mathrm{d}s \\
&- \frac{2k^2}{k^2-\lambda^2}e^{\frac{k^2}{i\lambda}x_3}\int_0^{-x_3} e^{\frac{k^2}{i\lambda}s}\partial_j E_3(x',s)\mathrm{d}s\Bigg) \\
&+ 2i\lambda e^{-i\lambda x_3}\Big(\partial_3 E_j(x',0) + i\lambda E_j(x',0)\Big) \\
&+ \frac{2\lambda^2}{k^2-\lambda^2}e^{-i\lambda x_3}\partial_j\Big(\partial_3 E_3(x',0) + i\lambda\partial_j E_j(x',0)\Big) \\
&- \frac{2k^2}{k^2-\lambda^2}e^{\frac{k^2}{i\lambda}x_3}\partial_j\left(\partial_3 E_3(x',0) - \frac{k^2}{i\lambda}E_3(x',0)\right).
\end{aligned}
$$

This together with (3.6) and the boundary conditions

$$
\partial_3 E_3(x',0) - \frac{k^2}{i\lambda}E_3(x',0) = 0, \quad \partial_j E_3(x',0) - \partial_3 E_j(x',0) - i\lambda E_j(x',0) = 0
$$

leads to the relation $\Delta\widetilde{E}_j + k^2\widetilde{E}_j = 0$ in Ω^-.

Step 3. Prove that $\nabla\cdot\widetilde{E}=0$ in Ω. It follows from Step 1 that $\widetilde{E}\in C^1(\Omega)$. Hence, we only need to show the divergence-free condition in Ω^-. For $x\in\Omega^-$, we see

$$\begin{aligned}\nabla\cdot\widetilde{E}(x)&=\partial_1E_1(x',-x_3)+\partial_2E_2(x',-x_3)-\partial_3E_3(x',-x_3)\\&\quad+2i\lambda\int_0^{-x_3}e^{-i\lambda(s+x_3)}\left(\partial_1E_1+\partial_2E_2\right)(x',s)\mathrm{d}s\\&\quad+\frac{2\lambda^2}{k^2-\lambda^2}\int_0^{-x_3}e^{-i\lambda(s+x_3)}\left(\partial_1^2E_3+\partial_2^2E_3\right)(x',s)\mathrm{d}s\\&\quad-\frac{2k^2}{k^2-\lambda^2}\int_0^{-x_3}e^{\frac{k^2}{i\lambda}(s+x_3)}\left(\partial_1^2E_3+\partial_2^2E_3\right)(x',s)\mathrm{d}s\\&\quad+\frac{2k^4}{\lambda^2}\int_0^{-x_3}e^{\frac{k^2}{i\lambda}(s+x_3)}E_3(x',s)\mathrm{d}s+\frac{2k^2}{i\lambda}E_3(x',-x_3).\end{aligned}$$

Now using $\nabla\cdot E=0$ and $\Delta E_j+k^2E_j=0$ for $1\le j\le 3$ in Ω^+, we have

(3.13)

$$\begin{aligned}\nabla\cdot\widetilde{E}(x)&=-2\partial_3E_3(x',-x_3)\underbrace{-2i\lambda\int_0^{-x_3}e^{-i\lambda(s+x_3)}\partial_sE_3(x',s)\mathrm{d}s}_{J_1}\\&\quad\underbrace{-\frac{2\lambda^2}{k^2-\lambda^2}\int_0^{-x_3}e^{-i\lambda(s+x_3)}\partial_s^2E_3(x',s)\mathrm{d}s}_{J_2}-\frac{2\lambda^2k^2}{k^2-\lambda^2}\int_0^{-x_3}e^{-i\lambda(s+x_3)}E_3(x',s)\mathrm{d}s\\&\quad+\underbrace{\frac{2k^2}{k^2-\lambda^2}\int_0^{-x_3}e^{\frac{k^2}{i\lambda}(s+x_3)}\partial_s^2E_3(x',s)\mathrm{d}s}_{J_3}+\frac{2k^4}{k^2-\lambda^2}\int_0^{-x_3}e^{\frac{k^2}{i\lambda}(s+x_3)}E_3(x',s)\mathrm{d}s\\&\quad+\frac{2k^4}{\lambda^2}\int_0^{-x_3}e^{\frac{k^2}{i\lambda}(s+x_3)}E_3(x',s)\mathrm{d}s+\frac{2k^2}{i\lambda}E_3(x',-x_3).\end{aligned}$$

Using integration by parts, we can rewrite J_1, J_2, and J_3 as

$$\begin{aligned}J_2&=\frac{2\lambda^2}{k^2-\lambda^2}\partial_3E_3(x',-x_3)-\frac{2\lambda^2}{k^2-\lambda^2}e^{-i\lambda x_3}\partial_3E_3(x',0)+\frac{2i\lambda^3}{k^2-\lambda^2}E_3(x',-x_3)\\&\quad-\frac{2i\lambda^3}{k^2-\lambda^2}e^{-i\lambda x_3}E_3(x',0)-\frac{2\lambda^4}{k^2-\lambda^2}e^{-i\lambda x_3}\int_0^{-x_3}e^{-i\lambda s}E_3(x',s)\mathrm{d}s\\&\quad+\frac{2\lambda^2k^2}{k^2-\lambda^2}e^{-i\lambda x_3}\int_0^{-x_3}e^{-i\lambda s}E_3(x',s)\mathrm{d}s,\\J_1&=2i\lambda E_3(x',-x_3)-2i\lambda e^{-i\lambda x_3}E_3(x',0)-2\lambda^2e^{-i\lambda x_3}\int_0^{-x_3}e^{-i\lambda s}E_3(x',s)\mathrm{d}s,\\J_3&=\frac{2k^2}{k^2-\lambda^2}\partial_3E_3(x',-x_3)-\frac{2k^2}{k^2-\lambda^2}e^{\frac{k^2}{i\lambda}x_3}\partial_3E_3(x',0)+\frac{2ik^4}{\lambda\left(k^2-\lambda^2\right)}E_3(x',-x_3)\\&\quad-\frac{2ik^4}{\lambda\left(k^2-\lambda^2\right)}e^{\frac{k^2}{i\lambda}x_3}E_3(x',0)-\frac{2k^6}{\lambda^2\left(k^2-\lambda^2\right)}e^{\frac{k^2}{i\lambda}x_3}\int_0^{-x_3}e^{\frac{k^2}{i\lambda}s}E_3(x',s)\mathrm{d}s.\end{aligned}$$

Inserting them into (3.13), applying integration by parts, and rearranging terms, we get

$$\nabla \cdot \widetilde{E}(x) = 2e^{-i\lambda x_3}\left[\left(i\lambda + \frac{i\lambda^3}{k^2-\lambda^2}\right)E_3(x',0) + \frac{\lambda^2}{k^2-\lambda^2}\partial_3 E_3(x',0)\right] + \frac{2k^2}{k^2-\lambda^2}e^{\frac{k^2}{i\lambda}x_3}\left(-\partial_3 E_3(x',0) + \frac{k^2}{i\lambda}E_3(x',0)\right).$$

Recalling $\partial_3 E_3(x',0) - \frac{k^2}{i\lambda}E_3(x',0) = 0$, we finally get $\nabla \cdot \widetilde{E} = 0$ in Ω^-.

So far we have proved that the function $\widetilde{E}$ with components given by (3.4)–(3.6) is the extension of the solution E of the Maxwell's equations. □

4. Applications of the reflection principle. The main purpose of this section is to prove the uniqueness result for recovering convex polyhedral scatterers of impedance type, which was stated in Theorem 1.1 when the incident fields are given by either (1.10) or (1.11). This part also gives a new proof for electromagnetic plane waves.

Assuming two of such different scatterers generate identical far-field patterns, we shall prove via the reflection principle that the scattered electric field could be analytically extended into the whole space, which is impossible. Similar ideas were employed in [16, 20, 15] for proving uniqueness in inverse conductivity and elastic scattering problems. Later we shall remark why our approach cannot be applied to nonconvex polyhedral scatterers and compare our arguments with the uniqueness proof of [8] in the Helmholtz case. The corollaries below follow straightforwardly from Theorem 3.1.

COROLLARY 4.1. *Let* (E,H) *be a solution to the Maxwell's equations* (3.1) *in* $x_3 > 0$ *fulfilling the impedance boundary condition* (3.2) *on* $\Pi = \{x \in \mathbb{R}^3 : x_3 = 0\}$. *Then* (E,H) *can be extended from the upper half-space* $x_3 \geq 0$ *to the whole space. Moreover, the extended electric field* $\widetilde{E}$ *is given by*

$$\widetilde{E}(x) = \begin{cases} E(x) & \text{if } x_3 \geq 0, \\ \mathcal{D}E(x',-x_3) & \text{if } x_3 < 0, \end{cases}$$

where $\mathcal{D}$ *is the operator defined in Theorem* 3.1.

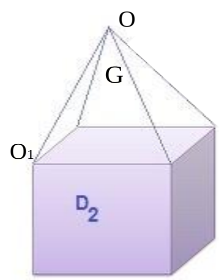
COROLLARY 4.2. *Let* $\Omega = \Omega^+ \cup \gamma \cup \Omega^-$ *be the domain defined in Theorem* 3.1. *Given a subset* $D \subset \Omega^-$, *suppose that* (E,H) *is a solution to the Maxwell's equations* (3.1) *in* $\Omega\backslash\overline{D}$ *fulfilling the impedance boundary condition* (3.2) *on* γ. *Then* (E,H) *can be analytically extended onto* $\overline{D}$.

The above results will be used in the proof of Theorem 1.1 to be carried out below.

4.1. Proof of Theorem 1.1 for electromagnetic Herglotz waves and point source waves. First, we proceed with the same arguments for electromagnetic plane waves. Assume that there are two different convex polyhedrons D_1 and D_2 which generate the same electric far-field pattern. We assume without loss of generality that there exists a vertex O of ∂D_1 and a neighborhood V_O of O such that $V_O \cap \overline{D}_2 = \emptyset$; see Figure 1. Denote by $\Lambda_j \subset \partial D_1$ $(j = 1, 2, \ldots, m)$ the $m \geq 3$ convex polygonal faces of ∂D_1 whose closure meet at O and by $\widetilde{\Pi}_j \supseteq \Lambda_j$ their maximum analytical extension in $\mathbb{R}^3\backslash\overline{D}_2$. Then we get

$$\nu \times (\nabla \times E_2) + i\lambda\nu \times (\nu \times E_2) = 0 \quad \text{on} \quad \widetilde{\Pi}_j, \quad j = 1, 2, \ldots, m$$

due to the analyticity of E_2 in the exterior of $\overline{D}_2$.



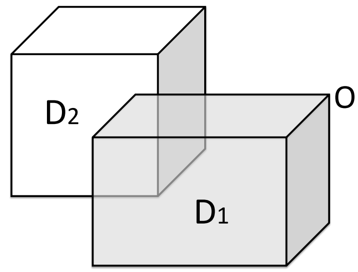


FIG. 1. *Illustration of two different convex polyhedral scatterers. Left:* D_2 *is a cube, and* D_1 *is the interior of* $\overline{D_2 \cup G}$*, where* G *denotes the gap domain between* D_1 *and* D_2*. There are four faces of* D_1 *around the vertex* O*; none of them extends to an entire plane in* $\mathbb{R}^3\backslash\overline{D}_2$*. Right:* D_1 *and* D_2 *are both cubes. There are three faces of* D_1 *around the vertex* O*; only one of them can be extended to an entire plane in* $\mathbb{R}^3\backslash\overline{D}_2$*.*

Proof of Theorem 1.1 *for electromagnetic Herglotz waves.* In this case, E_2 satisfies the Maxwell's equation in $\mathbb{R}^3\backslash\overline{D}_2$. We consider two cases.

Case (i): One of $\widetilde{\Pi}_j$ coincides with some hyperplane Π in $\mathbb{R}^3\backslash\overline{D_2}$ (see Figure 1, right). Since D_2 is convex, it must lie completely on one side of the plane Π. By Corollary 4.1, the electric field E_2 can be analytically extended to $\mathbb{R}^3$ as a solution to the Maxwell's equation. This implies that E_2^{sc} is an entire radiating solution to the Maxwell's equation. Consequently, we get $E_2^{sc} \equiv 0$, and thus the total field $E_2 = E^{in}$ satisfies the impedance boundary condition on ∂D_2.

Case (ii): None of $\widetilde{\Pi}_j$ coincides with an entire hyperplane in $\mathbb{R}^3\backslash\overline{D_2}$ (see Figure 1, left). Denote by $\Pi_j \supset \widetilde{\Pi}_j$ the hyperplane in $\mathbb{R}^3$ containing Λ_j. We shall prove via the reflection principle that E_2 satisfies the impedance boundary condition on each Π_j, which again leads to the relation $E_2 = E^{in}$ by repeating the same arguments in Case (i).

Without loss of generality, we take $j = 1$ and consider the plane $\Pi_1 \supset \widetilde{\Pi}_1 \supset \Lambda_1$. Recall that $\Lambda_1 \subset \partial D_1$ is a convex polygonal face and that the total field E_2 is analytic near the corner O of $\partial\Lambda_1$. It suffices to prove that E_2 is analytic on $\partial\Lambda_1$. Let $O_1 \in \partial\Lambda_1$ be a neighboring corner of O, which is also a vertex of D_2. By the convexity of D_2, there exists at least one face $\Lambda_{j'}$ with $j' \neq 1$ such that the finite line segment OO_1 lies completely on one side of the hyperplane $\Pi_{j'}$ and the projection of OO_1 onto $\Pi_{j'}$, which we denote by L, is a subset of $\widetilde{\Pi}_{j'}(\subset \Pi_{j'})$. See Figure 2 for an illustration of the proof in two dimensions. Since D_2 does not intersect with $\widetilde{\Pi}_{j'}$, one can always find a symmetric domain $\Omega \subset \mathbb{R}^3\backslash\overline{D}_2$ with respect to Π_j such that $OO_1 \subset \overline{\Omega}$ and $L \subset (\Omega \cap \widetilde{\Pi}_{j'})$. Recall that E_2 fulfills the impedance boundary condition on $\widetilde{\Pi}_{j'}$. Now applying Corollary 4.2 with $D = OO_1$ to E_2, we find that E_2 must be analytic on $\overline{OO}_1$, and in particular, E_2 is analytic near O_1. Here we have used the fact that the reflection of $\overline{OO_1}$ with respect to $\Pi_{j'}$ lies completely in $\mathbb{R}^3\backslash\overline{D}_2$ and E_2 is real-analytic in $\mathbb{R}^3\backslash\overline{D}_2$. Analogously, one can prove the analyticity of E_2 at another neighboring corner point O_2 to $O \in \partial\Lambda_1$ and also the analyticity on the line segment $OO_2 \subset \partial\Lambda_1$. Applying the same arguments to O_1 and O_2 in place of O, we can conclude that E_2 is analytic on the closure of Λ_1. This implies that E_2 satisfies the impedance boundary condition on the entire plane $\Pi_1 \supset \widetilde{\Pi}_1$ and thus $E_2 = E^{in}$ in $\mathbb{R}^3$.

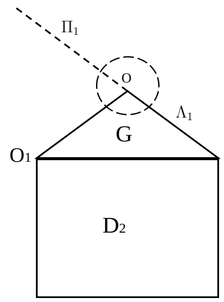


FIG. 2. *Illustration of two different convex polygonal scatterers: D_2 is a square, and D_1 is the interior of $\overline{D_2 \cup G}$, where G denotes the gap domain between D_1 and D_2. There are two sides of D_1 around the corner O; both of them cannot be extended to a straight line in $\mathbb{R}^2\backslash\overline{D}_2$.*

To continue the proof, we recall from Cases (i) and (ii) that $E^{sc} \equiv 0$ and the incident field E^{in} satisfies the following boundary value problem in D_2:

$$\nabla \times \left(\nabla \times E^{in}\right) - k^2 E^{in} = 0 \text{ in } D_2,$$
$$\nu \times \left(\nabla \times E^{in}\right) + i\lambda\nu \times \left(\nu \times E^{in}\right) = 0 \text{ on } \partial D_2.$$

Taking the inner product with $\overline{E^{in}}$, integrating over D_2, and using integration by parts, we obtain

$$\int_{D_2} |\nabla \times E^{in}|^2 - k^2|E^{in}|^2 \,\mathrm{d}x + i\lambda \int_{\partial D_2} |\nu \times E^{in}|^2 \mathrm{d}s = 0.$$

Taking the imaginary parts in the above integral gives $\nu \times E^{in} = 0$ on ∂D_2, which together with the impedance boundary condition implies further that $\nu \times (\nabla \times E^{in}) = 0$ on ∂D_2. Finally, using the analogue of Holmgren's theorem for Maxwell's equations (see, e.g., [10, Theorem 6.5]), we get $E^{in} \equiv 0$, which is a contradiction. Therefore, we have $D_1 = D_2$. This proves Theorem 1.1 for incident fields given by electromagnetic Herglotz waves. ∎

Proof of Theorem 1.1 *for electromagnetic point source waves.* Suppose that $E^{in} = E^{in}(x, y)$ is an electric point source incited by a magnetic dipole located at $y \in \mathbb{R}^3\backslash\overline{D}_j$ $(j = 1, 2)$. By the previous proof for electromagnetic Herglotz waves, one can always find $m \geq 3$ entire planes Π_j $(j = 1, 2, \ldots, m)$ meeting at the vertex $O \in \partial D_1$ such that E_2 fulfills the impedance boundary condition on Π_j, $j = 1, 2, \ldots, m$. Further, for each $j = 1, 2, \ldots, m$, Π_j does not pass through the source position y, and the convex polyhedron D_2 lies in one side of Π_j. Repeating the same arguments in Case (ii) of the proof for plane waves, one can prove that E_2 is analytic on the closure of each face of D_2. In particular, E_2 fulfills the impedance boundary condition on the entire plane Π_Λ, which extends a face Λ of ∂D_2. By the arbitrariness of Λ, we can always find a face $\Lambda' \subset \partial D_2$ such that the reflection of y with respect to $\Pi_{\Lambda'}$ belongs to $\mathbb{R}^3\backslash\overline{D}_2$. By the reflection principle (see Corollary 4.2), E_2 must be analytic at y, which is a contradiction to the singularity of E_2 at the source position. ∎

4.2. Remarks and corollaries. Below we present several remarks concerning the proof of Theorem 1.1.

Remark 4.3. (i) Using the reflection principle for the Helmholtz equation (see Theorem 3.1 or [8]), the idea in the proof of Theorem 1.1 can be used to prove unique determination of a convex polyhedral or polygonal scatterer of acoustically impedance type with a single incoming wave; see Figure 2 for an illustration of the uniqueness proof in two dimensions. This improves the result of [8], where two incident directions were used in two dimensions. (ii) For nonconvex polyhedral scatterers, one cannot find a vertex O around which the total field is analytic under the assumption (1.12). Hence, our uniqueness proof to inverse electromagnetic scattering does not apply to nonconvex polyhedrons of impedance type. However, this might be possible if one can establish a reflection principle by removing the geometrical assumption of Ω made in Theorem 3.1 (see, e.g., [15] in the elastic case).

As a consequence of the proof of Theorem 1.1, we have the following corollaries. In particular, the "singularity" of E^{sc} at vertices motivates us to design a data-driven scheme (see section 5 below) to locate all vertices of D so that the position and shape of D can be recovered from a single measurement datum.

Corollary 4.4. *Let $D \subset \mathbb{R}^3$ be a convex polyhedron, and let $E = E^{in} + E^{sc}$ be the solution to* (1.2)–(1.5)*. Then E cannot be analytically extended from $\mathbb{R}^3\backslash\overline{D}$ to the interior of D across a vertex of ∂D, or, equivalently, E cannot be analytic on the vertices of D.*

Corollary 4.5. *Let D be a perfectly conduction polyhedron such that $\mathbb{R}^3\backslash\overline{D}$ is connected. Suppose that $E = E^{in} + E^{sc}$ is a solution to* (1.2)–(1.4) *with the boundary condition $\nu \times E = 0$ on ∂D. If E^{in} is an incident Herglotz wave, we suppose additionally that k^2 is not the eigenvalue of the operator $\nabla \times \nabla \times$ over D with the boundary condition of vanishing tangential components on ∂D. Then ∂D can be uniquely determined by a single electric far-field pattern E^∞ over all observation directions. Moreover, E cannot be analytically extended from $\mathbb{R}^3\backslash\overline{D}$ to the interior of D across a vertex of ∂D.*

Proof. Let E^{in} be an incident plane wave with the incident direction $d \in \mathbb{S}^2$ and polarization direction $p \in \mathbb{S}^2$. Suppose that two perfect polyhedral conductors D_1 and D_2 generate identical electric far-field patterns but $D_1 \neq D_2$. Combining the path arguments of [29] and the uniqueness proof in Theorem 1.1 for incoming waves given by electromagnetic Herglotz waves, one can always find a perfectly conducting hyperplane $\Pi \subset \mathbb{R}^3$ such that D_j ($j = 1$ or $j = 2$) lies completely on one side of Π. In fact, such a plane Π can be found by applying the point-to-point reflection principle with the perfectly conducting boundary condition. This implies that the total electric field E can be analytically extended into the whole space, leading to $E^{sc} \equiv 0$ in $\mathbb{R}^3$ and thus $\nu \times E^{in} = 0$ on ∂D_j. Hence, we get $\nu \times p = 0$ for any normal direction on ∂D, which is impossible.

If E^{in} is an electric Herglotz function, by the assumption of k^2 one can also get the vanishing of E^{in}. In the case that $E^{in} = E^{in}(x, y)$ is an electric point source wave emitting from the source position $y \in \mathbb{R}^3\backslash\overline{D}_j$, one can prove that E_2 satisfies the Dirichlet boundary condition on the entire plane, which extends a face of D_2; see the previous section in the impedance case. Using the point-to-point reflection principle together with the path argument, this could lead to the analyticity of E_2 at $x = y$, which is a contradiction to the singularity of E_2 at the source position; see [19] in the acoustic case.

The impossibility of analytical extension across a vertex can be proved analogously. ∎

Note that the perfectly conducting polyhedron in Corollary 4.5 is allowed to be nonconvex but cannot contain two-dimensional screens on its closure. The incident wave appearing in Corollaries 4.4 and 4.5 can be a plane wave, a Herglotz wave function, or a point source wave.

4.3. Green's tensor to the Maxwell's equations in a half-space with the impedance boundary condition. As another application of the reflection principle, we derive Green's tensor $G_I(x,y) \in \mathbb{C}^{3\times 3}$ for the Maxwell's equations in the half-space $\mathbb{R}^3_+ := \{x : x_3 > 0\}$ with the impedance boundary condition enforcing on $\Pi := \{x : x_3 = 0\}$; that is, for any constant vector $\vec{a} \in \mathbb{R}^3$,

$$\begin{aligned} &\nabla \times (\nabla \times G_I(x,y)\vec{a}) - k^2 G_I(x,y)\vec{a} = \delta(x-y)\vec{a} && \text{in } x_3 > 0,\\ &\nu \times (\nabla \times G_I(x,y)\vec{a}) + i\lambda\nu \times (\nu \times G_I(x,y)\vec{a}) = 0 && \text{on } x_3 = 0. \end{aligned} \tag{4.1}$$

For this purpose, we need the free-space Green's tensor given by

$$G(x,y) := \Phi(x,y)\mathbf{I} + \frac{1}{k^2}\nabla_y\nabla_y\Phi(x,y), \quad x \neq y,$$

where $\mathbf{I}$ is the 3×3 identity matrix and $\nabla_y\nabla_y\Phi(x,y)$ is the Hessian matrix for Φ defined by

$$(\nabla_y\nabla_y\Phi(x,y))_{l,m} = \frac{\partial^2\Phi(x,y)}{\partial_{y_l}\partial_{y_m}}, \quad 1 \le l,m \le 3, \quad y = (y_1,y_2,y_3) \in \mathbb{R}^3.$$

Note that here $\Phi(x,y) := \frac{1}{4\pi}\frac{e^{ik|x-y|}}{|x-y|}$ is the fundamental solution to the Helmholtz equation in three dimensions.

Following the arguments from [15, Corollary 2.2]), we can prove the following.

Lemma 4.6. *Denote by R_Π the reflection with respect to the plane Π and by $\mathcal{D}_x$ the action with respect to x of the operator $\mathcal{D} := (\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3)$ defined in* (3.5) *and* (3.6)*. Then the impedance Green's tensor $G_I(x,y)$ can be represented as*

$$G_I(x,y) = G(x,y) + \mathcal{D}_x G(R_\Pi x, y), \ x \neq y, \quad x,y \in \mathbb{R}^3_+.$$

Here the action of $\mathcal{D}_x$ on the tensor G is understood columnwisely.

5. A data-driven imaging scheme. The aim of this section is to establish a data-driven inversion scheme for imaging arbitrarily convex polyhedral scatterers. Motivated by the one-wave factorization method in inverse elastic scattering [15], we shall propose a domain-defined indicator functional to characterize an inclusion relationship between a test domain and our target; see also [18] in the acoustic case. Being different from other domain-defined sampling approaches [24, 25, 31, 32, 33] arising from inverse scattering, our scheme will be interpreted as a data-driven method because it relies on measurement data corresponding to a priori given test domains. In this paper, we shall take for simplicity perfectly conducting balls with different centers and radii as test domains. Similar techniques were used in the extended linear sampling method [30] for extracting information of a sound-soft obstacle from a single far-field pattern.

Consider the scattering of an incident plane wave $E^{in} = ik\,(d\times p)\times de^{ikx\cdot d}$ by a ball $B_h(z) := \{x \in \mathbb{R}^3 : \ |x-z| < h\}$ with $h > 0, z \in \mathbb{R}^3$, where $d \in \mathbb{S}^2$ is the incident direction and $p \in \mathbb{R}^3$ is a polarization vector. Then the total field $E = E^{in} + E^{sc}$ satisfies

$$\begin{cases} \nabla\times(\nabla\times E) - k^2 E = 0 & \text{in} \quad |x-z| > h,\\ (\nu\times E)\times\nu = 0 & \text{on} \quad |x-z| = h,\\ \lim_{|x|\to\infty}\left(E^{sc}\times\widehat{x} + \frac{1}{ik}\nabla\times E^{sc}\right)|x| = 0, & \widehat{x} = \frac{x}{r}. \end{cases} \tag{5.1}$$

It is well known that (5.1) has a series solution $E(\widehat{x};d,p,h,z)$ for a given $E^{in}(x;d,p)$ [35]. For notational convenience, we will omit the dependence of solutions on d, p, and k (all of them are fixed in our arguments) and only indicate the dependence on the center $z\in\mathbb{R}^3$ and radius $h>0$ of the ball $B_h(z)$. Denote by $E^\infty(\widehat{x};h,z)$ the electric far-field pattern of the scattered electric field E^{sc}. We expand $E^\infty(\widehat{x};h,z)$ into a series by using vector spherical harmonics. For any orthonormal system Y_n^m, $m=-n,\dots,n$ of spherical harmonics of order $n>0$, the tangential fields defined on the unit sphere

$$U_n^m(\widehat{x}):=\frac{1}{\sqrt{n(n+1)}}\,\mathrm{Grad}\,Y_n^m(\widehat{x}),\quad V_n^m(\widehat{x}):=\widehat{x}\times U_n^m(\widehat{x})$$

are called vector spherical harmonics of order n. By coordinate translation, it is easy to check that E^∞ can be expanded into the convergent series [11, 35]

$$E^\infty(\widehat{x};h,z)e^{ikz\cdot\widehat{x}}=4\pi\sum_{n=1}^{\infty}\sum_{m=-n}^{n}\left(u_n^{(h)}\,[\overline{U_n^m}(\widehat{x})\cdot p]\,U_n^m+v_n^{(h)}\,[\overline{V_n^m}\cdot p]V_n^m\right), \tag{5.2}$$

where

$$u_n^{(h)}:=\frac{\psi_n'(kh)}{(\zeta_n^{(1)})'(kh)}\in\mathbb{C},\quad v_n^{(h)}:=-\frac{\psi(kh)}{\zeta_n^{(1)}(kh)}\in\mathbb{C},$$

with $\psi_n(t):=tj_n(t)$ and $\zeta_n^{(1)}(t):=th_n^{(1)}(t)$. Here j_n is the spherical Bessel function of order n, and $h_n^{(1)}$ is the spherical Hankel function of first kind of order n. Denote the far-field operator $F^{(z,h)}:T(\mathbb{S}^2)\mapsto T(\mathbb{S}^2)$ by

$$\left(F^{(z,h)}g\right)(\widehat{x}):=\int_{\mathbb{S}^2}E^\infty(\widehat{x},d,g(d),h,z)\mathrm{d}s(d), \tag{5.3}$$

where $T(\mathbb{S}^2):=\{g\in L^2(\mathbb{S}^2)^3: g(\widehat{x})\cdot\widehat{x}=0 \text{ for all } \widehat{x}\in\mathbb{S}^2\}$ denotes the tangential space defined on $\mathbb{S}^2$. The expression (5.2) shows that $F^{(z,h)}$ is diagonal in the basis

$$\widetilde{U}_{m,n}^{(z)}(\widehat{x}):=e^{-ikz\cdot\widehat{x}}U_n^m(\widehat{x}),\quad \widetilde{V}_{m,n}^{(z)}(\widehat{x}):=e^{-ikz\cdot\widehat{x}}V_n^m(\widehat{x}).$$

It can be verified that $(4\pi u_n^{(h)},4\pi v_n^{(h)})$ and $(\widetilde{U}_{m,n}^{(z)},\widetilde{V}_{m,n}^{(z)})$ are eigenvalues and the associated eigenvectors of $F^{(z,h)}$. Note that the eigenvalues depend on the radius h only and the eigenfunctions depend on the location z only. We refer to [11] for detailed analysis when the ball is located at the origin. The general case can be easily justified via coordinate translation.

To proceed, we suppose that $w^\infty\in T(\mathbb{S}^2)$ is the electric field pattern of some radiating electric field w^{sc} in $|x|>b$ for some $b>0$ sufficiently large. Introduce the function

$$I_{w^\infty}(z,h):=\frac{1}{4\pi}\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left(\frac{|\langle w^\infty,\widetilde{U}_{m,n}^{(z)}\rangle|^2}{|u_n^{(h)}|}+\frac{|\langle w^\infty,V_{m,n}^{(z)}\rangle|^2}{|v_n^{(h)}|}\right), \tag{5.4}$$

where $z\in\mathbb{R}^3$ and $h>0$ will be referred to as sampling variables in this paper. Equation (5.4) can be regarded as a functional defined on the test domain $B_h(z)$. If the above series is convergent, we shall prove below that the radiated electric field w^{sc} can be analytically extended at least to the exterior of the test domain $B_h(z)$. For simplicity, we still denote by w^{sc} the extended solution.

LEMMA 5.1. *Suppose that k^2 is not the Dirichlet eigenvalue (that is, the tangential component of the electric field vanishes) of the operator* curlcurl *over the ball $B_h(z)$. We have $I_{w^\infty}(z,h)<\infty$ if and only if w^∞ is the far-field pattern of the radiating field w^{sc} which satisfies*

$$\nabla\times(\nabla\times w^{sc})-k^2w^{sc}=0\quad in\ |x-z|>h,\qquad \nu\times w^{sc}\times\nu\in H^{-1/2}_{\rm curl}(\partial B_h(z)). \tag{5.5}$$

Here $H^{-1/2}_{\rm curl}(\partial D)$ denotes the trace space of

$$H(\mathrm{curl},\mathrm{D})=\{\phi\in(L^2(D))^3:\ \nabla\times\phi\in(L^2(D))^3\}$$

of a bounded Lipschitz domain $D\subset\mathbb{R}^3$, given by

$$H^{-1/2}_{\rm curl}(\partial D):=\left\{u\in\left(H^{-1/2}(\partial D)\right)^3:\ \nu\cdot u=0,\nabla\times u\in\left(H^{-1/2}(\partial D)\right)^3\ on\ \partial D\right\}.$$

Proof. Without loss of generality, we may assume that $B_h(z)$ is located at the origin, so that $\widetilde{U}^{(z)}_{m,n}=U^m_n$ and $\widetilde{V}^{(z)}_{m,n}=V^m_n$. Since the assumption on the wave number k ensures that (see [37, Chapter 5] for related discussions)

$$j_n(t)\neq 0\quad\text{and}\quad j_n(t)+tj_n'(t)\neq 0\qquad\text{for}\quad t=kh,\ n=1,2,\ldots,$$

we have $|u_n^{(h)}|\neq 0$ and $|v_n^{(h)}|\neq 0$ for all n. By [10, equation (6.73)], it follows that w^{sc} can be expressed as

$$w^{sc}(x)=\sum_{n=1}^{\infty}\frac{1}{n(n+1)}\sum_{m=-n}^{n}\Big[a_n^m q_n^m(x)+b_n^m\nabla\times q_n^m(x)\Big]\quad\text{in}\quad |x|>b,$$

with the coefficients $a_n^m,b_n^m\in\mathbb{C}$, and $q_n^m(x):=\nabla\times\{xh_n^{(1)}(k|x|)Y_n^m(\widehat{x})\}$. Correspondingly, the far-field pattern w^∞ is given by (see equation (6.74) on p. 219 in [10])

$$w^\infty(\widehat{x})=\frac{i}{k}\sum_{n=1}^{\infty}\frac{1}{i^{n+1}}\sum_{m=-n}^{n}\left(ikb_n^mU_n^m(\widehat{x})-a_n^mV_n^m(\widehat{x})\right).$$

Inserting the above expression into (5.4), we get

$$I_{w^\infty}(o,h)=\frac{1}{4\pi}\sum_{n=1}^{\infty}\sum_{m=-n}^{n}\left(\frac{|b_n^m|^2}{|u_n^{(h)}|}+\frac{|a_n^m|^2}{|v_n^{(h)}|}\right),\quad o=(0,0,0). \tag{5.6}$$

To analyze the convergence of the above series, we need the asymptotic behavior of $u_n^{(h)}$ and $v_n^{(h)}$ as $n\to+\infty$. Using the asymptotics of special functions for large orders, it is easy to observe that

$$\frac{1}{|u_n^{(h)}|}=\left|\frac{\left(\zeta_n^{(1)}\right)'}{\psi_n'}\right|=\left|\frac{\left(th_n^{(1)}(t)\right)'}{\left(tj_n(t)\right)'}\Bigg|_{t=kh}\right|\sim\left|\frac{\left(h_n^{(1)}\right)'(kh)}{j_n'(kh)}\right|\sim\frac{C_1}{|n|}\left|\left(h_n^{(1)}\right)'(kh)\right|^2,$$

$$\frac{1}{|v_n^{(h)}|}=\left|\frac{\zeta_n^{(1)}(kh)}{\psi_n(kh)}\right|=\left|\frac{h_n^{(1)}(kh)}{j_n(kh)}\right|\sim C_2\,|n|\,|h_n^{(1)}(kh)|^2$$

as $n \to \infty$, where $C_1, C_2 \in \mathbb{C}$ are fixed constants. Thus, it follows from (5.6) that

$$I_{w^\infty}(o,h) \sim \sum_{n=1}^{\infty}\sum_{m=-n}^{n}\left(\frac{C_1|b_n^m|^2|h_n^{(1)\prime}(kh)|^2}{|n|} + C_2|a_n^m|^2|n||h_n^{(1)}(kh)|^2\right). \tag{5.7}$$

On the other hand, it is seen from the expression of w^{sc} that on $|x| = h$,

$$(\widehat{x} \times w^{sc} \times \widehat{x}) = \sum_{n=1}^{\infty}\sum_{m=-n}^{n}\left\{\frac{b_n^m}{h}\left(th_n^{(1)}(t)\right)'\Big|_{t=kh} U_n^m(\widehat{x}) - a_n^m h_n^{(1)}(kh) V_n^m(\widehat{x})\right\}.$$

By definition of the $H_{\mathrm{curl}}^{-1/2}(\partial B_h(z))$ norm (see, e.g., [37, Chapter 5] and [35, Chapter 9.3.3]), we obtain

$$\begin{aligned}
&\|\widehat{x} \times w^{sc} \times \widehat{x}\|_{H_{\mathrm{curl}}^{-1/2}(\partial B_h(z))}\\
&= \sum_{n=1}^{\infty}\sum_{m=-n}^{n}\left(\frac{1}{\sqrt{n\,(n+1)}}\frac{|b_n^m|^2}{|h|^2}\left[\left(th_n^{(1)}(t)\right)'\Big|_{t=kh}\right]^2 + \sqrt{n\,(n+1)}|a_n^m|^2|h_n^{(1)}(kh)|^2\right)\\
&\sim \sum_{n=1}^{\infty}\sum_{m=-n}^{n}\left(\frac{1}{n}|b_n^m|^2|h_n^{(1)\prime}(kh)|^2 + n|a_n^m|^2|h_n^{(1)}(kh)|^2\right).
\end{aligned} \tag{5.8}$$

Obviously, (5.7) and (5.8) have the same convergence. In the same manner, one can prove that (5.7) has the same convergence with $||\nu \times w^{sc}||_{H_{\mathrm{div}}^{-1/2}(\partial B_h(z))}$, where

$$H_{\mathrm{div}}^{-1/2}(\partial D) := \{u \in \left(H^{-1/2}(\partial D)\right)^3 : \ \nu \cdot u = 0 \text{ on } \partial D \text{ and } (\mathrm{Div\,u}) \in H^{-1/2}(\partial D)\}.$$

Using the relation

$$\begin{aligned}
&||\nu \times w^{sc}||_{L^2(\partial B_h(z))} + ||\nu \times w^{sc} \times \nu||_{L^2(\partial B_h(z))}\\
&\le C\left(||\nu \times w^{sc}||_{H_{\mathrm{div}}^{-1/2}(\partial B_h(z))} + ||\nu \times w^{sc} \times \nu||_{H_{\mathrm{curl}}^{-1/2}(\partial B_h(z))}\right),
\end{aligned}$$

we conclude that the tangential components of w^{sc} on $\partial B_h(z)$ are convergent in the L^2-sense if $I_{w^\infty}(o,h) < \infty$. This together with [10, Theorem 6.27] implies that w^{sc} is a solution to the Maxwell's equation in $|x| > h$. The proof of Lemma 5.1 is thus complete. ∎

Combining Lemma 5.1 and Corollary 4.4, we may characterize an inclusion relation between D and $B_h(z)$ through the measurement data E^∞ of our target and the spectra of the far-field operator $F^{(z,h)}$ corresponding to the test ball.

THEOREM 5.2. *Let E^∞ be the electric far-field pattern of a convex polyhedral scatterer D with a constant impedance coefficient. Suppose that k^2 is not the Dirichlet eigenvalue of the operator* curlcurl *over the ball $B_h(z)$. It holds that*

$$I_{E^\infty}(z,h) < \infty \qquad \textit{if and only if} \qquad D \subset \overline{B_z(h)}.$$

Hence, we have

$$D = \bigcap_{I_{E^\infty}(z,h)<\infty}^{(z,h)} B_h(z).$$

TABLE 1
Data-driven scheme for imaging convex polyhedral scatterers.

Step 1	Collect the measurement data $E^\infty(\hat{x})$ for all $\hat{x} \in \mathbb{S}^2$, and suppose that $D \subset B_R := \{x : \lvert x\rvert < R\}$ for some large $R > 0$.
Step 2	Choose sampling variables $z_j \in \{x : \lvert x\rvert = R\}$ and $h_i \in (0, 2R)$ to get the spectra of the far-field operator $F^{(z,h)}$ corresponding to testing balls $B_{h_j}(z_j) \subset B_R$.
Step 3	Calculate the domain-defined indicator function $I_{E^\infty}(z_j, h_i)$ by (5.4) with $w^\infty = E^\infty$. In particular, it follows from Theorem 5.2 that $h_i < \max_{y\in\partial D} \lvert z_j - y\rvert \longrightarrow I_{E^\infty}(z_j, h_i) = \infty,$ $h_i \geq \max_{y\in\partial D} \lvert z_j - y\rvert \longrightarrow I_{E^\infty}(z_j, h_i) < \infty.$
Step 4	Image D as the intersection of all test balls $B_{h_i}(z_j)$ such that $I_{E^\infty}(z_j, h_i) < \infty$.

Proof. If $D \subset \overline{B_z(h)}$, then the scattered electric field E^{sc} is well defined in $\{x : |x-z| > h\}$, which lies in the exterior of D. Hence, E^{sc} satisfies (5.5), and by Lemma 5.1 it holds that $I_{E^\infty}(z, h) < \infty$. On the other hand, suppose that $I_{E^\infty}(z, h) < \infty$ but the relation $D \subset \overline{B_z(h)}$ does not hold. Since D is a convex polyhedron, there must exist at least one vertex O of ∂D such that $|O - z| > h$. Again using Lemma 5.1, we conclude that E^{sc} can be extended from $\mathbb{R}^3\backslash\overline{D}$ to the exterior of $B_h(z)$. This implies that E^{sc} is analytic at O, which contradicts Corollary 4.4. ∎

By Theorem 5.2, the function $h \to I_{E^\infty}(z, h)$ for fixed $|z| = R$ will blow up when $h \geq \max_{y\in\partial D} |y-z|$, indicating a rough location of D with respect to $z \in \mathbb{R}^3$. In Table 1, we describe an inversion procedure for imaging an arbitrary convex polyhedron D by taking both $z \in \partial B_R$ and h as sampling variables. The mesh for discretizing $h \in (0, 2R)$ should be finer than the mesh for $z \in \partial B_R$. To avoid the assumption that k^2 is not a Dirichlet eigenvalue of curl curl over $B_h(z)$, one may use coated balls by a thin dielectric layer (which can be modeled by the impedance boundary condition) as test domains in place of our choice of perfectly conducting balls. We refer the reader to [11, section 3.1] for a description of the spectra of the far-field operator corresponding to such coated balls centered at the origin. If the impedance coefficient is a positive constant, one can prove that k^2 cannot be an impedance eigenvalue of curl curl over any boundary Lipschitz domain. It should be remarked that the test domains can also be taken as penetrable balls under the assumption that k^2 is not an interior transmission eigenvalue. Both Theorem 5.2 and Lemma 5.1 can be carried over to these test domains. Finally, it is worth mentioning that a regularization scheme should be employed to truncate the series (5.6) because the eigenvalues $u_n^{(h)}$ and $v_n^{(h)}$ decay very fast and the calculation of the inner product between E^∞ and the eigenfunctions $(\widetilde{U}_{n,m}^{(z)}, \widetilde{V}_{n,m}^{(z)})$ is usually polluted by data noise and numerical errors. We refer the reader to [34] for numerical examples in inverse acoustic scattering. Numerical tests for Maxwell's equations will be reported in our forthcoming publications.

Remark 5.3. In [33], the no-response test was discussed for reconstructing convex perfectly conducting polyhedrons with two or a few incident electromagnetic plane waves. In comparison with [33], our inversion scheme uses only a single incoming wave within a more general class of plane waves, Herglotz wave functions, and point source waves. Although both of them belong to the class of domain-defined sampling methods, the computational criterion explored here (see (5.4) and Theorem 5.2) involves simple inner product calculations and new sampling schemes due to the special choice of testing balls.

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