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Optimal reinsurance-investment strategy with thinning dependence and delay factors under mean-variance framework

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ABSTRACT

In this paper, we study the optimal time-consistent reinsurance-investment problem for a risk model with the thinning-dependence structure. The insurer's wealth process is described by a jump-diffusion risk model with two dependent classes of insurance business. We assume that the insurer is allowed to purchase per-loss reinsurance and invest its surplus in a financial market consisting of a risk-free asset and a risky asset. Also, the performance-related capital inflow or outflow feature is introduced, and the wealth process is modeled by a stochastic delay differential equation. Under the time-inconsistent mean-variance criterion, we derive the explicit optimal reinsurance-investment strategy and value function under the expected value premium principle as well as the variance premium principle by solving the extended Hamilton–Jacobi–Bellman (HJB) delay system. In particular, we prove the existence and uniqueness of the optimal strategy under the expected value premium principle. Finally, some numerical examples are provided to illustrate the influence of model parameters on the optimal strategy.

1. Introduction

In the past decade, due to its practical importance, the study of optimal reinsurance has drawn great attention in the field of actuarial science. To achieve their management goals, insurers adjust the retention level continuously to cope with the insurance risks of their portfolios according to the market and internal conditions.

Since the work of Markowitz (1952), the mean-variance criterion has become one of the milestones in mathematical finance. For example, Li & Ng (2000) developed an embedding technique to change the original mean-variance problem into a stochastic linear-quadratic (LQ) control problem in a discrete-time setting. Zhou & Li (2000) extended the technique of Li & Ng (2000) to a continuous-time case and derived the optimal mean-variance portfolio. Bäuerle (2005) adopted the mean-variance approach to investigate an optimal proportional reinsurance problem. Bi & Guo (2013) obtained the optimal investment or reinsurance strategy with constraints for the mean-variance insurer in a jump-diffusion financial market. Ming, Liang, & Zhang (2016) considered an insurance portfolio with the common shock dependence under the mean-variance framework. There are many other results with mean-variance analysis in the actuarial literature; for example, see Fu, Lari-Lavassani, & Li (2010); Gao, Xiong, & Li (2016); Shen & Zeng (2014); Yuan, Mi, & Chen (2022c); Zeng, Li, & Gu (2016) and references therein.

It is well known that the dynamic reinsurance problems under the mean-variance criterion face the issue of time-inconsistency, that is, the Bellman optimality principle does not hold due to the failure of the iterated-expectation property. Specifically, the optimal control law for some fixed initial point is no longer optimal at some later point. To find the time-consistent strategy, Björk & Murgoci (2010) formulated the problem in a game theoretic framework, and derived the extended Hamilton–Jacobi–Bellman (HJB) equation through a verification theorem. Based on their work, the game theoretic approach for time-inconsistent problems have been extended in many directions by researchers. Among others, Björk, Murgoci, & Zhou (2014) assumed that the insurer's risk aversion is inversely proportional to the current wealth, and obtained the time-consistent strategy; Zeng et al. (2016) analyzed the equilibrium investment-reinsurance strategy for an ambiguity-averse insurer who worries about model uncertainty; and Yuan, Liang, & Han (2022b) designed a robust reinsurance contract for the insurer and reinsurer with mean-variance preference in the Stackelberg differential game. For the other papers on reinsurance problems under the mean-variance criterion, we refer to Chen & Shen (2019); Li & Young (2021) and the references therein.

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Although lots of research on the time-inconsistent mean-variance problems have been carried out in the past few years, many interesting issues remain to be investigated. Most of the literature considers the optimal reinsurance problems in which the insurer is restricted to purchase either pure quota-share reinsurance (Chen, Qian, Shen, & Wang, 2016; Guan & Liang, 2019; Zhang & Liang, 2021) or excess of loss reinsurance (Bai, Guo, & Zhang, 2010; Li, Zeng, & Yang, 2018; Zhao, Rong, & Zhao, 2013). However, it is more meaningful to determine optimal reinsurance strategy without a pre-specified form of reinsurance. In the literature, many optimal reinsurance strategies have been obtained for various optimization problems without restricting the form of reinsurance. For example, Zhang, Meng, & Zeng (2016) obtained the general form of optimal reinsurance under the criteria of maximizing the expected utility function of terminal wealth and minimizing the probability of ruin; Liang & Young (2018) computed the optimal per-loss strategy for the general mean-variance premium in a compound Poisson model with the objective of minimizing the probability of ruin; and Han, Liang, & Young (2020) determined the optimal reinsurance strategy to minimize the probability of drawdown, that is, the probability that the insurer's surplus process reaches some fixed fraction of its maximum value to date.

In a financial market, sophisticated investors not only pay close attention to the current stock price but also care about the trend of the stock price in the past periods. Therefore, it is more practical to consider the historical information in a certain period, which is commonly called delay factor or bounded memory. In fact, the performance of the past wealth does have influence on the decision maker in rational terms. With the method of dynamic programming principle, Chang, Pang, & Yang (2011) modeled the price of stock by a stochastic system with delay, and transferred the original problem into a finite dimensional space. Agram, Haadem, Øksendal, & Proske (2013) derived the optimal strategy for the stochastic delay system in the portfolio problem by the method of maximum principle. Shen & Zeng (2014) first introduced the delay factors into the time-inconsistent mean-variance problem, and obtained the optimal pre-committed investment/reinsurance strategy. For more applications to insurance, we refer the readers to A & Li (2015); Bai, Zhou, Xiao, Gao, & Zhong (2022) and references therein.

Most of the literature mentioned above assumes that the insurance risk model has one kind of insurance claim only or that claims of different types are independent of each other. However, it is believed that various claims in a book of insurance business tend to be dependent in some way. A typical example is that a severe car accident may cause not only the loss of the damaged car but also the medical expenses of injured driver and passengers. To depict such a dependence structure among several classes of insurance business, the so-called common shock and thinning-dependence risk models are often used; see, for example, Chen, Yuen, & Wang (2021); Liang, Bi, Yuen, & Zhang (2016); Yuen & Wang (2002); Zhang & Liang (2016) and Han, Liang, Yuan, & Zhang (2021).

Inspired by the aforementioned works, this paper extends the study of optimal reinsurance and investment with delay factors to a risk model with two classes of insurance business which are correlated through the thinning process of a single stochastic risk source. The stochastic source may cause a claim in each insurance class with a certain probability. To control the risk, the insurer can purchase per-loss reinsurance, and invest its surplus into a financial market consisting of one risky asset and one risky-free asset. The price process of the risky asset is described by a jump-diffusion model. In addition, it is assumed that there exists capital inflow into or outflow from the insurer's current wealth, and that the corresponding wealth process of the insurer is modeled by a stochastic delay differential equation. In this setup, our objective is to seek the optimal time-consistent strategy for the mean-variance problem within a game theoretic framework. Recently, Li, Yuan, & Chen (2023) considered an optimal mean-variance investment and reinsurance problem with delay and common-shock dependence in a jump-diffusion process. An efficient strategy and the efficient frontier were derived by Lagrange dual method. Unlike their work, we investigate the optimal reinsurance form which is not limited to proportional reinsurance, formulate the problem within a non-cooperative game theoretic framework proposed by Björk & Murgoci (2010), and derive the equilibrium strategy of the game. The synergy of the three features of our model, namely per-loss reinsurance, time consistency and thinning dependence, can help us better understand the decision making process of an insurance company.

Applying the technique of stochastic control theory and the corresponding extended HJB delay system, we first prove the existence and uniqueness of the optimal strategy for our optimization problem under the expected value principle. By constructing some useful auxiliary functions, the closed-form expressions for the optimal reinsurance-investment strategy and the corresponding value function are also derived explicitly. Furthermore, we obtain the optimal results under the variance principle and find that the optimal retention level naturally falls into the interval [0,1] when the same safety loadings are applied to both classes of insurance business. Finally, sensitivity analyses and several numerical simulations are provided to further illustrate the influence of model parameters on the optimal results.

The purpose of the present paper is threefold. Firstly, we incorporate both the thinning-dependence structure and the effect of bounded memory into the optimization problem, and prove the existence as well as the uniqueness of the optimal strategy. Secondly, the reinsurance form under consideration is not limited to quota-share or excess-of-loss reinsurance but a general reinsurance policy, which makes the optimization problem more challenging and practical. Nevertheless, we find that the optimal per-loss strategy is exactly in either of the two most popular forms, that is, the excess-of-loss reinsurance under the expected value principle, and the proportional reinsurance under the variance premium principle. Thirdly, some interesting properties of the optimal reinsurance and investment strategies under the two premium principles are investigated in detail, and the corresponding proofs are presented rigorously. In particular, the incorporation of the delay factors into the wealth process makes the insurer more aggressive, and results in a higher retention level and a larger risky investment amount.

The rest of the paper is organized as follows. In Section 2, the model and optimization problem are presented. In Section 3, we provide the verification theorem, which is used to find our value function. Under the expected value principle, explicit expressions for the optimal strategies and the corresponding value function are derived in Section 4. Optimal results under the variance premium principle are given in Section 5. In Section 6, several numerical examples are given to illustrate the impact of some model parameters on the optimal results. Finally, we conclude the paper in Section 7.

2. The risk model

Let $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$ be a filtered probability space satisfying the usual assumptions of completeness and right continuity, and $T > 0$ be a finite time horizon.

2.1. Thinning-dependence structure

We first introduce the thinning risk model proposed in Yuen & Wang (2002). Assume that the insurer has two dependent classes of business such as motor insurance and health insurance. In the

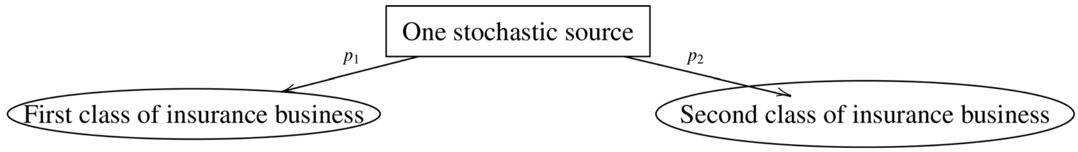

Fig. 1. The thinning-dependence structure.

structure of the thinning risk model, each event (such as flood or fire) may cause a claim in the ith class with probability $p_i \in [0,1]$ for $i = 1, 2$ (see Fig. 1). Let $\{Z_{ij}, j = 1, 2, \ldots\}$ be the claim size random variables for class $i \in \{1, 2\}$, and Z_i be a generic random variable which has the same distribution as Z_{ij}. We denote by $F_{Z_i}(z_i)$ the common cumulative distribution of Z_i with $F_{Z_i}(z_i) = 0$ for $z_i \leqslant 0$ and $0 < F_{Z_i}(z_i) \leqslant 1$ for $z_i > 0$. Furthermore, it is assumed that the first and second raw moments of Z_i are finite, i.e., $a_i = \mathrm{E}[Z_i] < \infty$ and $b_i = \mathrm{E}[Z_i^2] < \infty$. Let $N = \{N(t)\}_{t\in[0,T]}$ be a homogeneous Poisson process with intensity $\lambda > 0$. Without purchasing any reinsurance, the total reserve of the insurer up to time t is given by

$$U(t) = u + ct - \sum_{i=1}^{N^{p_1}(t)} Z_{1i} - \sum_{i=1}^{N^{p_2}(t)} Z_{2i},$$

where $u \geqslant 0$ is the initial surplus, c is the insurance premium rate, and $N^{p_i}(t)$ is a homogenous Poisson process with intensity λp_i representing the cumulative claim number of the ith class of insurance business. Mathematically, $N^{p_i}(t)$ is the p_i-thinning process of $N(t)$ with intensity λ. In other words, the occurrence of stochastic event may cause a claim in the ith class with probability p_i.

Remark 1. The thinning-dependence structure widely exists in insurance business. A typical example is that a traffic accident may cause a property claim and a medical claim with certain probabilities p_1 and p_2 at the same time. The dependence structure was also studied in Han, Liang, & Yuen (2018) and Chen et al. (2021). In particular, by setting $\lambda = \lambda_1 + \lambda_2 + \lambda_0$ and $\lambda p_i = \lambda_i + \lambda_0$ for $i \in \{1, 2\}$, the thinning-dependence structure is reduced to the classical model with the common-shock dependence; see, for example, Han, Liang, & Zhang (2019); Liang & Wang (2012) and Yuan, Liang, & Han (2022a). Thus, the thinning-dependence structure is a more general risk model which includes the common-shock structure as a special case.

In this paper, we allow the insurance company to continuously reinsure its claim with per-loss reinsurance. Let $\mathcal{H}_i = \{\mathcal{H}_i(t, z_i)\}_{t\geqslant 0}$ denote the retained claim at time t which is a function of the (possible) claim size $Z_i = z_i$ at that time for $i \in \{1, 2\}$. Thus, $z_i - \mathcal{H}_i(t, z_i)$ is the amount of each claim transferred to the reinsurer.[1] Apparently, this reinsurance protection is not for free and has a cost. Suppose that the insurer pays reinsurance premium to the reinsurer continuously, which is denoted by $\delta(\mathcal{H}_1(t, Z_1), \mathcal{H}_2(t, Z_2))$ at time t. Thus, in the presence of per-loss reinsurance, the wealth process of the insurer $X = \{X_t\}_{t\geqslant 0}$ is given by

$$\mathrm{d}X(t) = [c - \delta(\mathcal{H}_1(t, Z_1), \mathcal{H}_2(t, Z_2))]\mathrm{d}t - \mathrm{d}\sum_{i=1}^{N^{p_1}(t)} \mathcal{H}_1(t, Z_{1i}) - \mathrm{d}\sum_{i=1}^{N^{p_2}(t)} \mathcal{H}_2(t, Z_{2i}).$$

Note that $\delta(0, 0)$ denotes the reinsurance premium rate when the insurer transfers all the claims to the reinsurer. To avoid the trivial case, we usually assume that

$$\lambda p_1 a_1 + \lambda p_2 a_2 < c < \delta(0, 0),$$

which implies that the insurer's premium income is greater than the expected value of the claims but less than the premium for full reinsurance. Otherwise, the insurer can transfer all the risk to the reinsurer and make profit.

According to Grandell (1991); Promislow & Young (2005) and Bai, Cai, & Zhou (2013), we can approximate the claim process according to a Brownian motion with drift as

$$\mathrm{d}\sum_{i=1}^{N^{p_1}(t)} \mathcal{H}_1(t, Z_{1i}) \approx \lambda p_1 \mathrm{E}[\mathcal{H}_1(t, Z_1)]\mathrm{d}t - \sqrt{\lambda p_1 \mathrm{E}[\mathcal{H}_1(t, Z_1)^2]}\mathrm{d}W_1(t),$$

and

$$\mathrm{d}\sum_{i=1}^{N^{p_2}(t)} \mathcal{H}_2(t, Z_{2i}) \approx \lambda p_2 \mathrm{E}[\mathcal{H}_2(t, Z_2)]\mathrm{d}t - \sqrt{\lambda p_2 \mathrm{E}[\mathcal{H}_2(t, Z_2)^2]}\mathrm{d}W_2(t),$$

in which $W_1 = \{W_1(t)\}_{t\geqslant 0}$ and $W_2 = \{W_2(t)\}_{t\geqslant 0}$ are two correlated standard Brownian motions with correlation coefficient

$$\rho(t) = \frac{\lambda p_1 p_2 \mathrm{E}[\mathcal{H}_1(t, Z_1)]\mathrm{E}[\mathcal{H}_2(t, Z_2)]}{\sqrt{\lambda p_1 \mathrm{E}[\mathcal{H}_1(t, Z_1)^2]}\sqrt{\lambda p_2 \mathrm{E}[\mathcal{H}_2(t, Z_2)^2]}}.$$

In addition to the reinsurance business, we assume that the insurer is able to invest its surplus into a risky asset (stock or mutual fund) and a risk-free asset (bond or bank account) with interest rate $r > 0$. Specifically, the price process of the risky asset follows a jump-diffusion process given by

$$\mathrm{d}S(t) = S(t-)\left(\mu \mathrm{d}t + \sigma \mathrm{d}W_3(t) + \mathrm{d}\sum_{i=1}^{N_3(t)} L_i\right), \tag{2.1}$$

where $\mu > r$, $\sigma > 0$, $W_3 = \{W_3(t)\}_{t\geqslant 0}$ is a standard Brownian motion, $N_3 = \{N_3(t)\}_{t\geqslant 0}$ is a homogeneous Poisson process with intensity $\lambda_3 > 0$, and L_i is the ith jump amplitude of the risky asset price. It is assumed that $L_i, i = 1, 2, \ldots$, are independent and identically distributed random variables with distribution function $F_L(l)$, finite first moment $\mathrm{E}[L_i] = \mu_L$ and finite second moment $\mathrm{E}\left[L_i^2\right] = \sigma_L^2$. It follows from Chapter V of Protter (2004) that the stochastic differential Eq. (2.1) admits a unique solution. Similar to Li et al. (2018); Zeng et al. (2016) and Zhang & Chen (2020), we further assume that $W_3(t)$ and $\sum_{i=1}^{N_3(t)} L_i$ are mutually independent and independent of $W_i (i = 1, 2)$, and that $P\{L_i \geqslant -1 \text{ for all } i \geqslant 1\} = 1$ so that the risky asset price remains positive. Generally, the expected return of the risky asset is larger than the risk-free interest rate, so we assume that $\mu + \lambda_3 \mu_L > r$.

Let $\pi = \{\pi(t)\}_{t\geqslant 0}$ be the amount invested in the risky asset at time t. The rest of the surplus is invested in the risk-free asset. Then, for a chosen combination of controls $\nu = \{\nu(t)\}_{t\geqslant 0} = (\mathcal{H}_1, \mathcal{H}_2, \pi)$, we have the wealth process $X^\nu(t)$ given by

$$\begin{aligned}\mathrm{d}X^\nu(t) = \Big(& rX^\nu(t) + c - \delta(\mathcal{H}_1(t, Z_1), \mathcal{H}_2(t, Z_2)) \\ & - \lambda p_1 \mathrm{E}[\mathcal{H}_1(t, Z_1)] - \lambda p_2 \mathrm{E}[\mathcal{H}_2(t, Z_2)] \\ & + (\mu - r)\pi(t)\Big)\mathrm{d}t + \sqrt{\lambda p_1 \mathrm{E}[\mathcal{H}_1(t, Z_1)^2]}\mathrm{d}W_1(t) \\ & + \sqrt{\lambda p_2 \mathrm{E}[\mathcal{H}_2(t, Z_2)^2]}\mathrm{d}W_2(t) \\ & + \sigma\pi(t)\mathrm{d}W_3(t) + \pi(t)\mathrm{d}\sum_{i=1}^{N_3(t)} L_i,\end{aligned} \tag{2.2}$$

[1] In some insurance optimization problems, researchers impose the condition that retained and transferred claims are non-decreasing functions of the underlying claim. However, in this paper, we can obtain the same monotonicity of $\mathcal{H}_i(t, Z_i)$ and $Z_i - \mathcal{H}_i(t, Z_i)$ without requiring that condition a priori.

where $X^\nu(0) = x_0 > 0$ is the initial wealth.

In the literature, the problem of optimal investment and reinsurance under various dependence structures has also been investigated. Cai & Wei (2012) minimized risk measures of the retained loss of an insurer in a single-period model, and showed that the excess-of-loss treaty is the optimal reinsurance form when the risks are correlated through stochastic ordering. See also Chi, Lin, & Tan (2017) for a related work. Brachetta & Schmidli (2020) proposed a risk model in which the insurance framework is affected by some environmental factors, and the aggregate claims and stock prices are dependent. Ceci, Colaneri, & Cretarola (2022) extended the study of Brachetta & Schmidli (2020) by further assuming that the claim arrival intensities of both business lines are modeled as functions of additional exogenous stochastic factors. In contrast, we focus on studying the reinsurance and investment optimization problem under a continuous-time risk model, in which the two kinds insurance business possess the thinning dependence structure.

2.2. The wealth process with delay

In this subsection, we present the influence of historical performance on the insurer's wealth process. In practice, due to the memory feature, the insurer's retention level is always dependent on the exogenous capital inflow into or outflow from the current wealth. Similar to Federico (2011), we denote the integrated, average and pointwise performance of the wealth in the past horizon $[t-h, t]$ as

$$Y^\nu(t) = \int_{-h}^{0} e^{\varphi s} X^\nu(t+s)\mathrm{d}s, \quad \overline{Y}^\nu(t) = \frac{Y^\nu(t)}{\int_{-h}^{0} e^{\varphi s}\mathrm{d}s},$$
$$M^\nu(t) = X^\nu(t-h), \;\; \forall t \in [0, T],$$

in which $\varphi \geqslant 0$ is an average parameter, and $h > 0$ is the delay time. Then the differential form of $Y^\nu(t)$ can be expressed as

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}Y^\nu(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{-h}^{0} e^{\varphi s}X^\nu(t+s)\mathrm{d}s\right] = \frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{t-h}^{t} e^{\varphi(u-t)}X^\nu(u)\mathrm{d}u\right] \\ &= X^\nu(t) - e^{-\varphi h}X^\nu(t-h) - \varphi\int_{t-h}^{t} e^{\varphi(u-t)}X^\nu(u)\mathrm{d}u \\ &= X^\nu(t) - e^{-\varphi h}X^\nu(t-h) - \varphi\int_{-h}^{0} e^{\varphi\theta}X^\nu(t+\theta)\mathrm{d}\theta \\ &= X^\nu(t) - \varphi Y^\nu(t) - e^{-\varphi h}M^\nu(t).\end{aligned}$$

Note that $\overline{Y}^\nu(t)$ is defined as the weighted average of the wealth process $X^\nu(\cdot)$ over the period $[t-h, t]$ with the exponential decaying factor $e^{\varphi s}$, $s \in [-h, 0]$, as the weight. It is clear that the exponential weight $e^{\varphi s}$ is a strictly increasing function with respect to (w.r.t.) time s, which implies that more weight is put on recent wealth. For the special case of $\varphi = 0$, $\overline{Y}^\nu(t)$ is just the moving average. The parameter h is the duration of the past that the insurer usually cares about. We then formulate the function $f(t, X^\nu(t) - \overline{Y}^\nu(t), X^\nu(t) - M^\nu(t))$ to represent the capital inflow or outflow amount of the insurer. The term $X^\nu(t) - \overline{Y}^\nu(t)$ implies the average performance in the horizon $[t-h, t]$, while $X^\nu(t) - M^\nu(t)$ accounts for the absolute performance between the two time points $t-h$ and t. For the solvability of the optimization problem, similar to Chang et al. (2011); Shen & Zeng (2014); Zhang & Chen (2020) and Bai et al. (2022), we assume that the amount of the capital inflow or outflow is proportional to the past performance of the wealth, i.e.,

$$\begin{aligned} & f\Big(t, X^\nu(t) - \overline{Y}^\nu(t), X^\nu(t) - M^\nu(t)\Big) \\ &\quad = \overline{B}\Big(X^\nu(t) - \overline{Y}^\nu(t)\Big) + C(X^\nu(t) - M^\nu(t)), \end{aligned} \tag{2.3}$$

where $\overline{B}$ and C are two non-negative constants. Eq. (2.3) shows that the instantaneous capital inflow or outflow function is a weighted sum of $X^\nu(t) - \overline{Y}^\nu(t)$ and $X^\nu(t) - M^\nu(t)$. Such a capital inflow/outflow is related to the performance of the wealth in the past, and may arise in various situations. For instance, if the current wealth is higher than the average performance in the given history time horizon, the insurer may use a proportion of the excess wealth to pay dividend to shareholders or bonus to its management. This is an example of capital outflow. On the contrary, when the current wealth is lower than the average performance in the given history time horizon, the insurer may go for financing from the capital market to make up the loss so that the final performance objective is still achievable. This corresponds to capital inflow.

Let $A = r - \overline{B} - C$ and $B = \frac{\overline{B}}{\int_{-h}^{0} e^{\varphi s}ds}$. Then it is easy to see that $BY^\nu(t) = \overline{B}\overline{Y}^\nu(t)$. Taking account of the capital inflow/outflow function f, the wealth process of the insurer is governed by the following stochastic delay differential system (SDDS):

$$\begin{cases} \mathrm{d}X^\nu(t) = \Big[AX^\nu(t) + c - \delta(\mathcal{H}_1(t, Z_1), \mathcal{H}_2(t, Z_2)) \\ \qquad -\lambda p_1 \mathrm{E}(\mathcal{H}_1(t, Z_1)) - \lambda p_2 \mathrm{E}(\mathcal{H}_2(t, Z_2)) \\ \qquad +BY^\nu(t) + CM^\nu(t) + (\mu - r)\pi(t)\Big]\mathrm{d}t \\ \qquad +\sqrt{\lambda p_1 \mathrm{E}[\mathcal{H}_1(t, Z_1)^2]}\mathrm{d}W_1(t) \\ \qquad +\sqrt{\lambda p_2 \mathrm{E}[\mathcal{H}_2(t, Z_2)^2]}\mathrm{d}W_2(t) \\ \qquad +\sigma\pi(t)\mathrm{d}W_3(t) + \pi(t)\mathrm{d}\sum_{i=1}^{N_3(t)} L_i, \\ \mathrm{d}Y^\nu(t) = (X^\nu(t) - \varphi Y^\nu(t) - e^{-\varphi h}M^\nu(t))\mathrm{d}t, \\ \mathrm{d}M^\nu(t) = \mathrm{d}X^\nu(t-h). \end{cases} \tag{2.4}$$

It is obvious that $f > 0$ implies capital outflow while $f < 0$ means capital inflow. Furthermore, we assume that the insurer is endowed with initial wealth x_0 at time $-h$, and does not start any business (investment/insurance/reinsurance) until time 0. Correspondingly, the initial conditions for the SDDS are

$$X^\nu(0) = x_0 > 0, \quad Y^\nu(0) = \frac{x_0}{\varphi}\left(1 - e^{-\varphi h}\right), \quad M^\nu(0) = x_0.$$

To end the subsection, we give the definition of admissible strategies formally.

Definition 1 (Admissible strategy)**.** A strategy $\nu = (\mathcal{H}_1, \mathcal{H}_2, \pi)$ is said to be admissible if the following conditions are satisfied:

(i) it is adapted to the filtration $\mathbb{F}$;
(ii) $\mathcal{H}_i(t, z_i)$ $(i = 1, 2)$ is a function of the possible claim size $Z_i = z_i$ at time t, and satisfies $0 \leqslant \mathcal{H}_i(t, z_i) \leqslant z_i$, for all $t \geqslant 0$ and $z_i \geqslant 0$;
(iii) $\pi(t)$ satisfies $\int_0^t \pi(s)^2\mathrm{d}s < \infty$ with probability one for $0 \leqslant t \leqslant T$;
(iv) the state equation of $X^\nu(t)$ has a unique strong solution.

Let $\mathcal{A}$ be the set of all admissible strategies.

3. Problem formulation and extended HJB delay system

In this section, we consider the optimization problem under the time-inconsistent mean-variance framework. In our study, the insurer is concerned with both the terminal wealth $X^\nu(T)$ and the historical average performance $\overline{Y}^\nu(T)$. As a result, we formulate the reward function with delay under the mean-variance criterion as

$$\begin{aligned} J(t, x, y, m; \nu) = \mathrm{E}_{t,x,y,m}\Big[X^\nu(T) + \overline{\kappa}\overline{Y}^\nu(T)\Big] \\ - \frac{\gamma}{2}\mathrm{Var}_{t,x,y,m}\Big[X^\nu(T) + \overline{\kappa}\overline{Y}^\nu(T)\Big], \end{aligned} \tag{3.1}$$

where γ is the risk-averse parameter of the insurer, and the constant $\bar{\kappa} \geqslant 0$ is the weight of $\overline{Y}^{\nu}(T)$ indicating the degree of impact of the historical average performance on the final performance. The notations $\mathrm{E}_{t,x,y,m}$ and $\mathrm{Var}_{t,x,y,m}$ stand for the conditional expectation and variance given $X^{\nu}(t)=x$, $Y^{\nu}(t)=y$ and $M^{\nu}(t)=m$, respectively.

Remark 2. By using the weight $\bar{\kappa}$ in the performance functional (3.1), we incorporate both the terminal wealth $X^{\nu}(T)$ and the average delayed wealth $\overline{Y}^{\nu}(t)$ over the period $[T-h,T]$ into the final mean-variance performance measure. It is noted that if we only take into account the terminal wealth $X^{\nu}(T)$, the management will likely be tempted to adopt short-term risk taking behavior in order to manipulate the final performance measure so as to achieve shinning performance at an instant. Incorporating the average delayed wealth $\overline{Y}^{\nu}(t)$ into the final performance measure makes the insurer pay more attention to the wealth over a period of time instead of a single time point. This certainly mitigates the risk of having an imprudent short-term risk taking behavior. Moreover, in our model framework, the term $X^{\nu}(t)+\bar{\kappa}\overline{Y}^{\nu}(T)$ is consistent with those in Shen, Meng, & Shi (2014); Shen & Zeng (2014) and Zhang & Chen (2020).

Since the differential form of $Y^{\nu}(T)$ is available in (2.4), we consider replacing the average delayed wealth $\overline{Y}^{\nu}(T)$ in (3.1) with the integrated delayed wealth $Y^{\nu}(T)$. Recall that $\overline{Y}^{\nu}(t)=\frac{Y^{\nu}(t)}{\int_{-h}^{0}e^{\varphi s}\mathrm{d}s}$. By letting $\kappa=\frac{\bar{\kappa}}{\int_{-h}^{0}e^{\varphi s}ds}$, the value function can be rewritten as

$$V(t,x,y,m)=\sup_{\nu\in\mathcal{A}}\Big\{\mathrm{E}_{t,x,y,m}[X^{\nu}(T)+\kappa Y^{\nu}(T)] -\frac{\gamma}{2}\mathrm{Var}_{t,x,y,m}[X^{\nu}(T)+\kappa Y^{\nu}(T)]\Big\}, \tag{3.2}$$

with the boundary condition $V(T,x,y,m)=x+\kappa y$.

Remark 3. If we take a convex combination of $X^{\nu}(T)$ and $Y^{\nu}(T)$ in the reward function, the value function $V(t,x,y,m)$ can be written as

$$\begin{aligned}
V&(t,x,y,m)\\
&=\sup_{\nu\in\mathcal{A}}\Big\{\mathrm{E}_{t,x,y,m}[(1-\tilde{\kappa})X^{\nu}(T)+\tilde{\kappa}Y^{\nu}(T)]\\
&\quad-\frac{\gamma}{2}\mathrm{Var}_{t,x,y,m}[(1-\tilde{\kappa})X^{\nu}(T)+\tilde{\kappa}Y^{\nu}(T)]\Big\},\\
&=(1-\tilde{\kappa})\cdot\sup_{\nu\in\mathcal{A}}\Big\{\mathrm{E}_{t,x,y,m}\Big[X^{\nu}(T)+\frac{\tilde{\kappa}}{1-\tilde{\kappa}}Y^{\nu}(T)\Big]\\
&\quad-\frac{\gamma(1-\tilde{\kappa})}{2}\mathrm{Var}_{t,x,y,m}\Big[X^{\nu}(T)+\frac{\tilde{\kappa}}{1-\tilde{\kappa}}Y^{\nu}(T)\Big]\Big\},
\end{aligned}$$

where $\tilde{\kappa}\in(0,1)$ represents the weight in the convex combination. Thus, one can simply replace the parameters κ and γ in (3.2) with $\frac{\tilde{\kappa}}{1-\tilde{\kappa}}$ and $\gamma(1-\tilde{\kappa})$, respectively, to obtain the optimal results.

We transform this optimization problem to the one with finite-dimensional space by imposing the following assumption of the model parameters.

Assumption 1. Throughout the paper, we assume that

$$C=\kappa e^{-\varphi h},\quad Be^{-\varphi h}=(\varphi+A+\kappa)C. \tag{3.3}$$

Recall that $A=r-\overline{B}-C$ and $B=\frac{\overline{B}}{\int_{-h}^{0}e^{\varphi s}ds}$, together with the condition (3.3), it is straightforward to show that

$$\begin{cases}A=r-\frac{(\varphi+r+\kappa-\kappa e^{-\varphi h})\kappa(1-e^{-\varphi h})}{\varphi+\kappa(1-e^{-\varphi h})}-\kappa e^{-\varphi h},\\ B=\frac{(\varphi+r+\kappa-\kappa e^{-\varphi h})\kappa\varphi}{\varphi+\kappa(1-e^{-\varphi h})},\\ C=\kappa e^{-\varphi h}.\end{cases}$$

The condition (3.3) plays a crucial role in deriving the explicit solution for the reinsurance-investment problem with delay. Under (3.3), we can derive the value function V, which does not depend on m. Note that if the value function V depends on m, the differential form

$$\begin{aligned}\frac{\mathrm{d}M^{\nu}}{\mathrm{d}t}&=\lim_{\triangle t\to0}\frac{M^{\nu}(t+\triangle t)-M^{\nu}(t)}{\triangle t}\\&=\lim_{\triangle t\to0}\frac{X^{\nu}(t-h+\triangle t)-X^{\nu}(t-h)}{\triangle t}\end{aligned}$$

does not exist due to the non-differentiable path of Brownian motion. In this case, it is unable to write the extended HJB delay system w.r.t. (t,x,y,m); see Eqs. (3.4) and (3.5). Furthermore, it is one of the sufficient conditions to guarantee the closed-form solution to the extended HJB delay system. In fact, we will show later that a candidate solution independent of m is indeed our value function. In short, under Assumption 1, we actually assume that the value function depends on (t,x,y) only, that is,

$$V(t,x,y,m)=V(t,x,y).$$

Similar assumptions can also be found in A & Li (2015); Bai et al. (2022); Deng, Bian, & Wu (2020); Shen & Zeng (2014) and Li et al. (2023).

Remark 4. Although Assumption 1 seems kind of restrictive, it not only makes the control problem solvable, but also has some sort of economic interpretation. In a certain situation, we can investigate the effect of the delay factors on the strategy with rational parameter settings. As mentioned in Deng et al. (2020), the insurer can first select the weighting coefficient φ and the delay time h to calculate the integrated delayed wealth $Y^{\nu}(t)$ and the pointwise wealth $M^{\nu}(t)$. Then the insurer chooses the weighting parameter κ in the mean-variance performance measure. Finally, using the condition (3.3), the insurer determines the parameters $\overline{B}$ and C to adjust the exogenous rates of inflow and outflow accordingly. In particular, when $\kappa=h=0$, the optimal control problem reduces to the one without delay.

As we know, one of the useful methods to solve the stochastic control problems is the dynamic programming principle (DPP). However, the optimization problem (3.2) is time-inconsistent. Specifically, we have

$$\begin{aligned}
J(t,x,y;\nu)&=\mathrm{E}_{t,x,y}\Big[X^{\nu}(T)+\kappa Y^{\nu}(T)-\frac{\gamma}{2}(X^{\nu}(T)+\kappa Y^{\nu}(T))^2\Big]\\
&\quad+\frac{\gamma}{2}(\mathrm{E}_{t,x,y}[X^{\nu}(T)+\kappa Y^{\nu}(T)])^2\\
&=\mathrm{E}_{t,x,y}\Big[\mathrm{E}_{s,x_s,y_s}\Big[X^{\nu}(T)+\kappa Y^{\nu}(T)-\frac{\gamma}{2}(X^{\nu}(T)+\kappa Y^{\nu}(T))^2\Big]\\
&\quad+\frac{\gamma}{2}(\mathrm{E}_{s,x_s,y_s}[X^{\nu}(T)])^2\Big]\\
&\quad-\mathrm{E}_{t,x,y}\Big[\frac{\gamma}{2}(\mathrm{E}_{s,x_s,y_s}[X^{\nu}(T)+\kappa Y^{\nu}(T)])^2\Big]\\
&\quad+\frac{\gamma}{2}(\mathrm{E}_{t,x,y}[X^{\nu}(T)+\kappa Y^{\nu}(T)])^2\\
&=\mathrm{E}_{t,x,y}[J(s,x_s,y_s;\nu)]\\
&\quad-\mathrm{E}_{t,x,y}\Big[\frac{\gamma}{2}(\mathrm{E}_{s,x_s,y_s}[X^{\nu}(T)+\kappa Y^{\nu}(T)])^2\Big]\\
&\quad+\frac{\gamma}{2}(\mathrm{E}_{t,x,y}[X^{\nu}(T)+\kappa Y^{\nu}(T)])^2\\
&\neq\mathrm{E}_{t,x,y}[J(s,x_s,y_s;\nu)],
\end{aligned}$$

for all $t<s<T$ with $X^{\nu}(s)=x_s$ and $Y^{\nu}(s)=y_s$. Hence, the Bellman optimality principle fails in our model because of the nonlinear term of conditional expectation (Björk & Murgoci, 2010; Björk

et al., 2014; Pun, 2018). In other words, we encounter a time-inconsistent problem, which implies that the optimal control law for some fixed initial point is no longer optimal at some later point. To tackle this problem, one can simply disregard the "time inconsistency", and find the pre-commitment control by the DPP and HJB equation. There is a large body of work on mean-variance portfolio selection studying pre-committed strategies; for instance, see Bi & Guo (2013); Bielecki, Jin, Pliska, & Zhou (2005); Ming et al. (2016); Zhou & Li (2000) and Yuan et al. (2022c).

To seek the time-consistent equilibrium strategy for the optimization problem (3.2), we formulate the time-inconsistent dynamic optimization problem into a noncooperative game theoretic framework proposed by Björk & Murgoci (2010), and then find the subgame perfect Nash equilibrium point. The game can be interpreted in the way that we have a fictitious player for each point at time t, and that each player can only control the state process at time t. A time-consistent strategy is subgame perfect Nash equilibrium if for any $t \in [0, T]$, the strategy chosen by Player $s \in [t, T]$ is also optimal for Player t. As such, our objective is to solve for equilibrium risk-bearing strategies which are defined as follows.

Definition 2 (Equilibrium strategy and value function)**.** For any chosen initial state $(t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R}$, a fixed strategy $\nu = (\mathcal{H}_1, \mathcal{H}_2, \pi)$ and an admissible strategy $\nu^* = (\mathcal{H}_1^*, \mathcal{H}_2^*, \pi^*)$, we define a perturbed strategy $\nu^\varepsilon = (\mathcal{H}_1^\varepsilon, \mathcal{H}_2^\varepsilon, \pi^\varepsilon)$ as

$$\nu^\varepsilon = (\mathcal{H}_1^\varepsilon(s, z_1), \mathcal{H}_2^\varepsilon(s, z_2), \pi^\varepsilon(s)) = \begin{cases} (\mathcal{H}_1(s, z_1), \mathcal{H}_2(s, z_2), \pi(s)), & s \in [t, t+\varepsilon], \\ (\mathcal{H}_1^*(s, z_1), \mathcal{H}_2^*(s, z_2), \pi^*(s)), & s \in [t+\varepsilon, T], \end{cases}$$

for any $t \in [0, T)$ and a fixed real number $\varepsilon > 0$. If

$$\liminf_{\varepsilon \to 0^+} \frac{J(t, x, y; \nu^*) - J(t, x, y; \nu^\varepsilon)}{\varepsilon} \geqslant 0$$

holds for any admissible control $\nu = (\mathcal{H}_1, \mathcal{H}_2, \pi) \in \mathcal{A}$, then $\nu^* = (\mathcal{H}_1^*, \mathcal{H}_2^*, \pi^*)$ is called an equilibrium control. The resulting equilibrium value function $V(t, x, y)$ is then given by

$$V(t, x, y) = J(t, x, y; \nu^*).$$

Let $\mathcal{D} = [0, T] \times \mathbb{R}^2$ and

$$C^{1,2,1}(\mathcal{D}) = \left\{ \phi(t, x, y) \;\middle|\; \begin{array}{l} \phi(t, \cdot, \cdot) \text{ is once continuously differentiable on } [0, T], \\ \phi(\cdot, \cdot, y) \text{ is once continuously differentiable on } \mathbb{R}, \\ \text{and } \phi(\cdot, x, \cdot) \text{ is twice continuously differentiable on } \mathbb{R}. \end{array} \right\}$$

For any $\phi(t, x, y) \in C^{1,2,1}(\mathcal{D})$, the variational operator $\mathcal{L}^\nu$ is defined as

$$\begin{aligned} \mathcal{L}^\nu \phi(t, x, y) = \phi_t + \Big(& Ax + By + Cz + c - \delta(\mathcal{H}_1(t, Z_1), \mathcal{H}_2(t, Z_2)) \\ & - \lambda p_1 \mathrm{E}[\mathcal{H}_1(t, Z_1)] \\ & - \lambda p_2 \mathrm{E}[\mathcal{H}_2(t, Z_2)] + (\mu - r)\pi(t) \Big) \phi_x \\ & + \big(x - \varphi y - e^{-\varphi h} m\big) \phi_y + \frac{1}{2} \Big(\lambda p_1 \mathrm{E}[\mathcal{H}_1(t, Z_1)^2] \\ & + \lambda p_2 \mathrm{E}\big[\mathcal{H}_2(t, Z_2)^2\big] + 2\lambda p_1 p_2 \mathrm{E} \\ & [\mathcal{H}_1(t, Z_1)] \mathrm{E}[\mathcal{H}_2(t, Z_2)] + \sigma^2 \pi(t)^2 \Big) \phi_{xx} \\ & + \lambda_3 \mathrm{E}[\phi(t, x + \pi l, y) - \phi(t, x, y)], \end{aligned} \tag{3.4}$$

where the notations ϕ_t, ϕ_x, ϕ_y and ϕ_{xx} represent the first and second-order partial derivatives w.r.t. the corresponding variables.

We next present the extended HJB delay system for the characterization of the value function V and the corresponding equilibrium strategy in Theorem 1. The proof of this theorem is standard, and thus we simply omit it. The readers are referred to the Theorem 5.2 of Björk, Khapko, & Murgoci (2017) for more details.

Theorem 1 (Verification Theorem)**.** *For the optimization problem (3.2), suppose that there are two functions $V(t, x, y) \in C^{1,2,1}(\mathcal{D})$ and $g(t, x, y) \in C^{1,2,1}(\mathcal{D})$ satisfying the following extended HJB delay system:*

$$\begin{cases} \sup_{\nu \in \mathcal{A}} \left\{ \mathcal{L}^\nu V(t, x, y) - \frac{\gamma}{2} \mathcal{L}^\nu [g(t, x, y)^2] + \gamma g(t, x, y) \cdot \mathcal{L}^\nu g(t, x, y) \right\} = 0, \\ \mathcal{L}^{\nu^*} g(t, x, y) = 0, \\ V(T, x, y) = x + \kappa y, \\ g(T, x, y) = x + \kappa y, \end{cases} \tag{3.5}$$

where

$$\nu^* = \arg\sup_{\nu \in \mathcal{A}} \left\{ \mathcal{L}^\nu V(t, x, y) - \frac{\gamma}{2} \mathcal{L}^\nu [g(t, x, y)^2] + \gamma g(t, x, y) \cdot \mathcal{L}^\nu g(t, x, y) \right\}.$$

Then ν^* is the equilibrium strategy, and $V(t, x, y)$ is the corresponding value function.

Based on the terminal conditions of V and g, we conjecture that the solution to (3.5) has the following form:

$$\begin{cases} V(t, x, y) = e^{(A+\kappa)(T-t)}(x + \kappa y) + Q(t), \\ g(t, x, y) = e^{(A+\kappa)(T-t)}(x + \kappa y) + q(t), \end{cases}$$

where $Q(T) = q(T) = 0$. Let Q_t and q_t denote the first-order derivatives of $Q(t)$ and $q(t)$, respectively. A direct calculation yields

$$\begin{cases} V_t = -(A+\kappa) e^{(A+\kappa)(T-t)}(x + \kappa y) + Q_t, \quad g_t = V_t + q_t - Q_t, \\ V_x = g_x = e^{(A+\kappa)(T-t)}, \qquad V_y = g_y = \kappa e^{(A+\kappa)(T-t)}. \end{cases} \tag{3.6}$$

Note that the choice of the reinsurance premium rate δ determines the explicit expressions for the uncertain functions $Q(t)$ and $q(t)$. In the next two sections, we derive the value function under the expected value principle as well as the variance principle.

4. Optimal results under expected value principle

In this section, we derive the explicit solution to the equilibrium strategy and value function for the optimization problem (3.2) under the expected value principle, i.e., the reinsurance premium rate at time t is given by

$$\begin{aligned} \delta(\mathcal{H}_1(t, Z_1), \mathcal{H}_2(t, Z_2)) = & (1+\eta_1)\lambda p_1 \mathrm{E}[Z_1 - \mathcal{H}_1(t, Z_1)] \\ & + (1+\eta_2)\lambda p_2 \mathrm{E}[Z_2 - \mathcal{H}_2(t, Z_2)], \end{aligned}$$

where η_1 and η_2 are reinsurer's safety loadings for the two classes of the insurance business. Without loss of generality, we assume that $\eta_1 \geqslant \eta_2$ since the results for the case of $\eta_1 < \eta_2$ can be obtained along the same lines.

Plugging the derivatives in (3.6) into the first equation of (3.5), we have

$$\begin{aligned} Q_t \; - \; & (A+\kappa) e^{(A+\kappa)(T-t)}(x + \kappa y) + e^{(A+\kappa)(T-t)} \Big(Ax + By + Cm \\ & + c - (1+\eta_1)\lambda p_1 a_1 - (1+\eta_2)\lambda p_2 a_2 \Big) \\ & + \big(x - \varphi y - e^{-\varphi h} m\big) \kappa e^{(A+\kappa)(T-t)} \\ & + e^{(A+\kappa)(T-t)} \sup_{\nu \in \mathcal{A}} \Big\{ \eta_1 \lambda p_1 \mathrm{E}[\mathcal{H}_1(t, Z_1)] \\ & + \eta_2 \lambda p_2 \mathrm{E}[\mathcal{H}_2(t, Z_2)] + (\mu - r)\pi(t) \\ & - \frac{\gamma}{2} e^{(A+\kappa)(T-t)} \Big(\lambda p_1 \mathrm{E}[\mathcal{H}_1(t, Z_1)^2] + \lambda p_2 \mathrm{E}[\mathcal{H}_2(t, Z_2)^2] \\ & + 2\lambda p_1 p_2 \mathrm{E}[\mathcal{H}_1(t, Z_1)] \mathrm{E}[\mathcal{H}_2(t, Z_2)] + \sigma^2 \pi(t)^2 + \lambda_3 \sigma_L^2 \pi(t)^2 \Big) \Big\} \end{aligned}$$

$$= 0.$$

Using the condition (3.3), we have

$$Q_t + e^{(A+\kappa)(T-t)}(c-(1+\eta_1)\lambda p_1 a_1-(1+\eta_2)\lambda p_2 a_2) + e^{(A+\kappa)(T-t)}\sup_{\nu\in\mathcal{A}}\Big\{\eta_1\lambda p_1 \mathrm{E}[\mathcal{H}_1(t,Z_1)] + \eta_2\lambda p_2\mathrm{E}[\mathcal{H}_2(t,Z_2)]+(\mu-r)\pi(t) - \frac{\gamma}{2}e^{(A+\kappa)(T-t)}\Big(\lambda p_1\mathrm{E}[\mathcal{H}_1(t,Z_1)^2]+\lambda p_2\mathrm{E}[\mathcal{H}_2(t,Z_2)^2] + 2\lambda p_1p_2\mathrm{E}[\mathcal{H}_1(t,Z_1)]\mathrm{E}[\mathcal{H}_2(t,Z_2)]+\sigma_1^2\pi(t)^2\Big)\Big\}=0,$$

with $\sigma_1=\sqrt{\sigma^2+\lambda_3\sigma_L^2}$. It is clear that the above equation has a solution which does not depend on m under Assumption 1. Define a related function G as[2]

$$\begin{aligned}G(t,\mathcal{H}_1,\mathcal{H}_2,\pi) &= \eta_2\lambda p_2\mathrm{E}[\mathcal{H}_2]-\frac{\gamma}{2}\lambda p_2 e^{(A+\kappa)(T-t)}\mathrm{E}[\mathcal{H}_2^2] \\ &+(\mu-r)\pi-\frac{\gamma}{2}e^{(A+\kappa)(T-t)}\sigma_1^2\pi^2 \\ &+\int_0^\infty\Big(\lambda p_1\eta_1\mathcal{H}_1(z_1)-\frac{\gamma}{2}e^{(A+\kappa)(T-t)}\big(\lambda p_1\mathcal{H}_1(z_1)^2 \\ &+2\lambda p_1p_2\mathcal{H}_1(z_1)\mathrm{E}[\mathcal{H}_2]\big)\Big)\mathrm{d}F_{Z_1}(z_1) \\ &=\eta_1\lambda p_1\mathrm{E}[\mathcal{H}_1]-\frac{\gamma}{2}\lambda p_1 e^{(A+\kappa)(T-t)}\mathrm{E}[\mathcal{H}_1^2] \\ &+(\mu-r)\pi-\frac{\gamma}{2}e^{(A+\kappa)(T-t)}\sigma_1^2\pi^2 \\ &+\int_0^\infty\Big(\lambda p_2\eta_2\mathcal{H}_2(z_2)-\frac{\gamma}{2}e^{(A+\kappa)(T-t)}\big(\lambda p_2\mathcal{H}_2(z_2)^2 \\ &+2\lambda p_1p_2\mathcal{H}_2(z_2)\mathrm{E}[\mathcal{H}_1]\big)\Big)\mathrm{d}F_{Z_2}(z_2). \end{aligned}\quad (4.1)$$

From the integral representation of G, we can deduce that maximizing the function G is equivalent to maximizing the integrand point-by-point, subject to $0\leqslant\mathcal{H}_i(t,z_i)\leqslant z_i$ $(i=1,2)$. As a function of $\mathcal{H}_1$, $\mathcal{H}_2$ and π, the integrand is a parabola, so it is maximized by

$$\mathcal{H}_1^*(t,z_1)=d_1^*(t)\wedge z_1,\qquad \mathcal{H}_2^*(t,z_2)=d_2^*(t)\wedge z_2, \quad (4.2)$$

and

$$\pi^*(t)=\frac{u-r}{\gamma\sigma_1^2}e^{-(A+\kappa)(T-t)}, \quad (4.3)$$

where

$$\begin{cases} d_1^*(t)=\frac{\eta_1}{\gamma}e^{-(A+\kappa)(T-t)}-p_2\mathrm{E}[\mathcal{H}_2^*(t,z_2)],\\ d_2^*(t)=\frac{\eta_2}{\gamma}e^{-(A+\kappa)(T-t)}-p_1\mathrm{E}[\mathcal{H}_1^*(t,z_1)].\end{cases} \quad (4.4)$$

Here, $d_1^*(t)$ and $d_2^*(t)$ represent the risk retention levels, and are admissible if both $d_1^*(t)$ and $d_2^*(t)$ are non-negative.

We see from the expressions for $\mathcal{H}_1^*(t,z_1)$ and $\mathcal{H}_2^*(t,z_2)$ that the strategy (0,0) can never be optimal. This can be proved by reduction to absurdity. If $\mathcal{H}_1^*(t,z_1)=0$, then it follows from (4.4) that $d_2^*(t)=\frac{\eta_2}{\gamma}e^{-(A+\kappa)(T-t)}>0$ such that $\mathcal{H}_2^*(t,z_2)=d_2^*(t)\wedge z_2>0$, and vice versa.

For notational convenience, we define the following auxiliary functions

$$h_{Z_1}(d)=\mathrm{E}[d\wedge Z_1]=\int_0^d S_{Z_1}(z_1)\mathrm{d}z_1,$$

$$h_{Z_2}(d)=\mathrm{E}[d\wedge Z_2]=\int_0^d S_{Z_2}(z_2)\mathrm{d}z_2,$$

where $S_{Z_1}(z_1)=1-F_{Z_1}(z_1)$ and $S_{Z_2}(z_2)=1-F_{Z_2}(z_2)$. Then it follows from (4.2) that the extreme point (d_1^*,d_2^*) should satisfy the following equations

$$\begin{cases}\eta_1-\gamma[e^{(A+\kappa)(T-t)}d_1^*+p_2e^{(A+\kappa)(T-t)}h_{Z_2}(d_2^*)]=0,\\ \eta_2-\gamma[e^{(A+\kappa)(T-t)}d_2^*+p_1e^{(A+\kappa)(T-t)}h_{Z_1}(d_1^*)]=0.\end{cases} \quad (4.5)$$

Note that the Hessian matrix of G at point $(\mathcal{H}_1^*,\mathcal{H}_2^*,\pi^*)$ is given by

$$-\gamma e^{(A+\kappa)(T-t)}\begin{pmatrix}\lambda p_1S_{Z_1}(d_1^*) & \lambda p_1p_2S_{Z_1}(d_1^*)S_{Z_2}(d_2^*) & 0\\ \lambda p_1p_2S_{Z_1}(d_1^*)S_{Z_2}(d_2^*) & \lambda p_2S_{Z_2}(d_2^*) & 0\\ 0 & 0 & \sigma_1^2\end{pmatrix},$$

which is clearly negative definite. Hence, if we can find the point (d_1^*,d_2^*) such that the Eq. (4.5) holds, then the point (d_1^*,d_2^*,π^*) is the extreme maximum point of G. The following lemma shows the existence and uniqueness of the solution to the system of equations (4.5).

Lemma 1. *There exists a point (d_1^*,d_2^*) solving the system of equations (4.5), and this point is unique.*

Proof. See Appendix A. □

Although the existence and uniqueness have been given in Lemma 1, the specific form of the solution to the optimization problem remains to be investigated. In order to determine the extreme maximum point (d_1^*,d_2^*) from the system of Eq. (4.5), we transform (4.5) into

$$\frac{d_1^*+p_2h_{Z_2}(d_2^*)}{d_2^*+p_1h_{Z_1}(d_1^*)}=\frac{\eta_1}{\eta_2}.$$

A simple algebraic manipulation yields

$$\eta_2d_1^*-\eta_1p_1h_{Z_1}(d_1^*)=\eta_1d_2^*-\eta_2p_2h_{Z_2}(d_2^*).$$

To continue our analysis, we need the following auxiliary functions

$$\psi_{Z_1}(x)=\eta_2x-\eta_1p_1h_{Z_1}(x),\qquad \psi_{Z_2}(x)=\eta_1x-\eta_2p_2h_{Z_2}(x), \quad (4.6)$$

and

$$K(x)=\psi_{Z_2}^{-1}(\psi_{Z_1}(x))+p_1h_{Z_1}(x)-\frac{\eta_2}{\gamma}e^{-(A+\kappa)(T-t)}. \quad (4.7)$$

One can verify that $\psi_{Z_1}(d_1^*)=\psi_{Z_2}(d_2^*)$ and $\psi_{Z_1}(0)=\psi_{Z_2}(0)$. Since $\eta_1\geqslant\eta_2$, $\psi_{Z_2}(x)$ is a strictly increasing function of x, and thus the inverse function $\psi_{Z_2}^{-1}(x)$ exists and strictly increases with x. Then it follows from $\psi_{Z_1}(d_1^*)=\psi_{Z_2}(d_2^*)$ that

$$d_2^*=\psi_{Z_2}^{-1}(\psi_{Z_1}(d_1^*)). \quad (4.8)$$

Substituting (4.8) into the second equation of (4.5) yields $K(d_1^*)=0$. In fact, we only need to show that the equation $K(x)=0$ has a solution on $(0,\infty)$, and that the solution is indeed $d_1^*(t)$. The next lemma gives the monotonicity of the function $\psi_{Z_1}(x)$.

Lemma 2. *Under the assumption of $\eta_1\geqslant\eta_2$, the following two statements hold:*

(i) *when $0\leqslant p_1\leqslant\frac{\eta_2}{\eta_1}$, $\psi_{Z_1}(x)$ is a strictly increasing function on $(0,\infty)$;*
(ii) *when $\frac{\eta_2}{\eta_1}<p_1\leqslant 1$, there exists a point $\bar{x}$ such that $\psi_{Z_1}(x)$ is strictly increasing on $(\bar{x},\infty)$.*

Proof. See Appendix B. □

Noting that $\psi_{Z_1}(0)=0$ and $\psi_{Z_1}(\infty)=\infty$, we define

$$x_0=\sup\{x\geqslant 0,\psi_{Z_1}(x)=0\}, \quad (4.9)$$

with $0\leqslant x_0<\infty$. In fact, whether x_0 equals to 0 or not depends on the relation between p_1 and $\frac{\eta_2}{\eta_1}$. Then it follows from (4.6), (4.7) that $h_{Z_1}(x_0)=\frac{\eta_2}{\eta_1p_1}x_0$ and

$$K(x_0)=\eta_2\left[\frac{x_0}{\eta_1}-\frac{1}{\gamma}e^{-(A+\kappa)(T-t)}\right]. \quad (4.10)$$

[2] To simplify our notation, we suppress the argument t of the strategies.

To summarize, we give the equilibrium strategy and the corresponding value function of the problem (3.2) in Theorem 2.

Theorem 2. *Suppose that $\eta_1 \geqslant \eta_2$. Recall the functions G of (4.1), ψ_{Z_1} and ψ_{Z_2} of (4.6) and the point x_0 of (4.9). The optimal reinsurance-investment strategy for the problem (3.2) is given by*

$$\begin{cases} \mathcal{H}_1^*(t,z_1) = d_1^*(t) \wedge z_1, \quad \mathcal{H}_2^*(t,z_2) = \psi_{Z_2}^{-1}(\psi_{Z_1}(d_1^*(t))) \wedge z_2, \\ \pi^*(t) = \frac{u-r}{\gamma\sigma_1^2} e^{-(A+\kappa)(T-t)} \end{cases} \tag{4.11}$$

for $x_0 \leqslant \frac{\eta_1}{\gamma} e^{-(A+\kappa)(T-t)}$, and

$$\begin{cases} \mathcal{H}_1^*(t,z_1) = \frac{\eta_1}{\gamma} e^{-(A+\kappa)(T-t)} \wedge z_1, \quad \mathcal{H}_2^*(t,z_2) = 0, \\ \pi^*(t) = \frac{u-r}{\gamma\sigma_1^2} e^{-(A+\kappa)(T-t)} \end{cases} \tag{4.12}$$

for $x_0 > \frac{\eta_1}{\gamma} e^{-(A+\kappa)(T-t)}$. Here $d_1^*(t) \in (x_0, \infty)$ uniquely solves the equation

$$\psi_{Z_2}^{-1}(\psi_{Z_1}(d_1)) + p_1 h_{Z_1}(d_1) - \frac{\eta_2}{\gamma} e^{-(A+\kappa)(T-t)} = 0.$$

Furthermore, the value function is

$$V(t,x,y) = e^{(A+\kappa)(T-t)}(x+\kappa y) + Q(t), \tag{4.13}$$

where

$$\begin{aligned} Q(t) = &\frac{1}{A+\kappa}[c - (1+\eta_1)\lambda p_1 a_1 - (1+\eta_2)\lambda p_2 a_2] \\ &+ \int_t^T e^{(A+\kappa)(T-s)} \Big\{ \eta_1 \lambda p_1 \mathrm{E}[\mathcal{H}_1^*(s,Z_1)] \\ &+ \eta_2 \lambda p_2 \mathrm{E}[\mathcal{H}_2^*(s,Z_2)] - \frac{\gamma}{2} e^{(A+\kappa)(T-s)} \Big(\lambda p_1 \mathrm{E}[\mathcal{H}_1^*(s,Z_1)^2] \\ &+ \lambda p_2 \mathrm{E}[\mathcal{H}_2^*(s,Z_2)^2] \\ &+ 2\lambda p_1 p_2 \mathrm{E}[\mathcal{H}_1^*(s,Z_1)] \mathrm{E}[\mathcal{H}_2^*(s,Z_2)] \Big) \Big\} \mathrm{d}s \\ &- \frac{1}{\gamma} \frac{(\mu - r)^2}{2\sigma_1^2} (T-t). \end{aligned}$$

Proof. In view of the fact that the optimal investment strategy (4.3) is independent of the reinsurance strategy, we focus on the optimal reinsurance strategies in two different cases.

(i) From (4.10), we know that if $x_0 \leqslant \frac{\eta_1}{\gamma} e^{-(A+\kappa)(T-t)}$, the inequality $K(x_0) \leqslant 0$ holds. Note that $\lim_{x\to\infty} K(x) = \infty$ and that when $x > x_0$, we have

$$\frac{\partial K(x)}{\partial x} = \frac{\partial \psi_{Z_2}^{-1}(\psi_{Z_1}(x))}{\partial \psi_{Z_1}(x)} \cdot \frac{\partial \psi_{Z_1}(x)}{\partial x} + p_1 \frac{\partial h_{Z_1}(x)}{\partial x} > 0,$$

which means that $K(x)$ is an increasing function on $[x_0, \infty)$. Combining these with $K(x_0) \leqslant 0$ shows that the equation $K(x) = 0$ admits a unique solution $d_1^*(t) \in [x_0, \infty)$. Furthermore, d_2^* is given by (4.8).

(ii) If $x_0 > \frac{\eta_1}{\gamma} e^{-(A+\kappa)(T-t)}$, then the inequality $K(x_0) > 0$ holds. Since $K(x)$ is strictly increasing on $[x_0, \infty)$ and $K(x) > 0$ on $[x_0, \infty)$, the equation $K(x) = 0$ has no solution on $[x_0, \infty)$, i.e., there does not exist $d_1^* \in [x_0, \infty)$ and $d_2^* \in [0, \infty)$ satisfying the system of Eq. (4.5). If the solution to $K(x) = 0$ exists, it can occur on $[0, x_0)$ only. It follows from Lemma 2 and the definition of x_0 that $\psi_{Z_1}(x)$ is a convex function of x and $\psi_{Z_1}(x_0) = 0$. As a result, we have $\psi_{Z_1}(x) < 0$ on $[0, x_0)$. Then one can show that $d_2^* = 0 \vee \psi_{Z_2}^{-1}(\psi_{Z_1}(d_1^*)) = 0$. Putting $\mathcal{H}_2^* = 0$ and π^* of (4.3) into the extended HJB delay system (3.5) and using the first-order condition, we conclude that

$$\mathcal{H}_1^*(t,z_1) = \frac{\eta_1}{\gamma} e^{-(A+\kappa)(T-t)} \wedge z_1.$$

Combining the results of (i) and (ii), one can obtain the optimal reinsurance and investment strategies given by (4.11) and (4.12).

It is straightforward to verify that the strategies given by (4.11) and (4.12) satisfy the conditions (i)–(iii) in Definition 1. The remaining item to show is that, under the reinsurance and investment strategies in (4.11) and (4.12), the stochastic differential Eq. (2.2) has a unique strong solution. It follows from Theorem 5.2.9 in Karatzas & Shreve (1991) that this holds if the drift and volatility of (2.2) under these strategies have bounded derivatives w.r.t. x. Note that the drift and volatility are bounded for any strategies defined in (4.11) and (4.12) with $0 < T < \infty$. As all these strategies are independent of the wealth process, one can easily show that the drift and volatility of (2.2) have bounded derivatives equal to A and 0, respectively. Thus, the drift and volatility of the optimally controlled diffusion of the wealth process are bounded and Lipschitz. This gives the desired results.

Substituting $(\mathcal{H}_1^*, \mathcal{H}_2^*, \pi^*)$ into the first equation of the extended HJB delay system (3.5) yields

$$\begin{aligned} Q_t &+ e^{(A+\kappa)(T-t)}[c - (1+\eta_1)\lambda p_1 a_1 - (1+\eta_2)\lambda p_2 a_2 \\ &+ G(t, \mathcal{H}_1^*, \mathcal{H}_2^*, \pi^*)] = 0. \end{aligned}$$

By simple integration, one can derive the value function of (4.13) directly. This completes the proof. □

Remark 5. Note that when $\eta_1 = \eta_2$, i.e., the reinsurance safety loadings are the same for the two classes of insurance business, it follows from Lemma 2 that $p_1 \leqslant \frac{\eta_2}{\eta_1}$, which implies that $\psi_{Z_1}(x)$ is a strictly increasing function on $(0, \infty)$ with $x_0 = 0$. Thus, the inequality $x_0 \leqslant \frac{\eta_1}{\gamma} e^{-(A+\kappa)(T-t)}$ always holds, and the optimal results for $\eta_1 = \eta_2$ are given in (4.11).

Remark 6. It is clear that the optimal reinsurance strategies in (4.11) and (4.12) are in the form of excess-of-loss reinsurance. This observation coincides with the one in Han et al. (2021) even though a different criterion is considered in their paper. Moreover, we can see that the retained and transferred claims are non-decreasing functions of the underlying claim, which ensures that the insurer and reinsurer would not confront with the situation of moral hazard. Similar to Basak & Chabakauri (2010); Björk et al. (2014); Zhang & Liang (2016) and Chen & Shen (2019), we would like to point out that the optimal strategy is independent of the wealth level x and the claim intensity λ. In relation to these works, we generalize the corresponding results to the case with the inclusion of the thinning-dependence structure and historical information. In particular, by setting $p_2 = B = C = \kappa = 0$ and $p_1 = 1$, the risk model is reduced to the one without dependent risks and delay factors. In this case, the optimal results can be found in Proposition 4.2 of Chen & Shen (2019).

In the following propositions, we present some important properties of the optimal reinsurance-investment strategy. Here the expressions for $(\mathcal{H}_1^*, \mathcal{H}_2^*, \pi^*)$ are given in (4.11) and (4.12). When writing "increases" or "decreases", we mean in the weak, or non-strict sense.

Proposition 1. *Let $(\mathcal{H}_1^*, \mathcal{H}_2^*, \pi^*)$ be given by (4.11) and (4.12). The following statements hold:*

(i) π^* increases with μ but decreases with σ_1 and γ;
(ii) $\mathcal{H}_i^*$ $(i = 1, 2)$ increases with η_i, while the monotonic direction of η_j $(j = 1, 2, j \neq i)$ is opposite;
(iii) $\mathcal{H}_i^*$ $(i = 1, 2)$ increases with γ.

Proof. See Appendix C. □

We see from (4.11), (4.12) that the optimal investment strategy is independent of the parameters of the insurance market. An obvious explanation is that the insurance market and risky asset are

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not correlated. However, the return and volatility do have influence on $\pi^*(t)$. The larger the μ, the more aggressive the insurer invests into the risky asset. Meanwhile, a higher value of σ_1 gives a smaller amount of the wealth invested into the risky asset. Note that $\sigma_1 = \sqrt{\sigma^2 + \lambda_3\sigma_L^2}$ implies that π decreases w.r.t. σ, λ_3 and σ_L. This can be explained by the fact that larger values of σ, σ_L and λ_3 imply greater uncertainty. Besides, when η_i increases, the reinsurer charges a relative expensive reinsurance premium for class *i*, and hence the insurer tends to decrease the purchase of reinsurance for class *i*. Meanwhile, when the insurer keeps buying less reinsurance for one class, it eventually needs to reduce the risk of the whole insurance portfolio by buying a bit more reinsurance for the other class. Moreover, a larger value of γ implies that the insurer is more risk averse, and becomes more cautious about the underlying risks. As a result, the insurer is inclined to take a conservative reinsurance strategy by buying more reinsurance, and to invest less into the risky asset to reduce the investment risk.

In the following proposition, we investigate the monotonicity of the optimal strategy w.r.t. the delay parameters κ and *h*. As was noted before, when $\kappa = h = 0$, the optimization problem becomes the one without delay.

Proposition 2. *Under the assumption of $\eta_1 \geqslant \eta_2$, the three optimal strategies in $(\mathcal{H}_1^*, \mathcal{H}_2^*, \pi^*)$ are all increasing w.r.t. the parameters κ and h.*

Proof. See Appendix D. □

Proposition 2 reveals that both the reinsurance strategy and the investment strategy increase as the delay parameters *h*, and κ increase. The parameter *h* is the duration of the past that the insurer usually cares about, and the parameter κ indicates the degree of impact of the historical average performance on the final performance. Intuitively, a larger value of *h* accounts for a longer time horizon under consideration. This in turn yields a relatively stable average delayed wealth, and hence the insurer's ability to handle the underlying risks will be enhanced. As a result, the insurer is able to increase the amount invested in risky assets as well as the retention levels $\mathcal{H}_1^*$ and $\mathcal{H}_2^*$. On the other hand, when κ increases, a larger weight attached to the average delayed wealth $Y^\nu(t)$ would reduce the overall risk of the insurer. Then the insurer can choose a relatively risky strategy, i.e., purchasing less reinsurance business and/or investing a larger amount into the risky asset to achieve the same level of final performance.

5. Optimal results under variance principle

In this section, we investigate the same optimization problem under the variance principle based on which the reinsurance premium rate can be expressed as

$$\begin{aligned}\delta(\mathcal{H}_1(t,Z_1),\mathcal{H}_2(t,Z_2)) &= \lambda p_1\mathrm{E}[Z_1-\mathcal{H}_1(t,Z_1)]\\ &+\lambda p_2\mathrm{E}[Z_2-\mathcal{H}_2(t,Z_2)]\\ &+\frac{\Lambda}{2}\Big(\lambda p_1\mathrm{E}[(Z_1-\mathcal{H}_1(t,Z_1))^2]+\lambda p_2\mathrm{E}[(Z_2-\mathcal{H}_2(t,Z_2))^2]\\ &+2\lambda p_1p_2\mathrm{E}[Z_1-\mathcal{H}_1(t,Z_1)]\mathrm{E}[Z_2-\mathcal{H}_2(t,Z_2)]\Big),\end{aligned}$$

where Λ is the common safety loading of the two classes of insurance business. Note that when using different safety loadings, say Λ_1 and Λ_2, the reinsurance premium rate is given by

$$\begin{aligned}\delta(\mathcal{H}_1(t,Z_1),\mathcal{H}_2(t,Z_2)) &= \lambda p_1\mathrm{E}[Z_1-\mathcal{H}_1(t,Z_1)]\\ &+\lambda p_2\mathrm{E}[Z_2-\mathcal{H}_2(t,Z_2)]\\ &+\frac{\Lambda_1}{2}\lambda p_1\mathrm{E}[(Z_1-\mathcal{H}_1(t,Z_1))^2]+\frac{\Lambda_2}{2}\lambda p_2\mathrm{E}[(Z_2-\mathcal{H}_2(t,Z_2))^2].\end{aligned}$$

As was pointed out by Han et al. (2018) and Han et al. (2021), the uniqueness and existence of the extreme maximum point and the corresponding value function are very difficult to solve analytically in this case. Therefore, in the following context, we assume that the two classes of insurance business are repackaged and that a portion of the repackaged business is transferred to a reinsurance company. So, there is only one safety loading Λ for the repackaged business. Plugging $\delta(\mathcal{H}_1(t,Z_1),\mathcal{H}_2(t,Z_2))$ into the first equation of (3.5) yields

$$\begin{aligned}Q_t &+ \Big[c-\lambda p_1a_1-\lambda p_2a_2-\frac{\Lambda}{2}e^{(A+\kappa)(T-t)}(\lambda p_1b_1\\ &+\lambda p_2b_2+2\lambda p_1p_2a_1a_2)\Big]e^{(A+\kappa)(T-t)}\\ &+e^{(A+\kappa)(T-t)}\sup_{\nu\in\mathcal{A}}\Big\{(\mu-r)\pi(t)+\frac{\Lambda}{2}\Big(-\lambda p_1\mathrm{E}[\mathcal{H}_1(t,Z_1)^2]\\ &+2\lambda p_1\mathrm{E}[Z_1\mathcal{H}_1(t,Z_1)]\\ &-\lambda p_2\mathrm{E}[\mathcal{H}_2(t,Z_2)^2]+2\lambda p_2\mathrm{E}[Z_2\mathcal{H}_2(t,Z_2)]\\ &+2\lambda p_1p_2\big(a_1\mathrm{E}[\mathcal{H}_2(t,Z_2)]+a_2\mathrm{E}[\mathcal{H}_1(t,Z_1)]\\ &-\mathrm{E}[\mathcal{H}_1(t,Z_1)]\mathrm{E}[\mathcal{H}_2(t,Z_2)]\big)\Big)\\ &-\frac{\gamma}{2}e^{(A+\kappa)(T-t)}\Big(\lambda p_1\mathrm{E}[\mathcal{H}_1(t,Z_1)^2]+\lambda p_2\mathrm{E}[\mathcal{H}_2(t,Z_2)^2]\\ &+2\lambda p_1p_2\mathrm{E}[\mathcal{H}_1(t,Z_1)]\mathrm{E}[\mathcal{H}_2(t,Z_2)]+\sigma_1^2\pi(t)^2\Big)\Big\}=0.\end{aligned}$$

As before, we define a related function as

$$\begin{aligned}\overline{G}(t,\mathcal{H}_1,\mathcal{H}_2,\pi) &= -\frac{\Lambda}{2}\lambda p_2\big(\mathrm{E}[\mathcal{H}_2^2]-2\mathrm{E}[Z_2\mathcal{H}_2]-2p_1a_1\mathrm{E}[\mathcal{H}_2]\big)\\ &+(\mu-r)\pi-\frac{\gamma}{2}e^{(A+\kappa)(T-t)}\sigma_1^2\pi^2\\ &-\frac{\gamma}{2}\lambda p_2e^{(A+\kappa)(T-t)}\mathrm{E}[\mathcal{H}_2^2]\\ &+\int_0^\infty\Big[\frac{\Lambda}{2}\lambda p_1\big(-\mathcal{H}_1^2(z_1)+2z_1\mathcal{H}_1(z_1)+a_2\mathcal{H}_1(z_1)\\ &-\mathcal{H}_1(z_1)\mathrm{E}[\mathcal{H}_2]\big)-\frac{\gamma}{2}e^{(A+\kappa)(T-t)}\big(\lambda p_1\mathcal{H}_1^2(z_1)\\ &+2\lambda p_1p_2\mathcal{H}_1(z_1)\mathrm{E}[\mathcal{H}_2]\big)\Big]\mathrm{d}F_{Z_1}(z_1)\\ &=-\frac{\Lambda}{2}\lambda p_1\big(\mathrm{E}[\mathcal{H}_1^2]-2\mathrm{E}[Z_1\mathcal{H}_1]-2p_2a_2\mathrm{E}[\mathcal{H}_1]\big)\\ &+(\mu-r)\pi-\frac{\gamma}{2}e^{(A+\kappa)(T-t)}\sigma_1^2\pi^2\\ &-\frac{\gamma}{2}\lambda p_1e^{(A+\kappa)(T-t)}\mathrm{E}[\mathcal{H}_1^2]\\ &+\int_0^\infty\Big[\frac{\Lambda}{2}\lambda p_1\big(-\mathcal{H}_2^2(z_2)+2z_2\mathcal{H}_2(z_2)+a_1\mathcal{H}_2(z_2)\\ &-\mathcal{H}_2(z_2)\mathrm{E}[\mathcal{H}_1]\big)-\frac{\gamma}{2}e^{(A+\kappa)(T-t)}\big(\lambda p_2\mathcal{H}_2^2(z_2)\\ &+2\lambda p_1p_2\mathcal{H}_2(z_2)\mathrm{E}[\mathcal{H}_1]\big)\Big]\mathrm{d}F_{Z_2}(z_2).\end{aligned}$$

According to the first-order conditions, we can obtain the maximizer of $\overline{G}$ given by

$$\mathcal{H}_1^*(t,z_1)=\overline{d}_1^*(t,z_1)\wedge z_1,\qquad \mathcal{H}_2^*(t,z_2)=\overline{d}_2^*(t,z_2)\wedge z_2,$$

with

$$\begin{aligned}\overline{d}_1^*(t,z_1) &= -p_2\mathrm{E}[\mathcal{H}_2(t,Z_2)]+\frac{\Lambda p_2a_2}{\Lambda+\gamma e^{(A+\kappa)(T-t)}}\\ &+\frac{\Lambda}{\Lambda+\gamma e^{(A+\kappa)(T-t)}}z_1,\end{aligned}\tag{5.1}$$

$$\begin{aligned}\overline{d}_2^*(t,z_2) &= -p_1\mathrm{E}[\mathcal{H}_1(t,Z_1)]+\frac{\Lambda p_1a_1}{\Lambda+\gamma e^{(A+\kappa)(T-t)}}\\ &+\frac{\Lambda}{\Lambda+\gamma e^{(A+\kappa)(T-t)}}z_2,\end{aligned}\tag{5.2}$$

and

$$\pi^*(t) = \frac{u-r}{\gamma\sigma_1^2} e^{-(A+\kappa)(T-t)}.$$

Note that $\overline{d}_1^*(t)$ and $\overline{d}_2^*(t)$ represent the risk retention levels, and are admissible if both $\overline{d}_1^*(t)$ and $\overline{d}_2^*(t)$ are non-negative. We see from the expressions for $\mathcal{H}_1^*(t,z_1)$ and $\mathcal{H}_2^*(t,z_2)$ that the strategy (0,0) can never be optimal, and the proof is similar to that under the expected value premium principle.

Assume that $(\mathcal{H}_1^*, \mathcal{H}_2^*) = (\overline{d}_1^*, \overline{d}_2^*)$. Taking expectation on both sides of the Eqs. (5.1) and (5.2) gives

$$\mathrm{E}(\mathcal{H}_1^*(t,Z_1)) = -p_2\mathrm{E}[\mathcal{H}_2^*(t,Z_2)] + \frac{\Lambda p_2 a_2 + \Lambda a_1}{\Lambda + \gamma e^{(A+\kappa)(T-t)}},$$

$$\mathrm{E}(\mathcal{H}_2^*(t,Z_2)) = -p_1\mathrm{E}[\mathcal{H}_1^*(t,Z_1)] + \frac{\Lambda p_1 a_1 + \Lambda a_2}{\Lambda + \gamma e^{(A+\kappa)(T-t)}},$$

from which we get

$$\mathrm{E}[\mathcal{H}_1^*(t,Z_1)] = \frac{\Lambda a_1}{\Lambda+\gamma e^{(A+\kappa)(T-t)}}, \quad \mathrm{E}[\mathcal{H}_2^*(t,Z_2)] = \frac{\Lambda a_2}{\Lambda+\gamma e^{(A+\kappa)(T-t)}}. \tag{5.3}$$

Substituting (5.3) into (5.1) and (5.2) yields

$$\overline{d}_1^*(t,z_1) = \frac{\Lambda}{\Lambda + \gamma e^{(A+\kappa)(T-t)}} z_1, \quad \overline{d}_2^*(t,z_2) = \frac{\Lambda}{\Lambda + \gamma e^{(A+\kappa)(T-t)}} z_2.$$

It is clear that $0 < \overline{d}_i^*(t,z_i) < z_i$ for $i = 1,2$, and thus we indeed have $\mathcal{H}_i^*(t,z_i) = \overline{d}_i^*(t,z_i)$. Meanwhile, the Hessen matrix of the function $\overline{G}$ at point $(\mathcal{H}_1^*(t,z_1), \mathcal{H}_2^*(t,z_2), \pi^*(t))$ is given by

$$-\gamma e^{(A+\kappa)(T-t)} \begin{pmatrix} \lambda p_1 b_1 & \lambda p_1 p_2 a_1 a_2 & 0 \\ \lambda p_1 p_2 a_1 a_2 & \lambda p_2 b_2 & 0 \\ 0 & 0 & \sigma_1^2 \end{pmatrix},$$

which is negative definite. Therefore, the point $(\mathcal{H}_1^*(t,z_1), \mathcal{H}_2^*(t,z_2), \pi^*(t))$ is the extreme maximum point of $\overline{G}$. The next theorem presents the explicit solution to the optimization problem (3.2) under the variance principle with the common safety loading.

Theorem 3. *The optimal reinsurance-investment strategy for the problem (3.2) under the variance principle is given by*

$$\begin{cases} \mathcal{H}_1^*(t,z_1) = \frac{\Lambda}{\Lambda+\gamma e^{(A+\kappa)(T-t)}} z_1, \quad \mathcal{H}_2^*(t,z_2) = \frac{\Lambda}{\Lambda+\gamma e^{(A+\kappa)(T-t)}} z_2, \\ \pi^*(t) = \frac{u-r}{\gamma\sigma_1^2} e^{-(A+\kappa)(T-t)}. \end{cases} \tag{5.4}$$

Furthermore, the value function is

$$V(t,x,y) = e^{(A+\kappa)(T-t)}(x+\kappa y) + Q(t), \tag{5.5}$$

where

$$\begin{aligned} Q(t) = \frac{e^{(A+\kappa)(T-t)} - 1}{A+k} \Bigg(& c - \lambda p_1 a_1 - \lambda p_2 a_2 \\ & - \frac{\lambda\Lambda}{2}(p_1 b_1 + p_2 b_2 + p_1 p_2 a_1 a_2) \Bigg) \\ & + \frac{\lambda\Lambda^2}{2(A+\kappa)} (p_1 b_1 + p_2 b_2 + 2(p_1+p_2) a_1 a_2) \cdot \\ & \times \ln\left(\frac{\Lambda + \gamma e^{(A+\kappa)(T-t)}}{\Lambda+\gamma} \right) \\ & - \frac{\lambda\Lambda^2\gamma}{A+\kappa} p_1 p_2 a_1 a_2 \left(\frac{1}{\Lambda+\gamma} - \frac{1}{\Lambda + \gamma e^{(A+\kappa)(T-t)}} \right) \\ & - \frac{1}{\gamma} \frac{(\mu - r)^2}{2\sigma_1^2}(T-t). \end{aligned}$$

Proof. Following the steps in the proof of Theorem 2, one can show that (5.4) satisfies all the conditions in Definition 1. Substituting $(\mathcal{H}_1^*(t,z_1), \mathcal{H}_2^*(t,z_2), \pi^*(t))$ into the first equation of the extended HJB delay system (3.5), we have

$$Q_t + e^{(A+\kappa)(T-t)} \Bigg[c - \lambda p_1 a_1 - \lambda p_2 a_2 - \frac{\Lambda}{2}(\lambda p_1 b_1 + \lambda p_2 b_2 + \lambda p_1 p_2 a_1 a_2) + \overline{G}(t, \mathcal{H}_1^*, \mathcal{H}_2^*, \pi^*) \Bigg] = 0,$$

where

$$\begin{aligned} & \overline{G}(t, \mathcal{H}_1^*(t,z_1), \mathcal{H}_2^*(t,z_2), \pi^*(t)) \\ & = \frac{\lambda\Lambda^2}{2(\Lambda + \gamma e^{(A+\kappa)(T-t)})}(p_1 b_1 + p_2 b_2 + 2(p_1+p_2) a_1 a_2) \\ & - \frac{\lambda\Lambda^2 \gamma e^{(A+\kappa)(T-t)}}{(\Lambda + \gamma e^{(A+\kappa)(T-t)})^2} p_1 p_2 a_1 a_2 - \frac{1}{\gamma}\frac{(\mu-r)^2}{2\sigma_1^2}. \end{aligned}$$

By integration, the value function (5.5) can be derived. This completes the proof. □

From (5.4), we see that the form of the optimal reinsurance strategy under the variance premium principle is a pure quota-share reinsurance, and that both retention proportions fall into the interval [0,1] and are equal to each other. This phenomenon coincides with the results in Hipp & Taksar (2010) and Han et al. (2020), in which they derived the optimal reinsurance strategy to minimize the probability of ruin and the probability of drawdown, respectively. Besides, we argue that the retention proportions are independent of any information about the claim sizes but depend on the time t and the related delay parameters. This indicates that the optimal proportional reinsurance strategy under the variance reinsurance premium principle is a model-free solution to the mean-variance problem. Another characteristic is that if we set $B = C = \kappa = 0$, then the optimal reinsurance-investment strategy reduces to the one without delay, and has the form

$$\mathcal{H}_i^*(t,z_i) = \frac{\Lambda}{\Lambda + \gamma e^{r(T-t)}} z_i \quad \text{for } i = 1,2, \qquad \pi^*(t) = \frac{u-r}{\gamma\sigma_1^2} e^{-r(T-t)}.$$

By setting $r = A+\kappa$, these results are consistent with those in the presence of the delay factors, i.e., those in (5.4).

To end this section, we give the following proposition to state the effects of some important model parameters on the optimal reinsurance-investment strategy. The proof is straightforward, so we omit the details.

Proposition 3. *When the reinsurance safety loadings for the two classes of insurance business are the same, the following statements hold:*

(i) *π^* increases with μ, κ and h, but decreases with σ_1 and γ_1;*
(ii) *$\mathcal{H}_i^*$ $(i = 1,2)$ increases with Λ, t, κ and h, but decreases with γ and r.*

Remark 7. Comparing with the results under the expected value principle, the influence of model parameters on the optimal reinsurance under the variance principle is more intuitional as a result of applying the same safety loadings for the two kinds of insurance business. In addition to some similarities as shown in Propositions 1 and 2, we observe that the insurer tends to retain more insurance business as t increases. This observation can be explained by the fact that the uncertainty of the insurance risk is decreasing as the decision time t approaches the ending time T. Also, a higher risk-free rate r indicates a larger earning from a risk-free asset which allows the company to buy more reinsurance so as to reduce the risk of the potential insurance loss.

Table 1
Values of basic parameters.

claim probability	insurer's safety loadings	reinsurer's safety loadings	claim intensity
$p_1 = 0.3,\ \ p_2 = 0.4$	$\theta_1 = 0.15,\ \ \theta_2 = 0.1$	$\eta_1 = 0.3,\ \ \eta_2 = 0.2$	$\lambda = 5$
interest rate	risk-averse parameter	initial time	terminal time
$r = 0.05$	$\gamma = 0.1$	$t = 0$	$T = 20$
delay parameter	weight parameter	average parameter	risky asset parameters
$h = 2$	$\kappa = 0.05$	$\varphi = 0.5$	$\mu = 0.2,\ \sigma_1 = 0.5$
claim amounts			
$z_1 = 1.5,\ z_2 = 1.2$			

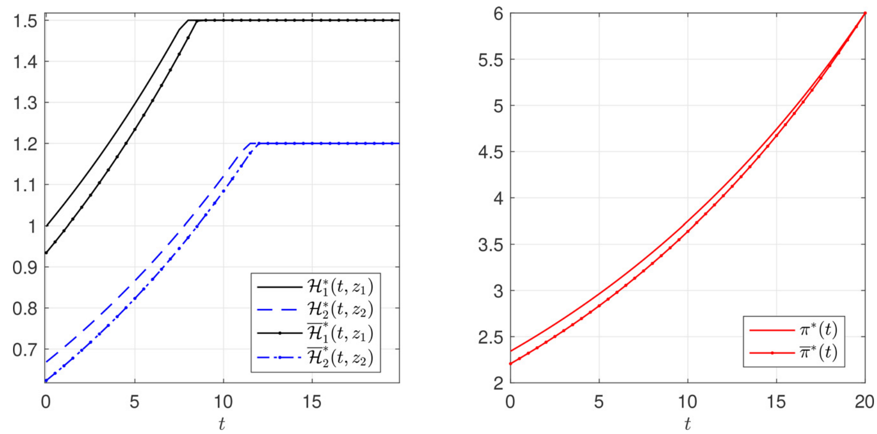

Fig. 2. The effect of t on reinsurance strategy and investment strategy.

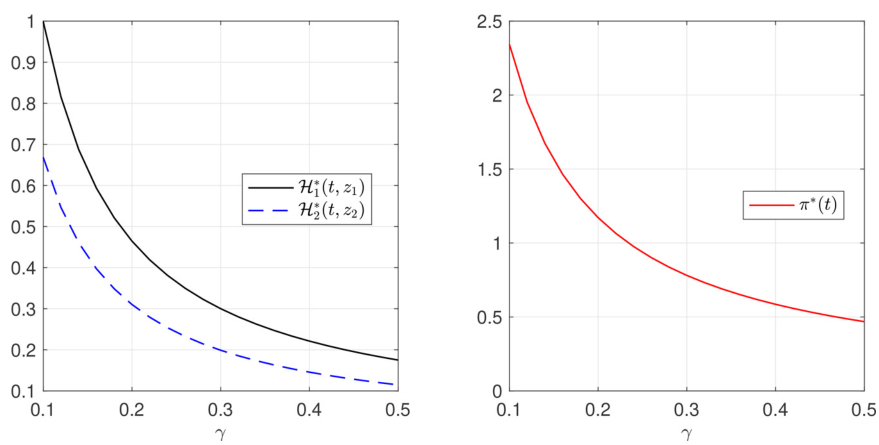

Fig. 3. The effect of γ on reinsurance strategy and investment strategy.

6. Sensitivity analysis and numerical examples

In this section, we assume that one class of insurance business has heavy-tailed risk with small arrival intensity, and the other is light-tailed risk with large arrival intensity. Let

$$F_{Z_1}(z_1) = 1 - \frac{1}{(z_1+1)^3},\ z_1 \geqslant 0; \qquad F_{Z_2}(z_2) = 1 - e^{-2z_2},\ z_2 \geqslant 0,$$

with the first and second moments given by $a_1 = \frac{1}{2}$, $a_2 = \frac{1}{2}$, $b_1 = 1$ and $b_2 = \frac{1}{2}$, respectively.

Here we carry out several numerical examples to show the influence of the model parameters on the optimal results under the expected value principle. The sensitivity analysis under the variance principle can be found in Proposition 3 and Remark 7. Assume that the insurer's insurance premium under the expected value principle is given by

$$c = (1+\theta_1)\lambda p_1 a_1 + (1+\theta_2)\lambda p_2 a_2,$$

where θ_1 and θ_2 are the insurer's safety loadings for the two classes of the insurance business. Unless stated otherwise, the values of the basic parameters are given in Table 1. In particular, we take $\kappa \in [0, 0.1]$ (also see Fig. 4). This implies that the insurer pays more attention to the wealth value at the terminal time T than to that before T.

Recall the optimal reinsurance-investment strategies of (4.11) and (4.12). Under the parameter setting, we first examine the monotonicity of the reinsurance-investment strategies $(\mathcal{H}_1^*, \mathcal{H}_2^*, \pi^*)$ over time t. By setting $h = \kappa = 0$, we also compare the optimal strategies with those ignoring the delay factors. Then, with a fixed decision time $t = 0$, we further investigate the effect of the risk-averse parameter γ on the optimal strategies. At last, we examine the impact of the delay parameters κ and h as well as the reinsurance parameters η_1, η_2, p_1 and p_2.

In Fig. 2, we use the notations $(\overline{\mathcal{H}}_1^*(t,z_1), \overline{\mathcal{H}}_2^*(t,z_2))$ and $\overline{\pi}^*(t)$ to denote the strategies without delay. Fig. 2 shows that both the

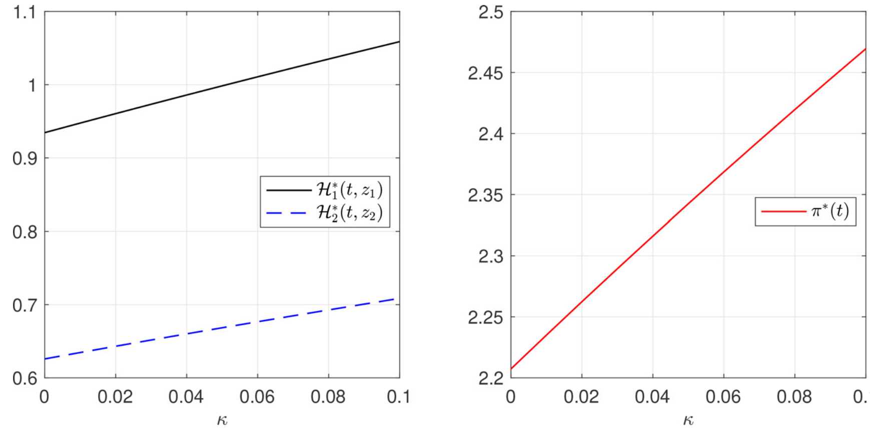

Fig. 4. The effect of κ on reinsurance strategy and investment strategy.

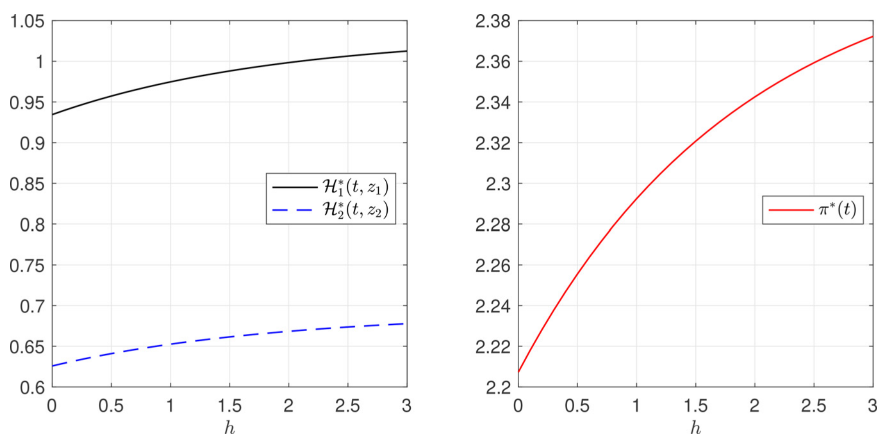

Fig. 5. The effect of h on reinsurance strategy and investment strategy.

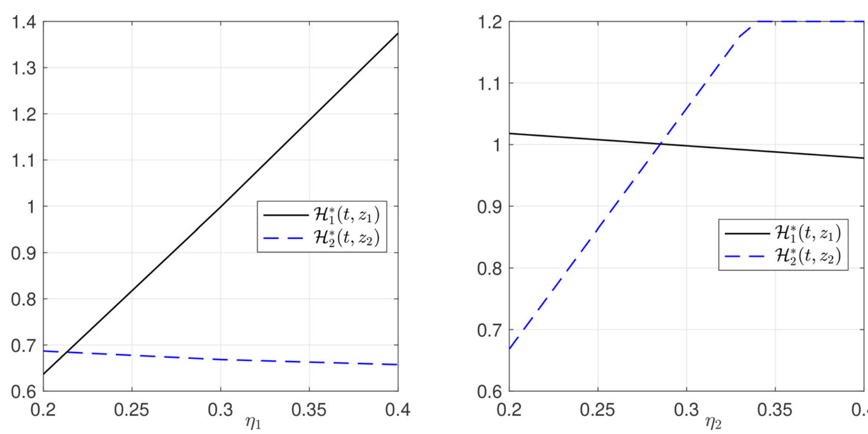

Fig. 6. The effects of η_1 and η_2 on reinsurance strategy.

reinsurance strategy and the investment strategy are increasing functions w.r.t. the time t, which coincides with the observation under the variance premium principle. As mentioned in Remark 7, this is due to the fact that the uncertainty of the insurance and investment risks decreases as the decision time t approaches the ending time T. This allows the insurer to choose a relatively risky strategy that bears more claims and invests a larger amount into the risky asset. On the other hand, in the case without delay factors, we can see that the insurer becomes more cautious about the underlying risks.

Fig. 3 reflects that both the reinsurance and investment strategies are decreasing functions w.r.t. the risk-averse parameter γ. This observation is in line with Propositions 1 and 2. For the mean-variance problem, the parameter γ represents the insurer's risk preference. As γ increases, the insurer pays more attention to the insurance risk, and is willing to transfer more business to the reinsurer. Meanwhile, the insurer would like to reduce the amount invested into the risky asset to alleviate the investment risk.

Figs. 4 and 5 show that both the reinsurance and investment strategies are increasing functions w.r.t. κ and h. The intu-

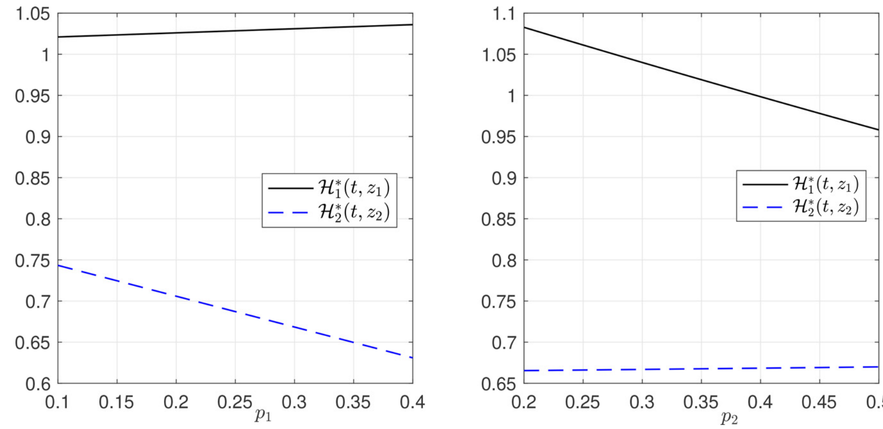

Fig. 7. The effects of p_1 and p_2 on reinsurance strategy.

itive reasons behind these increasing trends have already given in Section 4.

Since the optimal investment strategy is independent of the parameters of the insurance risk model, we only show the effects of η_1, η_2, p_1 and p_2 on the reinsurance strategy.

Fig. 6 shows that for $i, j = 1, 2$ and $i \neq j$, $\mathcal{H}_i$ increases but $\mathcal{H}_j$ decreases as η_i increases. On one hand, when the reinsurance premium for class i becomes much more expensive as η_i increases, the insurer would rather retain a greater share of each claim by purchasing less reinsurance for class i. On the other hand, when the company keeps buying less and less reinsurance for one class, it eventually needs to reduce the whole risk of its insurance portfolios by buying a bit more reinsurance for the other class.

From Fig. 7, we can see that a greater value of p_i yields a greater value of $\mathcal{H}_i^*$ but a smaller value of $\mathcal{H}_j^*$ for $i, j = 1, 2$ and $i \neq j$. For class i, if p_i increases, then reinsurance becomes more expensive, and the insurer optimally purchases less reinsurance. Intuitively, a larger value of p_i implies a larger insurance risk in class i on average, so one might expect that the insurer would retain less of each claim. However, the increase in the reinsurance premium dominates the insurer's optimal decision.

7. Conclusion

In this paper, we investigate an optimal reinsurance-investment problem for an insurer, who has two classes of insurance business with thinning dependence. The wealth process of the insurer is described by a jump-diffusion model with delay. Under the mean-variance criterion, we solve the extended HJB delay system, and derive the time-consistent equilibrium strategies not only for the expected value principle but also for the variance principle. In particular, the existence and uniqueness of the optimal reinsurance strategy is first proved by using several auxiliary functions. Based on the optimal results, we have the following conclusions: (*i*) the optimal per-loss reinsurance strategy is in the form of pure excess-of-loss reinsurance strategy under the expected value principle, and in the form of pure quota-share reinsurance under the variance premium principle; (*ii*) the delay factors makes the performance function tend to be more stable, and the insurer becomes more aggressive to achieve its target; (*iii*) the thinning dependence structure leads to an interaction effect on the optimal reinsurance strategy under the expected value principle; and (*iv*) the reinsurance strategy is sensitive to its own safety loading but robust to its own claim probability.

We remark that, to make the optimization problem tractable, it is assumed that the process of the risky asset and the claims are independent in this paper. There is no doubt that the optimization problem would become more practical if we assume that the claim sizes are correlated, the stock price process and the aggregate claims process depend on each other, and/or the arrival intensities are not constants. These certainly introduce further technical complexity to the problem, and needs a separate study.

For the other further research on this topic, one may consider different choices of risk preference specifications (for example, Value at Risk, Conditional Value at Risk and expected utility). It is also an appealing future work to investigate the model uncertainty for the optimization problem and examine the influence of the ambiguous components such as diffusion and jump risks. All in all, these modeling features may result in more challenging problems and greatly enrich our current work.

Declaration of Competing Interest

Authors declare that they have no conflict of interest.

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Appendix A. The proof of Lemma 1

Proof. We first show the existence of the solution to (4.5). From the second line of (4.5), we obtain

$$d_2^* = \frac{\eta_2}{\gamma} e^{-(A+\kappa)(T-t)} - p_1 h_{Z_1}(d_1^*).$$

Substituting d_2^* into the first equation of (4.5), we have

$$\eta_1 - \gamma \Big[e^{(A+\kappa)(T-t)} d_1^* + p_2 e^{(A+\kappa)(T-t)} h_{Z_2} \times \Big(\frac{\eta_2}{\gamma} e^{-(A+\kappa)(T-t)} - p_1 h_{Z_1}(d_1^*) \Big) \Big] = 0. \tag{A.1}$$

Define

$$\zeta(d_1^*) = \eta_1 - \gamma\Big[e^{(A+\kappa)(T-t)}d_1^* + p_2 e^{(A+\kappa)(T-t)}h_{Z_2} \times\Big(\frac{\eta_2}{\gamma}e^{-(A+\kappa)(T-t)} - p_1 h_{Z_1}(d_1^*)\Big)\Big].$$

It is clear that $\zeta(\infty) = -\infty$ and

$$\begin{aligned}\zeta(0) &= \eta_1 - \gamma p_2 e^{(A+\kappa)(T-t)}h_{Z_2}\Big(\frac{\eta_2}{\gamma}e^{-(A+\kappa)(T-t)}\Big)\\ &= \gamma p_2 e^{(A+\kappa)(T-t)}\Big[\frac{\eta_1}{\gamma p_2}e^{-(A+\kappa)(T-t)} - h_{Z_2}\Big(\frac{\eta_2}{\gamma}e^{-(A+\kappa)(T-t)}\Big)\Big]\\ &\geqslant \gamma p_2 e^{(A+\kappa)(T-t)}\Big[\frac{\eta_1}{\gamma p_2}e^{-(A+\kappa)(T-t)} - \frac{\eta_2}{\gamma}e^{-(A+\kappa)(T-t)}\Big] \geqslant 0.\end{aligned}$$

The last inequality holds due to the assumption of $\eta_1 \geqslant \eta_2$. Therefore, the Eq. (A.1) has a positive root. This shows the existence of (d_1^*, d_2^*).

We now prove the uniqueness of the point. Assume that there exists another point $(\tilde{d}_1, \tilde{d}_2)$ satisfying the Eq. (4.5), i.e.,

$$\begin{cases}\eta_1 - \gamma\Big[e^{(A+\kappa)(T-t)}\tilde{d}_1 + p_2 e^{(A+\kappa)(T-t)}h_{Z_2}(\tilde{d}_2)\Big] = 0,\\ \eta_2 - \gamma\Big[e^{(A+\kappa)(T-t)}\tilde{d}_2 + p_1 e^{(A+\kappa)(T-t)}h_{Z_1}(\tilde{d}_1)\Big] = 0.\end{cases} \tag{A.2}$$

Without loss of generality, we suppose that $\tilde{d}_1 > d_1^*$, and thus $\tilde{d}_2 < d_2^*$ holds. Combining (4.5) and (A.2), it is not difficult to show that

$$\begin{cases}(\tilde{d}_1 - d_1^*) - p_2\int_{\tilde{d}_2}^{d_2^*} S_{Z_2}(z_2)dz_2 = 0,\\ (\tilde{d}_2 - d_2^*) + p_1\int_{d_1^*}^{\tilde{d}_1} S_{Z_1}(z_1)dz_1 = 0;\end{cases}$$

or equivalently, there exists some $\xi_1 \in (d_1^*, \tilde{d}_1)$ and $\xi_2 \in (\tilde{d}_2, d_2^*)$ such that

$$\begin{cases}(\tilde{d}_1 - d_1^*) + p_2 S_{Z_2}(\xi_2)(\tilde{d}_2 - d_2^*) = 0,\\ (\tilde{d}_2 - d_2^*) + p_1 S_{Z_1}(\xi_1)(\tilde{d}_1 - d_1^*) = 0.\end{cases} \tag{A.3}$$

Note that the matrix

$$\begin{pmatrix}1 & p_2 S_{Z_2}(\xi_2)\\ p_1 S_{Z_1}(\xi_1) & 1\end{pmatrix}$$

is invertible, and thus the solution to the system of Eq. (A.3) equals $(\tilde{d}_1 - d_1^*, \tilde{d}_2 - d_2^*) = (0, 0)$, which completes the proof. □

Appendix B. The proof of Lemma 2

Proof. The proof is straightforward. Differentiating $\psi_{Z_1}(x)$ w.r.t. x, we obtain

$$\frac{\partial\psi_{Z_1}(x)}{\partial x} = \eta_2 - \eta_1 p_1 S_{Z_1}(x), \qquad \frac{\partial^2\psi_{Z_1}(x)}{\partial x^2} = \eta_1 p_1 f_{Z_1}(x) > 0.$$

If $0 \leqslant p_1 \leqslant \frac{\eta_2}{\eta_1}$, we can see that $\eta_2 - \eta_1 p_1 S_{Z_1}(x) \geqslant 0$. Hence, $\psi_{Z_1}(x)$ is a strictly increasing function on $(0, \infty)$. If $\frac{\eta_2}{\eta_1} < p_1 \leqslant 1$, it follows that $\eta_2 - \eta_1 p_1 S_{Z_1}(0) \leqslant 0$, and that $\lim_{x\to\infty}(\eta_2 - \eta_1 p_1 S_{Z_1}(x)) = \eta_2 > 0$. Let $\bar{x} = S_{Z_1}^{-1}(\frac{\eta_2}{\eta_1 p_1})$. Then the function $\psi_{Z_1}(x)$ is strictly decreasing on $(0, \bar{x})$ and increasing on $(\bar{x}, \infty)$. □

Appendix C. The proof of Proposition 1

Proof. The monotonicity in (i) can be easily verified. Now, we prove (ii). It follows from the closed form of the optimal reinsurance strategy (4.12) that the monotonicity properties in (ii) hold. We only need to think about the one in (4.11). From Theorem 2, we know that $d_1^*(t) \in (x_0, \infty)$ uniquely solves the equation $K(d_1^*) = 0$ where $K(x)$ is defined in (4.7). Recall $\psi_{Z_1}(x)$ and $\psi_{Z_2}(x)$ of (4.6). By abusing the notations a bit, we write

$$\psi_{Z_1}(d_1, \eta_1) = \eta_2 d_1 - \eta_1 p_1 h_{Z_1}(d_1),$$
$$\psi_{Z_2}(d_1, \eta_1) = \eta_1 d_1 - \eta_2 p_2 h_{Z_2}(d_1),$$

and

$$K(d_1, \eta_1) = \psi_{Z_2}^{-1}(\psi_{Z_1}(d_1^*, \eta_1), \eta_1) + p_1 h_{Z_1}(d_1^*) - \frac{\eta_2}{\gamma}e^{-(A+\kappa)(T-t)}.$$

Without loss of generality, we give the proof for $\eta_1 \geqslant \eta_2$, and the results for $\eta_1 < \eta_2$ can be derived symmetrically. On one hand, it is clear that

$$\frac{\partial\psi_{Z_1}(d_1, \eta_1)}{\partial\eta_1} < 0, \quad \text{and} \quad \frac{\partial\psi_{Z_2}^{-1}(d_1, \eta_1)}{\partial\eta_1} > 0,$$

and thus $K(d_1, \eta_1)$ eventually decreases with η_1. On the other hand, $K(d_1, \eta_1)$ increases as d_1 increases. To ensure $K(d_1^*, \eta_1) = 0$, it can be seen that d_1^* increases as η_1 increases. Then it follows from the system equations in (4.5) that

$$\frac{\partial d_2^*}{\partial d_1^*} = \frac{\partial\psi_{Z_2}^{-1}(\psi_{Z_1}(d_1^*))}{\partial d_1^*} < 0,$$

which implies that d_2^* is a decreasing function w.r.t. η_1. Along the same lines above, we can prove the corresponding properties for d_2^*.

We next prove (iii). Analogous to the proofs of (ii), one can show that $K(d_1^*, \gamma)$ is an increasing function w.r.t. γ. Also, to ensure $K(d_1^*, \gamma) = 0$, d_1^* (d_2^*, respectively) must decrease (increase, respectively) as γ increases. □

Appendix D. The proof of Proposition 2

Proof. It is clear from (4.11), (4.12) that the monotonicity of the strategy w.r.t. κ and h depends on the term $g(\kappa, h) = -(A+\kappa)$. After some algebraic manipulation, we get

$$\begin{aligned}g(\kappa, h) &= -r - \kappa + \kappa e^{-\varphi h} + \frac{(\varphi + r + \kappa - \kappa e^{-\varphi h})\kappa(1 - e^{-\varphi h})}{\varphi + \kappa(1 - e^{-\varphi h})}\\ &= -\frac{r\varphi}{\varphi + \kappa(1 - e^{-\varphi h})},\end{aligned}$$

which is an increasing function w.r.t. κ and h. Thus, the optimal reinsurance strategy $(\mathcal{H}_1^*, \mathcal{H}_2^*)$ and the investment strategy π^* are both increasing functions w.r.t. κ and h. □

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