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Optimal reinsurance contract in a Stackelberg game framework: a view of social planner

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ABSTRACT

In this paper, we consider an optimal reinsurance contract under a mean-variance criterion in a Stackelberg game theoretical framework. The reinsurer is the leader of the game and decides on an optimal reinsurance premium to charge, while the insurer is the follower of the game and chooses an optimal per-loss reinsurance to purchase. The objective of the insurer is to maximize a given mean-variance criterion, while the reinsurer adopts the role of social planner balancing its own interests with those of the insurer. That is, we assume that the reinsurer determines the reinsurance premium by maximizing a weighted sum of the insurer's and reinsurer's mean-variance criteria. Under the general mean-variance premium principle, we derive the optimal reinsurance contract by solving the extended Hamilton–Jacobi–Bellman (HJB) systems. Moreover, we provide an intuitive way to set the weight of each party in the reinsurer's objective. Finally, we consider some special cases to illustrate our main results.

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Decision analysis; reinsurance contract design; Stackelberg game; social planner view; mean-variance premium principle

1. Introduction

Reinsurance is an important risk management tool available to insurers to manage risk. An insurer can employ reinsurance to control the amount of risk exposure, and a reinsurer can generate income and control risk by adjusting the premium for reinsurance contracts. Subject to a control on reinsurance, optimization problems related to minimizing the probability of ruin (see Schmidli 2002, Bai & Guo 2008, Liang & Young 2018, Tan et al. 2020), maximizing the expected utility of terminal wealth (see Liu & Ma 2009, Gu et al. 2012, Liang & Yuen 2016), and maximizing other mean-variance criteria (see Fu et al. 2010, Yi et al. 2015, Zeng et al. 2016) have become a popular research topic in the actuarial literature. The techniques of stochastic control theory and the corresponding Hamilton–Jacobi–Bellman (HJB) equation(s) are widely used to cope with these problems.

It is worth noting that most of the aforementioned works are carried out by optimizing the reinsurance design from the exclusive perspective of the insurer. However, since any reinsurance contract is a mutual agreement between two parties, a reinsurance strategy that is optimal for one party may be unacceptable to the other. Intuitively speaking, a more expensive reinsurance contract will undoubtedly restrain the insurer's demand for risk transfer, while a relatively cheaper reinsurance premium will naturally incentivize the insurer to seek more reinsurance protection. Balancing the interests of

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both parties is critical to settle on the terms of a reinsurance contract. With the increasingly fierce market competition, the stochastic differential reinsurance game problems have been widely studied in recent years. Some scholars focus on the zero-sum stochastic differential reinsurance game between two insurers (see, e.g. Taksar & Zeng 2011), while others investigate the non-zero-sum stochastic differential reinsurance game between two insurers (see, e.g. Bensoussan et al. 2014, Meng et al. 2015, Pun & Wong 2016, Deng et al. 2018). However, the optimal strategies derived from the above studies only consider the exclusive interests of insurers.

Recently, there are also some scholars taking the perspective of the reinsurer into consideration when designing the reinsurance contract. They apply the Stackelberg game framework to the reinsurance design problem. Under the utility maximization criterion, Chen & Shen (2018) propose the continuous-time Stackelberg game framework to analyze an optimal reinsurance problem from joint interests of the insurer and the reinsurer. Hu et al. (2018a, 2018b) study a similar problem to Chen & Shen (2018), in which the reinsurer is concerned about potential model ambiguity in the claim process. Following in the footstep of Hu et al. (2018a, 2018b) and Gu et al. (2019) discuss an optimal excess-of-loss reinsurance contract in a continuous-time Stackelberg game framework, where an insurer's surplus is modeled by the classical Cramer-Lundberg (C-L) model. Bai et al. (2021) and Gu et al. (2022) investigate a hybrid stochastic differential reinsurance and investment game between one reinsurer and two insurers, including a stochastic Stackelberg differential subgame and a non-zero-sum stochastic differential subgame. Under the mean-variance criterion, Chen & Shen (2019) consider a Stackelberg differential game by assuming that the reinsurance premium principle is calculated by either the expected value principle or the variance premium principle. Li & Young (2021) compute the solution of a one-period, mean-variance Stackelberg game by assuming that the seller's insurance premium is calculated according to the mean-variance premium principle. Yuan et al. (2022) investigate a robust optimal reinsurance contract from joint interests of the insurer and reinsurer under the stochastic Stackelberg differential game, who both seek the optimal strategy to maximize their respective mean-variance cost functionals.

In this paper, we focus on a reinsurance problem in which a risk-averse insurer is the follower and a risk-averse reinsurer is the leader in the Stackelberg game framework. The insurer transfers part of its claims to the reinsurer in exchange for the payment of a reinsurance premium. At the same time, the reinsurer adjusts the reinsurance premium based on the reinsurance purchased by the insurer. We assume that the reinsurance premium is calculated by the mean-variance premium principle, and the risk loadings vary with the amount of reinsurance. Then the risk loadings can describe the price of the reinsurance, and the reinsurer can choose an appropriate risk loading to optimize its objective function. We solve this two-player game in an order that the reinsurer first generally states its premium rule, and the insurer chooses the optimal reinsurance strategy given the rule to maximize its mean-variance functional of the terminal wealth. Then the reinsurer chooses the optimal risk loading strategies of the mean-variance premium principle to maximize its objective. The reinsurer adopts the role of social planner, that is, its objective is to maximize a combination of its mean-variance functional of the terminal wealth and that of the insurer. We therefore rely on a differential game theoretic point of view to study the reinsurance-premium control problem arising in optimal reinsurance policy design, and pay attention to the interaction between the insurer and the reinsurer.

As in Chen & Shen (2019) and Yuan et al. (2022), we consider the optimal reinsurance problem with mean-variance criterion under the Stackelberg game framework in a continuous-time model, but with three main differences. Firstly, we consider a general reinsurance premium principle, namely the mean-variance premium principle, which includes the expected value principle and the variance premium principle as special cases. Using a pair of risk loadings as control, we show that there exists an optimal reinsurance contract between the insurer and the reinsurer. Secondly, the reinsurer adopts the role of social planner balancing its own interests with those of the insurer in the determination of the reinsurance premium. It can be interpreted as the reinsurer stands at a higher position to consider the reinsurance problem and acts as a social planner to maximize the joint welfare of the insurer and the reinsurer. Moreover, we provide a novel and intuitive way to set the insurer's weight in the reinsurer's objective function by linking the reinsurer's optimization problem to an original Markowitz's model. Thirdly, in most research papers on optimal reinsurance arrangements under the expected value principle or the variance premium principle, few of them show which premium principle is actually better for the reinsurer. By assuming that the claim size follows some special distributions, we further compare the corresponding results under the two different reinsurance premium principles and try to provide some guidance on which premium principle the reinsurer should choose in different situations. We find that it is better for the reinsurer to choose the expected value principle when the ratio of the reinsurer's risk-averse parameter to the insurer's risk-averse parameter is relatively small; otherwise, the variance premium principle is better for the reinsurer. Therefore, our model is an extension of those in most existing papers about optimal reinsurance problems under the Stackelberg game framework (see e.g. Chen & Shen 2019).

The rest of the paper is organized as follows. In Section 2, we introduce our model framework. In Section 3, we derive the optimal reinsurance contract under the general mean-variance premium principle. In Section 4, we state the results under two special cases of the reinsurance premium principle, namely the expected value principle and the variance premium principle. In Section 5, we provide a sensitivity analysis of the effect of the weight parameter on the reinsurance strategy and premium strategy. Interesting comparisons are later drawn. Some concluding remarks are made in Section 6. All technical proofs are relegated to an appendix.

2. Model setup

Let $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$ be a filtered probability space satisfying the usual assumptions of completeness and right continuity, and $T > 0$ be a finite time horizon. Also, on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, let $N = \{N(t)\}_{t\in[0,T]}$ be a homogeneous Poisson process with intensity $\lambda > 0$, and $\{Y_i\}_{i\in\mathbb{N}^+}$ be a sequence of iid positive rv's (also, independent of N) with common cumulative distribution function F_Y, finite mean a_Y, and finite second moment σ_Y^2.

Without reinsurance, we assume an insurer's aggregate loss is modeled by the Cramér-Lundberg risk model

$$\sum_{i=1}^{N(t)} Y_i = \int_0^t \int_0^\infty yN(\mathrm{d}s, \mathrm{d}y), \quad t \in [0, T],$$

where $N(t)$ counts the number of claims occurring in the time horizon $[0, t]$, and the rv Y_i denotes the ith claim size. Here, $N(\mathrm{d}t, \mathrm{d}y)$ is a Poisson random measure with compensator given by $\nu(\mathrm{d}y)\mathrm{d}t = \lambda \mathrm{d}F(y)\mathrm{d}t$. The compensated measure of N is denoted by $\tilde{N}(\mathrm{d}t, \mathrm{d}y) = N(\mathrm{d}t, \mathrm{d}y) - \nu(\mathrm{d}y)\mathrm{d}t$.

In what follows, we assume that the insurer can transfer its claims to a reinsurer under a per-loss reinsurance arrangement via a continuously payable premium. We use $l(t, y)$ to denote the part of claims retained by the insurer at time t, as a function of the (possible) claim $Y = y$, thus, $y - l(t, y)$ is the amount ceded to the reinsurer. We suppose that the reinsurer charges a reinsurance premium at the rate of $p(t)$. With the reinsurance protection, the dynamics of the controlled wealth process of the insurer evolves according to

$$\mathrm{d}X_I(t) = (c - p(t))\mathrm{d}t - \int_0^\infty l(t, y)N(\mathrm{d}t, \mathrm{d}y), \tag{1}$$

in which $X_I(0) = x_I > 0$ is the insurer's initial surplus and $c > 0$ is the premium rate. Here, we assume that $c > \lambda a_Y$ which means that the insurer's premium income exceeds the expected claim account per unit time. Correspondingly, the reinsurer's surplus follows the dynamics

$$\mathrm{d}X_R(t) = p(t)\mathrm{d}t - \int_0^\infty \big(y - l(t, y)\big) N(\mathrm{d}t, \mathrm{d}y), \tag{2}$$

in which $X_R(0) = x_R > 0$ is the reinsurer's initial surplus. Throughout this paper, we call $l = \{l(t,y)\}_{(t,y)\in[0,T]\times\mathbb{R}^+}$ a reinsurance strategy and $p = \{p(t)\}_{t\in[0,T]}$ a reinsurance premium strategy. Let $\mathcal{A}_I$ and $\mathcal{A}_R$ denote the spaces of all admissible reinsurance strategies and premium strategies, respectively, which will be defined in Definition 2.1.

To account for the interests of both parties in the reinsurance arrangement, we formulate the optimal reinsurance problem under a Stackelberg differential game, in which the reinsurer plays the leadership role and moves on to setting a reinsurance premium first, while the insurer is the follower and cedes reinsurance risk sequentially in response to the reinsurance premium.

Denote $\mathbb{E}_{t,x}[\cdot] = \mathbb{E}[\cdot \mid \mathcal{F}_t, X(t) = x]$, $\mathrm{Var}_{t,x}[\cdot] = \mathrm{Var}[\cdot \mid \mathcal{F}_t, X(t) = x]$, where $X(t) = x$ stands for $(X_I(t), X_R(t)) = (x_I, x_R)$. For any $t \in [0,T]$ and reinsurance premium contract $p(t)$, the insurer's optimization problem is defined as

$$\sup_{l\in\mathcal{A}_I} J_I(t,x;l,p) = \sup_{l\in\mathcal{A}_I} \left\{ \mathbb{E}_{t,x}[X_I(T)] - \frac{\gamma_I}{2}\mathrm{Var}_{t,x}[X_I(T)] \right\}, \tag{3}$$

where $\gamma_I > 0$ is the insurer's risk-aversion coefficient. Moreover, the objective of the reinsurer is given by

$$\begin{aligned}\sup_{p\in\mathcal{A}_R} J_R(t,x;l^*(p),p) = \sup_{p\in\mathcal{A}_R} &\left\{ \mathbb{E}_{t,x}[X_R(T)] - \frac{\gamma_R}{2}\mathrm{Var}_{t,x}[X_R(T)] \right. \\ &\left. +\alpha \left(\mathbb{E}_{t,x}[X_I(T)] - \frac{\gamma_I}{2}\mathrm{Var}_{t,x}[X_I(T)] \right) \right\},\end{aligned} \tag{4}$$

in which $\gamma_R > 0$ is the reinsurer's risk-aversion coefficient. Here, $\alpha \in [0,1]$ can be viewed as a measure of the insurer's importance in the reinsurer's objective function. We assume $0 \le \alpha \le 1$, that is, the reinsurer takes parts of insurer's interests into consideration when pricing reinsurance premium but does not exceed its own weight. It can be interpreted as the reinsurer acts as a social planner and aims to maximize the joint welfare.

Remark 2.1: Note that the optimal reinsurance problem is formulated in a continuous-time model. In reality, although the insurer and the reinsurer do not continuously modify reinsurance demand and price, our study still has practical significance. On the one hand, the reinsurer can adopt the continuous reinsurance price to guide uniform pricing, interval pricing, and differential pricing, such as using the middle price as the real price in a period. On the other hand, the insurer can use the continuous reinsurance demand as reference when determining the actual demand.

Remark 2.2: The term 'social planner' comes from economics literature. Different from the regulator or an investor (the regulator may require companies to hold minimum capital reserves even if it goes against the maximality of the economic welfare and an investor would only pay attention to his own welfare behavior), the social planner is a decision-maker concerning with maximizing the total welfare. Since the reinsurance market in reality is not competitive but oligopolistic due to the less reinsurance companies and sufficient capital, it makes sense for the reinsurer to stand at a higher position and adopt the role of social planner balancing its own interests with those of the insurer in the determination of the reinsurance premium. To the best of our knowledge, our paper is the first one to consider an optimal reinsurance contract problem from the view of social planer in a Stackelberg game theoretical framework. For other criteria where problems are formulated from the view of the social planner, we refer to Assa (2015) and Cao et al. (2022).

Remark 2.3: One may wonder how to determine the value of α. As argued by Golubin (2006), one way to determine α is given by 'experts' exogenously, according to empirical studies. Other ways may be based on theories of barging games or cooperative games (see, e.g. Suijs et al. 1998, Golubin 2006, Zeng & Luo 2013 and references therein). In Proposition 3.3, we provide an intuitive way to set the value of α based on the expected annual return of the reinsurer.

In the following definition, we define the set of admissible strategies (l, p).

Definition 2.1: A pair of strategy (l, p) is said to be *admissible* if (i) l is adapted to the filtration $\mathbb{F}$, and satisfies $0 \leq l(t, y) \leq y$ for all $0 \leq t \leq T$ and $y \geq 0$; (ii) p is adapted to the filtration $\mathbb{F}$; (iii) the state Equations (1) and (2) associated with (l, p) have strong solutions X_I and X_R such that they are càdlàg, $\mathbb{F}$-adapted processes satisfying $\mathbb{E}[\sup_{t\in[t,T]} |X_I(t)|] < \infty$ and $\mathbb{E}[\sup_{t\in[t,T]} |X_R(t)|] < \infty$. The set of all admissible strategies (l, p) is denoted by $\mathcal{A} = \mathcal{A}_I \times \mathcal{A}_R$.

3. Reinsurance contract under mean-variance premium principle

In this section, we derive the optimal reinsurance contract under the mean-variance premium principle in a Stackelberg differential game framework. We first demonstrate a verification theorem to solve the insurer's and reinsurer's problems given by (3) and (4), respectively. Then, we show the existence of the equilibrium strategy. Finally, we discuss the determination of the weight parameter α.

3.1. Verification theorem

The dynamic mean-variance criterion has the well-known issue of time inconsistency in which a strategy that is optimal today may not be optimal tomorrow. We tackle the problem from a non-cooperative game point of view by defining an equilibrium strategy and its corresponding equilibrium value function; see for example, Basak & Chabakauri (2010), Björk & Murgoci (2010), and Björk et al. (2014). Such a solution is formally referred to as an intra-personal equilibrium strategy. The following gives the definition of the equilibrium strategy.

Definition 3.1: Let $l^*(\cdot, p(\cdot)) \in \mathcal{A}_I$ be a reinsurance strategy associated with any given premium strategy $p(\cdot)$. For any $t \in [0, T], y \in \mathbb{R}^+$ and a fixed constant $\epsilon > 0$, we define a perturbed risk-sharing strategy as

$$l^\epsilon(s, y, p(\cdot)) = \begin{cases} \tilde{l}(y), & s \in [t, t+\epsilon], \\ l^*(s, y, p(\cdot)), & s \in [t+\epsilon, T]. \end{cases}$$

If

$$\liminf_{\epsilon \to 0^+} \frac{J_I(t, x; l^*, p) - J_I(t, x; l^\epsilon, p)}{\epsilon} \geq 0,$$

for all deterministic functions $\tilde{l}(y) \in [0, y]$, then $l^*(\cdot, p(\cdot))$ is called an equilibrium reinsurance strategy, and the equilibrium value function of the insurer's optimization problem (3) for a given premium strategy p is $V_I(t, x; p) = J_I(t, x; l^*, p)$.

Let $p^*(\cdot) \in \mathcal{A}_R$ be a premium strategy and $l^*(\cdot, p(\cdot))$ be the equilibrium reinsurance strategy given above. For any $t \in [0, T]$ and a fixed constant $\epsilon > 0$, we define a perturbed premium strategy as

$$p^\epsilon(s) = \begin{cases} \tilde{p}, & s \in [t, t+\epsilon], \\ p^*(s), & s \in [t+\epsilon, T]. \end{cases}$$

If

$$\liminf_{\epsilon \to 0^+} \frac{J_R(t, x; l^*(p^*), p^*) - J_R(t, , x; l^*(p^\epsilon), p^\epsilon)}{\epsilon} \geq 0,$$

for all $\tilde{p} \in \mathbb{R}$, then p^* is called an equilibrium premium strategy, and the equilibrium value function of the reinsurer's optimization problem (4) is $V_R(t, x) = J_R(t, x; l^*, p^*)$. Finally, when there is no risk of confusion, $l^*(p^*)$ is the equilibrium reinsurance strategy, and $V_I(t, x) = J_I(t, x; l^*(p^*), p^*)$ is the equilibrium value function of the insurer's optimization problem.

Note that if the reinsurance premium principle is left unspecified, the game problem is intractable since a map from $\mathcal{A}_R$ to $\mathcal{A}_I$ cannot be identified to link the insurer's strategy to that of the reinsurer's. Thus, in the following, we assume that the reinsurance premium is computed according to a so-called mean-variance premium principle, a combination of the expected value and variance premium principles (see, e.g. Zhang et al. 2016, Han et al. 2020). Then the mean-variance premium rate at time t associated with $l(t,y)$ is given by

$$p(t) = (1+\theta)\int_0^\infty \big(y - l(t,y)\big)\,\nu(\mathrm{d}y) + \frac{\eta}{2}\int_0^\infty \big((y - l(t,y))^2\big)\,\nu(\mathrm{d}y), \tag{5}$$

in which θ and η are the non-negative risk loading parameters. If $\theta(\eta) \equiv 0$, (5) reduces to the variance (expected value) premium principle. This parametric form of p allows us to consider θ and η as the reinsurer's controls, and we call $\xi = (\theta, \eta)$ the risk loading strategy. A risk loading strategy ξ is said to be admissible if the associated reinsurance premium strategy p given in (5) is admissible. We denote the admissible risk loading strategies by $\mathcal{D}_R$.

In the following proposition, we demonstrate important properties of the equilibrium strategy and the corresponding value functions, which play an important role in proving the verification theorem.

Proposition 3.1: *Under the mean-variance criterion, the insurer's and the reinsurer's equilibrium strategies are both independent of the wealth level of both parties. Moreover, the value function of the insurer is independent of the reinsurer's initial surplus, but the value function of the reinsurer depends on the insurer's initial surplus.*

Proof: See Appendix A.1. ■

According to the result of Proposition 3.1, the insurer's value function can be written as $V_I(t, x_I)$. We define the variational operator for the insurer's and the reinsurer's problems as

$$\begin{aligned}\mathcal{L}_I^{(l,\xi)}\psi_I(t,x_P) =& \frac{\partial \psi_I}{\partial t} + \frac{\partial \psi_I}{\partial x_I}\left(c - \int_0^\infty \left((1+\theta)(y-l(t,y)) + \frac{\eta}{2}(y-l(t,y))^2\right)\nu(\mathrm{d}y)\right)\\ &+ \int_0^\infty \big(\psi_I(t,x_I - l(t,y)) - \psi_I(t,x_I)\big)\,\nu(\mathrm{d}y),\end{aligned}$$

and

$$\begin{aligned}\mathcal{L}_R^{(l,\xi)}\psi_R(t,x_I,x_R) =& \frac{\partial \psi_R}{\partial t} + \frac{\partial \psi_R}{\partial x_I}\left(c - \int_0^\infty \left((1+\theta)(y-l(t,y)) + \frac{\eta}{2}(y-l(t,y))^2\right)\nu(\mathrm{d}y)\right)\\ &+ \frac{\partial \psi_R}{\partial x_R}\left(\int_0^\infty \left((1+\theta)(y-l(t,y)) + \frac{\eta}{2}(y-l(t,y))^2\right)\nu(\mathrm{d}y)\right)\\ &+ \int_0^\infty \big(\psi_R(t,x_I - l(t,y), x_R - (y - l(t,y))) - \psi_R(t,x_I,x_R)\big)\,\nu(\mathrm{d}y),\end{aligned}$$

respectively. The following verification theorem gives the extended HJB equations of problems (3) and (4). We track the optimization problems by solving the extended HJB equations. Here, we omit the proof because it is similar to the proof of Theorem 4.1 in Björk & Murgoci (2010). Also, see the discussion in Björk et al. (2014) about the application of the verification theorem under the mean-variance criterion.

Theorem 3.1: *For a reinsurance premium of the form* (5)*, the extended HJB equation for the insurer's problem* (3) *is*

$$\sup_{l\in\mathcal{A}_I}\left\{\mathcal{L}_I^{(l,\xi)}V_I(t,x_I) - \frac{\gamma_I}{2}\mathcal{L}_I^{(l,\xi)}g_I^2(t,x_I) + \gamma_I g_I(t,x_I)\mathcal{L}_I^{(l,\xi)}g_I(t,x_I)\right\} = 0, \tag{6}$$

$$\mathcal{L}_I^{(l^*,\xi)} g_I(t,x_I) = 0, \quad V_I(T,x_I) = x_I, \quad g_I(T,x_I) = x_I, \tag{7}$$

where l^* is given by

$$l^*(t,y,\xi) = \arg\sup_{l\in\mathcal{A}_I} \left\{ \mathcal{L}_I^{(l,\xi)} V(t,x_I) - \frac{\gamma_I}{2}\mathcal{L}_I^{(l,\xi)} g_I^2(t,x_I) + \gamma_I g_I(t,x_I)\mathcal{L}_I^{(l,\xi)} g_I(t,x_I) \right\}.$$

The extended HJB equation for the reinsurer's problem (4) *is*

$$\sup_{\xi\in\mathcal{D}_R} \left\{ \mathcal{L}_R^{(l^*(\xi),\xi)} V_R(t,x_I,x_R) - \frac{\gamma_R}{2}\mathcal{L}_R^{(l^*(\xi),\xi)} g_R^2(t,x_I,x_R) + \gamma_R g_R(t,x_I,x_R)\mathcal{L}_R^{(l^*(\xi),\xi)} g_R(t,x_I,x_R) \right.$$
$$\left. -\alpha\frac{\gamma_I}{2}\mathcal{L}_R^{(l^*(\xi),\xi)} \bar{g}_I^2(t,x_I,x_R) + \alpha\gamma_I \bar{g}_I(t,x_I,x_R)\mathcal{L}_R^{(l^*(\xi),\xi)} \bar{g}_I(t,x_I,x_R) \right\} = 0, \tag{8}$$

$$\begin{aligned} &\mathcal{L}_R^{(l^*(\xi^*),\xi^*)} g_R(t,x_I,x_R) = 0, \quad \mathcal{L}_R^{(l^*(\xi^*),\xi^*)} \bar{g}_I(t,x_I,x_R) = 0, \\ &V_R(T,x_I,x_R) = \alpha x_I + x_R, \quad g_R(T,x_I,x_R) = x_R, \quad \bar{g}_I(T,x_I,x_R) = x_I, \end{aligned} \tag{9}$$

where ξ^* is given by

$$\xi^*(t) = -\arg\sup_{\xi\in\mathcal{D}_R} \left\{ \mathcal{L}_R^{(l^*(\xi),\xi)} V_R(t,x_I,x_R) - \frac{\gamma_R}{2}\mathcal{L}_R^{(l^*(\xi),\xi)} g_R^2(t,x_I,x_R) \right.$$
$$+ \gamma_R g_R(t,x_I,x_R)\mathcal{L}_R^{(l^*(\xi),\xi)} g_R(t,x_I,x_R) - \alpha\frac{\gamma_I}{2}\mathcal{L}_R^{(l^*(\xi),\xi)} g_I^2(t,x_I,x_R)$$
$$\left. + \alpha\gamma_I g_I(t,x_I,x_R)\mathcal{L}_R^{(l^*(\xi),\xi)} g_I(t,x_I,x_R) \right\}.$$

3.2. Solution to the game

When the reinsurance premium is calculated by the general mean-variance premium principle, it is difficult to get the explicit expression of the risk loading strategy. Thus, in this section, we first prove the existence of the optimal strategy (l^*,ξ^*) with $\xi^* = (\theta^*,\eta^*)$ by restricting ξ^* into a meaningful bounded region, and then analysis the case where the optimal reinsurance premium is obtained at ∞. Note that both θ and η take non-negative values.

Given the non-negative constants $\theta_1 \le \theta_2$ and $\eta_1 \le \eta_2$, we define that $\theta_m = \theta_1$, $\theta_M = \theta_2$, $\eta_m = 2\eta_1 a_Y/\sigma_Y^2$ and $\eta_M = 2\eta_2 a_Y/\sigma_Y^2$. If the reinsurance premium strategy p is specified by (5), we restrict θ and η into $[\theta_m,\theta_M]$ and $[\eta_m,\eta_M]$, respectively. In fact, if $\theta = \theta_m$ and $\eta = \eta_m$, the reinsurer's premium rate achieves the lower bound at $c_1 = (1+\delta_1)\lambda a_Y$ with $\delta_1 = \theta_1 + \eta_1$; if $\theta = \theta_M$ and $\eta = \eta_M$, the reinsurer's premium rate attains the upper bound $c_2 = (1+\delta_2)\lambda a_Y$ with $\delta_2 = \theta_2 + \eta_2$. We denote by $\bar{\mathcal{D}}_R$ the space of all bounded risk loading strategies ξ that are admissible.

For convenience, we denote

$$\Pi(\theta,\eta) = \int_0^\infty \left((1-\alpha)\theta\left(y - \frac{\eta y+\theta}{\eta+\gamma_I}\right)_+ - \alpha\frac{\gamma_I}{2}\left(\frac{\eta y+\theta}{\eta+\gamma_I}\wedge y\right)^2 \right.$$
$$\left. + \left((1-\alpha)\frac{\eta}{2} - \frac{\gamma_R}{2}\right)\left(y - \frac{\eta y+\theta}{\eta+\gamma_I}\right)_+^2 \right) \mathrm{d}F(y). \tag{10}$$

The following proposition is used to conform with the existence of equilibrium strategies.

Proposition 3.2: *The function $\Pi(\theta,\eta)$ is Lipschitz continuous in (θ,η) over $[\theta_m,\theta_M]\times[\eta_m,\eta_M]$.*

Proof: See Appendix A.2. ■

Together with the results of Proposition 3.2, we show that there exists an optimal reinsurance contract for the optimization objectives (3) and (4) in the following theorem.

Theorem 3.2: *The optimal reinsurance contract is given by*

$$l^*(y) = \frac{\eta^* y + \theta^*}{\eta^* + \gamma_I} \wedge y, \tag{11}$$

and the risk loading strategy $\xi^* = (\theta^*, \eta^*)$ *is the maximizer of* $\Pi(\theta, \eta)$ *given by* (10) *which depends on the value of* α. *Under this optimal reinsurance contract, the insurer's value function is given by*

$$\begin{aligned} V_I(t, x_I) = x_I + (c - \lambda a_Y)(T - t) - \Big(\theta^* \mathbb{E}[Y - l^*(Y)]. \\ + \frac{\eta^*}{2}\mathbb{E}[(Y - l^*(Y))^2] + \frac{\gamma_I}{2}\mathbb{E}[l^*(Y)^2]\Big)\lambda(T - t), \end{aligned} \tag{12}$$

and the reinsurer's value function is given by

$$\begin{aligned} V_R(t, x_I, x_R) = \alpha x_I + x_R + \alpha(c - \lambda a_Y)(T - t) + \Big((1 - \alpha)\theta^* \mathbb{E}[Y - l^*(Y)] \\ - \alpha \frac{\gamma_I}{2}\mathbb{E}[l^*(Y)^2] + \left((1 - \alpha)\frac{\eta^*}{2} - \frac{\gamma_R}{2}\right)\mathbb{E}[(Y - l^*(Y))^2]\Big)\lambda(T - t). \end{aligned} \tag{13}$$

Proof: See Appendix A.3. ∎

Remark 3.1: In Proposition 3.2, by assuming ξ is bounded by $[\theta_m, \theta_M] \times [\eta_m, \eta_M]$, we show that $\Pi(\theta, \eta)$ is Lipschitz continuous, and then the optimal reinsurance strategy is given by (11). In fact, if ξ is not bounded and the maximum of $\Pi(\theta, \eta)$ is obtained at $\theta^* = \infty$ or $\eta^* = \infty$, then we have $l^* = y$, which means that both the insurer and the reinsurer do not participate in the reinsurance business. Since $p \equiv 0$ belongs to the admissible set $\mathcal{A}_R$, the reinsurer's optimal value function should be no less than the one in which no reinsurance business is undertaken, and thus the participation constraint holds. Since the optimal reinsurance contract can be derived explicitly when the premium is calculated by the variance premium principle or the expected value principle, in the next section, the optimization problem is discussed without the constraint to the premium. Note that adding a constraint will not increase the difficulty of solving the problem. With the constraint, we only need to compare the values of the corresponding candidate strategy with the boundaries, and thus several cases should be discussed to derive the final optimal reinsurance contract; see for example, Bai et al. (2021) and Chen & Shen (2019).

Remark 3.2: The expression of (11) shows that the insurer's transferred claims are equal to

$$y - l^*(y) = \frac{\gamma_I}{\eta^* + \gamma_I}\left(y - \frac{\theta^*}{\gamma_I}\right)_+,$$

which is a composition of a constant deductible followed by a constant coinsurance proportion. In particular, given a pair of (θ, η), the insurer's choices of the deductible and the proportion are determined separately by θ and η. A similar result is also obtained by Li & Young (2021). Not surprisingly, the optimal reinsurance strategy is in the form of pure excess-of-loss reinsurance strategy under the expected value premium principle ($\eta \equiv 0$); similarly, under the variance premium principle ($\theta \equiv 0$), the optimal reinsurance strategy is in the form of pure proportion reinsurance. We refer the readers to Section 3 for more discussions on these two special cases. Moreover, we see that the equilibrium strategy (11) is a time-homogeneous strategy which coincides with the observations in Li & Young (2021) and Zhang & Li (2021). Time homogeneity is a practical feature because the parameters in real (re)insurance contracts are typically fixed within a contract period. It also helps to avoid the conflict of interest in case the insurer and the reinsurer have different preferences on contract duration.

Remark 3.3: We remark that under the general premium principle (5), we investigate a Stackelberg stochastic differential reinsurance game under the time-inconsistent mean-variance framework in which the reinsurer also takes parts of the insurer's interests into consideration when setting the reinsurance premium. Compared to Chen & Shen (2019), both the criterion and the premium principle are more general, and in particular when $\alpha = 0$, the results in Chen & Shen (2019) can be derived directly by setting $\theta = 0$ or $\eta = 0$, respectively.

As mentioned in Remark 2.3, the reinsurer's optimization problem in (4) requires the specification of the weight parameter α in the formulation, which is less intuitive. Inspired by Dai et al. (2020), we consider an alternative dynamic mean-variance formulation for the reinsurer based on the original Markowitz' model

$$\begin{aligned} &\inf_{p\in\mathcal{A}_R}\left\{\frac{\gamma_R}{2}\mathrm{Var}_{t,x}[X_R(T)]+\alpha\frac{\gamma_I}{2}\mathrm{Var}_{t,x}[X_I(T)]\right\} \\ &\text{subjectto}\ \frac{1}{T-t}\mathbb{E}_{t,x}[X_R(T)+\alpha X_I(T)-X_R(t)-\alpha X_I(t)]\geq R(t), \end{aligned} \tag{14}$$

where $R(t)$ is a predetermined adaptive process, representing the reinsurer's expected annual target return at time t. The above formulation indicates that the reinsurer dynamically minimizes its risk subject to a predetermined target annual return $R(t)$. We have an interesting link from formulation (14) to formulation (4).

Proposition 3.3: *Let p^* be an optimal strategy to* (4) *and $X_R^*(t)+\alpha X_I^*(t)$ be the associated optimal return. Then p^* must be an optimal strategy to* (14) *with*

$$R(t)=\frac{1}{T-t}\mathbb{E}_{t,x}[X_R^*(T)+\alpha X_I^*(T)-X_R^*(t)-\alpha X_I^*(t)]. \tag{15}$$

Proof: See Appendix A.4. ■

With a given constant target annual return R, problem (14) shares the same optimal strategy as problem (4) with R computed by (15). This observation suggests a simple and intuitive way, which can be employed to identify the weight parameter α used in our dynamic mean-variance criterion (4). In the following section, we calculate the target annual return associated with the corresponding equilibrium strategy under two special premium principles, that is, the variance premium principle and the expected value principle. Using the idea of Proposition 3.3, we can estimate the weight parameter α given any proper target annual return, which is more intuitive.

4. Optimal results for two special cases

Since it is difficult to derive the explicit form of the equilibrium reinsurance strategy and the corresponding value functions under the general mean-variance premium principle, we consider two special cases of the reinsurance premium principle by setting $\theta \equiv 0$ and $\eta \equiv 0$ in (5), respectively. Since the reinsurance premium reduces to the variance premium principle and the expected value principle, the closed-form solutions and the corresponding properties are much more convenient to discuss for both parties.

4.1. The variance premium principle

In this subsection, we set $\theta \equiv 0$ in (5), then the reinsurance premium is computed according to the variance premium principle given by

$$p(t) = \int_0^\infty \big(y - l(t,y)\big)\,\nu(\mathrm{d}y) + \frac{\eta}{2}\int_0^\infty \big(y - l(t,y)\big)^2\,\nu(\mathrm{d}y), \tag{16}$$

in which η is the non-negative risk loading strategy to be determined. According to Remark 3.1, both the insurer and the reinsurer satisfy the participation constraints.

In the following theorem, according to the proofs of Proposition 3.1 and Theorem 3.2, we derive the explicit expressions of the optimal strategies and the corresponding value functions. Since the proof of Theorem 4.1 is similar to Theorem 3.2, we only present the major steps that highlight the differences.

Theorem 4.1: *When the variance premium principle* (16) *is adopted, the optimal reinsurance contract is given by*

$$l^*(y) = \frac{2\gamma_R + (1-\alpha)\gamma_I}{2(\gamma_R+\gamma_I)}\,y, \tag{17}$$

and

$$\eta^* = \frac{2\gamma_R + (1-\alpha)\gamma_I}{1+\alpha}. \tag{18}$$

Under this optimal reinsurance contract, the insurer's value function is given by

$$V_I(t,x_I) = x_I + (c - \lambda a_Y)(T-t) - \frac{2\gamma_R\gamma_I + (1-\alpha)\gamma_I^2}{4(\gamma_R+\gamma_I)}\sigma_Y^2\lambda(T-t), \tag{19}$$

and the reinsurer's value function is given by

$$V_R(t,x_I,x_R) = \alpha x_I + x_R + \alpha(c-\lambda a_Y)(T-t) + \frac{(1-\alpha)^2\gamma_I^2 - 4\alpha\gamma_R\gamma_I}{8(\gamma_R+\gamma_I)}\sigma_Y^2\lambda(T-t). \tag{20}$$

Proof: See Appendix B.1. ■

Remark 4.1: Even though we have argued in Remark 3.1 that both the insurer and the reinsurer are better off from participating in risk sharing, we can still verify that the constraints are established by direct calculation according to the expressions in (19) and (20). From (18), we can see that the reinsurer's optimal risk loading η^* is independent of a_Y and σ_Y^2 as well as any other information of claim sizes, and so is the insurer's optimal reinsurance strategy. This reveals that for the proportional reinsurance and the variance reinsurance premium principle, the equilibrium is a model-free solution to the game. Besides, it is clear that both l^* and η^* are decreasing functions with respect to α, which implies that when the weight of the insurer becomes higher, the reinsurer will price the reinsurance premium much cheaper, and then it is to be expected that the insurer would like to transfer more risks to the reinsurer.

In the above-mentioned differential reinsurance game, the insurer's problem is solved in the first step, and the reinsurer's is solved in the second step. This is different from the procedure of only solving a Pareto-optimal problem, where the reinsurance strategy and the risk loading strategies are determined simultaneously, i.e.

$$V(t,x) = \sup_{(l,\eta)}\left\{\mathbb{E}_{t,x}[X_R(T)] - \frac{\gamma_R}{2}\mathrm{Var}_{t,x}[X_R(T)] + \alpha\left(\mathbb{E}_{t,x}[X_I(T)] - \frac{\gamma_I}{2}\mathrm{Var}_{t,x}[X_I(T)]\right)\right\} \tag{21}$$

with $\alpha > 0$. We can derive a pair of optimal strategies to problem (21), which turns to be a bang-bang control and obviously depends on the value of α.

Proposition 4.1: *Assume that $\eta \in [\eta_m, \eta_M]$. Then under the variance premium principle, the Pareto-optimal strategy for problem* (21) *is*

(i) if $\alpha \geq 1$, $\eta^* = \eta_m$ and

$$l^*(y) = \frac{(\alpha - 1)\eta_m + \gamma_R}{(\alpha - 1)\eta_m + \gamma_R + \alpha\gamma_I} y;$$

(ii) if $0 \leq \alpha < 1$, $\eta^* = \eta_M$ and

$$l^*(y) = 0\mathbb{I}_{\left\{\eta_M \geq \frac{\gamma_R}{1-\alpha}\right\}} + \frac{(\alpha - 1)\eta_M + \gamma_R}{(\alpha - 1)\eta_M + \gamma_R + \alpha\gamma_I} y\mathbb{I}_{\left\{\eta_M < \frac{\gamma_R}{1-\alpha}\right\}}.$$

Proof: See Appendix B.2. ■

Remark 4.2: Proposition 4.1 shows that the optimal risk loading strategy either equals to η_m or η_M, and its value depends on α. It implies that the optimal risk loading strategy is a bang-bang control. Specifically, if we have a larger weight of the insurer in the optimization problem, then the optimal risk loading equals to the smallest value, and when the weight of the reinsurer is larger, then the risk loading equals to the largest one. Thus, compared with the profound implications offered by the Stackelberg differential game, problem (21) is much less interesting. A similar observation also applies to the Pareto-optimal strategy under the expected premium principle.

In the following proposition, using the idea of Proposition 3.3, we calculate the target annual return associated with the corresponding strategies under the variance premium principle. Then we can estimate the weight parameter α given any proper target annual return, which is more intuitive.

Proposition 4.2: *Under the variance premium principle, the target annual return associated with the strategy η^* given by* (18) *is*

$$R(\alpha) = \frac{\gamma_I^2(1-\alpha^2)(2\gamma_R + (1-\alpha)\gamma_I)}{8(\gamma_R + \gamma_I)^2}\lambda\sigma_Y^2 + \alpha(c - \lambda a_Y).$$

Moreover, R is always non-negative and satisfies

$$R(0) = \frac{\gamma_I^2(2\gamma_R + \gamma_I)}{8(\gamma_R + \gamma_I)^2}, \quad R(1) = \alpha(c - \lambda a_Y).$$

Proof: See Appendix B.3. ■

4.2. The expected value principle

In this subsection, we set $\eta \equiv 0$ in (5), then the reinsurance premium is computed according to the expected value principle given by

$$p(t) = (1 + \theta) \int_0^\infty \big(y - l(t, y)\big)\, \nu(\mathrm{d}y),$$

in which θ is the non-negative risk loading parameter to be determined.

Similarly, under some proper conditions, we derive the explicit expression of the equilibrium strategies and the corresponding value functions under the expected value principle in Theorem 4.2.

Theorem 4.2: *Define*

$$\Phi(z) = \mathbb{E}[(Y-z)_+|Y>z] - \frac{z}{1-\alpha+\frac{\gamma_R}{\gamma_I}}. \tag{22}$$

Assume $\Phi(z)$ *has at most one root over* $(0,\infty)$ *denoted by* z_0*. Then the optimal reinsurance contract is given by* $l^*(y) = z_0 \wedge y$ *and* $\theta^* = z_0\gamma_I$.[1] *The insurer's value function is given by*

$$\begin{aligned} V_I(t,x_I) = x_I + (c-\lambda a_Y)(T-t) - \bigg(& z_0\gamma_I \int_{z_0}^{\infty} y\,\mathrm{d}F(y) + \frac{\gamma_I}{2}\int_0^{z_0} y^2\mathrm{d}F(y) \\ & -\frac{\gamma_I}{2}z_0^2(1-F(z_0))\bigg)\lambda(T-t), \end{aligned} \tag{23}$$

and the reinsurer's value function is given by

$$\begin{aligned} V_R(t,x_I,x_R) = \alpha x_I + x_R + (c-\lambda a_Y)(T-t) + \bigg(& (\gamma_I(1-\alpha)+\gamma_R)z_0\int_{z_0}^{\infty} y\,\mathrm{d}F(y) \\ & -\frac{\gamma_R}{2}\int_{z_0}^{\infty} y^2\,\mathrm{d}F(y) + \left(\frac{\gamma_I}{2}\alpha - \frac{\gamma_R}{2} - \gamma_I\right) z_0^2(1-F(z_0)) \\ & -\alpha\frac{\gamma_I}{2}\int_0^{z_0} y^2\mathrm{d}F(y)\bigg)\lambda(T-t). \end{aligned} \tag{24}$$

Proof: See Appendix B.4. ■

Remark 4.3: Note that $\Phi(z)$ in (22) is in a similar form as the one in Chen & Shen (2019) (see Equation (24) for details). We refer to Proposition 4.4 and Remark 4.3 in their work, in which four sufficient conditions are provided for the assumption of Theorem 4.2. Also, we can see that $\Phi(0) = \mathbb{E}Y > 0$, thus $\theta^* \to \infty$ when $\Phi(z)$ has no root over $(0,\infty)$, which implies that both the insurer and the reinsurer would not participate in the reinsurance business. Also, as mentioned in Remark 3.1, the insurer's and reinsurer's optimal value functions should be larger than the values if they do not purchase any insurance because $(l,p) = (0,0)$ belongs to the admissible set $\mathcal{A}_I \times \mathcal{A}_R$. Compared with the results under the variance premium principle, we can see from (22) that the solutions to the differential reinsurance game are model-dependent for the excess-of-loss reinsurance and the expected value reinsurance premium principle. Besides, it is clear that the root z_0 decreases as α increases, resulting in a lower retention level l^* and a cheaper risk loading strategy θ^*, which is consistent with intuition.

Under the expected value principle, we also consider the Pareto-optimal problem (21). The optimal risk loading strategy is still a bang-bang control and depends on the value of α. Since the proof is similar to Proposition 4.1, we omit the details here.

Proposition 4.3: *Assume that* $\theta \in [\theta_m, \theta_M]$*. Then under the expected value principle, the optimal Pareto-optimal strategy for the problem* (21) *is*

(i) if $\alpha \geq 1$, $\theta^* = \theta_m$ *and*

$$l^*(y) = \frac{(\alpha-1)\theta_m + \gamma_R y}{\gamma_R + \alpha\gamma_I} \wedge y;$$

[1] In particular, if $\Phi(z)$ has no root, $z_0 = \infty$, and thus $(l^*, \eta^*) = (y, \infty)$.

(ii) *if* $0 \leq \alpha < 1, \theta^* = \theta_M$ *and*

$$l^*(y) = 0\mathbb{I}_{\left\{\theta_M \geq \frac{\gamma_R}{1-\alpha}y\right\}} + \frac{(\alpha-1)\theta_M + \gamma_R y}{\gamma_R + \alpha\gamma_I}\mathbb{I}_{\left\{\theta_M < \frac{\gamma_R}{1-\alpha}y\right\}}.$$

Analogous to the proof of Proposition 4.2, the target annual return under the expected value principle is derived in the following proposition.

Proposition 4.4: *Assume that there exists a unique solution z_0 to the algebraic equation $\Phi(z) = 0$ with Φ given by* (22). *Under the expected value principle, the target annual return associated with the equilibrium strategy $\theta^* = z_0\gamma_I$ is given by*

$$R(\alpha) = \alpha(c - \lambda\mathbb{E}Y) + \gamma_I z_0(1-\alpha)\mathbb{E}[Y - z_0]_+.$$

Moreover, R is always non-negative and satisfies

$$R(0) = \gamma_I z_0 \mathbb{E}[Y - z_0]_+, \quad R(1) = \alpha(c - \lambda a_Y).$$

Proof: See Appendix B.5. ■

5. Numerical examples

In this section, we first provide a numerical example for theoretical results under the general mean-variance premium principle in Section 3. We mainly focus on the effects of the weight parameter α, which is an important contribution in our paper. We consider two specific distributions of the claim size: the uniform distribution and the exponential distribution. Suppose that $\gamma_I = 0.25$ and $\gamma_R = 0.1$. In each of the two cases, a unique maximizer exists and can be found explicitly by maximizing the function

$$\Pi(\theta,\eta) = \int_0^\infty \left((1-\alpha)\theta\left(y - \frac{\eta y + \theta}{\eta + \gamma_I}\right)_+ - \alpha\frac{\gamma_I}{2}\left(\frac{\eta y + \theta}{\eta + \gamma_I} \wedge y\right)^2 + \left((1-\alpha)\frac{\eta}{2} - \frac{\gamma_R}{2}\right)\left(y - \frac{\eta y + \theta}{\eta + \gamma_I}\right)_+^2\right) \mathrm{d}F(y).$$

Figure 1 shows the effects of α on equilibrium reinsurance strategy and premium strategy for the uniform distribution in the interval $[0, 2]$, and Figure 2 provides the results for the exponential distribution with the parameter 1. We find that l^* and θ^* decrease with α, which implies that if the reinsurer pays more attention to the interests of the insurer, the reinsurer will charge a lower reinsurance premium, and the insurer prefers to share more risks to the reinsurer and undertake less risks by itself. Meanwhile, η^* is an increasing function of α under the uniform distribution and keeps a relatively high value under the exponential distribution. This is mainly because the tail risk when the distribution of the claim size is the exponential distribution is heavier than that of the uniform distribution. As a result, the reinsurer would like to adopt a higher level of the parameter in the variance term under the case of the exponential distribution.

Since the premium strategy is chosen by the reinsurer, a natural question is which premium principle is better for the reinsurer. In the following, under the two special premium principles, we first calculate the explicit form of the optimal reinsurance contracts and the corresponding value functions based on the results in Section 4. By assuming that the claim size follows some special distributions, we further compare the corresponding results and try to provide some guidance on which premium principle the reinsurer should choose in different situations. For simplicity, we assume $\alpha = 0$.

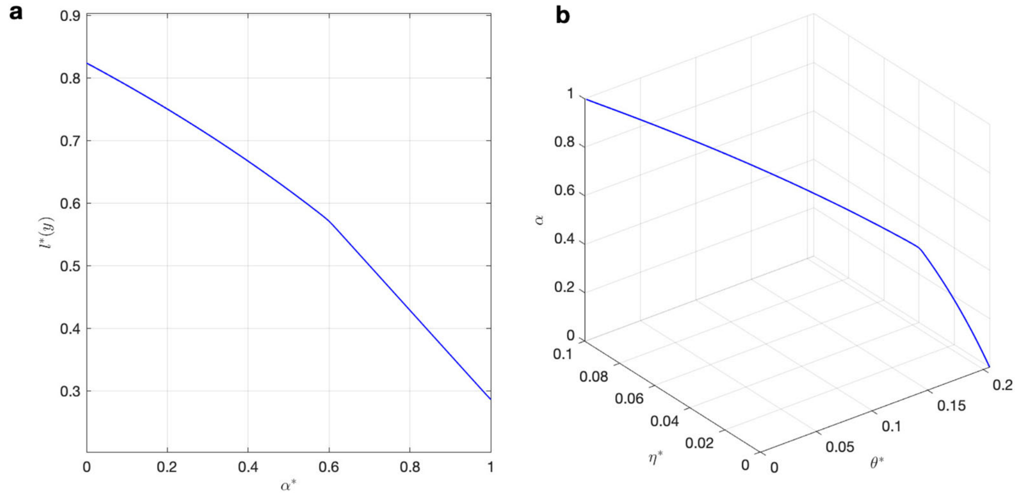

Figure 1. The effects of α on l^* and (θ^*, η^*) with $y = 1$ (uniform distribution).

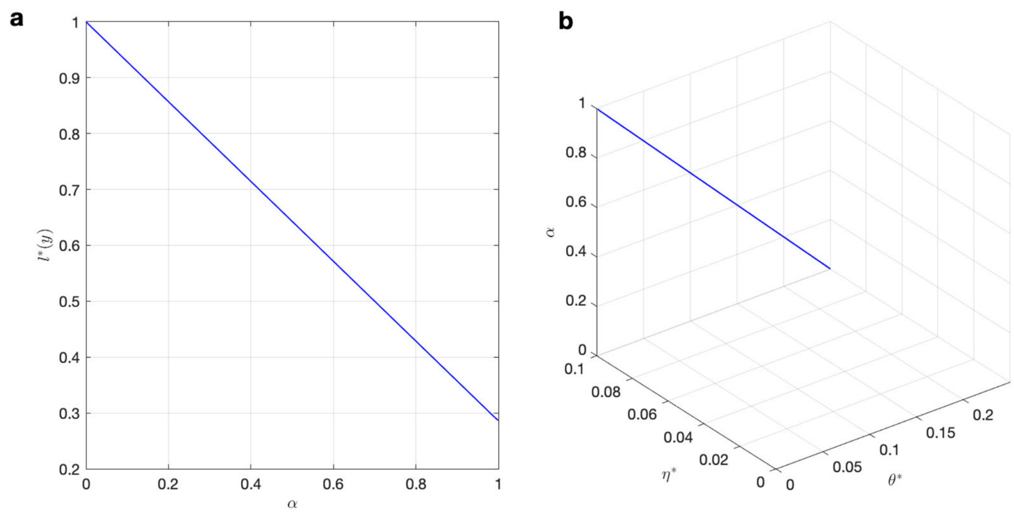

Figure 2. The effects of α on l^* and (θ^*, η^*) with $y = 5$ (exponential distribution).

Example 5.1: We assume that the claim size is uniformly distributed in the interval $[0, b]$. Then we have $a_Y = b/2$ and $\sigma_Y^2 = b^2/3$. Under the variance premium principle, the value function is

$$V_R(t, x_R; \eta^*) = x_R + \frac{b^2}{3}\lambda(T - t)\frac{\gamma_I^2}{8(\gamma_R + \gamma_I)}.$$

Under the expected value principle, by solving $\Phi(z) = 0$, we have $z_0 = b(\gamma_I + \gamma_R)/(3\gamma_I + \gamma_R)$. The value function for the reinsurer is

$$V_R(t, x_R; \theta^*) = x_R + \frac{b^2}{3}\lambda(T - t)\frac{2\gamma_I^2(9\gamma_I^2 - \gamma_I\gamma_R)}{(3\gamma_I + \gamma_R)^3}.$$

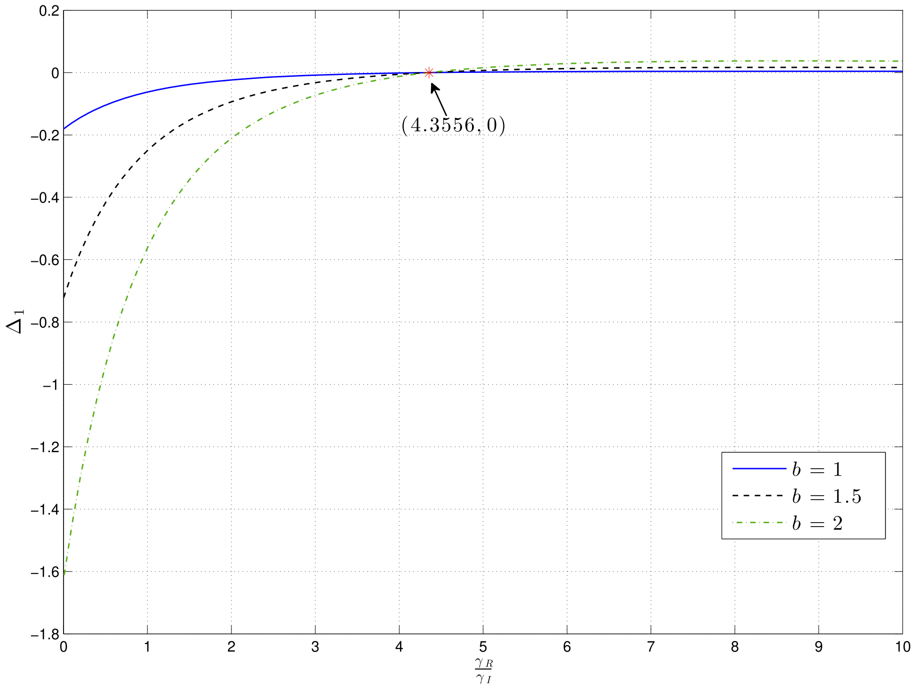

Figure 3. The effect of $\frac{\gamma_R}{\gamma_I}$ and b on the value of Δ_1.

Thus, we have

$$\Delta_1 = V_R(t, x_R; \eta^*) - V_R(t, x_R; \theta^*) = \lambda(T-t)\sigma_Y^2 \gamma_I \left(\frac{1}{8(1+\frac{\gamma_R}{\gamma_I})} - \frac{2(9-\frac{\gamma_R}{\gamma_I})}{(3+\frac{\gamma_R}{\gamma_I})^3} \right).$$

Denote by

$$f_1(x) = \frac{1}{8(1+x)} - \frac{2(9-x)}{(3+x)^3}.$$

It is not difficult to prove that there exists a unique zero x_1 such that $f(x_1) = 0$. Note the x_1 is independent of b. Also, we have $f_1(0) < 0$ and $f_1(\infty) \to 0^+$. Therefore, we can conclude that when $\gamma_R/\gamma_I \le x_1$, it is better for reinsurer to choose the expected value principle; when $\gamma_R/\gamma_I > x_1$, it is better for reinsurer to choose the variance premium principle. In particular, by setting $t = 0$, $T = 10$, $\lambda = 1$, and $\gamma_I = 0.1$, we show the effects of γ_R/γ_I and b on the value of Δ_1 in Figure 3. We can see that $x_1 = 4.3556$, which is independent of b.

Example 5.2: We assume that the claim size is exponentially distributed with the parameter rate $\gamma > 0$. Then we have $a_Y = 1/\gamma$ and $\sigma_Y^2 = 2/\gamma^2$. Under the variance premium principle, the value function is

$$V_R(t, x_R; \eta^*) = x_I + \frac{\gamma_R^2}{8(\gamma_R + \gamma_I)} \lambda \sigma_Y^2 (T-t).$$

Under the expected value principle, by solving $\Phi(z) = 0$, we have $z_0 = (\gamma_I + \gamma_R)/(\gamma_I \gamma_R)$. The value function for the reinsurer is

$$V_R(t, , x_R; \theta^*) = x_R + \gamma_I e^{-\frac{\gamma_I + \gamma_R}{\gamma_I}} \lambda \sigma_Y^2 (T-t).$$

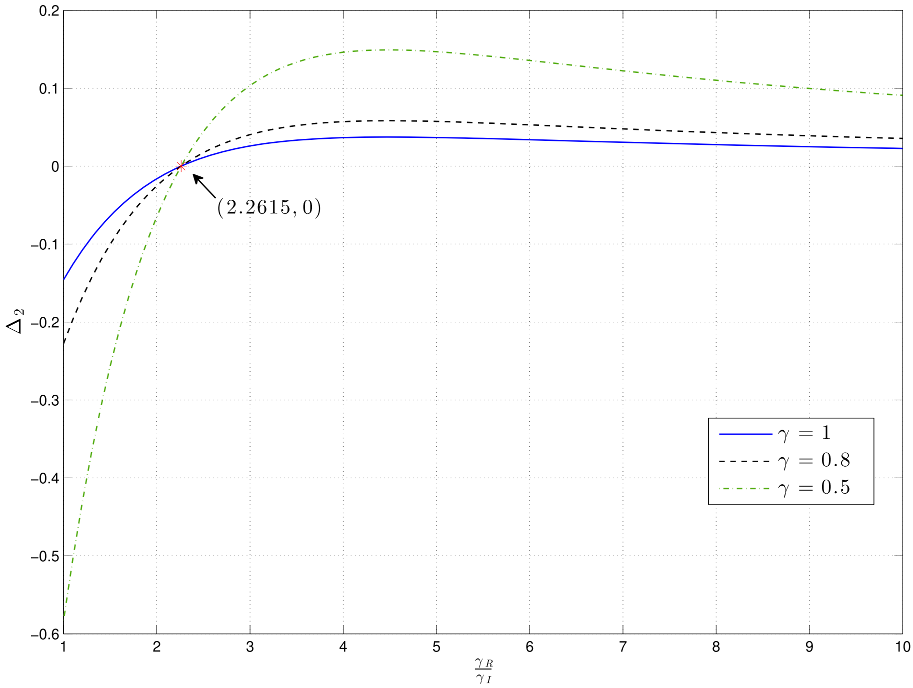

Figure 4. The effect of $\frac{\gamma_R}{\gamma_I}$ and γ on the value of Δ_2.

Thus, we have

$$\Delta_2 = V_R(t, x_R; \eta^*) - V_R(t, x_R; \theta^*) = \lambda(T-t)\sigma_Y^2\gamma_I \left(\frac{1}{8(1+\frac{\gamma_R}{\gamma_I})} - e^{-(1+\frac{\gamma_R}{\gamma_I})} \right).$$

Denote by

$$f_2(x) = \frac{1}{8(1+\frac{\gamma_R}{\gamma_I})} - e^{-(1+\frac{\gamma_R}{\gamma_I})}.$$

Similarly to the analysis of Example 5.1, we can also show that there exists a unique zero x_2 such that $f_2(x_2) = 0$ and satisfies $f_2(0) < 0$ and $f_2(\infty) \to 0^+$. Thus, the observations in Example 5.1 are applied to the exponential distribution. In particular, by setting $t = 0$, $T = 10$, $\lambda = 1$, and $\gamma_I = 0.1$, we show the effects of γ_R/γ_I and b on the value of Δ_2 in Figure 4. We can see that $x_1 = 2.2615$, which is independent of γ.

Example 5.3: We assume that the claim size is Pareto distributed and the distribution function is given by $F(y) = 1 - (1/(y+1))^\beta$ with the shape parameter $\beta > 2$. Then we have $a_Y = 1/(\beta-1)$ and $\sigma_Y^2 = 2/((\beta-1)(\beta-2))$. Under the variance premium principle, the value function is

$$V_R(t, x_R; \eta^*) = x_R + \frac{\gamma_I^2}{8(\gamma_R+\gamma_I)}\lambda\sigma_Y^2(T-t).$$

Under the expected value principle, by solving $\Phi(z) = 0$, we have $z_0 = (\gamma_I + \gamma_R)/(\gamma_I(\beta-2) - \gamma_R)$ for $\beta > 2 + \gamma_R/\gamma_I$, and $z_0 = \infty$ for $\beta \le 2 + \gamma_R/\gamma_I$. The value function for the reinsurer is $V_R(t, x_R, \theta^*) = x_R$ for $\beta \le \gamma_R/\gamma_I + 2$; and for $\beta > \gamma_R/\gamma_I + 2$,

$$V_R(t, x_R; \theta^*) = x_R + \frac{(\beta-2)\gamma_I^2\gamma_R + 2(\beta-2)\gamma_I\gamma_R^2 + (\beta-1)(\beta-2)\gamma_I^3}{2(\gamma_I(\beta-2) - \gamma_R)^2}.$$

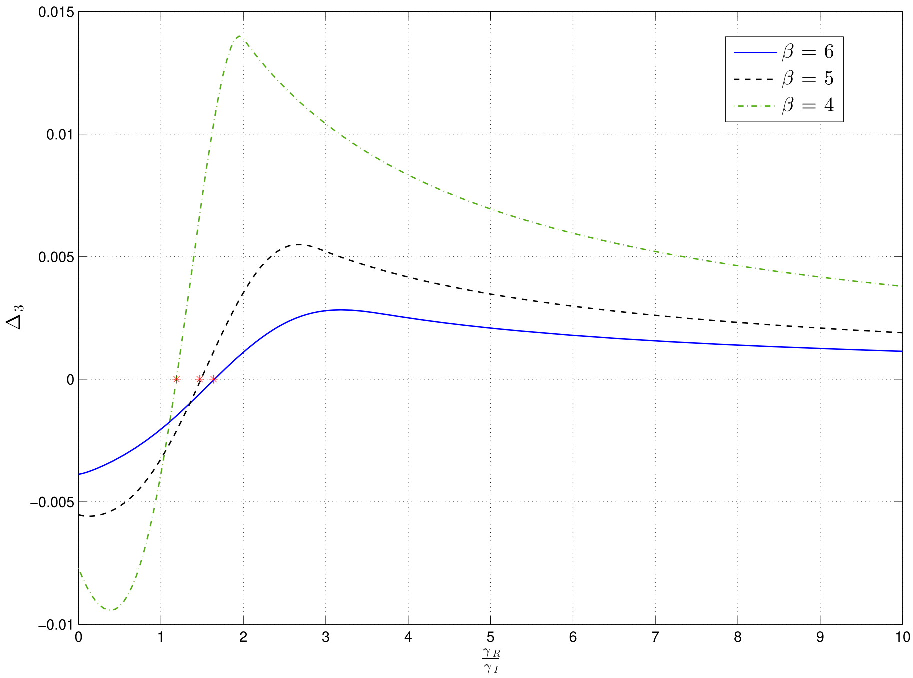

Figure 5. The effect of $\frac{\gamma_R}{\gamma_I}$ and β on the value of Δ_3.

$$\left(\frac{(\beta-2)\gamma_I-\gamma_R}{(\beta-1)\gamma_I}\right)^{\beta}\lambda\sigma_Y^2(T-t).$$

If $\beta \leq \gamma_R/\gamma_I + 2$, we have

$$\Delta_3 = V_R(t, x_R; \eta^*) - V_R(t, x_R; \theta^*) = \frac{1}{8(1+\frac{\gamma_R}{\gamma_I})}\lambda(T-t)\sigma_Y^2\gamma_I > 0,$$

which implies that if β is small enough, the variance premium principle is always better than the expected premium principle. This observation is to be expected since a smaller value of β leads to more claims on average, and thus it is always better for reinsurer to choose the variance premium principle. If $\beta > \gamma_R/\gamma_I + 2$,

$$\begin{aligned}\Delta_3 &= V_R(t, x_R; \eta^*) - V_R(t, x_R, \theta^*)\\ &= \left(\frac{1}{8(1+\frac{\gamma_R}{\gamma_I})} - \frac{(\beta-2)\frac{\gamma_R}{\gamma_I} + 2(\beta-2)(\frac{\gamma_R}{\gamma_I})^2 + (\beta-1)(\beta-2)}{2(\beta-2-\frac{\gamma_R}{\gamma_I})^2}\cdot\right.\\ &\quad\left.\left(\frac{\beta-2-\frac{\gamma_R}{\gamma_I}}{\beta-1}\right)^{\beta}\right)\lambda(T-t)\sigma_Y^2\gamma_I.\end{aligned}$$

We can see clearly that the sign of the expression depends on the value of β. Even though the relation of the value functions is not obvious for $\beta > \gamma_R/\gamma_I + 2$, it is easy to see that when β is larger enough or the insurer tends to be risk-neutral, the result under the expected value principle is much better than that under the variance premium principle. In particular, by setting $t = 0$, $T = 10$, $\lambda = 1$, and $\gamma_I = 0.1$, we show the effects of γ_R/γ_I and β on the value of Δ_1 in Figure 5. We obtain a similar observation as in Example 5.1 except that the intersections with x-axis are different as β varies.

Examples 5.1–5.3 show that if the value of γ_R/γ_I is relative small, it is better for reinsurer to choose the expected value principle; but if the value of γ_R/γ_I is relative large, it is better for reinsurer to choose the variance premium principle. This is because a larger value of γ_R implies a higher degree of risk aversion. If the reinsurer is relatively risk-averse, it is better for the reinsurer to choose the variance premium principle which takes the fluctuation of claims into consideration. If the reinsurer is less risk-averse or risk-neutral, it is better for the reinsurer to choose the expected value principle, which only considers the mean of the claims. Therefore, when the reinsurance premium is calculated by either the expected value principle or the variance premium principle, we may suggest that when the risk-averse parameter of the reinsurer and the average claims are relatively large, the variance premium principle is a good choice for pricing the insurance contract; otherwise, the expected premium principle is much better than the variance premium principle.

6. Conclusion

In this paper, we use the mean-variance framework to consider an optimal reinsurance contract problem between the insurer and the reinsurer, and the reinsurance premium is calculated by the mean-variance premium principle. Specifically, the insurer as the follower of the game aims to find the optimal retention level to maximize its mean-variance criterion, while the reinsurer as the leader of the game needs to determine the optimal reinsurance premium to maximize the weighted sum of the insurer's and reinsurer's mean-variance criteria. To the best of our knowledge, our paper is the first one to treat the reinsurer as a social planer when pricing the reinsurance premium in a stochastic Stackelberg differential game. It turns out that the optimal reinsurance contract depends on the weight of the insurer that the reinsurer considers. When the reinsurance premium is calculated by either the expected value principle or the variance premium principle, we also provide some guidance on which premium principle the reinsurer should use. In particular, we suggest adopting the expected value principle when the ratio of the reinsurer's risk-averse parameter to the insurer's risk-averse parameter is relatively small; otherwise, the variance premium principle is better for the reinsurer to choose. Potential future research on differential games is still of great interest, such as to investigate a hybrid differential game among multiple insurers and reinsurers and take the investment and model uncertainty into consideration simultaneously.

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Technical appendices

Appendix 1. Proofs of Section 3

A.1 The Proof of Proposition 3.1

Proof: This proof is adapted from Chen & Shen (2019). Given the insurer's and reinsurer's surplus processes in (1) and (2), we have

$$\begin{aligned} X_I(T) - X_I(t) &= \int_t^T (c - p(s))\mathrm{d}s - \int_t^T \int_0^\infty l(s,y)N(\mathrm{d}s,\mathrm{d}y) \\ &= (c - \lambda a_Y)(T-t) - \int_t^T \int_0^\infty \left(\theta(y - l(s,y)) + \frac{\eta}{2}(y - l(s,y))^2\right)\nu(\mathrm{d}y)\mathrm{d}s \\ &\quad - \int_t^T \int_0^\infty l(s,y)\tilde{N}(\mathrm{d}s,\mathrm{d}y), \end{aligned}$$

and

$$\begin{aligned} X_R(T) - X_R(t) &= \int_t^T p(s)\mathrm{d}s - \int_t^T \int_0^\infty (y - l(s,y))N(\mathrm{d}s,\mathrm{d}y) \\ &= \int_t^T \int_0^\infty \left(\theta(y - l(s,y)) + \frac{\eta}{2}(y - l(s,y))^2\right)\nu(\mathrm{d}y)\mathrm{d}s \\ &\quad - \int_t^T \int_0^\infty (y - l(s,y))\tilde{N}(\mathrm{d}s,\mathrm{d}y). \end{aligned}$$

Then it follows that

$$\begin{aligned} &\mathbb{E}_{t,x}[X_I(T)] - \frac{\gamma_I}{2}\mathrm{Var}_{t,x}[X_I(T)] \\ &= x_I + (c - \lambda a_Y)(T-t) - \int_t^T \int_0^\infty \left(\theta(y - l(t,y)) + \frac{\eta}{2}(y - l(s,y))^2 + \frac{\gamma_I}{2} l(s,y)^2\right)\nu(\mathrm{d}y)\mathrm{d}s, \end{aligned}$$

and

$$\begin{aligned} &\mathbb{E}_{t,x}[X_R(T)] - \frac{\gamma_R}{2}\mathrm{Var}_{t,x}[X_R(T)] \\ &= x_R + \int_t^T \int_0^\infty \left(\theta(y - l(s,y)) + \frac{\eta - \gamma_R}{2}(y - l(s,y))^2\right)\nu(\mathrm{d}y)\mathrm{d}s. \end{aligned}$$

Thus, we can compute the insurer's and reinsurer's value functions as

$$\begin{aligned} &\sup_{l \in \mathcal{A}_I}\left\{\mathbb{E}_{t,x}[X_I(T)] - \frac{\gamma_I}{2}\mathrm{Var}_{t,x}[X_I(T)]\right\} \\ &= x_I + (c - \lambda a_Y)(T-t) - \inf_{l \in \mathcal{A}_I}\left\{\int_t^T \int_0^\infty \left(\theta(y - l(s,y)) + \frac{\eta}{2}(y - l(s,y))^2 + \frac{\gamma_I}{2} l^2(s,y)\right)\nu(\mathrm{d}y)\mathrm{d}s\right\}, \end{aligned} \tag{A1}$$

and

$$\begin{aligned} &\sup_{(\theta,\eta)\in\mathcal{D}_R}\left\{\mathbb{E}_{t,x}[X_R(T)] - \frac{\gamma_R}{2}\mathrm{Var}_{t,x}[X_R(T)] + \alpha\left(\mathbb{E}_{t,x}[X_I(T)] - \frac{\gamma_I}{2}\mathrm{Var}_{t,x}[X_I(T)]\right)\right\} \\ &= x_R + \alpha x_I + \alpha(c - \lambda a_Y)(T-t) + \sup_{(\eta,\theta)\in\mathcal{D}_R}\left\{\int_t^T \int_0^\infty \left((1-\alpha)\theta(y - l^*(s,y))\right.\right. \\ &\quad \left.\left. -\alpha\frac{\gamma_I}{2} l^*(s,y)^2 + \left(\frac{\eta}{2}(1-\alpha) - \frac{\gamma_R}{2}\right)(y - l^*(s,y))^2\right)\nu(\mathrm{d}y)\mathrm{d}s\right\}. \end{aligned} \tag{A2}$$

Clearly, the strategies l and $\xi = (\theta,\eta)$ only appear in the mean and variance terms on the right hand sides of (A.1) and (A.2), which are independent of the surpluses, their optimal trajectories must be state-independent. Besides, since the reinsurer considers part of the insurer's interests, the reinsurer's value function also depends on the insurer's surplus process. ■

A.2 The Proof of Proposition 3.2

Proof: We only show that $\Pi(\theta,\eta)$ is Lipschitz continuous in θ over $[\theta_m,\theta_M]$ since the continuity of η can be obtained along the same lines but the proof is more straightforward. Fix $\eta\in[\eta_m,\eta_M]$, for any $\theta_1,\theta_2\in[\theta_m,\theta_M]$, we have

$$|\Pi(\theta_1,\eta)-\Pi(\theta_2,\eta)|\le\bar{\mathbb{I}}_1+\bar{\mathbb{I}}_2+\bar{\mathbb{I}}_3,$$

in which

$$\bar{\mathbb{I}}_1=|1-\alpha|\int_0^\infty\left|\theta_1\left(y-\frac{\theta_1+\eta y}{\eta+\gamma_I}\right)_+-\theta_2\left(y-\frac{\theta_2+\eta y}{\eta+\gamma_I}\right)_+\right|\mathrm{d}F(y),$$

$$\bar{\mathbb{I}}_2=\left|\frac{\eta}{2}(1-\alpha)-\frac{\gamma_R}{2}\right|\int_0^\infty\left|\left(y-\frac{\theta_1+\eta y}{\eta+\gamma_I}\right)_+^2-\left(y-\frac{\theta_2+\eta y}{\eta+\gamma_I}\right)_+^2\right|\mathrm{d}F(y),$$

and

$$\bar{\mathbb{I}}_3=\alpha\frac{\gamma_I}{2}\int_0^\infty\left|\left(\frac{\theta_1+\eta y}{\eta+\gamma_I}\wedge y\right)^2-\left(\frac{\theta_2+\eta y}{\eta+\gamma_I}\wedge y\right)^2\right|\mathrm{d}F(y).$$

Without loss of generality, we assume that $\theta_1<\theta_2$. Then, it follows that

$$\begin{aligned}\bar{\mathbb{I}}_1&\le|1-\alpha|\left(\int_0^{\frac{\theta_1}{\gamma_I}}0\,\mathrm{d}F(y)+\int_{\frac{\theta_1}{\gamma_I}}^{\frac{\theta_2}{\gamma_I}}\theta_1\left(y-\frac{\theta_1+\eta y}{\eta+\gamma_I}\right)\mathrm{d}F(y)\right.\\&\quad\left.+\int_{\frac{\theta_2}{\gamma_I}}^\infty\left|\theta_1\left(y-\frac{\theta_1+\eta y}{\eta+\gamma_I}\right)-\theta_2\left(y-\frac{\theta_2+\eta y}{\eta+\gamma_I}\right)\right|\mathrm{d}F(y)\right)\\&\le|1-\alpha|\left(\theta_M\int_{\frac{\theta_1}{\gamma_I}}^{\frac{\theta_2}{\gamma_I}}\left|\frac{\theta_2}{\gamma_I}-\frac{\theta_1+\eta\frac{\theta_1}{\gamma_I}}{\eta+\gamma_I}\right|]\mathrm{d}F(y)+|\theta_1-\theta_2|\int_{\frac{\theta_2}{\gamma_I}}^\infty y\,\mathrm{d}F(y)\right.\\&\quad\left.+\frac{2\theta_M}{\eta+\gamma_I}|\theta_1-\theta_2|\mathbb{P}(Y\ge\frac{\theta_2}{\gamma_I})+\frac{\eta}{\eta+\gamma_I}|\theta_1-\theta_2|\int_{\frac{\theta_2}{\gamma_I}}^\infty y\,\mathrm{d}F(y)\right)\\&\le|1-\alpha|\,|\theta_1-\theta_2|\left(\frac{\theta_M}{\gamma_I}+\frac{2\theta_M}{\eta+\gamma_I}+\frac{2\eta+\gamma_I}{\eta+\gamma_I}a_Y\right).\end{aligned}$$

In the same vein, we can derive

$$\bar{\mathbb{I}}_2\le\left|(1-\alpha)\frac{\eta}{2}-\frac{\gamma_R}{2}\right||\theta_1-\theta_2|\left(\frac{2\theta_M}{\gamma_I^2}+\frac{2a_Y}{\eta+\gamma_I}+\frac{2\theta_M+2\eta a_Y}{(\eta+\gamma_I)^2}\right),$$

and

$$\bar{\mathbb{I}}_3\le\alpha\frac{\gamma_I}{2}|\theta_1-\theta_2|\left(\frac{2\theta_M}{\gamma_I^2}+\frac{2\theta_M+2\eta a_Y}{(\eta+\gamma_I)^2}\right).$$

Then there exists a finite constant M such that $|\Pi(\theta_1,\eta)-\Pi(\theta_2,\eta)|\le M|\theta_1-\theta_2|$. Therefore, the function $\Pi(\theta,\eta)$ is Lipschitz continuous in θ over $[\theta_m,\theta_M]$. ■

A.3 The Proof of Theorem 3.2

Proof: For the insurer's problem (3), we try the following ansatz for the value functions

$$V_I(t,x_I)=x_I+B_I(t),\quad g_I(t,x_I)=x_I+b_I(t)\tag{A3}$$

with the terminal conditions $B_I(T)=b_I(T)=0$. Substituting (A3) back into the extended HJB equation (6) yields

$$B_I'(t)+(c-\lambda a_Y)-\inf_{l\in\mathcal{A}_I}\left\{\int_0^\infty\left(\theta(y-l(t,y))+\frac{\eta}{2}(y-l(t,y))^2+\frac{\gamma_I}{2}l^2(t,y)\right)\nu(\mathrm{d}y)\right\}=0.\tag{A4}$$

From this integral representation, we can obtain the minimizer by minimizing the integrand y-by-y, subject to $0\le l(t,y)\le y$. As a function of $l(t,y)$, the integrand is a parabola, so it is minimized by

$$l^*(y,\theta,\eta)=\frac{\eta y+\theta}{\eta+\gamma_I}\wedge y.\tag{A5}$$

For the reinsurer's problem (4), we try the following ansatz for the value functions

$$V_R(t,x_I,x_R) = x_R + \alpha x_I + B_R(t), \quad g_R(t,x_I,x_R) = x_R + b_R(t), \quad \bar{g}_I(t,x_I,x_R) = x_I + \bar{b}_I(t) \tag{A6}$$

with the terminal conditions $B_R(T) = b_R(T) = \bar{b}_I(T) = 0$. By plugging (A6) back into (8), we have

$$\begin{aligned} B_R'(t) + \alpha(c - \lambda a_Y) + \sup_{(\theta,\eta)\in\bar{\mathcal{D}}_R} \Bigg\{ \int_0^\infty & \left((1-\alpha)\theta\left(y - l^*(y,\theta,\eta)\right)\right. \\ & \left. -\alpha\frac{\gamma_I}{2} l^*(y,\theta,\eta)^2 + \left((1-\alpha)\frac{\eta}{2} - \frac{\gamma_R}{2}\right)\left(y - l^*(y,\theta,\eta)\right)^2\right)\nu(\mathrm{d}y)\Bigg\} = 0. \end{aligned} \tag{A7}$$

We need to solve the following optimization problem

$$\begin{aligned} \sup_{(\theta,\eta)\in\bar{\mathcal{D}}_R} \Bigg\{ \int_0^\infty & \left((1-\alpha)\theta(y - l^*(y,\theta,\eta)) - \alpha\frac{\gamma_I}{2}(l^*(y,\theta,\eta))^2\right. \\ & \left. + \left((1-\alpha)\frac{\eta}{2} - \frac{\gamma_R}{2}\right)(y - l^*(y,\theta,\eta))^2\right)\nu(\mathrm{d}y)\Bigg\} = \lambda \sup_{(\theta,\eta)\in\bar{\mathcal{D}}_R} \Pi(\theta,\eta). \end{aligned} \tag{A8}$$

Proposition 3.2 has proved that $\Pi(\theta,\eta)$ is Lipschitz continuous on the interval $[\theta_m,\theta_M]\times[\eta_m,\eta_M]$. Therefore, a maximum must be achieved in (A8) at $\xi^* = (\theta^*,\eta^*) \in \bar{\mathcal{D}}_R$. Once we get the optimal reinsurance contract $(l^*(y),\xi^*)$, by the terminal conditions $B_I(T) = B_R(T)$, one obtains

$$B_I(t) = \left((c - \lambda a_Y) - \lambda\left(\theta^*\mathbb{E}[Y - l^*(Y)] + \frac{\eta^*}{2}\mathbb{E}[(Y - l^*(Y))^2] + \frac{\gamma_I}{2}\mathbb{E}[l^*(Y)^2]\right)\right)(T-t), \tag{A9}$$

and

$$\begin{aligned} B_R(t) = \alpha(c - \lambda a_Y)(T-t) + & \left((1-\alpha)\theta^*\mathbb{E}[Y - l^*(Y)] - \alpha\frac{\gamma_I}{2}\mathbb{E}[l^*(Y)^2]\right. \\ & \left. + \left((1-\alpha)\frac{\eta}{2} - \frac{\gamma_R}{2}\right)\mathbb{E}[(Y - l^*(Y))^2]\right)\lambda(T-t). \end{aligned} \tag{A10}$$

With the terminal conditions $b_I(T) = b_R(T) = \bar{b}_I(T) = 0$, we have

$$b_I(t) = \bar{b}_I(t) = (c - \lambda a_Y)(T-t) - \left(\theta^*\mathbb{E}[Y - l^*(Y)] + \frac{\eta^*}{2}\mathbb{E}[(Y - l^*(Y))^2]\right)\lambda(T-t),$$

and

$$b_R(t) = \left(\theta^*\mathbb{E}[Y - l^*(Y)] + \frac{\eta^*}{2}\mathbb{E}[(Y - l^*(Y))^2]\right)\lambda(T-t).$$

Combining (A9) and (A10) with the trial solutions given in (A3) and (A6) gives the value functions (12) and (13). ■

A.4 The Proof of Proposition 3.3

Proof: Suppose for the purpose of contradiction that there exists a perturbation ϵ at time t such that

$$\liminf_{\epsilon\to 0^+} -\frac{\frac{\gamma_R}{2}(\mathrm{Var}_{t,x}[X_R^*(T)] - \mathrm{Var}_{t,x}[X_R^\epsilon(T)]) + \alpha\frac{\gamma_I}{2}(\mathrm{Var}_{t,x}[X_I^*(T)] - \mathrm{Var}_{t,x}[X_I^\epsilon(T))}{\epsilon} < 0,$$

it then follows that

$$\begin{aligned} & \liminf_{\epsilon\to 0^+} \frac{J_R(t,x;l^*(\cdot,p^*(\cdot)),p^*(\cdot)) - J_R(t,,x;l^*(\cdot,p^\epsilon(\cdot)),p^\epsilon(\cdot))}{\epsilon} \\ & = \liminf_{\epsilon\to 0^+} \frac{\mathbb{E}_{t,x}[X_R^*(T) + \alpha X_I^*(T)] - \mathbb{E}_{t,x}[X_R^\epsilon(T) - \alpha X_I^\epsilon(T)]}{\epsilon} \\ & \quad - \frac{\frac{\gamma_R}{2}(\mathrm{Var}_{t,x}[X_R^*(T)] - \mathrm{Var}_{t,x}[X_R^\epsilon(T)]) + \alpha\frac{\gamma_I}{2}(\mathrm{Var}_{t,x}[X_I^*(T)] - \mathrm{Var}_{t,x}[X_I^\epsilon(T)]}{\epsilon} \\ & < -\frac{\frac{\gamma_R}{2}(\mathrm{Var}_{t,x}[X_R^*(T)] - \mathrm{Var}_{t,x}[X_R^\epsilon(T)]) + \alpha\frac{\gamma_I}{2}(\mathrm{Var}_{t,x}[X_I^*(T)] - \mathrm{Var}_{t,x}[X_I^\epsilon(T))}{\epsilon} < 0, \end{aligned}$$

which contradicts the fact that p^* is an optimal strategy for problem (4). Thus, p^* must be an optimal strategy to (14). ■

Appendix 2. Proofs of Section 4

A.5 The Proof of Theorem 4.1

Proof: When $\theta \equiv 0$, Equation (A4) reduces to

$$B_I'(t) + (c - \lambda a_Y) - \inf_{l \in \mathcal{A}_I} \left\{ \int_0^\infty \left(\frac{\eta}{2}(y - l(t,y))^2 + \frac{\gamma_I}{2} l(t,y)^2 \right) \nu(\mathrm{d}y) \right\} = 0, \tag{A11}$$

and it is minimized by

$$l^*(y,\eta) = \frac{\eta y}{\eta + \gamma_I}. \tag{A12}$$

Meanwhile, Equation (A7) reduces to

$$B_R'(t) + \alpha(c - \lambda a_Y) + \sup_{\eta \ge 0} \left\{ \int_0^\infty \left((1-\alpha)\frac{\eta}{2} - \frac{\gamma_R}{2} \right)(y - l^*(y,\eta))^2 - \alpha \frac{\gamma_I}{2} l^*(y,\eta)^2 \right\} \nu(\mathrm{d}y) = 0. \tag{A13}$$

By plugging (A12) into (A13), we need to solve the following optimization problem

$$\sup_{\eta \ge 0} \Pi_1(\eta) = \sup_{\eta \ge 0} \left\{ \frac{1}{2} \frac{\gamma_I}{(\eta + \gamma_I)^2} \left((1-\alpha)\eta\gamma_I - \alpha\eta^2 - \gamma_R\gamma_I \right) \right\}.$$

The first order of $\Pi_1(\eta)$ is given by

$$\Pi_1'(\eta) = \frac{\gamma_I^2(1+\alpha)}{2(\eta + \gamma_I)^3} (\eta_0 - \eta)$$

with

$$\eta_0 = \frac{2\gamma_R + (1-\alpha)\gamma_I}{1+\alpha}.$$

It is not difficult to see that $\Pi_1'(\eta) > 0$ if $0 < \eta < \eta_0$; $\Pi_1'(\eta) < 0$ if $\eta > \eta_0$; $\Pi_1'(\eta_0) = 0$. So, the solution to the optimization problem (A13) is given by

$$\eta^* = \frac{2\gamma_R + (1-\alpha)\gamma_I}{1+\alpha}. \tag{A14}$$

Bringing (A14) back into (A12) yields (17).

Substituting (A12) and (A14) back into (A11) and (A13) and using the terminal conditions $B_I(T) = B_R(T) = 0$, we can easily obtain the explicit expressions of $B_I(t)$ and $B_R(t)$ given as

$$B_I(t) = (c - \lambda a_Y)(T-t) - \frac{2\gamma_I\gamma_R + (1-\alpha)\gamma_I^2}{4(\gamma_R + \gamma_I)} \sigma_Y^2 \lambda (T-t), \tag{A15}$$

and

$$B_R(t) = \alpha(c - \lambda a_Y)(T-t) + \frac{(1-\alpha)^2\gamma_I^2 - 4\alpha\gamma_R\gamma_I}{8(\gamma_R + \gamma_I)} \sigma_Y^2 \lambda (T-t). \tag{A16}$$

Similarly, the terminal conditions $b_I(T) = \bar{b}_I(T) = b_R(T)$ lead to

$$b_I(t) = \bar{b}_I(t) = (c - \lambda a_Y)(T-t) - \frac{\gamma_I^2(2\gamma_R + (1-\alpha)\gamma_I)(1+\alpha)}{8(\gamma_R + \gamma_I)^2} \sigma_Y^2 \lambda (T-t), \tag{A17}$$

and

$$b_R(t) = \frac{\gamma_I^2(2\gamma_R + (1-\alpha)\gamma_I)(1+\alpha)}{8(\gamma_R + \gamma_I)^2} \lambda \sigma_Y^2 (T-t). \tag{A18}$$

■

A.6 The Proof of Proposition 4.1

Proof: To derive the Pareto-optimal strategy for problem (21), we need to solve the following optimization problem

$$\sup_{(l,\eta) \in [0,y] \times [\eta_m, \eta_M]} \bar{\Pi}_1(l,\eta), \tag{A19}$$

where

$$\bar{\Pi}_1(l,\eta) = \int_0^\infty \left(\left(\frac{\eta}{2}(1-\alpha) - \frac{\gamma_R}{2} \right)(y - l(y))^2 - \frac{\gamma_I}{2}\alpha l(y)^2 \right) \nu(\mathrm{d}y)$$

$$= \frac{1}{2}\int_0^\infty \left(((1-\alpha)\eta - \alpha\gamma_I - \gamma_R)\, l(y)^2 - 2\,((1-\alpha)\eta - \gamma_R)\, l(y)\, y + ((1-\alpha)\eta - \gamma_R)\, y^2\right) \nu(\mathrm{d}y).$$

We discuss the optimal strategy for (A19) in two cases:

(i) If $\alpha \geq 1$, $\bar{\Pi}_1$ decreases with η, and thus $\eta^* = \eta_m$. In particular, if $\alpha = 1$, the expression above is independent of η. Moreover, it is clear that $(1-\alpha)\eta_m - \alpha\gamma_I - \gamma_R < 0$ and $(1-\alpha)\eta_m - \gamma_R < 0$ for $\alpha \geq 1$, and thus the maximizer of $\bar{\Pi}_1(l, \eta^*)$ is obtained at

$$l^*(y) = \frac{(\alpha-1)\eta_m + \gamma_R}{(\alpha-1)\eta_m + \gamma_R + \alpha\gamma_I} y.$$

(ii) If $0 \leq \alpha < 1$, $\bar{\Pi}_1$ increases with η, and thus $\eta^* = \eta_M$. For $0 \leq \alpha < 1$, if $(1-\alpha)\eta_M - \alpha\gamma_I - \gamma_R \geq 0$, it follows that $(1-\alpha)\eta_M - \gamma_R > 0$, and we can verify that $l^*(y) = 0$; if $(1-\alpha)\eta_M - \alpha\gamma_I - \gamma_R < 0$ and $(1-\alpha)\eta_M - \gamma_R \geq 0$, then $l^*(y) = 0$; and if $(1-\alpha)\eta_M - \alpha\gamma_I - \gamma_R < 0$ and $(1-\alpha)\eta_M - \gamma_R < 0$, we have

$$l^*(y) = \frac{(\alpha-1)\eta_M + \gamma_R}{(\alpha-1)\eta_M + \gamma_R + \alpha\gamma_I} y.$$

We complete the proof. ■

A.7 The Proof of Proposition 4.2

Proof: Bringing (18) back into (15) gives

$$R(\alpha) = \frac{1}{T-t}\mathbb{E}_{t,x}[X_R^*(T) + \alpha X_I^*(T) - X_R^*(t) - \alpha X_I^*(t)]$$
$$= \frac{\gamma_I^2(1-\alpha^2)(2\gamma_R + (1-\alpha)\gamma_I)}{8(\gamma_R+\gamma_I)^2}\lambda\sigma_Y^2 + \alpha(c - \lambda a_Y). \tag{A20}$$

It is not difficult to verify that R decreases with α, and

$$R(1) = c - \lambda\mathbb{E}Y, \quad R(0) = \frac{\gamma_I^2(2\gamma_R+\gamma_I)}{8(\gamma_R+\gamma_I)^2}.$$

Therefore, under the variance premium principle, we can solve Equation (A20) to estimate the weight parameter α given a constant target annual R when considering the mean-variance criterion (4). We complete the proof. ■

A.8 The Proof of Theorem 4.2

Proof: When $\eta \equiv 0$, Equation (A4) reduces to

$$B_I'(t) + (c - \lambda a_Y) - \inf_{l\in\mathcal{A}_I}\left\{\int_0^\infty \left(\theta(y - l(t,y)) + \frac{\gamma_I}{2} l^2(t,y)\right)\nu(\mathrm{d}y)\right\} = 0, \tag{A21}$$

and it is minimized by

$$l^*(y,\theta) = \frac{\theta}{\gamma_I} \wedge y. \tag{A22}$$

Also, Equation (A7) reduces to

$$B_R'(t) + \alpha(c - \lambda a_Y) + \sup_{\theta\geq 0}\left\{\int_0^\infty \left((1-\alpha)\theta(y - l^*(y,\theta)) - \alpha\frac{\gamma_I}{2}(l^*(y,\theta))^2 - \frac{\gamma_R}{2}(y - l^*(y,\theta))^2\right)\nu(\mathrm{d}y)\right\} = 0. \tag{A23}$$

Substituting (A22) back into (A23), then we need to solve the following optimization problem

$$\sup_{\theta\geq 0}\Pi_2(\theta)=\sup_{\theta\geq 0}\left\{\int_0^\infty\left((1-\alpha)\theta\left(y-\frac{\theta}{\gamma_I}\right)_+-\alpha\frac{\gamma_I}{2}\left(\frac{\theta}{\gamma_I}\wedge y\right)^2-\frac{\gamma_R}{2}\left(y-\frac{\theta}{\gamma_I}\right)^2\right)\mathrm{d}F(y)\right\}.$$

Taking the first order of $\Pi_2(\theta)$ yields

$$\Pi_2'(\theta)=\left(1-\alpha+\frac{\gamma_R}{\gamma_I}\right)\int_{\frac{\theta}{\gamma_I}}^\infty y\mathrm{d}F(y)-\left(2-\alpha+\frac{\gamma_R}{\gamma_I}\right)\frac{\theta}{\gamma_I}\left(1-F\left(\frac{\theta}{\gamma_I}\right)\right).$$

Let $z=\theta/\gamma_I$, it follows that

$$\begin{aligned}\Pi'(z\gamma_I)&=\left(1-\alpha+\frac{\gamma_R}{\gamma_I}\right)\int_z^\infty y\,\mathrm{d}F(y)-\left(2-\alpha+\frac{\gamma_R}{\gamma_I}\right)z\,(1-F(z))\\&=\left(1-\alpha+\frac{\gamma_R}{\gamma_I}\right)(1-F(z))\left(\frac{\int_z^\infty y-z\mathrm{d}F(y)}{1-F(z)}-\frac{z}{1-\alpha+\frac{\gamma_R}{\gamma_I}}\right)\\&=\left(1-\alpha+\frac{\gamma_R}{\gamma_I}\right)(1-F(z))\left(\mathbb{E}[(Y-z)_+|Y>z]-\frac{z}{1-\alpha+\frac{\gamma_R}{\gamma_I}}\right).\end{aligned}$$

On one hand, if $\Phi(z)$ in (22) has a unique root over $(0,\infty)$ denoted by z_0, then $\theta_0=\gamma_I z_0$ must be the unique finite root to $\Pi_2'(\theta)=0$, and thus we have $\theta^*=\theta_0$. One the other hand, if $\Phi(z)$ has no root over $(0,\infty)$, because of $\Pi'(0)>0$, we have $z_0\to\infty$, and thus we derive $(l^*,\theta^*)=(y,\infty)$.

Substituting (A22) and $\theta^*=\gamma_I z_0$ back into (A21) and (A23) and using the terminal conditions $B_I(T)=B_R(T)=0$, we have

$$\begin{aligned}B_I(t)=(c-\lambda a_Y)(T-t)-\Bigg(&z_0\gamma_I\int_{z_0}^\infty y\,\mathrm{d}F(y)+\frac{\gamma_I}{2}\int_0^{z_0}y^2\mathrm{d}F(y)\\&-\frac{\gamma_I}{2}z_0^2\,(1-F(z_0))\Bigg)\lambda(T-t),\end{aligned}$$

and

$$\begin{aligned}B_R(t)=\alpha(c-\lambda a_Y)(T-t)+\Bigg(&(\gamma_I(1-\alpha)+\gamma_R)z_0\int_{z_0}^\infty y\,\mathrm{d}F(y)-\frac{\gamma_R}{2}\int_{z_0}^\infty y^2\,\mathrm{d}F(y)\\&-\alpha\frac{\gamma_I}{2}\int_0^{z_0}y^2\mathrm{d}F(y)+\left(\frac{\gamma_I}{2}\alpha-\frac{\gamma_R}{2}-\gamma_I\right)z_0^2(1-F(z_0))\Bigg)\lambda(T-t).\end{aligned}$$

Similarly, the terminal conditions $b_I(T)=\bar{b}_I(T)=b_R(T)$ lead to

$$b_I(t)=\bar{b}_I(t)=(c-\lambda a_Y)(T-t)+\left(z_0^2\gamma_I(1-F(z_0))-z_0\gamma_I\int_{z_0}^\infty y\,\mathrm{d}F(y)\right)\lambda(T-t),$$

and

$$b_R(t)=\left(z_0\gamma_I\int_{z_0}^\infty y\,\mathrm{d}F(y)-z_0^2\gamma_I(1-F(z_0))\right)\lambda(T-t).$$

■

A.9 The Proof of Proposition 4.4

Proof: Bringing $\theta^*=\gamma_I z_0$ back into (15) yields

$$R(\alpha)=\alpha(c-\lambda\mathbb{E}Y)+\gamma_I z_0(1-\alpha)\mathbb{E}[Y-z_0]_+.\tag{A24}$$

It is not difficult to see that R is non-negative and satisfies $R(0)=\gamma_I z_0\mathbb{E}[Y-z_0]_+$ and $R(1)=c-\lambda\mathbb{E}Y$. Therefore, under the expected value principle, we can solve Equation (A24) to estimate the weight parameter α given a constant target annual R when considering the mean-variance criterion (4). ■