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# Minimizing the Probability of Absolute Ruin Under Ambiguity Aversion

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## Abstract

In this paper, we consider an optimal robust reinsurance problem in a diffusion model for an ambiguity-averse insurer, who worries about ambiguity and aims to minimize the robust value involving the probability of absolute ruin and a penalization of model ambiguity. It is assumed that the insurer is allowed to purchase per-claim reinsurance to transfer its risk exposure, and that the reinsurance premium is computed according to the mean-variance premium principle which is a combination of the expected-value and variance premium principles. The optimal reinsurance strategy and the associated value function are derived explicitly by applying stochastic dynamic programming and by solving the corresponding boundary-value problem. We prove that there exists a unique point of inflection which relies on the penalty parameter greatly such that the robust value function is strictly concave up to the unique point of inflection and is strictly convex afterwards. It is also interesting to observe that the expression of the optimal robust reinsurance strategy is independent of the penalty parameter and coincides with the one in the benchmark case without ambiguity. Finally, some numerical examples are presented to illustrate the effect of ambiguity aversion on our optimal results.

**Keywords** Absolute ruin probability · Ambiguity aversion · Mean-variance premium principle · Per-claim reinsurance · Robust optimization

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## 1 Introduction

Risk models taking reinsurance into consideration have received a great deal of attention in the literature because reinsurance is an effective approach for insurers to manage their risk exposures so as to achieve their financial objectives. Subject to reinsurance control with or without investment control, optimization problems under various objective functions have become a popular research topic in the actuarial literature. Schmidli [1], Bai and Guo [2], and Liang and Young [3] considered the optimal reinsurance-investment problems in which an insurance company wishes to minimize the probability of ruin. Under the criterion of maximizing the expected utility of terminal wealth, the optimal reinsurance or investment strategies were studied in Liu and Ma [4], Liang and Bayraktar [5], and Liang and Yuen [6]. It is worth noting that the technique of stochastic control theory and the corresponding Hamilton–Jacobi–Bellman (HJB) equation are widely used to cope with these problems.

Although much research on optimal reinsurance has been carried out in the past decade, many interesting issues remain to be investigated. An important issue is to account for model uncertainty. A notorious fact in the practice of portfolio management is that return levels of risky assets are difficult to estimate with precision. In the context of insurance and reinsurance, similar uncertainty also holds regarding expected surpluses. As a consequence, decision makers need to develop a methodology to deal with this uncertainty. Rather than making ad-hoc judgement on estimation errors of return levels of risky assets and surpluses, the decision makers may instead consider some alternative models that are, in some sense, close to the estimation model. This so-called robust optimization method has been successfully implemented over the last twenty years in quantitative finance, in particular, portfolio selection and asset pricing with model uncertainty or model misspecification. For instance, Maenhout [7] analyzed the optimal intertemporal portfolio problem of an investor who worries about model misspecification and insists on robust decision rules when facing a mean-reverting risk premium; Bayraktar and Zhang [8] derived the optimal robust investment strategy of an individual who targets a given rate of consumption and seeks to minimize the probability of lifetime ruin. Other than financial applications, this approach has been applied to insurance problems in recent years. Among others, Yi et al. [9] considered a robust optimal reinsurance and investment problem under Heston's stochastic volatility model for an ambiguity-averse insurer who aims to maximize the expected utility of terminal wealth within a fixed time horizon; Zeng et al. [10] investigated the equilibrium strategy of an optimal robust reinsurance and investment problem under the mean-variance criterion in a model with jumps; and Luo et al. [11] used the method of exponential transformation to obtain the optimal robust investment-reinsurance strategy and the associated value function under the criterion of maximizing the goal-reaching probability.

Much of the literature considers problems of optimal reinsurance in which insurance companies are restricted to buy either pure quota-share reinsurance [12,13], pure excess-of-loss reinsurance [14,15], or a combination of the two [16,17]. Some other researchers found the optimal reinsurance strategies for various optimization problems without restricting the form of reinsurance [3,18,19]. However, to the best of our knowledge, only Li and Young [20] allowed the insurer to buy per-claim reinsurance and solved an optimal reinsurance problem for an ambiguity-averse insurer who worries about ambiguity in the rate of claim occurrence under the criterion of minimizing the penalized discounted probability of ruin. Therefore, it is worthwhile to investigate other robust optimization problems without restricting the form of the reinsurance.

Absolute ruin probability is an important risk measure and has been considered in some research works. Unlike the study of traditional ruin probability with ruin level zero, absolute ruin probability allows us to investigate the behavior of a company when it is in deficit. In some papers, absolute ruin occurs when the surplus rate level drops below a critical level (the premium received are not sufficient to make the interest payments on the debt); see for example, Dassios and Embrechts [21], Cai [22], Gerber and Yang [23], and Zhou and Cai [24]. However, if there is a diffusion term in the surplus process, typically, when investment in a Black-Scholes risky asset is introduced, or the surplus process itself is modeled or perturbed by a Brownian motion, then the surplus process has a positive probability to bounce back from any negative level. In this spirit, Luo and Taksar [13] defined the absolute ruin as an event that lim inf of the surplus process is negative infinity, and apply the HJB method to obtain the optimal proportional reinsurance and investment strategy that minimizes the probability of absolute ruin. Liang and Long [25] investigated the same optimization problem under the assumption that the insurer's liabilities and capital gains in financial market are negatively correlated. We note here that absolute ruin is called infinite-time ruin in Schmidli [26]; there the corresponding infinite-time ruin probability is obtained explicitly in a diffusion approximation model.

Inspired by the above-mentioned works, we incorporate model uncertainty into an insurer's controlled surplus, and solve a novel optimal robust reinsurance problem under the objective of minimizing the probability of absolute ruin defined in Luo and Taksa [13]. In our model set-up, the insurer can purchase per-loss reinsurance of which the premium is computed according to the mean-variance premium principle. To make the optimization problem tractable, we use a diffusion approximation of the classical Cramér–Lundberg model to describe the insurer's surplus. Here, we assume that the insurer is ambiguity-averse and thus wants to guard herself against worst-case scenarios. We use the idea of model ambiguity to describe this phenomenon. We would like to point out that the model ambiguity under consideration is different from the one in Li and Young [20]. Similar to the robust value approach of Bayraktar and Zhang [8] and Luo et al. [11], we consider a reference probability measure (benchmark) and a set of equivalent measures which have absolute continuous Radon–Nikodym derivatives with respect to the benchmark. To capture the insurer's ambiguity aversion, we penalize the absolute ruin probability with a term based on relative entropy, and thus a new robust value is defined by employing the criterion of minimizing the penalized probability of absolute ruin.

In this paper, we use the technique of stochastic dynamic programming to derive the corresponding HJB equation. A verification theorem is then proved to show that a decreasing $\mathcal{C}^2$ solution to the HJB equation coincides with the robust value function. The corresponding optimal reinsurance strategy and the optimal drift distortion of the equivalent measure are also obtained in explicit feedback forms. It is natural to expect that the model ambiguity affects the value function and the selection of the equivalent probability measure. However, we see a rather surprising result that the optimal robust reinsurance strategy with ambiguity penalization is independent of the penalty parameter and coincides with the one without model ambiguity. Besides, we prove that there exists a unique point of inflection such that the robust value function is strictly concave up to the unique point of inflection, and is strictly convex afterwards. This result is similar to the one in the non-robust case. Due to the fact that the robust value function depends on the penalty parameter greatly, the resulting position of the inflection point is different from the one without model ambiguity.

The rest of the paper is organized as follows. In Sect. 2, we formulate our model and define the robust optimization problem faced by the ambiguity-averse insurer. In Sect. 3, the optimal robust reinsurance strategy and the associated value function are derived explicitly by solving the corresponding boundary-value problem. Numerical examples and economic analysis are given in Sect. 4. Finally, we conclude the paper in Sect. 5.

## 2 Model and Problem Formulation

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with filtration $\{\mathcal{F}_t\}_{t\geq 0}$ containing all the objects defined in the following. Here, we do not complete the filtrations because later on, we would like to include measures that are only locally equivalent to $\mathbb{P}$ as part of our consideration.[1]

According to the classical Cramér–Lundberg model, the surplus process of the insurer $X = \{X_t\}_{t\geq 0}$ can be described by the following dynamics

$$dX_t = c dt - Y_{N_t} dN_t, \tag{1}$$

in which $c > 0$ is the premium rate, $N = \{N_t\}_{t\geq 0}$ is a homogeneous Poisson process with intensity $\lambda > 0$, and the claim sizes $Y_1, Y_2, \ldots$ are independent and identically distributed positive random variables with common cumulative distribution function $F_Y$, finite expectation, and finite second moment.

Suppose that the insurer can reinsure its claims by per-loss reinsurance with premium computed by means of the mean-variance premium principle, that is, a combination of the expected-value and variance premium principles. Let $R_t(y)$ denote the retained claim at time $t$ as a function of the (possible) claim $Y = y$ at that time. Thus, $y - R_t(y)$ is the amount of each claim transferred to the reinsurer. Then the mean-variance premium rate at time $t$ associated with $R_t$ is given by

$$(1+\theta)\lambda\mathbb{E}\big(Y - R_t\big) + \frac{\eta}{2}\lambda\mathbb{E}\big((Y - R_t)^2\big), \tag{2}$$

[1] By locally equivalent, we mean the measures are equivalent on $\mathcal{F}_t$ for all $t \geq 0$. Note that the stochastic integral can still be defined and has all the usual properties even though the filtrations in our setup are not complete. See, for example, Chapter 1 in Jacod and Shiryaev [27] and Sect. 2 in Bayraktar and Zhang [8].

in which $\theta$ and $\eta$ are the non-negative risk loading parameters. If $\theta = 0$, then (2) reduces to the variance premium principle; similarly, if $\eta = 0$, then (2) reduces to the expected-value premium principle. A retention strategy $R = \{R_t\}_{t\geq 0}$ is said to be *admissible* if it (i) is adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$, (ii) is a non-decreasing function with respect to $y$, and (iii) satisfies $0 \leq R_t \leq Y$ for all $t \geq 0$. The set of all admissible strategies $R$ is denoted by $\mathcal{D}$.

Let $U^R = \{U_t^R\}_{t\geq 0}$ denote the surplus process associated with a reinsurance strategy $R = \{R_t\}_{t\geq 0}$, that is, $U_t^R$ is the surplus of the insurer at time $t$ under the strategy $R$. Furthermore, we assume that the insurer can invest its surplus in a risk-free asset (bond or bank account) with rate $r > 0$. Then the controlled surplus process has the following dynamics

$$dU_t^R = \big(rU_t^R + c - (1+\theta)\lambda\mathbb{E}\big(Y - R_t\big) - \frac{\eta}{2}\lambda\mathbb{E}\big((Y - R_t)^2\big)\big)dt - R_t dN_t. \quad (3)$$

Throughout the paper, we assume that

$$\lambda\mathbb{E}Y < c < (1+\theta)\lambda\mathbb{E}Y + \frac{\eta}{2}\lambda\mathbb{E}\big(Y^2\big),$$

that is, the insurer's premium income is greater than the expected value of the claims but less than the premium for full reinsurance. Let $\kappa$ denote the positive difference

$$\kappa = (1+\theta)\lambda\mathbb{E}Y + \frac{\eta}{2}\lambda\mathbb{E}\big(Y^2\big) - c. \quad (4)$$

To get explicit solutions, similar to Bai et al. [14], Grandell [28], and Liang and Yuen [6], we deal with the optimization problem of study by using the diffusion approximation of the jump process in (3). That is,

$$R_t\, dN_t \approx \lambda\mathbb{E}\big(R_t\big)\, dt - \sqrt{\lambda\mathbb{E}\big(R_t^2\big)}\, dB_t, \quad (5)$$

where $B = \{B_t\}_{t\geq 0}$ is a standard Brownian motion. The resulting process $\hat{U} = \{\hat{U}_t\}_{t\geq 0}$ then evolves according to the dynamics

$$d\hat{U}_t^R = \big(r\hat{U}_t^R - \kappa + \theta\lambda\mathbb{E}\big(R_t\big) + \eta\lambda\mathbb{E}\big(YR_t\big) - \frac{\eta}{2}\lambda\mathbb{E}\big(R_t^2\big)\big)dt + \sqrt{\lambda\mathbb{E}\big(R_t^2\big)}\, dB_t, \quad (6)$$

with initial surplus $\hat{U}_0 = u$. Note that even though we are using the diffusion approximation of the compound Poisson model, one can consider the random claim severity $Y$ as existing in reality. In other words, we use the diffusion approximation to obtain the optimal retention strategy for that model, but an insurer could apply the obtained per-loss reinsurance strategy to claims within the original compound Poisson model. See Liang and Young [3] for numerical comparisons of the optimal retention strategies under the diffusion and compound Poisson models when reinsurance is priced according to the expected-value premium principle with the optimal strategy equal to excess-of-loss reinsurance. They showed that the optimal deductibles are close under the two models when the surplus is not too small, even when $\lambda$ is small. Besides, Liang et al. [29] also investigated the connection between the results of compound Poisson model and its diffusion model. They derived that, under an appropriate scaling of the classical risk process, the minimum probability of ruin converges to the minimum probability of ruin under the diffusion approximation.

The insurer seeks to minimize the probability of absolute ruin, namely, the probability that the event where the lim inf of the surplus is equal to negative infinity; for example see Luo and Taskar [13] and Liang and Long [25]. The event of absolute ruin under policy $R$ is defined as

$$\Theta^R = \{\omega \in \Omega : \liminf_{t\to\infty} \hat{U}_t^R = -\infty\}. \tag{7}$$

Then the corresponding minimum probability of absolute ruin $V_0$ is given as

$$V_0(u) = \inf_{R\in\mathcal{D}} \mathbb{P}_u(\Theta^R), \tag{8}$$

where $\mathbb{P}_u$ denotes the probability conditional on $\hat{U}_0^R = u$. Therefore, the optimization problem is an infinite-time horizon stochastic problem. Note that if the value of the surplus is greater than or equal to

$$u_s = \frac{\kappa}{r}, \tag{9}$$

then the insurer can buy full reinsurance via income from the riskless asset, and therefore the surplus will never drop below its current value. For this reason, we call $u_s$ the *safe level*.

***Remark 1*** We point out that in some papers including Dassios and Embrechts [21] and Cai [22], absolute ruin is defined as the first time when the drift coefficient in (6) turns negative, or in other words, the premiums received by the insurers are not sufficient to cover its debt. Then the value function associated with this control problem is denoted by

$$\widetilde{V}_0(u) = \inf_{\mathbb{R}\in\mathcal{D}} \mathbb{P}_u\{\tau_0^R < \infty\} \tag{10}$$

with

$$\tau_0^R = \inf\left\{t \geq 0 : r\hat{U}_t^R - \kappa + \theta\lambda\mathbb{E}\big(R_t\big) + \eta\lambda\mathbb{E}\big(YR_t\big) - \frac{\eta}{2}\lambda\mathbb{E}\big(R_t^2\big) < 0\right\}.$$

Note that if the insurer does not take any reinsurance, it turns out that the drift coefficient of the controlled process equals $r\hat{U}_t^R + c - \lambda\mathbb{E}Y$, which is the maximum value of the drift coefficient for all admissible reinsurance policies. Define a critical level $u_1$ by

$$u_1 = \frac{\lambda\mathbb{E}Y - c}{r}. \tag{11}$$

It is not difficult to see that once the current surplus drops below $u_1$, we cannot find an admissible strategy to keep the drift coefficient positive, and hence absolute ruin occurs. Thus, the ultimate time of absolute ruin under the policy $R$ can be written as

$$\tau_0^R = \inf\left\{t \geq 0 : \hat{U}_t^R < u_1\right\}.$$

It is interesting to find that $u_1$ defined in (11) is exactly the inflection point of the value function without model ambiguity given in Proposition 1. Furthermore, the optimal results for this version of absolute ruin defined in (10) can be derived by using the analysis similar to that in Sect. 3.2, also see Remark 13 for details.

We now introduce the ambiguity that appears in the stochastic process. In the traditional reinsurance model, the insurer is assumed to be ambiguity-neutral with objective function. However, it is reasonable to assume that the insurer is ambiguity-averse and thus wants to guard herself against worst-case scenarios. We assume that the knowledge about ambiguity for the ambiguity-averse insurer (AAI) is described by probability $\mathbb{P}$, namely, the reference probability (or model). However, she is sceptical about this reference model, and hopes to consider alternative models. So, instead of optimizing under the reference measure $\mathbb{P}$, she considers a set $\mathcal{Q}$ of candidate measures that are locally equivalent to $\mathbb{P}$, and penalizes their deviation from $\mathbb{P}$. Let $h(\mathbb{Q}|\mathbb{P}) = \mathbb{E}^{\mathbb{Q}}\left(\ln \frac{d\mathbb{Q}}{d\mathbb{P}}\right)$ be the relative entropic function. We penalize the deviation from $\mathbb{P}$ using a variant of $h$:

$$h(\mathbb{Q}_t|\mathbb{P}_t) = \mathbb{E}^{\mathbb{Q}}\left(\ln \frac{d\mathbb{Q}_t}{d\mathbb{P}_t}\right),$$

where $\mathbb{Q}_t$ is the probability measure $\mathbb{Q}$ restricted to $\mathcal{F}_t$ and $\mathbb{P}_t$ is the probability measure $\mathbb{P}$ restricted to $\mathcal{F}_t$.

We now give the precise definition of the set $\mathcal{Q}$ of candidate measures. A probability measure $\mathbb{Q} \in \mathcal{Q}$ if

$$\frac{d\mathbb{Q}_t}{d\mathbb{P}_t} = \exp\left\{-\frac{1}{2}\int_0^t \phi_s^2 ds + \int_0^t \phi_s dB_s\right\}, \quad t \geq 0, \tag{12}$$

for some $\mathbb{F}$-progressively measurable process $\phi$ satisfying $\mathbb{E}[e^{\frac{1}{2}\int_0^t \phi_s^2 ds}] < \infty$ and $\mathbb{E}^{\mathbb{Q}} \int_0^t \phi_s^2 ds < \infty$ for all $t \geq 0$. Conversely, given any $\mathcal{F}$-progressively measurable process $\phi$ satisfying $\mathbb{E}[e^{\frac{1}{2}\int_0^t \phi_s^2 ds}] < \infty$ for all $t \geq 0$, we can define a consistent family of measures $\mathbb{Q}_t \sim \mathbb{P}_t$ on $(\Omega, \mathcal{F}_t)$ by (12). According to Lemma 4.2 in Stroock [30] (also see Proposition 1 in Huang and Pages [31]), there exists a probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ such that $\mathbb{Q}|_{\mathcal{F}_t} = \mathbb{Q}_t$ for all $t \geq 0$. [2]

Note that $\mathbb{P} \in \mathcal{Q}$ with $\phi_t \equiv 0$. We define $\{B_t^{\mathbb{Q}}\}_{t\geq 0}$ as

$$dB_t^{\mathbb{Q}} = dB_t - \phi_t dt.$$

By Girsanov's theorem, $\{B_t^{\mathbb{Q}}\}_{t\geq 0}$ is a standard Brownian motion adapted to information filtration $\{\mathcal{F}\}_{t\geq 0}$ in a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$. Therefore, under measure $\mathbb{Q}$, the surplus process (6) follows the dynamics

[2] The existence of such a measure is not guaranteed if the filtration has been completed with respect to $\mathbb{P}$.

$$d\hat{U}_t^R = \Big(r\hat{U}_t^R - \kappa + \theta\lambda\mathbb{E}\big(R_t\big) + \eta\lambda\mathbb{E}\big(YR_t\big) - \frac{\eta}{2}\lambda\mathbb{E}\big(R_t^2\big) + \phi_t\sqrt{\lambda\mathbb{E}\big(R_t^2\big)}\Big)dt + \sqrt{\lambda\mathbb{E}\big(R_t^2\big)}\,dB_t^{\mathbb{Q}}. \tag{13}$$

Let $\mathbb{Q} \in \mathcal{Q}$. We have

$$\begin{aligned} h(\mathbb{Q}_t|\mathbb{P}_t) &= \mathbb{E}^{\mathbb{Q}}\left[-\frac{1}{2}\int_0^t \phi_s^2 ds + \int_0^t \phi_s dB_s\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{2}\int_0^t \phi_s^2 ds + \int_0^t \phi_s dB_s^{\mathbb{Q}}\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{2}\int_0^t \phi_s^2 ds\right] < \infty. \end{aligned} \tag{14}$$

***Remark 2*** Note that ambiguity aversion refers to distrust in reference measure $\mathbb{P}$. The insurer is concerned about the estimation of the misspecification errors in the reference model, and considers a set $\mathcal{Q}$ of candidate measures that are locally equivalent to $\mathbb{P}$, analogous to Maenhout [7] and Bayraktar and Zhang [8]. Hence, choosing an alternative $\mathbb{Q}$ is equivalent to determining a stochastic process $\phi_t$. On the other hand, we can rewrite (5) as

$$R_t\,dN_t \approx \sqrt{\lambda\mathbb{E}(R_t^2)}\left[\left(\frac{\lambda\mathbb{E}\big(R_t\big)}{\sqrt{\lambda\mathbb{E}(R_t^2)}} + \phi_t\right)dt - dB_t^{\mathbb{Q}}\right]$$

under measure $\mathbb{Q}$. Thus, in this sense, $\phi$ seems to be distorting the ratio $\frac{\lambda\mathbb{E}(R_t)}{\sqrt{\lambda\mathbb{E}(R_t^2)}}$ and the ambiguity aversion can also be illustrated as the insurer being uncertain about the impact of the retention strategy on the surplus process.

For the initial surplus $u \in (-\infty, u_s]$, define hitting times

$$\tau_\infty^R = \lim_{M\to-\infty} \inf\left\{t \geq 0 : \hat{U}_t^R = M\right\}, \qquad \tau_{u_s}^R = \inf\left\{t \geq 0 : \hat{U}_t^R = u_s\right\}. \tag{15}$$

We denote a performance function for any $\mathbb{Q} \in \mathcal{Q}$ as

$$V^{R,\mathbb{Q}}(u) = \mathbb{Q}_u(\Theta^R) - \frac{1}{\epsilon}h(\mathbb{Q}_{\tau^R}|\mathbb{P}_{\tau^R}), \tag{16}$$

with $\tau^R = \tau_\infty^R \wedge \tau_{u_s}^R$ and $\Theta^R$ defined in (7). The first term is the probability of absolute ruin under the probability measure $\mathbb{Q}$ given the initial surplus level $u$, and the second term with penalty parameter $\epsilon \geq 0$ is the rescaled entropy representing a penalization of ambiguity. Then the insurer faces the following robust optimization problem

$$V(u) = \inf_{R\in\mathcal{D}} \sup_{\mathbb{Q}\in\mathcal{Q}} V^{R,\mathbb{Q}}(u). \tag{17}$$

***Remark 3*** The non-negative penalization parameter $\epsilon$ reflects the insurer's level of ambiguity aversion. The case with $\epsilon = 0$ corresponds to the classical non-robust model, also known as the *reference model*. Larger values of $\epsilon$ means that the insurer is more ambiguity averse and believes less in the reference model. As $\epsilon \to \infty$, the optimization corresponds to the worst-case approach (causing the second term in (16) to vanish), that is, the insurer has equal belief in all candidate measures.

***Remark 4*** Note that the drift uncertainty is irrelevant to the safe level since the insurer can always be safe by buying full reinsurance. Specifically, if $u \geq u_s$, we have

$$V(u) = \inf_{R\in\mathcal{D}} \sup_{\mathbb{Q}\in\mathcal{Q}} V^{R,\mathbb{Q}}(u) \leq \sup_{\mathbb{Q}\in\mathcal{Q}} V^{0,\mathbb{Q}}(u) = \sup_{\mathbb{Q}\in\mathcal{Q}} \left\{0 - \frac{1}{\epsilon} h(\mathbb{Q}_{\tau^R} | \mathbb{P}_{\tau^R})\right\} \leq 0$$

on the one hand; and

$$V(u) = \inf_{R\in\mathcal{D}} \sup_{\mathbb{Q}\in\mathcal{Q}} V^{R,\mathbb{Q}}(u) \geq \inf_{R\in\mathcal{D}} V^{R,\mathbb{P}}(u) = \inf_{R\in\mathcal{D}} \mathbb{P}_u(\Theta^R) \geq 0$$

on the other hand. Thus, we have $V(u) = 0$ for any $u \geq u_s$. Therefore, we only consider the values of $V$ on $(-\infty, u_s)$ in following context.

## 3 Minimizing the Probability of Absolute Ruin

In this section, for the risk model (13), we aim at finding the optimal reinsurance strategy to minimize the penalized probability of absolute ruin. In Sect. 3.1, we prove a verification theorem, which is used to find $V$ on $(-\infty, u_s)$. Then, in Sect. 3.2, the optimal robust reinsurance strategy and the associated value function are derived explicitly by solving the corresponding boundary-value problem.

### 3.1 Verification Theorem

To prove the verification theorem, we need some auxiliary lemmas but omit the proofs because they are similar to Lemmas 2.3-2.4 in Luo and Taksar [13] and Lemmas 3.1-3.2 in Han et al. [32].

**Lemma 1** *Let $\hat{U}_t^R$ be given in (13). Let $-\infty < M < N < u_s$. Define $\tau_N^R = \inf\{t > 0 : \hat{U}_t^R = N\}$, $\tau_M^R = \inf\{t > 0 : \hat{U}_t^R = M\}$, and $\tau_{M,N}^R = \tau_M^R \wedge \tau_N^R$. Then, for any admissible strategy $R$ and any initial surplus $u \in (M, N)$, it holds that* $\mathbb{P}_u(\tau_{M,N}^R < \infty) = 1$.

**Lemma 2** *Let $\hat{U}_t^R$ be given in (13). For any admissible strategy $R$, if* $\liminf_{t\to\infty} \hat{U}_t^R = -\infty$, *then* $\lim_{t\to\infty} \hat{U}_t^R = -\infty$; *and if* $\limsup_{t\to\infty} \hat{U}_t^R = u_s$, *then* $\lim_{t\to\infty} \hat{U}_t^R = u_s$.

***Remark 5*** Lemma 2 shows that absolute ruin can be redefined as the event where the controlled process tends to negative infinity. As $M \to -\infty$ and $N \to u_s$, we see in

Lemma 1 that the surplus process diverges to either the safe level or negative infinity with probability 1. This shows the ergodicity of the controlled process.

For fixed $R \in \mathcal{D}$, $\phi \in \mathcal{R} := (-\infty, \infty)$ and any $\mathcal{C}^2$ function on $(-\infty, u_s]$, define the differential operator $\mathcal{A}^{R,\phi}$ on appropriately differentiable functions by

$$\mathcal{A}^{R,\phi} h(u) = \Big(ru - \kappa + \theta\lambda\mathbb{E}(R) + \eta\lambda\mathbb{E}(YR) - \frac{\eta}{2}\lambda\mathbb{E}(R^2) + \phi_t\sqrt{\lambda\mathbb{E}(R^2)}\Big)h_u + \frac{1}{2}\lambda\mathbb{E}(R^2)h_{uu} - \frac{1}{2\epsilon}\phi^2. \tag{18}$$

**Theorem 1** *Suppose $W : (-\infty, u_s] \to [0, 1]$ is a bounded, continuous function, which satisfies the following conditions:*

*(i) $W \in \mathcal{C}^2(-\infty, u_s)$ is a non-increasing function with bounded first derivative;*
*(ii) $W$ is a solution of*

$$\inf_{R\in\mathcal{D}} \sup_{\phi\in\mathcal{R}} \Big\{\mathcal{A}^{R,\phi} W(u)\Big\} = 0, \quad u \in (-\infty, u_s]; \tag{19}$$

*(iii) $W(-\infty) = 1$ and $W(u_s) = 0$;*
*(iv) $R^*(u, y)$ attains the infimum in (ii) for each $-\infty < u \le u_s$ and $y \ge 0$; and $\phi^*(R, u)$ is bounded and attains the supremum in (ii) for each $0 \le R \le Y$ and $-\infty < u \le u_s$.*

*Then the value function $V$ defined in (17) coincides with $W$. The policies $R^*$ and $\phi^*$ defined in feedback forms via $R^*_t = R^*_t(\hat{U}^*_t, Y)$ and $\phi^*_t = \phi^*_t(R^*_t(\hat{U}^*_t, Y), \hat{U}^*_t)$ are the optimal reinsurance strategy and optimal probability distortion, respectively. Here, $U^*_t$ denotes the controlled surplus process (13) under the policies $R^*$ and $\phi^*$. Moreover, we have*

$$V(u) = V^{R^*,\mathbb{Q}^*}(u) = \mathbb{Q}^*_u(\Theta^{R^*}) - \frac{1}{\epsilon}h(\mathbb{Q}^*_{\tau^{R^*}}|\mathbb{P}_{\tau^{R^*}}),$$

*where $\mathbb{Q}^*$ is defined by (12) using process $\phi^*$.*

***Proof*** See Appendix A. □

### 3.2 Probability of Absolute Ruin

In this subsection, we begin by computing $V$ in two extreme cases, that is, $\epsilon = 0$ and $\epsilon = \infty$. In the former case, the optimization problem reduces to the classical non-robust case, and the value function and the associated optimal retention strategy can be derived using the results of Han et al. [32] (they considered the optimal reinsurance-investment problem for an insurer who wishes to minimize the probability of absolute ruin without model ambiguity). We give the corresponding results in Proposition 1.

For notational convenience, we denote

$$\beta_2^*(u) = \frac{2\big(ru + c - \lambda\mathbb{E}Y\big)}{\lambda\mathbb{E}\big(Y^2\big)}. \tag{20}$$

**Proposition 1** *When $\epsilon = 0$, the optimal drift distortion $\phi^* \equiv 0$, so the non-robust value function $V_0$ equals $\inf_{R\in\mathcal{D}} \mathbb{P}_u(\Theta^R)$. Specifically,*

$$V_0(u) = \begin{cases} 1 - \dfrac{F_1(u)}{F_1(u_1) + F_2(u_s)}, & -\infty < u \le u_1, \\ 1 - \dfrac{F_1(u_1) + F_2(u)}{F_1(u_1) + F_2(u_s)}, & u_1 < u \le u_s, \end{cases} \tag{21}$$

*where*

$$F_1(u) = \int_{-\infty}^{u} \exp\left\{\int_{-\infty}^{y} -\beta_2^*(w)dw\right\} dy,$$

*and*

$$F_2(u) = \int_{u_1}^{u} \exp\left\{\int_{-\infty}^{u_1} -\beta_2^*(w)\,dw + \int_{u_1}^{y} \big(\eta - \bar{\beta}_1^*(w)\big)\,dw\right\} dy.$$

*In (21), $u_1$ and $\beta_2^*$ are given by (11) and (20), respectively; $\bar{\beta}_1^*(u) > \eta$ uniquely solves*

$$\theta\mathbb{E}\big(R_0^*\big) + \eta\mathbb{E}\big(YR_0^*\big) - \frac{\bar{\beta}_1^*}{2}\mathbb{E}\big((R_0^*)^2\big) = \frac{\kappa - ru}{\lambda} \tag{22}$$

*for $u \in (u_1, u_s]$ with the optimal retention function $R_0^*$ given by*

$$R_0^*(u, y) = \begin{cases} y, & -\infty < u \le u_1, \\ \dfrac{\theta + \eta y}{\bar{\beta}_1^*(u)} \wedge y, & u_1 < u \le u_s. \end{cases} \tag{23}$$

***Remark 6*** As was mentioned in Han et al. [32], since $\beta_2^*(u) \le 0$ for any $u \le u_1$, it is not difficult to verify that the second derivative of $V_0$ is strictly negative when $u \in (-\infty, u_1)$ but strictly positive when $u \in (u_1, u_s]$. Thus, unlike the case of traditional ruin probability with the ruin level zero, the value function $V_0$ is no longer convex but is S-shaped with the unique point of inflection $u_1$.

The other extreme case $\epsilon = \infty$ gives us the worst-case value function $V_\infty$. For the worst-case problem, the optimal retention strategy is to reinsure all its losses since the insurer believes equally in all candidate measures. In this case, the insurer's surplus solves the deterministic differential equation $d\hat{U}_t = (r\hat{U} - \kappa)dt$. Since $ru - \kappa < 0$ for any $u \in (-\infty, u_s)$, the robust value function equals 1 almost surely. We summarize the results in Proposition 2.

**Proposition 2** *When $\epsilon = \infty$, the optimal retention strategy $R^*_\infty(u, y) = 0$ and the robust value function $V_\infty = 1$ for all $u \in (-\infty, u_s)$. As $R^*_\infty = 0$, it does not matter what drift distortion is chosen.*

***Remark 7*** Note that $\mathbb{P} \in \mathcal{Q}$. It follows from (8) and (17) that

$$V_0(u) = \inf_{R\in\mathcal{D}} \mathbb{P}_u(\Theta^R) = \inf_{R\in\mathcal{D}} V^{R,\mathbb{P}}(u) \le \inf_{R\in\mathcal{D}} \sup_{\mathbb{Q}\in\mathcal{Q}} V^{R,\mathbb{Q}}(u) = V(x) \le 1 = V_\infty.$$

This relation is naturally expected since $V$ of (17) is a non-decreasing function with respect to $\epsilon$. Therefore, we can treat the robust optimal value as a conservative ruin probability due to the fact that the robust value function $V$ is always no less than the non-robust value function $V_0$.

In the following context, we focus on finding the robust value function by solving the HJB equation of (19) for the case with $\epsilon \in (0, \infty)$. Notice that the operator $\mathcal{A}^{R,\mathbb{Q}}W(u)$ defined in (18) is quadratic in $\phi$ with negative leading coefficient. By the first order condition, the maximizer of $\phi$ given in $R$ takes the form $\epsilon\sqrt{\lambda\mathbb{E}\big(R^2\big)}W_u$. Substituting

$$\phi_t = \epsilon\sqrt{\lambda\mathbb{E}\big(R_t^2\big)}W_u$$

into (19) yields

$$\inf_{R\in\mathcal{D}} \left\{\Big(ru - \kappa + \theta\lambda\mathbb{E}\big(R_t\big) + \eta\lambda\mathbb{E}\big(YR_t\big)\Big)W_u + \frac{1}{2}\lambda\mathbb{E}\big(R_t^2\big)\left(\epsilon W_u^2 + W_{uu} - \eta W_u\right)\right\} = 0.$$

Define

$$\begin{aligned}\mathcal{L}(u, R) &= (ru - \kappa)W_u \\ &\quad + \Big(\theta\lambda\mathbb{E}\big(R_t\big) + \eta\lambda\mathbb{E}\big(YR_t\big)\Big)W_u + \frac{1}{2}\lambda\mathbb{E}\big(R_t^2\big)\Big(\epsilon W_u^2 + W_{uu} - \eta W_u\Big).\end{aligned} \tag{24}$$

By using the cumulative distribution function of $Y$, we can rewrite $\mathcal{L}$ as

$$\begin{aligned}\mathcal{L}(u, R) = (ru - \kappa)W_u + \int_0^\infty &\Big\{\Big(\theta\lambda R(y) + \eta\lambda y R(y)\Big)W_u \\ &+ \frac{1}{2}\lambda R^2(y)(\epsilon W_u^2 + W_{uu} - \eta W_u)\Big\}dF_Y(y).\end{aligned} \tag{25}$$

From this integral representation of $\mathcal{L}$, we can minimize $\mathcal{L}$ by minimizing the integrand $y$-by-$y$, subject to $0 \le R(y) \le y$. Write

$$\hat{R}(u, y) = \frac{\theta + \eta y}{\beta(u)} \wedge y, \tag{26}$$

with

$$\beta(u)=\eta-\epsilon W_u-\frac{W_{uu}}{W_u}, \tag{27}$$

which is used to determine the minimizer of the HJB equation. We define the following sets according to different control regions:

$$\begin{aligned}\mathcal{O}_{W_1}&=\{-\infty<u\le u_s:\epsilon W_u^2+W_{uu}-\eta W_u>0,\ \epsilon W_u^2+W_{uu}>0\},\\ \mathcal{O}_{W_2}&=\{-\infty<u\le u_s:\epsilon W_u^2+W_{uu}-\eta W_u>0,\ \epsilon W_u^2+W_{uu}\le 0\},\\ \mathcal{O}_{W_3}&=\{-\infty<u\le u_s:\epsilon W_u^2+W_{uu}-\eta W_u\le 0\},\end{aligned} \tag{28}$$

where the corresponding solutions are of various forms.

***Remark 8*** Liang and Young [33] studied a goal reaching problem when the individual is uncertain about the drift of the risky asset and her hazard rate of mortality. They investigated the optimization problem from a game theoretic point of view and verified that only when the value function keeps convex, the game has an equilibrium value, that is, it does not matter which player acts first, the individual choosing an investment strategy or nature choosing a measure $\mathbb{Q}$. Bayraktar and Zhang [8] proved that the Issacs condition does not hold for their robust problem without further restrictions since the value function can not be convex everywhere. For our risk model, if switching inf and sup in (17), we take the derivative of the operator (18) to obtain the minimizer firstly. However, it is not easy to write the expression of $R$ into the integral representation as (25) due to the existence of the term $\sqrt{\lambda\mathbb{E}\big(R_t^2\big)}$ with a radical sign. Thus, the method of minimizing the integrand $y$-by-$y$ to derive the global minimum is no longer applicable. According to the conclusion of the papers mentioned above, the Issacs condition may not hold for our robust problem either since we have verified that $V$ defined in (17) is S-shaped with a unique inflection point (see Proposition 3 for details).

To simplify our analysis, we define the following functions

$$g_1(\beta)=\mathbb{E}R=\int_0^{\frac{\theta}{\beta-\eta}}S_Y(y)dy+\frac{\eta}{\beta}\int_{\frac{\theta}{\beta-\eta}}^{\infty}S_Y(y)dy, \tag{29}$$

$$g_2(\beta)=\mathbb{E}(YR)=2\int_0^{\frac{\theta}{\beta-\eta}}yS_Y(y)dy+\frac{1}{\beta}\int_{\frac{\theta}{\beta-\eta}}^{\infty}(\theta+2\eta y)S_Y(y)dy, \tag{30}$$

and

$$g_3(\beta)=\mathbb{E}\big(R^2\big)=2\int_0^{\frac{\theta}{\beta-\eta}}yS_Y(y)dy+\frac{2\eta}{\beta^2}\int_{\frac{\theta}{\beta-\eta}}^{\infty}(\theta+\eta y)S_Y(y)dy, \tag{31}$$

where $S_Y=1-F_Y$. In following lemmas, we study the strategy of (26) for different regions and express the corresponding HJB equation in a more convenient form for each of the regions. Define

$$u_2 = \frac{\lambda \mathbb{E}Y - c - \frac{\eta}{2}\lambda \mathbb{E}\big(Y^2\big)}{r}. \tag{32}$$

Recall $u_1$ of (11). It is not difficult to see that $u_2 < u_1 < 0$.

**Lemma 3** *For region $\mathcal{O}_{W_1}$, the extreme minimum point of $\mathcal{L}$ is given by*

$$R^*(u, y) = \frac{\theta + \eta y}{\beta_1^*(u)} \wedge y, \tag{33}$$

*where $\beta_1^* > \eta$ uniquely solves $G(u, \beta_1^*) = 0$ with*

$$G(u, \beta) = ru - \kappa + \lambda\Big(\theta g_1(\beta) + \eta g_2(\beta) - \frac{\beta}{2}\, g_3(\beta)\Big). \tag{34}$$

*Under this strategy, the corresponding HJB equation becomes*

$$(\beta_1^*(u) - \eta) W_u(u) + \epsilon W_u^2 + W_{uu}(u) = 0, \tag{35}$$

*and we deduce that $\mathcal{O}_{W_1} = (u_1, u_s]$.*

***Proof*** Because of the condition (i) in Theorem 1, we have $W_u < 0$, then it follows that $\beta(u) = \eta - \epsilon W_u - \frac{W_{uu}}{W_u} > \eta$ in region $\mathcal{O}_{W_1}$. Thus, the function $\mathcal{L}(u, R)$ is minimized by $\hat{R}$ given in (26). Putting $R^* = \hat{R}$ into (24) yields

$$\mathcal{L}(u, R^*) = W_u(u) G(u, \beta) \tag{36}$$

with $G(u, \beta)$ given in (34). Firstly, recall $g_i$ $(i = 1, 2, 3)$ of (29)–(31), we have

$$\begin{aligned} G(u, \eta) &= ru - \kappa + \lambda\Big(\theta g_1(\eta) + \eta g_2(\eta) - \frac{\eta}{2}\, g_3(\eta)\Big) \\ &= ru - \kappa + \lambda\left(\mathbb{E}Y + \frac{\eta}{2}\mathbb{E}Y^2\right) = ru + c - \lambda \mathbb{E}Y. \end{aligned}$$

The third equality follows from the definition of $\kappa$ in (4). Then, from (11), we obtain $G(u, \eta) \le 0$ for any $u \le u_1$ and $G(u, \eta) > 0$ for any $u > u_1$. Secondly, we see that $\lim_{\beta\to\infty} G(u, \beta) = ru - \kappa \le 0$, and strictly negative for $u < u_s$. Finally,

$$\begin{aligned} \frac{\partial G(u, \beta)}{\partial \beta} &= \theta\lambda g_1'(\beta) + \eta\lambda g_2'(\beta) - \frac{1}{2}\lambda\, g_3(\beta) - \frac{\beta}{2}\lambda\, g_3'(\beta) \\ &= \frac{\beta}{2}\lambda\, g_3'(\beta) - \frac{1}{2}\lambda\, g_3(\beta) - \frac{\beta}{2}\lambda\, g_3'(\beta) \\ &= -\frac{1}{2}\lambda\, g_3(\beta) < 0. \end{aligned}$$

The second line follows from the fact that $\theta g_1'(\beta) + \eta g_2'(\beta) = \frac{\beta}{2} g_3'(\beta)$. Thus, according to the analysis above, it follows that $G$ has a unique zero at $\beta_1^*(u) > \eta$ for any $u \in (u_1, u_s]$. Specially, since $\lim_{\beta\to\infty} G(u_s, \beta) = 0$, we have $\beta_1^*(u) \to \infty$ as $u \to u_s$. Thus, we obtain the corresponding HJB equation of (35).

Besides, since $\beta_1^*(u)$ satisfies the equation $G(u,\beta_1^*(u))=0$, taking derivatives with respect to $u$ and simplifying the expression give

$$\frac{r}{\big(\beta_1^*(u)\big)'}=\frac{1}{2}\lambda\, g_3(\beta_1^*(u)),$$

which implies that $\beta_1^*(u)$ is an increasing function with respect to $u$ and $\beta_1^*(u)\to\eta$ as $u\to u_1$. Furthermore, because of $W_u<0$ and $\beta_1^*(u)=\eta-\epsilon W_u-\frac{W_{uu}}{W_u}>\eta$, it follows that $\epsilon W_u^2+W_{uu}>0$ for all $u_1<u\le u_s$. Thus, we have $(u_1,u_s]\subseteq\mathcal{O}_{W_1}$. On the other hand, we have $\beta(u)>\eta$ for any $u\in\mathcal{O}_{W_1}$. Since $G(u,\beta)$ is decreasing in $\beta$ with $G(u,\eta)=ru+c-\lambda\mathbb{E}Y$ and $\lim_{\beta\to\infty}G(u,\beta)=ru-\kappa\le 0$, then we must have $G(u,\eta)>0$. Hence, it follows that $u\in(u_1,u_s]$, which implies $\mathcal{O}_{W_1}\subseteq(u_1,u_s]$. Consequently, one can obtain $\mathcal{O}_{W_1}=(u_1,u_s]$. □

**Lemma 4** *For region $\mathcal{O}_{W_2}\cup\mathcal{O}_{W_3}$, the extreme minimum point of $\mathcal{L}$ is given by $R^*(u,y)=y$. Under this strategy, the corresponding HJB equation becomes*

$$\big(ru+c-\lambda\mathbb{E}Y\big)W_u(u)+\frac{1}{2}\lambda\mathbb{E}Y^2\Big(\epsilon W_u^2(u)+W_{uu}(u)\Big)=0, \tag{37}$$

*and we deduce that $\mathcal{O}_{W_2}=(u_2,u_1]$ and $\mathcal{O}_{W_3}=(-\infty,u_2]$.*

***Proof*** Since $W_u<0$, we have $\beta(u)=\eta-\epsilon W_u-\frac{W_{uu}}{W_u}\in(0,\eta]$ for region $\mathcal{O}_{W_2}$, and thus $\hat{R}$ of (26) equals $y$. Substituting $R^*=y$ into (24) yields (37). Note that $\epsilon W_u^2+W_{uu}-\eta W_u>0$, or equivalently, $\epsilon W_u^2+W_{uu}>\eta W_u$. It follows that

$$\begin{aligned}0&=\big(ru+c-\lambda\mathbb{E}Y\big)W_u(u)+\tfrac{1}{2}\lambda\mathbb{E}Y^2\Big(\epsilon W_u^2(u)+W_{uu}(u)\Big)\\&>\big(ru+c-\lambda\mathbb{E}Y\big)W_u(u)+\tfrac{1}{2}\lambda\mathbb{E}Y^2\eta W_u(u)\\&=\big(ru+c-\lambda\mathbb{E}Y+\tfrac{1}{2}\lambda\mathbb{E}Y^2\eta\big)W_u(u).\end{aligned} \tag{38}$$

Thus, we have $u>u_2$ with $u_2$ given in (32). Besides, because of the first equality in (38), to ensure $\epsilon W_u^2+W_{uu}\le 0$, the inequality $\big(ru+c-\lambda\mathbb{E}Y\big)W_u\ge 0$ needs to be held, which implies that $(u_2,u_1]\subseteq\mathcal{O}_{W_2}$. On the other hand, we can see from Lemma 3 that $G(u,\beta)$ is decreasing in $\beta$ with $G(u,0)=ru+c+\frac{\eta}{2}\lambda\mathbb{E}Y^2-\lambda\mathbb{E}Y$ and $G(u,\eta)=ru+c-\lambda\mathbb{E}Y$. Thus, due to the fact that $0<\beta(u)\le\eta$ for $u\in\mathcal{O}_{W_2}$, it follows that $G(u,0)>0$ and $G(u,\eta)\le 0$. Then we can deduce that $u\in(u_2,u_1]$, which implies $\mathcal{O}_{W_2}\subseteq(u_2,u_1]$. Therefore, one can conclude $\mathcal{O}_{W_2}=(u_2,u_1]$

For region $\mathcal{O}_{W_3}$, on the one hand, if the inequality $\epsilon W_u^2+W_{uu}-\eta W_u<0$, we see that $\beta(u)$ of (27) is negative because of $W_u(u)<0$. Then it is not difficult to verify that $R^*=y$ minimizes the expression (25). On the other hand, if $\epsilon W_u^2+W_{uu}-\eta W_u=0$, we can directly derive from (25) that $R^*=y$. Similarly, putting $R^*=y$ into (24), we obtain the HJB equation of (37). Moreover, to ensure $\epsilon W_u^2+W_{uu}-\eta W_u\le 0$, or equivalently, $\epsilon W_u^2+W_{uu}\le\eta W_u$, following the similar lines in (38), we deduce $\mathcal{O}_{W_3}\subseteq(-\infty,u_2]$. Conversely, since $\beta(u)<0$ for any $u\in\mathcal{O}_{W_3}$, from the discussions above, we must have $u\in(-\infty,u_2]$, from which we deduce $(-\infty,u_2]\subseteq\mathcal{O}_{W_3}$. Therefore, one can obtain $\mathcal{O}_{W_3}=(-\infty,u_2]$. □

***Remark 9*** Note that the second derivative $W_{uu}$ is strictly negative for any $u \in \mathcal{O}_{W_2} \cup \mathcal{O}_{W_3}$, but the sign of $W_{uu}$ in $\mathcal{O}_{W_1}$ is not clear since $W_{uu}$ can be either negative or positive as long as $\epsilon W_u^2 + W_{uu} > 0$. The detail analysis about the concavity of the robust value function is given in Proposition 3 from which one can deduce that there exists a unique point $u_0 \in \mathcal{O}_{W_1}$ such that $W_{uu} < 0$ for any $u \in (u_1, u_0)$ and $W_{uu} > 0$ for any $u \in (u_0, u_s]$.

We now consider the following boundary-value problem:

$$\begin{cases} \big(ru + c - \lambda \mathbb{E}Y\big) W_u(u) + \dfrac{1}{2}\lambda \mathbb{E}Y^2 \Big(\epsilon W_u^2(u) + W_{uu}(u)\Big) = 0, & -\infty < u \le u_1, \\ (\beta_1^*(u) - \eta) W_u(u) + \epsilon W_u^2 + W_{uu}(u) = 0, & u_1 < u \le u_s, \\ W(-\infty) = 1, \qquad W(u_s) = 0. \end{cases} \tag{39}$$

In Theorem 2, according to the analysis given in Lemmas 3–4, we find our candidate minimum probability of absolute ruin. It follows from Theorem 1 that the resulting expressions are indeed the robust value function $V$ on $(-\infty, u_s)$. We first give a lemma which play an important role in the proof of Theorem 2.

**Lemma 5** *Recall $u_1$ and $\beta_2^*$ of (11) and (20), respectively. Suppose that $\beta_1^*(u) > \eta$ uniquely solves $G(u, \beta_1^*) = 0$ where $G$ is given by (34). Define*

$$H_1(C_1) = 1 + \int_{u_1}^{u_s} f_1(C_1, z)dz + \int_{-\infty}^{u_1} f_2(C_1, z)dz, \tag{40}$$

*where*

$$f_1(C_1, z) = \frac{e^{\int_{u_1}^{z} (\eta - \beta_1^*(w))dw}}{C_1 + \epsilon \int_{u_1}^{z} e^{\int_{u_1}^{v} (\eta - \beta_1^*(w))dw} dv},$$

*and*

$$f_2(C_1, z) = \frac{1}{C_1 e^{\int_z^{u_1} -\beta_2^*(w)dw} - \epsilon \int_z^{u_1} e^{\int_z^{v} -\beta_2^*(w)dw} dv}.$$

*Assume that for every $\tilde{\epsilon} > 0$ there exists a constant $M$ such that*

$$\left| \int_{A_1}^{A_2} f_2(C_1, z)dz \right| < \tilde{\epsilon} \tag{41}$$

*for every $A_1, A_2 > M$ and every $-\infty < C_1 < -\epsilon \int_{u_1}^{u_s} e^{\int_{u_1}^{v} (\eta - \beta_1^*(w))dw} dv$.*[3] *Then there exists a unique solution $C_1$ satisfying the equation $H_1(C_1) = 0$.*

[3] Note that (41) is a sufficient and necessary condition in Cauchy criterion to ensure the improper integral $\int_{-\infty}^{u_1} f_2(C_1, z)dz$ is uniformly convergent. See, for example, Chapter 17.2.1 in Zorich [34].

***Proof*** Lemma 3 shows that $\beta_1^*(u)$ is an increasing function in $u$ with $\beta_1^*(u) \to \eta$ as $u \to u_1$ and $\beta_1^*(u) \to \infty$ as $u \to u_s$, then we have $\eta - \beta_1^*(u) \in (-\infty, 0)$ for any $u_1 < u < u_s$. It follows that $e^{\int_{u_1}^{y}(\eta-\beta_1^*(w))dw} \in [0, 1)$ for any $y \in (u_1, u_s]$, which implies the integrand $f_1$ is bounded for any $-\infty < C_1 < -\epsilon \int_{u_1}^{u_s} e^{\int_{u_1}^{v}(\eta-\beta_1^*(w))dw} dv$. Combining with the condition in (41), we obtain $\lim_{C_1 \to -\infty} H_1(C_1) = 1$ and $H_1(C_1) \to -\infty$ as $C_1 \to -\epsilon \int_{u_1}^{u_s} e^{\int_{u_1}^{v}(\eta-\beta_1^*(w))dw} dv$. Differentiating $H_1$ with respect to $C_1$ yields

$$
\begin{aligned}
H_1'(C_1) = &- \int_{u_1}^{u_s} \frac{e^{\int_{u_1}^{z}(\eta-\beta_1^*(w))dw}}{\left(C_1 + \epsilon \int_{u_1}^{z} e^{\int_{u_1}^{v}(\eta-\beta_1^*(w))dw} dv\right)^2} dz \\
&- \int_{-\infty}^{u_1} \frac{e^{\int_{z}^{u_1} -\beta_2^*(w)dw}}{\left(C_1 e^{\int_{z}^{u_1} -\beta_2^*(w)dw} - \epsilon \int_{z}^{u_1} e^{\int_{z}^{v} -\beta_2^*(w)dw} dv\right)^2} dz < 0.
\end{aligned}
$$

Therefore, there exists a unique solution

$$
-\infty < C_1 < -\epsilon \int_{u_1}^{u_s} e^{\int_{u_1}^{v}(\eta-\beta_1^*(w))dw} dv, \tag{42}
$$

such that $H_1(C_1) = 0$. □

**Theorem 2** *Let $C_1$ be the unique zero to (40). Then the robust value function $V(u)$ equals*

$$
V(u) = \begin{cases}
\displaystyle\int_{-\infty}^{u} \frac{e^{\int_{-\infty}^{y} -\beta_2^*(w)dw}}{C_3 + \epsilon \int_{-\infty}^{y} e^{\int_{-\infty}^{v} -\beta_2^*(w)dw} dv} dy + 1, & -\infty < u \le u_1, \\
\displaystyle\int_{u_1}^{u} \frac{e^{\int_{u_1}^{y}(\eta-\beta_1^*(w))dw}}{C_1 + \epsilon \int_{u_1}^{y} e^{\int_{u_1}^{v}(\eta-\beta_1^*(w))dw} dv} dy + C_2, & u_1 < u \le u_s,
\end{cases} \tag{43}
$$

*where $u_1$ and $\beta_2^*(u)$ are given in (11) and (20), $\beta_1^*(u) > \eta$ uniquely solves*

$$
\int_0^{\frac{\theta}{\beta-\eta}} \big(1 + (\beta - \eta)y\big) S_Y(y)dy + \int_{\frac{\theta}{\beta-\eta}}^{\infty} \left(1 + \frac{\beta - \eta}{\beta}(\theta + \eta y)\right) S_Y(y)dy = \frac{c + ru}{\lambda}, \tag{44}
$$

*and the constants $C_2$ and $C_3$ are given in (79). The optimal robust reinsurance strategy is given by $R^* = R^*(\hat{U}_t^*, Y)$, where*

$$
R^*(u, y) = \begin{cases}
y, & -\infty < u \le u_1, \\
\dfrac{\theta + \eta y}{\beta_1^*(u)} \wedge y, & u_1 < u \le u_s,
\end{cases} \tag{45}
$$

*and the optimal probability distortion is given by* $\phi^* = \phi^*(\hat{U}_t^*)$*, where*

$$\phi^*(u) = \begin{cases} \epsilon\sqrt{\lambda\mathbb{E}\big(Y^2\big)}V_u(u), & -\infty < u \le u_1, \\ \epsilon\sqrt{\lambda\mathbb{E}\left(\left(\frac{\theta+\eta Y}{\beta_1^*(u)} \wedge Y\right)^2\right)}V_u(u), & u_1 < u \le u_s. \end{cases} \tag{46}$$

*Here,* $\hat{U}_t^*$ *is the optimally controlled surplus at time* $t$*, and* $Y$ *is the possible claim size at that time.*

***Proof*** See Appendix B. □

So far, we have solved the robust optimal control problem with explicit solutions. In the following context, we further explore the concavity of the robust value function in Proposition 3. Then a few remarks are given to show some implications of the optimal results.

**Proposition 3** *The robust value function* $V$ *is strictly concave in* $(-\infty, u_0)$*, and strictly convex in* $(u_0, u_s]$ *where* $u_0$ *is the unique point in* $(u_1, u_s)$ *satisfying* $(\eta - \beta_1^*(u_0))V_u(u_0) - \epsilon V_u^2(u_0) = 0$.

***Proof*** Remark 9 together with the results given in Theorem 2 enable us to show that $V$ is strictly concave when $-\infty < u < u_1$. Thus, we only need to prove that there exists a unique inflection point $u_0 > u_1$ such that $V$ is strictly convex when $u > u_0$. Note that when $u \in (u_1, u_s]$, $\beta_1^*$ satisfies the equation $(\beta_1^*(u) - \eta)V_u + \epsilon V_u^2 + V_{uu} = 0$. Let $f(u) = (\eta - \beta_1^*(u))V_u$. Then we have $V_{uu} = f - \epsilon V_u^2$. Differentiating $f$ with respect to $u$ yields

$$f' = -(\beta_1^*(u))'V_u + (\eta - \beta_1^*(u))V_{uu} > (\eta - \beta_1^*(u))V_{uu}, \tag{47}$$

where the inequality is due to the fact that $\beta_1^*(u)$ is an increasing function with respect to $u$ and $V_u < 0$. Besides, from Lemma 3, we have $\beta_1^*(u) \to \infty$ as $u \to u_s$, then it is straightforward to show that $V_{uu}$ will be strictly positive in a neighborhood of $u_s$. Thus, according to the continuity of the value function, we can conclude that the function $V$ changes concavity at least once on $(u_1, u_s]$. We assume that $V$ changes concavity at the point $u_0$ for the first time. Then we have $V_{uu}(u_0) = 0$. From the inequality given in (47), we have $f'(u_0) > 0$, that is, $f$ is strictly increasing whenever $V$ changes concavity. Since $V_{uu} = f - \epsilon V_u^2$, we have $V_{uuu} = f' - 2\epsilon V_u V_{uu}$. Combining with $V_u(u_0) < 0$ and $V_{uu}(u_0) = 0$, it follows that $V_{uuu}(u_0) > 0$, which implies that $V$ can only change from concavity to convexity. As we have mentioned in the beginning of the proof that $V$ is strictly concave when $u \le u_1$ and strictly convex in a neighborhood of $u_s$, we come to the conclusion that $V$ changes concavity only once, and that it is strictly concave up to the unique point $u_0$, and is strictly convex afterwards.

***Remark 10*** It is shown that the non-robust value function $V_0$ is strictly concave when $u < u_1$, and strictly convex when $u > u_1$ in Proposition 1. However, one can see from Proposition 3 that $u_1$ is no longer the inflection point of the robust value function $V$. This is caused by the nonlinear term $\epsilon V_u^2$. In fact, when $\epsilon$ is nonzero, $V_{uu}$ can be negative as long as $\epsilon V_u^2 + V_{uu} > 0$ for any $u \in (u_1, u_s]$. The larger the $\epsilon$ is, the more concave the value can potentially be. It is rather interesting to prove that the robust value function is still S-shaped with the unique inflection point $u_0$, but the resulting position of the inflection point is different from the one without model ambiguity. Note that the sign of $u_0$ depends on the model parameters.

***Remark 11*** It follows from (45) that the optimal robust reinsurance strategy is a non-increasing function with respect to $u$. In particular, $\beta_1^*(u)$ tends to $\infty$ as the surplus value approaches $u_s$, and thus $R^*$ approaches zero retention. It is natural to see that when the value of the surplus increases towards $u_s$, the insurer can transfer all the risk to the reinsurer. As a result, wealth will never decrease and absolute ruin cannot happen. Note that the monotonicity of the optimal control is also seen in the case of non-robust ruin minimization problems. We observe that the optimal robust reinsurance strategy of (45) is independent of the penalty parameter $\epsilon$, and coincides with the one in the benchmark case without model ambiguity. Thus, in this sense, we can conclude that the insurer does not really need to worry about the model uncertainty under our model settings. Such an independence is nontrivial and not foreseen. It is rather an algebraic coincidence that $\bar{\beta}_1^*(u)$ and $\beta_1^*(u)$ solve the same equation $G(u, \beta) = 0$ with $G$ given in (34), which is independent with $\epsilon$ (see Proposition 1 and Lemma 3 for details). Bayraktar and Zhang [8] (when the hazard rate is zero) and Luo et al. [11] also reached a similar conclusion in their work. Besides, because of $u_1 < 0$, we expect that the optimal robust reinsurance strategy for $u \in (u_1, u_s]$ also minimizes the penalized probability of traditional ruin. This result is given in Corollary 1.

***Remark 12*** The expression (46) shows that the optimal distortion function $\phi^*$ is continuous and negative on $(-\infty, u_s]$. This phenomenon is expected since a measure $\mathbb{Q}$ with negative $\phi^*$ results in a negative drift term in the surplus process, yielding a higher absolute ruin probability under measure $\mathbb{Q}$. Besides, the penalty parameter affects the value of the robust absolute ruin probability and the selection of the equivalent probability measure for the optimization. Moreover, it is not difficult to verify that $V(u) \to 1$ as $\epsilon \to \infty$ for all $u \in (-\infty, u_s]$, which implies that the absolute ruin occurs almost surely when approaching the worst-case (maximal penalization on ambiguity). This conclusion coincides with the one given in Proposition 2.

Note that, for our risk model (6), the criterion of minimizing the penalized probability of traditional ruin $\bar{V}$ with the ruin level 0 has never been considered in other existing literature before. Corollary 1 provides an explicit solution to $\bar{V}$, which is not only interesting for its own sake, but also serves in the numerical examples when we analyze the effects of ambiguity aversion.

Define the hitting time

$$\bar{\tau}_0^R = \inf\left\{t \geq 0 : \hat{U}_t^R = 0\right\}.$$

For any $\mathbb{Q} \in \mathcal{Q}$ and the initial surplus $u \in [0, u_s]$, the minimum probability of traditional ruin with the ruin level 0 is defined by

$$\bar{V}(u) = \inf_{R \in \mathcal{D}} \sup_{\mathbb{Q} \in \mathcal{Q}} \left\{ \mathbb{Q}_u(\bar{\tau}_0^R < \infty) - \frac{1}{\epsilon} h(\mathbb{Q}_{\bar{\tau}^R} | \mathbb{P}_{\bar{\tau}^R}) \right\}, \tag{48}$$

where $\bar{\tau}^R = \bar{\tau}_0^R \wedge \tau_{u_s}^R$ with $\tau_{u_s}^R$ given in (15).

**Corollary 1** *The robust value function $\bar{V}(u)$ equals*

$$\bar{V}(u) = 1 + \int_0^u \frac{e^{\int_0^y (\eta - \beta_1^*(w))dw}}{C_0 + \epsilon \int_0^y e^{\int_0^v (\eta - \beta_1^*(w))dw} dv} dy, \tag{49}$$

*where $C_0$ uniquely solves the equation $H_0(C_0) = 0$ with*

$$H_0(C_0) = 1 + \int_0^{u_s} \frac{e^{\int_0^y (\eta - \beta_1^*(w))dw}}{C_0 + \epsilon \int_0^y e^{\int_0^v (\eta - \beta_1^*(w))dw} dv} dy. \tag{50}$$

*The robust optimal reinsurance strategy is given by*

$$\bar{R}^*(u, y) = \frac{\theta + \eta y}{\beta_1^*(u)} \wedge y, \tag{51}$$

*and the optimal probability distortion equals*

$$\bar{\phi}^*(u) = \epsilon \sqrt{\lambda \mathbb{E} \left( \left( \frac{\theta + \eta Y}{\beta_1^*(u)} \wedge Y \right)^2 \right)} \bar{V}_u(u). \tag{52}$$

***Proof*** The boundary-value problem of (39) becomes

$$\begin{cases} (\beta_1^*(u) - \eta) W_u(u) + \epsilon W_u^2 + W_{uu}(u) = 0, \\ W(0) = 1, \qquad W(u_s) = 0. \end{cases} \tag{53}$$

Analogous to the proofs of Lemma 5 and Theorem 2, it is not difficult to verify that there exists a unique solution $C_0$ such that $H_0(C_0) = 0$, and then one can derive (49), (51) and (52). □

***Remark 13*** From (43) and (49), it is not difficult to verify that the concavity of the value function for traditional ruin is identical to the one for absolute ruin when the surplus falls into the common region. Proposition 3 shows that the sign of $u_0$ is uncertain, and relies on the ambiguity parameter $\epsilon$ greatly. Hence, it is expected that the robust value function for minimizing the traditional ruin probability is S-shaped if $u_0 > 0$ but convex everywhere if $u_0 \leq 0$. Also, as mentioned in Remark 1, if the absolute ruin is defined as the event that the surplus falls below the critical level $u_1$ given by (11) as in Dassios and Embrechts [21], the robust value function for this modified version of absolute ruin can be derived directly by means of our original analysis, and because of $u_0 > u_1$, the corresponding value function turns to be S-shaped as well.

By setting $\eta = 0$ or $\theta = 0$ in (2), the mean-variance premium principle reduces to the expected-value or variance premium principle, respectively. In the following corollaries, we consider these two special cases. It is clear that the optimal results also hold for the traditional ruin probability when $u$ belongs to $[0, u_s]$.

**Corollary 2** *If $\eta = 0$, the optimal robust reinsurance strategy of (45) reduces to excess-of-loss reinsurance, that is,*

$$R^*(u, y) = \begin{cases} y, & -\infty < u \le u_1, \\ d^*(u) \wedge y, & u_1 < u \le u_s, \end{cases} \tag{54}$$

*where $d^*(u) > 0$ uniquely solves*

$$\theta \int_0^d \left(1 - \frac{y}{d}\right) S_Y(y) dy = \frac{\kappa - ru}{\lambda}. \tag{55}$$

***Proof*** By setting $\eta = 0$ and $d^*(u) = \frac{\theta}{\beta_1^*(u)}$ in (44) and (45), one can obtain (54) and (55). □

**Corollary 3** *If $\theta = 0$, the optimal robust reinsurance strategy in (45) reduces to quota-share reinsurance, that is,*

$$R^*(u, y) = \begin{cases} y, & -\infty < u \le u_1, \\ q^*(u) y, & u_1 < u \le u_s, \end{cases} \tag{56}$$

*where $q^*(u) \in [0, 1)$ equals*

$$q^*(u) = \frac{2}{\eta} \cdot \frac{\kappa - ru}{\lambda \mathbb{E}(Y^2)}. \tag{57}$$

***Proof*** By setting $\theta = 0$ and $q^*(u) = \frac{\eta}{\beta_1^*(u)}$ in (44) and (45), we obtain (56) and (57). □

## 4 Numerical Examples

In this section, we present two numerical examples to illustrate the effects of ambiguity aversion by showing the optimal solutions and discuss the economic implications therein. To keep things simple, we consider the special case of minimizing the penalized probability of traditional ruin and assume that the reinsurance premium is calculated by the expected-value principle and the variance premium principle, respectively, which corresponds to the optimal results in Corollaries 1–3. According to the properties given in Remark 13, the following observations can be used as a reference to the optimal results in Theorem 2. Here, we assume that the claim-size random variable $Y$ is uniformly distributed in the interval $[0, 2]$, and so we have $\mathbb{E}Y = 1$ and $\mathbb{E}Y^2 = 4/3$.

***Example 1*** In this example, we set $(\theta, \eta) = (0.4, 0)$ for the expected value principle and $(\theta, \eta) = (0, 0.6)$ for the variance premium principle. Also, we set $r = 0.05$, $c = 3.3$, $\lambda = 3$ and $\epsilon = 3$ which implies that the safe level $u_s = 18$. With these parameter values, we compute the robust value function $\bar{V}$, the optimal drift distortion $\bar{\phi}^*$, and the first order derivative of the value function $\bar{V}_u$. The optimal robust results are shown in Figs. 1 and 2.

Under the expected-value premium principle, the optimal deductible $d^*(u)$ of the excess-of-loss reinsurance is given by

$$d^*(u) = 3 - \sqrt{0.5u}, \tag{58}$$

and the robust value function $\bar{V}$ equals

$$\bar{V}(u) = 1 + \int_0^u \frac{e^{0.8\sqrt{2y}}(6-\sqrt{2y})^{4.8}}{C_0 + \epsilon\int_0^y e^{0.8\sqrt{2\tilde{y}}}(6-\sqrt{2\tilde{y}})^{4.8}d\tilde{y}}dy, \tag{59}$$

where $C_0$ uniquely solves the equation $H_0(C_0) = 0$ with

$$H_0(C_0) = 1 + \int_0^{18} \frac{e^{0.8\sqrt{2y}}(6-\sqrt{2y})^{4.8}}{C_0 + \epsilon\int_0^y e^{0.8\sqrt{2\tilde{y}}}(6-\sqrt{2\tilde{y}})^{4.8}d\tilde{y}}dy. \tag{60}$$

Then the optimal probability distortion $\bar{\phi}^*$ is given by

$$\bar{\phi}^*(u) = \frac{\epsilon\sqrt{\lambda}\big((3-\sqrt{0.5u})^2 - \frac{1}{3}(3-\sqrt{0.5u})^3\big)\, e^{0.8\sqrt{2u}}(6-\sqrt{2u})^{4.8}}{C_0 + \epsilon\int_0^u e^{0.8\sqrt{2\tilde{y}}}(6-\sqrt{2\tilde{y}})^{4.8}d\tilde{y}}. \tag{61}$$

Also, under the variance premium principle, the optimal quota-share reinsurance has retained proportion $q^*(u)$ given by

$$q^*(u) = \frac{18-u}{24}, \tag{62}$$

and the robust value function equals

$$\bar{V}(u) = 1 + \int_0^u \frac{e^{0.6y}(1-\frac{y}{18})^{14.4}}{C_0 + \epsilon\int_0^y e^{0.6\tilde{y}}(1-\frac{\tilde{y}}{18})^{14.4}d\tilde{y}}dy, \tag{63}$$

where $C_0$ uniquely solves the equation $H_0(C_0) = 0$ with

$$H_0(C_0) = 1 + \int_0^{18} \frac{e^{0.6y}(1-\frac{y}{18})^{14.4}}{C_0 + \epsilon\int_0^y e^{0.6\tilde{y}}(1-\frac{\tilde{y}}{18})^{14.4}d\tilde{y}}dy. \tag{64}$$

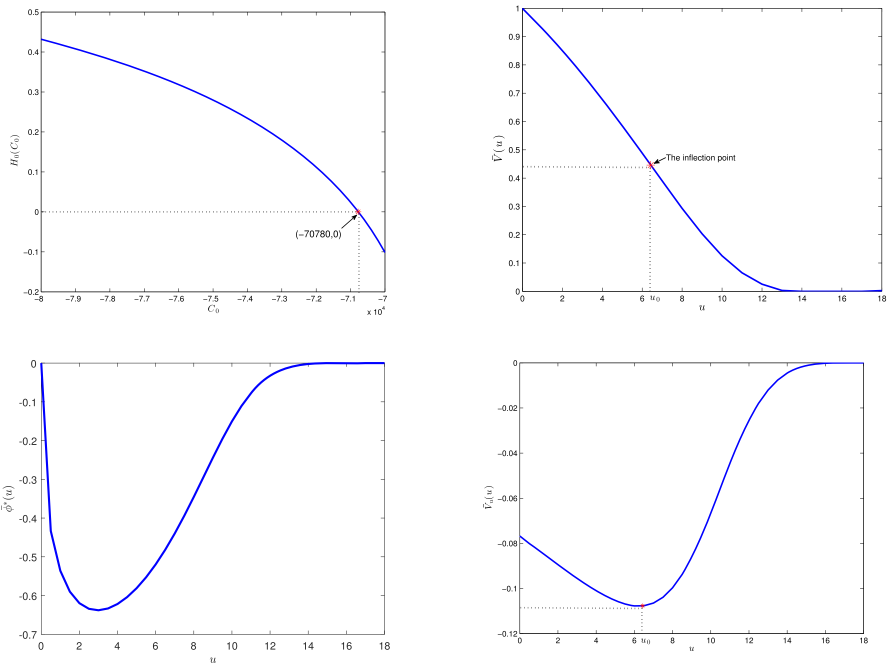


**Fig. 1** Values of $C_0$, $\bar{V}$, $\bar{\phi}^*$ and $\bar{V}_u$ under the expected value principle for $\epsilon = 3$

Then the optimal probability distortion equals

$$\bar{\phi}^*(u) = \frac{\epsilon\sqrt{\lambda}\left(\frac{18-u}{24}\right) e^{0.6u}(1-\frac{u}{18})^{14.4}}{C_0 + \epsilon \int_0^u e^{0.6\tilde{y}}(1-\frac{\tilde{y}}{18})^{14.4} d\tilde{y}}. \tag{65}$$

Figures 1 and 2 show that $H_0(C_0)$ in (60) and (64) is a decreasing function with respect to $C_0$, and there indeed exists a unique zero such that $H_0(C_0) = 0$. Besides, it is not difficult to see that both the robust value function and the associated reinsurance strategy decrease as $u$ increases. These observations are to be expected. When the value of the surplus increases towards $u_s$, the insurer can buy full reinsurance via income from the riskless asset. As a result, the surplus will never drop below its current value and ruin cannot happen. We can also see that the optimal probability distortion function first decreases and then increases as $u$ increases. This is because the insurer would rather choose a conservative probability measure with more negatively drifted distortion to avoid ruin when the surplus is relatively low. However, when the surplus becomes relatively high, the insurer may choose a risky probability measure with less negatively distortion so as to attain the safe level. Furthermore, we discover that the first order derivative $\bar{V}_u$ is not monotone with respect to $u$, and the robust value function is S-shaped with a unique point of inflection $u_0$ (see Remark 13).

***Example 2*** In this example, we show how the penalty parameter $\epsilon$ affects the robust value function $\bar{V}$ and the optimal probability distortion $\bar{\phi}^*$. Again, we set $(\theta, \eta) =$

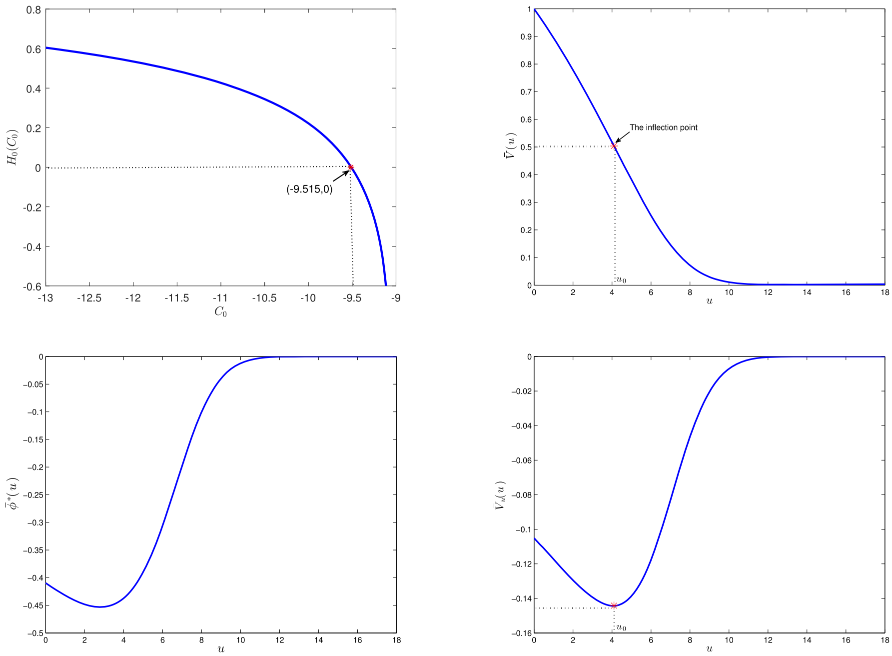


**Fig. 2** Values of $C_0$, $\bar{V}$, $\bar{\phi}^*$ and $\bar{V}_u$ under variance premium principle for $\epsilon = 3$

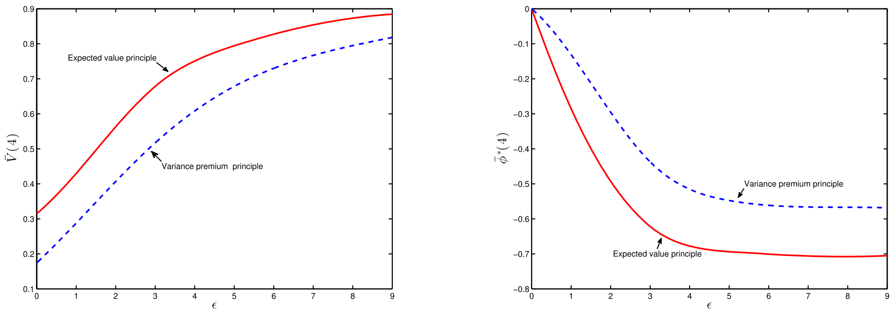


**Fig. 3** The influence of $\epsilon$ on the values of $\bar{V}(4)$ and $\bar{\phi}^*(4)$

(0.4, 0) for the expected value principle and $(\theta, \eta) = (0, 0.6)$ for the variance premium principle. Under such parameter settings, we have

$$(1+\theta)\lambda\mathbb{E}Y = \lambda\mathbb{E}Y + \frac{\eta}{2}\lambda\mathbb{E}Y^2, \tag{66}$$

that is, when the insurer transfer all its risk to the reinsurer with $\bar{R}^* = 0$, the reinsurance premiums under the two premium principles are equal. Also, we set $r = 0.05$, $c = 3.3$, $\lambda = 3$ and $u = 4$. The optimal robust results are shown in Fig. 3.

Figure 3 shows that the robust value function $\bar{V}$ is an increasing function with respect to $\epsilon$ but the optimal probability distortion $\bar{\phi}^*$ decreases as $\epsilon$ increases. In particular,

$\epsilon = 0$ corresponds to the non-robust case. These observations are reasonable. We can see from (48) that a larger value of $\epsilon$ means that the insurer put larger weight on the deviation of the ambiguous measure from the reference probability measure, reflecting the insurer's higher level of ambiguity aversion. As a result, insurers with higher ambiguity aversion adopt a more conservative probability measure with more negatively distortion, which in turn makes ruin more likely. Furthermore, with the condition in (66), we find that the optimal probability distortion is more negatively valued under the expected value principle than the variance premium principle. Then it is to be expected that the probability of absolute ruin under the former principle is always larger than the one under the latter principle since a more negative value is added to the drift of the surplus process.

## 5 Conclusion

In this paper, we investigate the optimal robust reinsurance problem of an insurer who seeks to minimize the probability of absolute ruin when the insurer is ambiguity-averse and want to guard herself against worst-case scenarios. By using stochastic control, we characterize the value function as the unique classical solution to the HJB equation, and obtain feedback forms for the optimal robust strategy and the optimal probability distortion. We prove that there exists a unique point of inflection $u_0$ such that the robust value function is strictly concave up to $u_0$, and is strictly convex afterwards. By comparing the optimal results with those in the benchmark case, we observe that the insurer's ambiguity aversion with penalization in our setting affects the evaluation of the robust value and the selection of the optimal ambiguous measure but does not affect the insurer's optimal strategy. It is also shown that the insurer chooses a measure with more negatively probability distortion to attain the robust value when the penalization on ambiguity increases. This in turn makes the minimized robust ruin probability increases.

In future research, we may consider the same optimization problem in which the insurer does not have perfect information on the rate of claim occurrence. Besides, we can generalize our results to a more general risk model where the risk-free bond and risky assets are involved, and the insurer does not have perfect information on the drift terms of the risky asset. It would be also interesting to impose some restrictions on investment. For example, the borrowing rate is higher than the saving rate, and short-selling and borrowing are based on the fraction of wealth. These modeling features may lead to more complex characterization of the corresponding robust value, and result in more significant effects of model ambiguity on control and decision-making.

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## Compliance with ethical standards

**Conflict of interest** The authors declare that they have no conflict of interest.

## Appendix

### A. The proof of Theorem 1

***Proof*** The proof of verification theorem is a modified version of Theorem 3.1 in Luo et al. [11] (also see Theorem 6.1 in Bayraktar and Zhang [8]). Firstly, under the optimal reinsurance policy $R^*$, the surplus process given in (6) becomes

$$d\hat{U}_t^{R^*} = \left(r\hat{U}_t^{R^*} - \kappa + \theta\lambda\mathbb{E}\big(R_t^*\big) + \eta\lambda\mathbb{E}\big(YR_t^*\big) - \frac{\eta}{2}\lambda\mathbb{E}\big((R_t^*)^2\big)\right)dt + \sqrt{\lambda\mathbb{E}\big((R_t^*)^2\big)}\,dB_t. \tag{67}$$

Let $\mathbb{Q} \in \mathcal{Q}$ be an arbitrary probability measure with a corresponding probability distortion process $\phi$. By the Girsanov theorem, $B_t^{\mathbb{Q}} = B_t - \int_0^t \phi_s ds$ is a $\mathbb{Q}$-Brownian motion. Then we have

$$d\hat{U}_t^{R^*} = \Big(r\hat{U}_t^{R^*} - \kappa + \theta\lambda\mathbb{E}\big(R_t^*\big) + \eta\lambda\mathbb{E}\big(YR_t^*\big) - \frac{\eta}{2}\lambda\mathbb{E}\big((R_t^*)^2\big) + \phi_t\sqrt{\lambda\mathbb{E}\big((R_t^*)^2\big)}\Big)dt + \sqrt{\lambda\mathbb{E}\big((R_t^*)^2\big)}\,dB_t^{\mathbb{Q}}.$$

Applying Ito's Lemma to $W(\hat{U}_t^{R^*})$, we have

$$\begin{aligned} dW(\hat{U}_t^{R^*}) &= \Big[\Big(r\hat{U}_t^{R^*} - \kappa + \theta\lambda\mathbb{E}\big(R_t^*\big) + \eta\lambda\mathbb{E}\big(YR_t^*\big) - \frac{\eta}{2}\lambda\mathbb{E}\big((R_t^*)^2\big) + \phi_t\sqrt{\lambda\mathbb{E}\big((R_t^*)^2\big)}\Big)\cdot \\ &\quad W_u(\hat{U}_t^{R^*}) + \frac{1}{2}\lambda\mathbb{E}\big((R_t^*)^2\big)W_{uu}(\hat{U}_t^{R^*})\Big]dt + \sqrt{\lambda\mathbb{E}\big((R_t^*)^2\big)}W_u(\hat{U}_t^{R^*})\,dB^{\mathbb{Q}} \\ &= \mathcal{A}^{R^*,\phi}W\big(\hat{U}_t^{R^*}\big)\,dt + \frac{1}{2\epsilon}\phi_t^2\,dt + \sqrt{\lambda\mathbb{E}\big((R_t^*)^2\big)}W_u(\hat{U}_t^{R^*})\,dB_t^{\mathbb{Q}}. \end{aligned}$$

By Lemma 1, we have $\tau_{M,N}^R < \infty$ almost surely for any $-\infty < M < N < u_s$. Integrating the above equation form 0 to $\tau_{M,N}^{R^*}$ and taking $\mathbb{Q}$-expectation on both sides, we get

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}\Big(W(\hat{U}_{\tau_{M,N}^{R^*}}^{R^*})\Big) - W(u) &= \mathbb{E}^{\mathbb{Q}}\int_0^{\tau_{M,N}^{R^*}} \mathcal{A}^{R^*,\phi}W\big(\hat{U}_t^{R^*}\big)dt + \mathbb{E}^{\mathbb{Q}}\int_0^{\tau_{M,N}^{R^*}} \frac{1}{2\epsilon}\phi_t^2\,dt \\ &\quad + \mathbb{E}^{\mathbb{Q}}\int_0^{\tau_{M,N}^{R^*}} \sqrt{\lambda\mathbb{E}\big((R_t^*)^2\big)}W_u(\hat{U}_t^{R^*})\,dB_t^{\mathbb{Q}}. \end{aligned}$$

Note that the expectation of the third intergral equals 0 because $W_u$ and $\sqrt{\lambda\mathbb{E}\big((R_t^*)^2\big)}$ are bounded. Besides, conditions (ii) and (iv) imply that

$$0 = \inf_{R\in\mathcal{D}} \sup_{\phi\in\mathcal{R}} \Big\{\mathcal{A}^{R^*,\phi} W(u)\Big\} = \sup_{\phi\in\mathcal{R}} \mathcal{A}^{R^*,\phi} W(\hat{U}_t^{R^*}) \geq \mathcal{A}^{R^*,\phi} W(\hat{U}_t^{R^*}). \tag{68}$$

Therefore, we obtain

$$W(u) \geq \mathbb{E}^{\mathbb{Q}}\Big(W(\hat{U}^{R^*}_{\tau^{R^*}_{M,N}})\Big) - \mathbb{E}^{\mathbb{Q}} \int_0^{\tau^{R^*}_{M,N}} \frac{1}{2\epsilon}\phi_t^2\, dt. \tag{69}$$

We write

$$\mathbb{E}^{\mathbb{Q}}\big(W(\hat{U}^{R^*}_{\tau^{R^*}_{M,N}})\big) = W(M)\mathbb{Q}_u(\tau_M^{R^*} < \tau_N^{R^*}) + W(N)\mathbb{Q}_u(\tau_M^{R^*} \geq \tau_N^{R^*}). \tag{70}$$

Because $W(u_s) = 0$, combining (69) with (70) and letting $N \to u_s$ yield

$$W(u) \geq W(M)\mathbb{Q}_u(\tau_M^{R^*} < \tau_{u_s}^{R^*}) - \mathbb{E}^{\mathbb{Q}} \int_0^{\tau^{R^*}_{M,u_s}} \frac{1}{2\epsilon}\phi_t^2\, dt.$$

Since the inequality holds for all $\mathbb{Q} \in \mathcal{Q}$, we have

$$\begin{aligned} W(u) &\geq \sup_{\mathbb{Q}\in\mathcal{Q}} \Big\{W(M)\mathbb{Q}_u(\tau_M^{R^*} < \tau_{u_s}^{R^*}) - \mathbb{E}^{\mathbb{Q}} \int_0^{\tau^{R^*}_{M,u_s}} \frac{1}{2\epsilon}\phi_t^2\, dt\Big\} \\ &= \inf_{R\in\mathcal{D}} \sup_{\mathbb{Q}\in\mathcal{Q}} \Big\{W(M)\mathbb{Q}_u(\tau_M^{R} < \tau_{u_s}^{R}) - \mathbb{E}^{\mathbb{Q}} \int_0^{\tau^{R}_{M,u_s}} \frac{1}{2\epsilon}\phi_t^2\, dt\Big\} \\ &= \inf_{R\in\mathcal{D}} \sup_{\mathbb{Q}\in\mathcal{Q}} \Big\{W(M)\mathbb{Q}_u(\tau_M^{R} < \infty) - \frac{1}{\epsilon} h(\mathbb{Q}_{\tau^R_{M,u_s}} | \mathbb{P}_{\tau^R_{M,u_s}})\Big\}. \end{aligned} \tag{71}$$

The equality in (71) follows from (14) and the fact that $\tau_M^R = \infty$ if $\tau_{u_s}^R \leq \tau_M^R$. Note that

$$\begin{aligned} V(u) &= \inf_{R\in\mathcal{D}} \sup_{\mathbb{Q}\in\mathcal{Q}} \Big\{\mathbb{Q}_u(\liminf_{t\to\infty} \hat{U}_t^R = -\infty) - \frac{1}{\epsilon} h(\mathbb{Q}_{\tau^R} | \mathbb{P}_{\tau^R})\Big\} \\ &= \inf_{R\in\mathcal{D}} \sup_{\mathbb{Q}\in\mathcal{Q}} \Big\{ \lim_{M\to-\infty} \mathbb{Q}_u(\tau_M^R < \infty) - \frac{1}{\epsilon} h(\mathbb{Q}_{\tau^R} | \mathbb{P}_{\tau^R})\Big\}. \end{aligned} \tag{72}$$

Letting $M \to -\infty$ in (71), we get $W(u) \geq V(u)$.

We next prove $W(u) \leq V(u)$. Let $R$ be any admissible reinsurance policy. Under the optimal strategy $\phi^* = \phi^*(R, u)$, there exists a measure $\mathbb{Q}^* \in \mathcal{Q}$ satisfying $\frac{d\mathbb{Q}_t^*}{d\mathbb{P}_t} = \varepsilon(\int_0^t \phi_t^* dB_s)$, where $\varepsilon$ denotes the stochastic exponential. It follows from the Girsanov theorem that $B_t^{\mathbb{Q}^*} = B_t - \int_0^t \phi_s^* ds$. Then the surplus process becomes

$$d\hat{U}_t^R = \Big(r\hat{U}_t^R - \kappa + \theta\lambda\mathbb{E}\big(R_t\big) + \eta\lambda\mathbb{E}\big(YR_t\big) - \frac{\eta}{2}\lambda\mathbb{E}\big(R_t^2\big) + \phi_t^*\sqrt{\lambda\mathbb{E}\big(R_t^2\big)}\Big)dt$$
$$+\sqrt{\lambda\mathbb{E}\big(R_t^2\big)}\,dB_t^{\mathbb{Q}^*}.$$

Applying Ito's lemma to $W(\hat{U}_t^R)$, we have

$$\begin{aligned}
dW(\hat{U}_t^R) &= \Big[\Big(r\hat{U}_t^R - \kappa + \theta\lambda\mathbb{E}\big(R_t\big) + \eta\lambda\mathbb{E}\big(YR_t\big) - \frac{\eta}{2}\lambda\mathbb{E}\big(R_t^2\big) + \phi_t^*\sqrt{\lambda\mathbb{E}\big(R_t^2\big)}\Big)W_u(\hat{U}_t^R) \\
&\quad + \frac{1}{2}\lambda\mathbb{E}\big(R_t^2\big)W_{uu}(\hat{U}_t^R)\Big]dt + \sqrt{\lambda\mathbb{E}\big(R_t^2\big)}W_u(\hat{U}_t^R)\,dB^{\mathbb{Q}^*} \\
&= \mathcal{A}^{R,\phi^*}W\big(\hat{U}_t^R\big)\,dt + \frac{1}{2\epsilon}(\phi_t^*)^2\,dt + \sqrt{\lambda\mathbb{E}\big(R_t^2\big)}W_u(\hat{U}_t^R)\,dB_t^{\mathbb{Q}^*}.
\end{aligned}$$

Integrating the above equation from 0 to $\tau_{M,N}^R$ and taking $\mathbb{Q}^*$-expectation on both sides yield

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}^*}\Big(W(\hat{U}_{\tau_{M,N}}^R)\Big) - W(u) &= \mathbb{E}^{\mathbb{Q}^*}\int_0^{\tau_{M,N}^R}\mathcal{A}^{R,\phi^*}W\big((\hat{U}_t^R)\big)dt + \mathbb{E}^{\mathbb{Q}^*}\int_0^{\tau_{M,N}^R}\frac{1}{2\epsilon}(\phi_t^*)^2\,dt \\
&\quad + \mathbb{E}^{\mathbb{Q}^*}\int_0^{\tau_{M,N}^R}\sqrt{\lambda\mathbb{E}\big(R_t^2\big)}W_u(\hat{U}_t^R)\,dB_t^{\mathbb{Q}^*}.
\end{aligned}$$

Similarly, the expectation of the third integral equals 0 due to the fact that $W_u$ and $\sqrt{\lambda\mathbb{E}\big(R_t\big)^2}$ are bounded. By conditions (ii), (iv) and our definition of $\phi^*$, we have

$$0 = \inf_{R\in\mathcal{D}}\sup_{\phi\in\mathcal{R}}\Big\{\mathcal{A}^{R,\phi}W(u)\Big\} \le \sup_{\phi\in\mathcal{R}}\mathcal{A}^{R,\phi}W(\hat{U}_t^R) = \mathcal{A}^{R,\phi^*}W(\hat{U}_t^R). \tag{73}$$

Therefore, we have

$$W(u) \le \mathbb{E}^{\mathbb{Q}^*}\big(W(\hat{U}_{\tau_{M,N}^R}^R)\big) - \mathbb{E}^{\mathbb{Q}^*}\int_0^{\tau_{M,N}^R}\frac{1}{2\epsilon}(\phi_t^*)^2\,dt. \tag{74}$$

Write

$$\mathbb{E}^{\mathbb{Q}^*}\big(W(\hat{U}_{\tau_{M,N}^R}^R)\big) = W(M)\mathbb{Q}_u^*(\tau_M^R < \tau_N^R) + W(N)\mathbb{Q}_u^*(\tau_M^R \ge \tau_N^R). \tag{75}$$

Because $W(u_s) = 0$, combining (74) with (75) and letting $N \to u_s$ yield

$$\begin{aligned}
W(u) &\le W(M)\mathbb{Q}_u^*\big(\tau_M^R < \tau_{u_s}^R\big) - \mathbb{E}^{\mathbb{Q}^*}\int_0^{\tau_{M,u_s}^R}\frac{1}{2\epsilon}(\phi_t^*)^2\,dt. \\
&= \sup_{\mathbb{Q}\in\mathcal{Q}}\Big\{W(M)\mathbb{Q}_u(\tau_M^R < \tau_{u_s}^R) - \mathbb{E}^{\mathbb{Q}}\int_0^{\tau_{M,u_s}^R}\frac{1}{2\epsilon}\phi_t^2\,dt\Big\} \\
&= \sup_{\mathbb{Q}\in\mathcal{Q}}\Big\{W(M)\mathbb{Q}_u(\tau_M^R < \infty) - \frac{1}{\epsilon}h(\mathbb{Q}_{\tau_{M,u_s}^R}|\mathbb{P}_{\tau_{M,u_s}^R})\Big\}.
\end{aligned} \tag{76}$$

This hold for any $R \in \mathcal{D}$, so we have

$$W(u) \leq \inf_{R\in\mathcal{D}} \sup_{\mathbb{Q}\in\mathcal{Q}} \left\{ W(M)\mathbb{Q}_u(\tau_M^R < \tau_{u_s}^R) - \mathbb{E}^{\mathbb{Q}} \int_0^{\tau_{M,u_s}^R} \frac{1}{2\epsilon}\phi_t^2\, dt \right\} \tag{77}$$

Letting $M \to -\infty$ in (77), it follows from (72) that $W(u) \leq V(u)$. Therefore, we conclude that $W(u) = V(u)$. Moreover, repeat the analysis above by using $R^*$ and $\phi^*$ in (73) and (74), (73) through (77) hold with equality, and hence

$$W(u) = V(u) = V^{R^*,\mathbb{Q}^*}(u).$$

□

## B. The proof of Theorem 2

***Proof*** Suppose that $W$ is a candidate solution to (19). When $u_1 < u \leq u_s$, the function $W_1(u) = W(u)$ satisfies the following ordinary differential equation (ODE):

$$(\beta_1^*(u) - \eta)W_u(u) + \epsilon W_u^2 + W_{uu}(u) = 0.$$

Note that (44) is a more explicit version of (34). By letting $y(u) = W_u$ and rearranging terms, the second order ODE is reduced to

$$y^{-2}\frac{dy}{du} = (\eta - \beta_1^*(u))y^{-1} - \epsilon, \tag{78}$$

which is in the form of Bernoulli differential equation. Furthermore, setting $z(u) = 1/y(u)$ yields

$$\frac{dz}{du} = (\beta_1^*(u) - \eta)z + \epsilon,$$

and thus has the following general solution

$$z(u) = e^{\int_{u_1}^u (\beta_1^*(w)-\eta)dw}\left(C_1 + \epsilon \int_{u_1}^u e^{\int_{u_1}^y (\eta-\beta_1^*(w)dw)} dy\right),$$

from which we obtain

$$W_1(u) = \int_{u_1}^u \frac{e^{\int_{u_1}^y (\eta-\beta_1^*(w))dw}}{C_1 + \epsilon \int_{u_1}^y e^{\int_{u_1}^v (\eta-\beta_1^*(w))dw} dv} dy + C_2.$$

Here, $C_1$ and $C_2$ are constants to be determined. When $-\infty < u \leq u_1$, the function $W_2(u) = W(u)$ satisfies the following ODE:

$$\left(ru + c - \lambda\mathbb{E}Y\right)W_u(u) + \frac{1}{2}\lambda\mathbb{E}Y^2\left(\epsilon W_u^2(u) + W_{uu}(u)\right) = 0.$$

Along the same lines, one can obtain

$$W_2(u) = \int_{-\infty}^{u} \frac{e^{\int_{-\infty}^{y} -\beta_2^*(w)dw}}{C_3 + \epsilon \int_{-\infty}^{y} e^{\int_{-\infty}^{v} -\beta_2^*(w)dw} dv} dy + C_4,$$

where $C_3$ and $C_4$ are constants to be determined. By using the boundary conditions in (39) and the smooth-fit conditions, that is,

$$W_1(u_1) = W_2(u_1), \qquad W_1'(u)|_{u=u_1} = W_2'(u)|_{u=u1},$$

we get

$$\begin{cases} C_4 = 1, \quad C_2 = -\int_{u_1}^{u_s} \frac{e^{\int_{u_1}^{y} (\eta - \beta_1^*(w))dw}}{C_1 + \epsilon \int_{u_1}^{y} e^{\int_{u_1}^{v} (\eta - \beta_1^*(w))dw} dv} dy, \\ C_2 = \int_{-\infty}^{u_1} \frac{e^{\int_{-\infty}^{y} -\beta_2^*(w)dw}}{C_3 + \epsilon \int_{-\infty}^{y} e^{\int_{-\infty}^{v} -\beta_2^*(w)dw} dv} dy + C_4, \\ \frac{1}{C_1} = \frac{e^{\int_{-\infty}^{u_1} -\beta_2^*(w)dw}}{C_3 + \epsilon \int_{-\infty}^{u_1} e^{\int_{-\infty}^{v} -\beta_2^*(w)dw} dv}. \end{cases} \tag{79}$$

Using these results and Lemma 5, we can show that $W$ equals the right-hand side of (43). In particular, due to the fact that $C_1$ satisfies conditions (42), it is straightforward to show that $W$ is a non-increasing function with bounded first derivative.

Besides, because of $\eta - \beta_1^*(u_1) = -\beta_2^*(u_1) = 0$, it is not difficult to verify that $W_1''(u)|_{u=u_1} = W_2''(u)|_{u=u_1}$. Therefore, the function $W$ satisfies all the conditions of Theorem 1. As a result, the probability of absolute ruin $V(u)$ is given by (43), the optimal robust reinsurance strategy is given by (45), and the optimal probability distortion function is given by (46). □

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