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Optimal reinsurance to minimize the probability of drawdown under the mean-variance premium principle

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ABSTRACT

In this paper, we determine the optimal reinsurance strategy to minimize the probability of drawdown, namely, the probability that the insurer's surplus process reaches some fixed fraction of its maximum value to date. We assume that the reinsurance premium is computed according to the mean-variance premium principle, a combination of the expected-value and variance premium principles. We derive closed-form expressions of the optimal reinsurance strategy and the corresponding minimum probability of drawdown. Then, under the variance premium principle, we show that the safe level can never be reached before drawdown under the optimally controlled surplus process. Finally, we present some numerical examples to show the impact of model parameters on the optimal results.

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Stochastic optimal control; probability of drawdown; optimal reinsurance; mean-variance premium principle

1. Introduction

In recent decades, optimal reinsurance problems have gained much interest in actuarial research. Much of the literature considers problems under which the insurance company is constrained to buy either pure quota-share reinsurance, pure excess-of-loss reinsurance, or a combination of the two. Zhang et al. (2007) minimized the probability of ruin by finding the optimal combination of quota-share and excess-of-loss reinsurance. Liang & Guo (2011) maximized expected exponential utility of terminal wealth, also using a combination of quota-share and excess-of-loss reinsurance. Bai et al. (2013) found the optimal excess-of-loss reinsurance to minimize the ruin probability for the diffusion approximation risk model. Under the mean-variance criterion, Liang et al. (2016) found the optimal proportional reinsurance and investment strategies in a financial market with a risky asset whose price process follows a jump-diffusion, in which the jumps in both the risky asset and insurance risk process are correlated through a common shock.

Other researchers have found optimal reinsurance strategies for various optimization problems without restricting the form of the reinsurance. For example, under the criterion of maximizing expected utility of terminal wealth and minimizing the probability of ruin, Zhang et al. (2016) investigated an optimal investment and reinsurance problem in which the insurer purchased a general reinsurance policy, in which reinsurance is priced according to the mean-variance premium principle, as in this paper. Liang & Young (2018) computed the optimal investment and per-loss reinsurance strategies for an insurance company facing a compound Poisson claim process; they assumed that the reinsurer used an expected-value premium principle.

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The expected-value principle is often used as a reinsurance premium principle due to its simplicity. See, for example, Bai & Zhang (2008), Liang & Bayraktar (2014), Han et al. (2018), and the references therein. Although the variance principle is a second important premium principle, it is not often used in a dynamic setting. Under the variance premium principle, Zhou & Yuen (2012) studied an optimal dividend and capital injection problem with reinsurance. Liang & Yuen (2016) considered the objective of maximizing expected exponential utility; they derived the optimal reinsurance strategy under the variance premium principle not only for the diffusion approximation risk model but also for the compound Poisson risk model. Hipp & Taksar (2010) determined optimal reinsurance under both the expected-value and the variance premium principles, separately, for the diffusion approximation risk model.

The mean-variance premium principle combines the expected-value and variance premium principles; therefore, it is more general than either and includes each as a special case. Under the mean-variance premium principle, Zhang et al. (2016) studied optimal investment and reinsurance problems, and Chen et al. (2018) studied a stochastic differential game between two insurers who invest in a financial market and adopt reinsurance to manage their claim risks.

Drawdown, which measures decline of the value of a portfolio from its historic high-water mark, is a frequently quoted risk metric to evaluate the performance of portfolio managers in the fund-management industry. A significant drawdown not only indicates large losses but may also trigger a long-term recession. In addition, investors tend to assess their investment success by comparing their current portfolio value to its historical maximum value.

Researchers are, thereby, motivated to study the optimization problem of minimizing the so-called *probability of drawdown*. Specifically, a decision-maker chooses a strategy that minimizes the probability that the value of surplus reaches some fixed proportion, say, $\alpha \in [0, 1)$, of its maximum value to date. It is more reasonable for the insurer to consider drawdown as a criterion instead of ruin, for which the wealth drops below a *fixed* level, namely, 0. Note that minimizing the probability of drawdown includes minimizing the probability of ruin as a special case by setting $\alpha = 0$.

Angoshtari et al. (2016b) and Han et al. (2018) minimized the probability of drawdown over an infinite-time horizon and showed that the strategy minimizing the probability of ruin also minimizes the probability of drawdown, where the relevant domains coincide. Also, Chen et al. (2015) and Angoshtari et al. (2016a) minimized the probability of lifetime drawdown for an individual investor. They found that the optimal strategy for a random (or finite) maturity setting such as lifetime drawdown is different from that of the corresponding ruin-minimization problem. In other research involving drawdown, such as Grossman & Zhou (1993), Cvitanić & Karatzas (1995), and Elie & Touzi (2008), drawdown is used as a constraint associated with maximizing expected utility of consumption and terminal wealth.

In this paper, under the criterion of minimizing the probability of drawdown, we find the optimal (general) reinsurance strategy under the mean-variance premium principle and discover that the optimal reinsurance strategy is *identical* to the one for minimizing the probability of ruin. We did not expect this coincidence *a priori*; Bäuerle & Bayraktar (2014) show that the two strategies are the same under a certain condition, but that condition does not hold for our problem. As in Zhang et al. (2016), we observe that the optimal reinsurance strategy is in the form of pure excess-of-loss reinsurance strategy under the expected-value premium principle; similarly, under the variance premium principle, the optimal reinsurance strategy is in the form of the pure quota-share reinsurance. Both of these results are expected from the work of Hipp & Taksar (2010). Furthermore, we investigate the behavior of the surplus process and prove that, under the variance premium principle, optimally controlled surplus never reaches the safe level u_s before drawdown.

The rest of the paper is organized as follows. In Section 2, we present the model and the optimization problem. In Section 3, under the mean-variance premium principle, we derive explicit expressions for the optimal strategy and the corresponding minimum probability of drawdown (Theorem 3.2), and we prove some properties of the optimal reinsurance strategy. In Section 4, we conclude the paper by presenting some numerical examples which show the impact of the model's parameters on the optimal results.

2. Model and problem formulation

Let $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}, \mathbb{P})$ be a probability space containing all the objects defined in the following. We first introduce the classical Cramér-Lundberg risk model for the uncontrolled surplus process $X = \{X_t\}_{t \geq 0}$:

$$X_t = u + ct - \sum_{i=1}^{N_t} Y_i, \tag{1}$$

in which $X_0 = u \geq 0$ is the initial surplus and c is the premium rate. Moreover, $N = \{N_t\}_{t \geq 0}$ is a homogeneous Poisson process with intensity $\lambda > 0$, Y_i represents the size of the ith claim, and the claim sizes $Y_1, Y_2, \ldots$ are independent and identically distributed, positive random variables, independent of N. Here, we assume that F_Y is the common cumulative distribution function of Y_i, $i = 1, 2, \ldots$, with $F_Y(0) = 0$ and $0 < F_Y(y) < 1$ for $y > 0$; thus, ess sup $Y = \infty$. Assume that $\mathbb{E}Y < \infty$ and $\mathbb{E}(Y^2) < \infty$.

In this paper, we suppose that the insurer can reinsure its claims with per-loss reinsurance via a continuously payable premium, computed according to the mean-variance premium principle. Let R_t denote the retained claim at time t, then $y - R_t$ is the amount of each claim transferred to the reinsurer. The mean-variance premium rate at time t associated with R_t is given by

$$(1+\theta)\lambda\mathbb{E}\big(Y - R_t\big) + \frac{\eta}{2}\lambda\mathbb{E}\big((Y - R_t)^2\big), \tag{2}$$

in which θ and η are the non-negative risk loading parameters. If $\theta = 0$, then (2) reduces to the variance premium principle; similarly, if $\eta = 0$, then (2) reduces to the expected-value premium principle.

Let $U = \{U_t\}_{t \geq 0}$ denote the surplus process associated with a retention strategy $R = \{R_t\}_{t \geq 0}$, that is, U_t is the surplus of the insurer at time t under the strategy R. Assume that the insurer invests all its surplus in a risk free asset (bond or bank account) with interest rate $r > 0$. Thus, the controlled surplus process has the following dynamics:

$$\mathrm{d}U_t = \left[rU_t + c - (1+\theta)\lambda\mathbb{E}\big(Y - R_t\big) - \frac{\eta}{2}\lambda\mathbb{E}\big((Y - R_t)^2\big)\right]\mathrm{d}t - R_t\,\mathrm{d}N_t. \tag{3}$$

To avoid mathematical trivialities, we assume that

$$\lambda\mathbb{E}Y < c < (1+\theta)\lambda\mathbb{E}Y + \frac{\eta}{2}\lambda\mathbb{E}\big(Y^2\big),$$

that is, the insurer's premium income is greater than the expected value of the claims but less than the premium for full reinsurance. Let κ denote the positive difference

$$\kappa = (1+\theta)\lambda\mathbb{E}Y + \frac{\eta}{2}\lambda\mathbb{E}\big(Y^2\big) - c. \tag{4}$$

A retention strategy $R = \{R_t\}_{t \geq 0}$ is said to be *admissible* if (i) $t \mapsto R_t$ is left-continuous and adapted to the filtration $\mathbb{F}$, (ii) R_t is a function of the possible claim size $Y = y$ at time t, that is, we can write $R_t = R_t(y)$, (iii) $0 \leq R_t(y) \leq y$, for all $t \geq 0$ and $y \geq 0$, and (iv) $R_t \equiv 0$ for all $t \geq t'$ if $U_{t'} \geq \kappa/r$. (In the proof of the Verification Theorem, condition (iv) makes the argument easier.) Let $\mathcal{D}$ denote the set of all admissible strategies.

We solve our optimization problem by approximating the jump process in (3) with a diffusion, as in Grandell (1991). We justify this diffusion approximation via the following argument: From

Lemma 7.2.2 in Lamberton & Lapeyre (2000), because Y's second moment is finite and $0 \leq R_t(y) \leq y$ for all $t \geq 0$ and $y \geq 0$, we deduce that the process $\mathcal{N} = \{\mathcal{N}_t\}_{t\geq 0}$ defined by

$$\mathcal{N}_t = \int_0^t \frac{R_s\,\mathrm{d}N_s - \lambda\mathbb{E}R_s\,\mathrm{d}s}{\sqrt{\lambda\mathbb{E}\left(R_s^2\right)}}$$

is a square-integrable martingale with $\mathbb{E}(\mathcal{N}_t^2) = t$ for all $t \geq 0$. Furthermore, if $\{R_t\}$ is uniformly bounded, then, as λ goes to infinity, $\mathcal{N}$ converges in law to B, in which $B = \{B_t\}_{t\geq 0}$ is a standard Brownian motion; see, for example, Theorem 7.1.4 in Ethier & Kurtz (2005). Although we do not assume that $\{R_t\}$ is uniformly bounded, it follows that, for large values of $L > 0$ and $\lambda > 0$,

$$\frac{(R_t \wedge L)\,\mathrm{d}N_t - \lambda\mathbb{E}(R_t \wedge L)\,\mathrm{d}t}{\sqrt{\lambda\mathbb{E}\left((R_t \wedge L)^2\right)}} \approx \mathrm{d}B_t,$$

in which $\approx$ indicates that the processes are approximate in law, with convergence as λ goes to ∞. Thus, we use the following approximation in this paper:

$$R_t\,\mathrm{d}N_t \approx \lambda\mathbb{E}R_t\,\mathrm{d}t - \sqrt{\lambda\mathbb{E}\left(R_t^2\right)}\,\mathrm{d}B_t, \tag{5}$$

which implies that the resulting process $\hat{U} = \{\hat{U}_t\}_{t\geq 0}$ evolves according to the dynamics

$$\mathrm{d}\hat{U}_t = \left[r\hat{U}_t - \kappa + \theta\lambda\mathbb{E}\left(R_t\right) + \eta\lambda\mathbb{E}\left(YR_t\right) - \frac{\eta}{2}\lambda\mathbb{E}\left(R_t^2\right)\right]\mathrm{d}t + \sqrt{\lambda\mathbb{E}\left(R_t^2\right)}\,\mathrm{d}B_t, \tag{6}$$

with initial surplus $\hat{U}_0 = u$.[1]

Note that if the surplus is greater than or equal to

$$u_s = \frac{\kappa}{r}, \tag{7}$$

with κ defined in (4), then the insurer can buy full reinsurance via income from the riskless asset and surplus will never drop below its current value. For this reason, we call u_s the *safe level*, and we see that condition (iv) in the definition of admissible retention strategies is not restrictive.

Define the maximum surplus process $M = \{M_t\}_{t\geq 0}$ by

$$M_t = \max\left\{\sup_{0\leq s\leq t} \hat{U}_s, M_{0-}\right\}, \tag{8}$$

with $M_0 = m \geq u$. We allow the surplus process to have a financial past, as embodied by the term M_{0-} in (8). Drawdown is the time when the surplus process reaches $\alpha \in [0, 1)$ times its maximum value, that is, at the hitting time τ_α given by

$$\tau_\alpha = \inf\{t \geq 0 : \hat{U}_t \leq \alpha M_t\}. \tag{9}$$

If $\alpha = 0$, then drawdown is the same as ruin for the ruin level 0; thus, minimizing the probability of drawdown generalizes the problem of minimizing the probability of ruin.

[1] Even though we are using the diffusion approximation of the compound Poisson model, one can consider the random claim severity Y as existing in reality. In other words, we use the diffusion approximation to obtain the optimal retention strategy for that model, but an insurer could apply the obtained per-loss reinsurance strategy to claims within the original compound Poisson model. See Liang & Young (2018) for numerical comparisons of the optimal retention strategies under the diffusion and compound Poisson models when reinsurance is priced according to the expected-value premium principle with the optimal strategy equal to excess-of-loss reinsurance. They showed that the optimal deductibles are close under the two models when the surplus is not close to ruin, even when λ is small.

The minimum probability of drawdown ϕ is defined by

$$\phi(u,m) = \inf_{R\in\mathcal{D}} \mathbb{P}^{u,m}(\tau_\alpha < \infty) = \inf_{R\in\mathcal{D}} \mathbb{E}^{u,m}\left(\mathbf{1}_{\{\tau_\alpha<\infty\}}\right), \tag{10}$$

in which $\mathbb{P}^{u,m}$ and $\mathbb{E}^{u,m}$ denote the probability and expectation, respectively, conditional on $\hat{U}_0 = u$ and $M_0 = m$. Here, we minimize over admissible strategies in $\mathcal{D}$, in which we replace condition (iv) with $R_t \equiv 0$ for all $t \geq t'$ if $\hat{U}_{t'} \geq u_s$. Note that, if $u \leq \alpha m$, then $\phi(u,m) = 1$, and if $u \geq u_s$ and $u > \alpha m$, then $\phi(u,m) = 0$. It remains for us to determine the minimum probability of drawdown ϕ on the domain

$$\mathcal{O} = \left\{(u,m) \in (\mathbb{R}^+)^2 : \alpha m \leq u \leq \min(m, u_s), \alpha m < u_s\right\}, \tag{11}$$

and that is the topic of the next section.[2]

3. Minimizing the probability of drawdown

In this section, for the risk model in (6), we compute the optimal reinsurance strategy to minimize the probability of drawdown on region $\mathcal{O}$ given in (11). To that end, in Section 3.1 we prove a verification theorem, which we use to find ϕ on $\mathcal{O}$. Then, in Sections 3.2 and 3.3, we find candidates for ϕ when $m \geq u_s$ and $m < u_s$, respectively. Finally, in Section 3.4, we combine the two candidates for ϕ into one function for all $m > 0$ with $\alpha m < u_s$; see Theorem 3.2.

3.1. Verification theorem

For a given retention function R such that $0 \leq R(y) \leq y$, define the differential operator $\mathcal{A}^R$ on appropriately differentiable functions by

$$\mathcal{A}^R h(u,m) = \left[ru - \kappa + \theta\lambda\mathbb{E}R + \eta\lambda\mathbb{E}(YR) - \frac{\eta}{2}\lambda\mathbb{E}\left(R^2\right)\right] h_u + \frac{1}{2}\lambda\mathbb{E}\left(R^2\right) h_{uu}. \tag{12}$$

The right side of (12) depends on the maximum surplus m only via h's dependence on m.

The proof of the verification theorem below follows readily from the corresponding proofs given in Chen et al. (2015) and Angoshtari et al. (2016a, 2016b), but for completeness, we include it.

Theorem 3.1 (Verification Theorem): *Suppose $h : \mathcal{O} \to [0,1]$ is a bounded, continuous function, which satisfies the following conditions:*

(i) $h(\cdot, m) \in \mathcal{C}^2((\alpha m, \min(m, u_s)))$ *is a non-increasing, convex function with bounded first derivative,*
(ii) $h(u, \cdot)$ *is continuously differentiable, except possibly at* u_s,
(iii) $h_m(m,m) \geq 0$ *if* $m < u_s$,
(iv) $h(\alpha m, m) = 1$,
(v) $h(u_s, m) = 0$ *if* $m \geq u_s$,
(vi) $\mathcal{A}^R h \geq 0$ *for all* $R \in \mathcal{D}$.

Then, $h \leq \phi$ *on* $\mathcal{O}$.

Furthermore, suppose that the function h satisfies all the above conditions in such a way that the inequalities in conditions (iii) *and* (vi) *hold with equality for some admissible strategy R^* defined in feedback form via $R^*_t = R^*(\hat{U}_t, M_t, y)$, such that* (6) *has a strong solution.*[3] *Then, $h = \phi$ on $\mathcal{O}$, and R^* is an optimal retention strategy.*

[2] Note that if $\alpha m = u_s$, then technically drawdown has occurred, but the insurer could keep its surplus at $\alpha m = u_s$ thereafter by purchasing full reinsurance. Therefore, we avoid this ambiguous case by assuming $\alpha m < u_s$ throughout.

[3] If $m \geq u_s$, then condition (iii) is moot, and we only require equality in condition (vi).

Proof: Assume that h satisfies the conditions specified in the statement of this theorem, and fix an admissible retention strategy R. Let $\hat{U}^R$ and M^R denote the surplus and the maximum surplus processes, respectively, when the insurer uses R. We split the problem into two parts: (1) $u_s \le m$, and (2) $m < u_s$.

Case 1: $u_s \le m$. Let $b \in (\alpha m, u_s)$, and define h^b by

$$h^b(u,m) = \frac{h(u,m) - h(b,m)}{1 - h(b,m)},$$

for $\alpha m \le u \le b$. Then, $h^b(\cdot, m)$ decreases from 1 to 0 as u increases from αm to b, and h^b satisfies conditions (i), (ii), and (vi) in the statement of the theorem.[4]

Also, define $\tau_b = \inf\{t \ge 0 : \hat{U}_t^R \ge b\}$, and $\tau_{\alpha b} = \tau_\alpha \wedge \tau_b$. In Lemma A.1 in the Appendix, we prove that $\tau_{\alpha b} < \infty$ with probability 1. Note that $M_t^R = m$ for all $0 \le t \le \tau_{\alpha b}$ because $b < u_s \le m$. By applying Itô's formula to $h^b(u,m)$, we have

$$\begin{aligned} h^b\big(\hat{U}^R_{\tau_{\alpha b}}, m\big) &= h^b(u,m) + \int_0^{\tau_{\alpha b}} \mathcal{A}^R h^b\big(\hat{U}_t^R, m\big)\,\mathrm{d}t \\ &\quad + \int_0^{\tau_{\alpha b}} h_u^b\big(\hat{U}_t^R, m\big)\sqrt{\lambda \mathbb{E}\big(R_t^2\big)}\,\mathrm{d}B_t + \int_0^{\tau_{\alpha b}} h_m^b\big(\hat{U}_t^R, m\big)\,\mathrm{d}M_t^R. \end{aligned} \tag{13}$$

The first integral in (13) is non-negative because of condition (vi) of the theorem. The expectation of the second integral equals 0 because h_u^b and $\sqrt{\lambda \mathbb{E}(R_t^2)}$ are bounded. Also, the third integral equals zero because $dM_t^R \equiv 0$ when $0 \le t \le \tau_{\alpha b}$ because $b < u_s \le m$. Thus, by taking expectations in (13), we have

$$\mathbb{E}^{u,m}\left[h^b\big(\hat{U}^R_{\tau_{\alpha b}}, m\big)\right] \ge h^b(u,m). \tag{14}$$

Because $\tau_{\alpha b} < \infty$ with probability 1, it follows from condition (iv) of the theorem, from the definition of h^b, and from inequality (14) that

$$\begin{aligned} h^b(u,m) &\le \mathbb{P}^{u,m}\big(\hat{U}^R_{\tau_{\alpha b}} = \alpha m\big) \cdot 1 + \mathbb{P}^{u,m}\big(\hat{U}^R_{\tau_{\alpha b}} = b\big) \cdot 0 \\ &= \mathbb{P}^{u,m}\big(\tau_\alpha < \tau_b\big) \cdot 1 + \mathbb{P}^{u,m}\big(\tau_\alpha > \tau_b\big) \cdot 0 \\ &= \mathbb{P}^{u,m}\big(\tau_\alpha < \tau_b\big) \le \mathbb{P}^{u,m}(\tau_\alpha < \infty), \end{aligned} \tag{15}$$

In the second line of (15), we purposefully omit $\mathbb{P}^{u,m}(\tau_\alpha = \tau_b)$ because that probability equals 0, which follows from $\mathbb{P}^{u,m}(\tau_{\alpha b} < \infty) = 1$ and $\alpha m < b$.

We now show that $\lim_{b \to (u_s)-}\{\tau_\alpha < \tau_b\} = \bigcup_{b \in (\alpha m, u_s)}\{\tau_\alpha < \tau_b\} = \{\tau_\alpha < \tau_s\}$, in which $\tau_s = \inf\{t \ge 0 : \hat{U}_t^R \ge u_s\}$, and we think of this limit in terms of sets of events. Because $\tau_b = \inf\{t \ge 0 : \hat{U}_t^R \ge b\}$, as b increases, the hitting time τ_b increases. Thus, we have

$$\lim_{b \to (u_s)-}\{\tau_\alpha < \tau_b\} = \bigcup_{b \in (\alpha m, u_s)}\{\tau_\alpha < \tau_b\} \subseteq \{\tau_\alpha < \tau_s\}. \tag{16}$$

To obtain the opposite inclusion, suppose $\omega \in \{\tau_\alpha < \tau_s\}$; then, among other things, τ_α occurs in finite time, say, at time t. Thus, $M_t' < u_s$, in which $M_t' = \max_{0 \le s \le t} \hat{U}_s^R$, which implies that $\omega \in \{\tau_\alpha < \tau_b\}$

[4] Note that condition (iii) is moot in this case because $u_s \le m$.

for all $b \in (M_t', u_s)$, and we have

$$\{\tau_\alpha < \tau_s\} \subseteq \bigcup_{b \in (\alpha m, u_s)} \{\tau_\alpha < \tau_b\} = \lim_{b \to (u_s)-} \{\tau_\alpha < \tau_b\}. \tag{17}$$

Equations (16) and (17) imply

$$\lim_{b \to (u_s)-} \{\tau_\alpha < \tau_b\} = \bigcup_{b \in (\alpha m, u_s)} \{\tau_\alpha < \tau_b\} = \{\tau_\alpha < \tau_s\}. \tag{18}$$

By applying the Dominated Convergence Theorem to (15) as we take the limit $b \to (u_s)-$ and by taking (18) into account, we obtain

$$h(u, m) \leq \mathbb{P}^{u,m}(\tau_\alpha < \tau_s) \leq \mathbb{P}^{u,m}(\tau_\alpha < \infty). \tag{19}$$

Moreover, we assert that $\{\tau_\alpha < \tau_s\} = \{\tau_\alpha < \infty\}$. Indeed, the inclusion $\{\tau_\alpha < \tau_s\} \subseteq \{\tau_\alpha < \infty\}$ is clear. To show the opposite inclusion, suppose $\omega \in \{\tau_\alpha < \infty\}$, and suppose, on the contrary, that $\omega \in \{\tau_s \leq \tau_\alpha\}$. Then, because $\omega \in \{\tau_s \leq \tau_\alpha < \infty\}$, it follows that $\omega \in \{\tau_\alpha = \infty\}$ because surplus reached the safe level in finite time, after which drawdown cannot occur. (Recall that, after time τ_s, any admissible retention strategy R is identically 0, which implies surplus remains at u_s and drawdown cannot occur.) Thus, we have a contradiction to our supposition that $\omega \in \{\tau_s \leq \tau_\alpha\}$, and we conclude that $\omega \in \{\tau_\alpha < \tau_s\}$. We have shown that $\{\tau_\alpha < \tau_s\} = \{\tau_\alpha < \infty\}$, and (19) becomes

$$h(u, m) \leq \mathbb{P}^{u,m}(\tau_\alpha < \tau_s) = \mathbb{P}^{u,m}(\tau_\alpha < \infty). \tag{20}$$

By taking the infimum over admissible strategies, we obtain $h \leq \phi$ on $\mathcal{O}$.

Finally, if we have equality in the inequalities in conditions (iii) and (vi) under an admissible strategy R^* in feedback form, then from the above proof, we have equality in (14), in the first inequality in (15), and in (20) when $R = R^*$. Thus, $h = \phi$ on $\mathcal{O}$, and R^* is an optimal retention strategy.

Case 2: $m < u_s$. Let $b \in (m, u_s)$, and define h^b by

$$h^b(u, m) = \frac{h(u, m) - h(b, b)}{1 - h(b, b)},$$

for $\alpha m \leq u \leq m < b$. Then, $h^b(\alpha m, m) = 1$, $h^b(b-, b-) = 0$, and h^b satisfies conditions (i), (ii), (iii), and (vi) in the statement of the theorem. Define τ_b and $\tau_{\alpha b}$ as in Case 1, and recall that $\tau_{\alpha b} < \infty$ with probability 1. Because $m < b$, then hitting b requires the maximum surplus to increase so that the drawdown level will also increase.

By applying Itô's formula to $h^b(u, m)$, we have

$$\begin{aligned} h^b\left(\hat{U}^R_{\tau_{\alpha b}}, M^R_{\tau_{\alpha b}}\right) &= h^b(u, m) + \int_0^{\tau_{\alpha b}} \mathcal{A}^R h^b\left(\hat{U}^R_t, M^R_t\right) \mathrm{d}t \\ &\quad + \int_0^{\tau_{\alpha b}} h^b_u\left(\hat{U}^R_t, M^R_t\right) \sqrt{\lambda \mathbb{E}(R_t^2)}\, \mathrm{d}B_t + \int_0^{\tau_{\alpha b}} h^b_m\left(\hat{U}^R_t, M^R_t\right) \mathrm{d}M^R_t. \end{aligned} \tag{21}$$

The first integral in (21) is non-negative because of condition (vi) of the theorem. The expectation of the second integral equals 0 because h^b_u and $\sqrt{\lambda \mathbb{E}(R_t^2)}$ are bounded. Also, the third integral is non-negative almost surely because $\mathrm{d}M^R_s$ is non-zero only when $M^R_s = \hat{U}^R_s$ and $h^b_m(m, m) \geq 0$ by condition (iii). Here, we also used the fact that M^R is non-decreasing; therefore, the first variation process associated with it is finite almost surely, and we conclude that the cross variation of M^R and $\hat{U}^R$ is zero almost surely. Thus, by taking expectations in (21), we have

$$\mathbb{E}^{u,m}\left[h^b\left(\hat{U}^R_{\tau_{\alpha b}}, M^R_{\tau_{\alpha b}}\right)\right] \geq h^b(u, m). \tag{22}$$

The remainder of the proof for Case 2 closely follows the proof for Case 1, so we omit it. ■

Remark 3.1: Because Hamilton-Jacobi-Bellman equation that results from $\min_R \mathcal{A}^R\phi = 0$ is independent of αm and u_s, the optimal reinsurance strategy which maximizes the probability of reaching the upper level $b < u_s$ before drawdown is identical to the one which minimizes the probability of drawdown. Taksar & Markussen (2003) observe a similar phenomenon in their setting; see their Remark 2.1.

In the next two sections, we use Theorem 3.1 to determine the minimum probability of drawdown.

3.2. Probability of drawdown when $m \geq u_s$

In this section, we consider the case for which $m \geq u_s$; recall, from the definition of the domain $\mathcal{O}$ in (11), that, in this case, we must have $u \leq u_s$. We hypothesize that optimally controlled surplus $\hat{U}^*$ is such that $\hat{U}^*_t \leq u_s$ with probability 1 for all $t \geq 0$; thus, the maximum level of surplus does not increase. Under this hypothesis, minimizing the probability of drawdown is equivalent to minimizing the probability of ruin with ruin level αm. More formally, we propose the following ansatz.

Ansatz 3.1: When $m \geq u_s$, we hypothesize that it is optimal for the insurer to purchase reinsurance as if it were minimizing the probability of ruin with ruin level αm. Pestien & Sudderth (1985) showed that, when controlling a diffusion towards a goal, the optimal strategy is the one that maximizes the drift divided by the square of the volatility.

In the following lemma, we determine the retention function that maximizes the drift of $\hat{U}$ divided by the square of its volatility, namely,

$$f(u, R) = \frac{ru - \kappa + \lambda \left(\theta \mathbb{E}R + \eta \mathbb{E}(YR) - \frac{\eta}{2}\mathbb{E}\big(R^2\big)\right)}{\mathbb{E}\big(R^2\big)}, \tag{23}$$

in which we ignore the factor of λ in the denominator.

Lemma 3.1: *Suppose $u \leq u_s \leq m$. Then, the retention function $R^* = R^*(u, y)$ that maximizes f in (23) is given by*

$$R^*(u, y) = \frac{\theta + \eta y}{\beta^*(u)} \wedge y, \tag{24}$$

in which $\beta^* > \eta$ uniquely solves

$$\theta \mathbb{E}R + \eta \mathbb{E}(YR) - \frac{\beta}{2}\mathbb{E}\big(R^2\big) = \frac{\kappa - ru}{\lambda}, \tag{25}$$

with $R = R^*$ given in (24).

Proof: To maximize f in (23), first fix $\mathbb{E}(R^2) = s^2 \in [0, \sigma^2]$, in which $\mathbb{E}(Y^2) = \sigma^2$. Then, maximizing f, subject to the restrictions $\mathbb{E}(R^2) = s^2$ and $0 \leq R(y) \leq y$, is equivalent to maximizing $\theta \mathbb{E}R + \eta \mathbb{E}(YR)$, with $\mathbb{E}(R^2) = s^2$ and $0 \leq R(y) \leq y$. To that end, we define the Lagrangian $\mathcal{L}$ by

$$\mathcal{L}(R) = \theta \mathbb{E}R + \eta \mathbb{E}(YR) - \frac{\beta}{2}\left(\mathbb{E}\big(R^2\big) - s^2\right),$$

in which $\beta \geq 0$ is the Lagrange multiplier. By using the cumulative distribution function of Y, we rewrite $\mathcal{L}(R)$ as follows.

$$\mathcal{L}(R) = \int_0^\infty \left[\theta R(y) + \eta y R(y) - \frac{\beta}{2} R^2(y)\right] \mathrm{d}F_Y(y) + \frac{\beta}{2} s^2.$$

From this integral representation of $\mathcal{L}(R)$, we deduce that we can maximize $\mathcal{L}(R)$ by maximizing the integrand y-by-y, subject to $0 \leq R(y) \leq y$. As a function of $R(y)$, the integrand is a parabola, so it is maximized by

$$R^*(y) = \frac{\theta + \eta y}{\beta} \wedge y.$$

Next, we show that, given $s^2 \in [0, \sigma^2]$, there exists a unique value of $\beta \geq \eta$ such that

$$s^2 = \mathbb{E}\left(\left(\frac{\theta + \eta Y}{\beta}\right)^2 \wedge Y^2\right),$$

or equivalently,

$$s^2 = 2\int_0^{\theta/(\beta-\eta)} y S_Y(y)\,\mathrm{d}y + \frac{2\eta}{\beta^2}\int_{\theta/(\beta-\eta)}^{\infty} (\theta + \eta y) S_Y(y)\,\mathrm{d}y, \tag{26}$$

in which $S_Y = 1 - F_Y$. It is straightforward to show that the right side of (26) decreases from σ^2 to 0 as β increases from η to ∞. It follows that (26) has a unique solution $\beta \geq \eta$.

Thus, we have reduced the infinite-dimensional problem of finding a function $R(y)$ to maximize f in (23) to the one-dimensional problem of finding the optimal value of s^2, or equivalently, of finding the optimal value of $\beta \geq \eta$ because the above argument shows that there is a one-to-one correspondence between $s^2 \in [0, \sigma^2]$ and $\beta \geq \eta$. Thus, we rewrite f and slightly abuse notation by replacing its argument R with β.

$$f(u, \beta) = \frac{ru - \kappa + \lambda\left(\theta g_1(\beta) + \eta g_2(\beta) - \frac{\eta}{2} g_3(\beta)\right)}{g_3(\beta)}, \tag{27}$$

in which

$$g_1(\beta) = \mathbb{E}R = \int_0^{\theta/(\beta-\eta)} S_Y(y)\,\mathrm{d}y + \frac{\eta}{\beta}\int_{\theta/(\beta-\eta)}^{\infty} S_Y(y)\,\mathrm{d}y, \tag{28}$$

$$g_2(\beta) = \mathbb{E}(YR) = 2\int_0^{\theta/(\beta-\eta)} y S_Y(y)\,\mathrm{d}y + \frac{1}{\beta}\int_{\theta/(\beta-\eta)}^{\infty} (\theta + 2\eta y) S_Y(y)\,\mathrm{d}y, \tag{29}$$

and

$$g_3(\beta) = \mathbb{E}\big(R^2\big) = 2\int_0^{\theta/(\beta-\eta)} y S_Y(y)\,\mathrm{d}y + \frac{2\eta}{\beta^2}\int_{\theta/(\beta-\eta)}^{\infty} (\theta + \eta y) S_Y(y)\,\mathrm{d}y. \tag{30}$$

By differentiating f in (27) with respect to β, we obtain

$$\begin{aligned}
\frac{\partial f}{\partial \beta} &\propto g_3(\beta)\big(\theta g_1'(\beta) + \eta g_2'(\beta)\big) - g_3'(\beta)\big(\theta g_1(\beta) + \eta g_2(\beta)\big) + \frac{\kappa - ru}{\lambda} g_3'(\beta) \\
&= \frac{\beta}{2} g_3(\beta) g_3'(\beta) - g_3'(\beta)\big(\theta g_1(\beta) + \eta g_2(\beta)\big) + \frac{\kappa - ru}{\lambda} g_3'(\beta) \\
&\propto \theta g_1(\beta) + \eta g_2(\beta) - \frac{\beta}{2} g_3(\beta) - \frac{\kappa - ru}{\lambda},
\end{aligned}$$

in which the second line follows from $\theta g_1'(\beta) + \eta g_2'(\beta) = (\beta/2) g_3'(\beta)$, and the third line follows from $g_3'(\beta) < 0$.

Define G by the third line above; specifically,

$$G(\beta) = \theta g_1(\beta) + \eta g_2(\beta) - \frac{\beta}{2} g_3(\beta) - \frac{\kappa - ru}{\lambda}.$$

We wish to show that G has a unique zero $\beta^* > \eta$. To that end, first, consider

$$G(\eta) = \theta \mathbb{E}Y + \eta \mathbb{E}\big(Y^2\big) - \frac{\eta}{2} \mathbb{E}\big(Y^2\big) - \frac{\kappa - ru}{\lambda} = \frac{ru + c - \lambda \mathbb{E}Y}{\lambda}.$$

Recall that we assume $c > \lambda \mathbb{E}Y$; thus, $G(\eta) > 0$. Next,

$$\lim_{\beta \to \infty} G(\beta) = -\frac{\kappa - ru}{\lambda} \le 0,$$

strictly negative for $u < \kappa/r$. Finally,

$$\begin{aligned} G'(\beta) &= \theta g_1'(\beta) + \eta g_2'(\beta) - \frac{1}{2} g_3(\beta) - \frac{\beta}{2} g_3'(\beta) \\ &= \frac{\beta}{2} g_3'(\beta) - \frac{1}{2} g_3(\beta) - \frac{\beta}{2} g_3'(\beta) = -\frac{1}{2} g_3(\beta) < 0. \end{aligned}$$

Thus, G has a unique zero $\beta^* > \eta$, from which it follows that f in (27) has a unique critical point $\beta^* > \eta$, which depends on the value of the surplus u via the term $(\kappa - ru)/\lambda$, which measures how far below the surplus is from the safe level $u_s = \kappa/r$.

Furthermore,

$$\frac{\partial f}{\partial \beta}(u, \eta) > 0 \quad \text{and} \quad \lim_{\beta \to \infty} \frac{\partial f}{\partial \beta}(u, \beta) < 0,$$

because $G(\eta) > 0$ and $\lim_{\beta \to \infty} G(\beta) < 0$, which implies that $f(u, \beta)$ is maximized at $\beta = \beta^*(u)$. Note that (25) is a restatement of $G(\beta) = 0$; thus, we have proved this lemma. ■

Remark 3.2: Note that as surplus approaches $u_s = \kappa/r$, then R^* in (24) approaches zero retention, which is equivalent to full reinsurance. Also, as surplus approaches u_s, then the probability of drawdown approaches 0, which we know from the discussion at the end of Section 2.

Via a verification theorem similar to Theorem 3.1 (see, for example, Zhang et al. 2016), if we find a smooth, decreasing, convex solution ξ of the following boundary-value problem (BVP) with ξ_u bounded, then that solution equals the minimum probability of ruin ψ with ruin level αm. For $\alpha m \le u \le u_s$,

$$\begin{cases} (ru - \kappa)\xi_u + \lambda \min\limits_{R} \left\{ \Big(\theta \mathbb{E}\big(R\big) + \eta \mathbb{E}\big(YR\big) - \frac{\eta}{2} \mathbb{E}\big(R^2\big) \Big) \xi_u + \frac{1}{2} \mathbb{E}\big(R^2\big) \xi_{uu} \right\} = 0, \\ \xi(\alpha m, m) = 1, \quad \xi(u_s, m) = 0. \end{cases} \tag{31}$$

In the next proposition, we show that the probability of ruin ψ, corresponding to the retention strategy given in Lemma 3.1 equals the minimum probability of ruin with ruin level αm.

Proposition 3.1: *If $\alpha m < u_s \le m$, then the minimum probability of ruin $\psi(u,m)$ with ruin level αm is given by*

$$\psi(u,m) = 1 - \frac{g(u,m)}{g(u_s,m)}, \quad \text{for } u \in [\alpha m, u_s], \tag{32}$$

in which g is defined by

$$g(u,m) = \int_{\alpha m}^{u} \exp\left\{-\int_{\alpha m}^{v} (\beta^*(w) - \eta)\,\mathrm{d}w\right\} \mathrm{d}v. \tag{33}$$

The associated optimal retention R^* is given in feedback form by

$$R_t^* = \frac{\theta + \eta Y}{\beta^*\big(\hat{U}_t^*\big)} \wedge Y, \tag{34}$$

for all $t \ge 0$. Here, $\beta^*(u) > \eta$ uniquely solves (25), *$\hat{U}_t^*$ is the optimally controlled surplus at time t, and Y is the possible claim size. Moreover, ψ equals the probability of drawdown associated with the retention strategy in* (34).

Proof: First, we calculate ξ, the probability of ruin under the retention strategy given in (34). Under this strategy, ξ solves the boundary-value problem

$$\big(\beta^*(u) - \eta\big)\xi_u + \xi_{uu} = 0,$$

with boundary conditions

$$\xi(\alpha m, m) = 1, \quad \xi(u_s, m) = 0,$$

and ψ in (32) equals the solution of this boundary-value problem. Furthermore, it is straightforward to show that ψ in (32) satisfies $\min_R \mathcal{L}^R \psi(u,m) = 0$ for $u \in [\alpha m, u_s]$; thus, ψ solves (31) and equals the *minimum* probability of ruin.

Additionally, ψ equals the probability of drawdown associated with the retention strategy in (34) because the probability of ruin with ruin level αm equals the probability of drawdown when $\alpha m < u_s \le m$. ∎

Remark 3.3: The minimum probability of ruin ψ in (32) satisfies conditions (i), (ii), (iv), (v), and (vi) of Theorem 3.1 for $\alpha m \le u \le u_s \le m$. In fact, condition (vi) is satisfied with equality. Thus, ψ is a candidate for the minimum probability of drawdown. (Recall that condition (iii) is moot when $m \ge u_s$.) After we find a candidate for the minimum probability of drawdown when $m < u_s$, then we will patch the two candidates together and verify that the resulting function satisfies the conditions of Theorem 3.1 on all of $\mathcal{O}$.

3.3. Probability of drawdown when $m < u_s$

In this section, we consider the case for which $m < u_s$. In this case, we hypothesize that the optimal retention is again given by (24), which allows maximum surplus to increase, thereby, allowing the drawdown level to increase. In the following proposition, we give the probability of drawdown ζ that corresponds to the retention strategy in (24).

Proposition 3.2: *If $m < u_s$, then the probability of drawdown $\zeta(u, m)$ corresponding to the retention function in* (24), *for $\alpha m \le u \le m$, equals*

$$\zeta(u, m) = 1 - \exp\left\{-\int_m^{u_s} k(y)\,\mathrm{d}y\right\} \cdot \frac{g(u, m)}{g(u_s, u_s)}, \tag{35}$$

in which

$$k(y) = \alpha\left[\frac{1}{g(y, y)} + \eta - \beta^*(\alpha y)\right]. \tag{36}$$

In (36), *g is defined in* (33) *and β^* uniquely solves (25).*

Proof: By a verification theorem similar to Theorem 3.1, if we find a classical solution ξ of the following BVP, then that solution equals the probability of drawdown ζ corresponding to the retention function in (24):

$$\begin{aligned} &\left(\beta^*(u) - \eta\right)\xi_u + \xi_{uu} = 0, \\ &\xi(\alpha m, m) = 1, \quad \xi(u_s, u_s) = 0, \\ &\xi_m(m, m) = 0, \end{aligned} \tag{37}$$

for $\alpha m \le u \le m < u_s$. It is straightforward to show that ζ in (35) solves (37); thus, it equals the desired probability of drawdown. ■

3.4. Main result and properties of the optimal retention strategy

In (35), we have our candidate minimum probability of drawdown for $\alpha m \le u \le m < u_s$ because ζ satisfies $\min_R \mathcal{L}^R \zeta(u, m) = 0$ in that region. We patch ζ together with the candidate ψ in (32) when $\alpha m \le u \le u_s \le m$ and verify that the resulting function, indeed, equals the minimum probability of drawdown on $\mathcal{O}$. In the following theorem, we do just that and present our main result of this paper.

Theorem 3.2: *The minimum probability of drawdown ϕ on $\mathcal{O} = \{(u, m) \in (\mathbb{R}^+)^2 : \alpha m \le u \le \min(m, u_s), \alpha m < u_s\}$ equals*

$$\phi(u, m) = \begin{cases} 1 - \dfrac{g(u, m)}{g(u_s, m)}, & \text{if } \alpha m \le u \le u_s \le m, \\ 1 - \exp\left\{-\displaystyle\int_m^{u_s} k(y)\,\mathrm{d}y\right\} \cdot \dfrac{g(u, m)}{g(u_s, u_s)}, & \text{if } \alpha m \le u \le m < u_s, \end{cases} \tag{38}$$

in which g is given in (33) *and k is given in* (36). *The corresponding optimal retention strategy is given by $R_t^* = R^*(\hat{U}_t^*, Y)$, in which*

$$R^*(u, y) = \frac{\theta + \eta y}{\beta^*(u)} \wedge y, \tag{39}$$

and $\beta^* > \eta$ uniquely solves

$$\int_0^{\theta/(\beta-\eta)} \left(1 + (\beta - \eta)y\right) S_Y(y)\,\mathrm{d}y + \int_{\theta/(\beta-\eta)}^{\infty} \left(1 + \frac{\beta - \eta}{\beta}(\theta + \eta y)\right) S_Y(y)\,\mathrm{d}y = \frac{c + ru}{\lambda}. \tag{40}$$

$\hat{U}_t^*$ is the optimally controlled surplus at time t, and Y is the possible claim size.

Proof: First, Equation (40) is a more explicit version of Equation (25). Second, in Remark 3.3, we noted that ϕ in (38) satisfies the conditions of Theorem 3.1 when $\alpha m \le u \le u_s \le m$; in particular, ϕ

satisfies condition (vi) with equality. Furthermore, the proof of Proposition 3.2 shows that ϕ satisfies the conditions of Theorem 3.1 when $\alpha m \le u \le m < u_s$; in particular, ϕ satisfies conditions (iii) and (vi) with equality. Also, note that the expression given in (38) is continuous at $m = u_s$, which is clear.

The remaining item to show is that, under the retention strategy defined via the retention function in (39), the stochastic differential equation (6) has a unique, strong solution. From Theorem 5.2.9 in Karatzas & Shreve (1991), this result will follow if we show that the drift and volatility of (6) under this retention strategy have bounded derivatives with respect to u. Note that the drift and volatility are bounded for any admissible retention strategy. Let h denote the drift of (6), as a function of u, under the retention strategy defined in feedback form via (39); then,

$$h(u) = ru - \kappa + \lambda \left(\theta g_1(\beta^*) + \eta g_2(\beta^*) - \frac{\eta}{2} g_3(\beta^*) \right),$$

in which $\beta^* = \beta^*(u)$ is given by (40), and g_1, g_2, and g_3 are given by (28), (29), and (30), respectively. By differentiating (40) with respect to u and simplifying the result, we obtain

$$\frac{d\beta^*}{du} = \frac{r}{\lambda g_3(\beta^*)}.$$

Then, by differentiating h and using $\theta g_1'(\beta^*) + \eta g_2'(\beta^*) = (\beta^*/2), g_3'(\beta^*)$, we obtain

$$h'(u) = r \left[1 + \frac{\beta^* - \eta}{2} \frac{g_3'(\beta^*)}{g_3(\beta^*)} \right],$$

which is bounded, except possibly as u goes to u_s, or equivalently as β^* goes to ∞, because $g_3 \to 0^+$ as $\beta^* \to \infty$. If we set $x = \theta/(\beta^* - \eta)$ in $g_3'(\beta^*)$ and $g_3(\beta^*)$, we get

$$\frac{\beta^* - \eta}{2} \frac{g_3'(\beta^*)}{g_3(\beta^*)} = \frac{\frac{-\theta}{\eta x + \theta} S_Y(x) - \frac{2\eta\theta}{(\eta x + \theta)^3} \int_x^\infty (\theta + \eta y) S_Y(y)\, dy}{\frac{2\int_0^x y S_Y(y)\, dy}{x^2} + \frac{2\eta}{(\eta x + \theta)^2} \int_x^\infty (\theta + \eta y) S_Y(y)\, dy} \to -1,$$

as $x \to 0^+$. Thus, $h'(u) \to 0$ as $u \to u_s^-$, and we deduce that $h'(u)$ is bounded on $[0, u_s]$. Similarly, one can show that the volatility of (6) has bounded derivative. Thus, the drift and volatility of the optimally controlled diffusion approximation of the surplus process are bounded and Lipschitz, and we are done. ■

In the following corollary, we note that $\phi_m \le 0$ and that R^* is independent of m. This corollary is easy to demonstrate, so we omit its proof.

Corollary 3.1: *The minimum probability of drawdown decreases with respect to m, strictly if* $u < \min(m, u_s)$, *and the optimal retention strategy is independent of m.*

Remark 3.4: The optimal retention strategy is identical to the one when minimizing the probability of ruin, the problem studied in Section 5 of Zhang et al. (2016). Bäuerle & Bayraktar (2014) showed that, if the maximum value of the drift divided by the square of the volatility is independent of the value of the state, then the strategy that minimizes the probability of ruin also minimizes the probability of drawdown and is given by the maximizer of the drift divided by the square of the volatility. For our problem, this means that if $\max_R f(u, R)$, in which f is given by (23), were independent of u, then we could have relied on the result of Bäuerle & Bayraktar (2014). However, $\max_R f(u, R)$ is not independent of u, so we did not know a priori that the optimal retention strategy to minimize the probability of drawdown is equal to the one that minimizes the probability of ruin.

Also, the optimal retention strategy in Theorem 3.2 is independent of the drawdown or ruin level. Instead, the optimal retention strategy depends on how far surplus is below the safe level $u_s = \kappa/r$, as one can see from (25) in Lemma 3.1, and this safe level is 'safe' for both minimizing the probability of drawdown and minimizing the probability of ruin. Angoshtari et al. (2016b) observed a similar phenomenon when minimizing the probability of drawdown for a general consumption function. Also, Bayraktar & Young (2007), when minimizing the probability of lifetime ruin by investing in a Black-Scholes financial market, showed that the optimal investment strategy was independent of the ruin level and only depended on how far wealth lies below the safe level.

In the following corollary, we present properties of the optimal retention strategy. Even though Zhang et al. (2016) found a similar retention strategy, they did not analyze it, and we deem it important to include these properties. To prove the properties in Corollary 3.2, we used the expression of R^* given in (39) and (40), but in the interest of space, we omit the proof. When we write 'increases' or 'decreases', we mean in the weak, or non-strict, sense.

Corollary 3.2: *The optimal retention strategy satisfies the following properties*:

(i) *$R^*(u,y)$ decreases to 0 as the surplus u increases to u_s. Also, $R^*(u,y)$ is bounded as the surplus approaches drawdown. For u close to u_s,*

$$R^*(u) \approx \frac{2(\theta + \eta y)(\kappa - ru)}{\theta^2 \lambda + \eta\big(2\theta\lambda \mathbb{E}Y + \eta\lambda\mathbb{E}(Y^2)\big)}. \tag{41}$$

(ii) *$R^*(u,y)$ and $y - R^*(u,y)$ both increase as the claim size y increases.*

(iii) *$R^*(u,y)$ increases with respect to θ, η, and λ.*

(iv) *$R^*(u,y)$ increases if Y increases in first-stochastic dominance. Specifically, if the claim size random variables are such that $Y_1 \leq_{FSD} Y_2$, that is, $S_{Y_1}(y) \leq S_{Y_2}(y)$ for all $y \geq 0$, then the optimal retention corresponding to Y_1 is less than the optimal retention corresponding to Y_2.*

Remark 3.5: In some insurance optimization problems, researchers impose the condition that retained and transferred claims are non-decreasing functions of the underlying claim. However, we obtain comonotonicity of $R^*(u,Y)$ and $Y - R^*(u,Y)$ without requiring that condition a priori.

If θ, η, or λ increases, then reinsurance becomes more expensive, and the insurer optimally purchases less reinsurance. On the other hand, a larger value of λ leads to more claims on average, so one might think that the insurer would retain less of each claim, but having to pay more in reinsurance premium dominates the insurer's optimal decision. As a result of purchasing less reinsurance, drawdown will be more likely. Note that the safe level also increases with increasing θ, η, or λ.

Finally, if the claim random variable becomes more risky, as measured by first-stochastic dominance, reinsurance becomes more expensive, and the insurer optimally purchases less reinsurance, even though the insurance risk has increased.

By setting $\eta = 0$ or $\theta = 0$ in (2), the mean-variance premium principle reduces to the expected-value or the variance premium principle, respectively. In the following corollaries of Theorem 3.2, we consider these two special cases in turn.

Corollary 3.3: *If $\eta = 0$, then the optimal retention strategy in* (39) *reduces to excess-of-loss reinsurance, specifically*,

$$R^*(u,y) = d^*(u) \wedge y, \tag{42}$$

in which $d^*(u) > 0$ uniquely solves

$$\theta \int_0^d \left(1 - \frac{y}{d}\right) S_Y(y)\,\mathrm{d}y = \frac{\kappa - ru}{\lambda}. \tag{43}$$

Moreover, the optimal deductible $d^*(u)$ decreases to 0 as surplus increases to $u_s = \kappa/r$, and $d^*(u)$ is convex with respect to u. Finally, for u close to u_s,

$$d^*(u) \approx \frac{2}{\theta} \cdot \frac{\kappa - ru}{\lambda}. \tag{44}$$

Proof: By setting $\eta = 0$ and $d^*(u) = \theta/\beta^*(u)$ in (39), (40), and (41), we obtain the expressions in (42), (43), and (44). It is easy to show that $d^*(u)$ decreases with u. As for the convexity of d^* with respect to u, one can show that the second derivative of d^* with respect to u is positively proportional to $h(d^*(u))$, in which h is given by

$$h(d) = 2\int_0^d yS_Y(y)\,\mathrm{d}y - d^2 S_Y(d),$$

and clearly $h(d) > 0$ for $d > 0$. ■

Corollary 3.4: *If $\theta = 0$, then the optimal retention strategy in* (39) *reduces to quota-share reinsurance, specifically,*

$$R^*(u,y) = q^*(u)y, \tag{45}$$

in which $q^*(u) \in [0,1)$ equals

$$q^*(u) = \frac{2}{\eta} \cdot \frac{\kappa - ru}{\lambda \mathbb{E}(Y^2)}, \tag{46}$$

which decreases linearly to 0 as u increases to u_s.

Proof: By setting $\theta = 0$ and $q^*(u) = \eta/\beta^*(u)$ in (39) and (40), we obtain the expressions in (45) and (46). ■

In the following proposition, for any quota-share reinsurance or excess-of-loss reinsurance, we compare the reinsurance premium under the expected-value premium principle with the one under the variance premium principle. We assume that when the insurer transfers all its risk to the reinsurer, that is, $R = 0$, then the reinsurance premiums under the two premium principles are equal.

Proposition 3.3: *Given η, set $\theta = \eta\mathbb{E}(Y^2)/2\mathbb{E}Y$ so that*

$$(1+\theta)\mathbb{E}Y = \mathbb{E}Y + \frac{\eta}{2}\mathbb{E}(Y^2), \tag{47}$$

that is, the premium for full reinsurance under the expected-value premium principle equals that under the variance premium principle. The condition in (47) *implies*

$$(1+\theta)\lambda\mathbb{E}\big((1-q)Y\big) > \lambda\mathbb{E}\big((1-q)Y\big) + \frac{\eta}{2}\lambda\mathbb{E}\big(((1-q)Y)^2\big) \tag{48}$$

for any quota-share reinsurance $R = qY$ with $0 < q < 1$; and

$$(1+\theta)\lambda\mathbb{E}\big((Y-d)_+\big) < \lambda\mathbb{E}\big((Y-d)_+\big) + \frac{\eta}{2}\lambda\mathbb{E}\big((Y-d)_+^2\big), \tag{49}$$

if and only if the coefficient of variation of $(Y-d)\,|\,(Y>d)$ is greater than 1 for any excess-of-loss reinsurance $R = d \wedge Y$ with $d > 0$.

Proof: Under condition (47), for any quota-share reinsurance $R = qY$ with $0 < q < 1$, we have

$$(1+\theta)\lambda\mathbb{E}\big((1-q)Y\big) = (1+\theta)\lambda(1-q)\mathbb{E}Y$$

$$
\begin{aligned}
&= (1-q)\lambda\mathbb{E}Y + \frac{\eta}{2}(1-q)\lambda\mathbb{E}(Y^2) \\
&> (1-q)\lambda\mathbb{E}Y + \frac{\eta}{2}(1-q)^2\lambda\mathbb{E}(Y^2) \\
&= \lambda\mathbb{E}\big((1-q)Y\big) + \frac{\eta}{2}\lambda\mathbb{E}\big(((1-q)Y)^2\big).
\end{aligned}
$$

Under condition (47), for any excess-of-loss reinsurance $R = d \wedge Y$ with $d > 0$, inequality (49) holds if and only if

$$
\frac{\mathbb{E}\big(Y^2\big)}{\mathbb{E}Y} < \frac{2\int_d^\infty (y-d)S_Y(y)\,\mathrm{d}y}{\int_d^\infty S_Y(y)\,\mathrm{d}y}. \tag{50}
$$

When $d = 0$, the two sides of (50) are equal, and the derivative of the right side of (50) is proportional to

$$
\begin{aligned}
&\int_d^\infty (y-d)\frac{S_Y(y)}{S_Y(d)}\,\mathrm{d}y - \left(\int_d^\infty \frac{S_Y(y)}{S_Y(d)}\,\mathrm{d}y\right)^2 \\
&= \frac{1}{2}\mathbb{E}\big((Y-d)^2 \,\big|\, Y > d\big) - \big(\mathbb{E}\big(Y-d \,\big|\, Y > d\big)\big)^2.
\end{aligned} \tag{51}
$$

Note that the expression in (51) is positive if and only if

$$
\frac{\sqrt{\mathrm{Var}\big(Y-d\big|Y>d\big)}}{\mathbb{E}\big(Y-d\,\big|\,Y>d\big)} > 1. \tag{52}
$$

that is, if and only if the coefficient of variation of $(Y-d) \mid (Y > d)$ is greater than 1. ■

Remark 3.6: When Y follows an exponential distribution, we can see from (52) that the coefficient of variation is identically 1 for all $d > 0$, which reflects the memorylessness of the exponential distribution; thus, the premium for excess-of-loss reinsurance under the expected-value principle equals the premium under the variance principle. When Y follows a Pareto distribution, this coefficient of variation is greater than 1; thus, the premium for excess-of-loss reinsurance is cheaper under the expected-value principle than under the variance principle. However, when Y follows a uniform distribution, this coefficient of variation is less than 1; thus, the premium for excess-of-loss reinsurance is more expensive under the expected-value principle than under the variance principle.

Therefore, if the distribution of Y has a heavier tail than the exponential, as measured by the coefficient of variation of $(Y-d) \mid (Y > d)$ for all $d > 0$, then the premium for excess-of-loss reinsurance is cheaper under the expected-value principle than under the variance principle, and vice versa.

To end this section, under the variance premium principle, we examine the behavior of the optimally controlled surplus value and show that the probability of reaching the safe level prior to drawdown is zero. This result is similar to the one proved by Browne (1997) using a model of constant consumption with investment in a Black-Scholes market; see Section 3.1 in that paper.

Because there is no explicit expression for the dynamics of the surplus process under the mean-variance premium principle when $\theta > 0$, we consider the special case for which the premium is calculated by the variance premium principle ($\theta = 0$), and we show that optimally controlled surplus does not reach the safe level u_s with positive probability in finite time.

Proposition 3.4: *Suppose $\theta = 0$. Let $\hat{U}^*$ denote the optimally controlled surplus process for minimizing the probability of drawdown. Define the hitting times*

$$\tau_s^* = \inf\{t \geq 0 : \hat{U}_t^* \geq u_s\},$$

and

$$\tau_\alpha^* = \inf\{t \geq 0 : \hat{U}_t^* \leq \alpha m\}.$$

Then, $\mathbb{P}^{u,m}(\tau_s^* < \tau_\alpha^*) = 0$ *for* $\alpha m < u < \min(m, u_s)$.

Proof: We use Feller's test for explosions to prove this proposition. Because we are only interested in whether the safe level can be reached before drawdown occurs, we may extend the domain of R^* given in (45) to all of $\mathbb{R}$ as follows

$$R^*(u,y) = \begin{cases} \dfrac{\eta\lambda\mathbb{E}(Y^2) - \sqrt{(\eta\lambda\mathbb{E}(Y^2))^2 - 2\eta\lambda\mathbb{E}(Y^2)(\kappa - ru)}}{\eta\lambda\mathbb{E}(Y^2)}\, y, & \text{if } u < \alpha m, \\ \dfrac{2}{\eta} \cdot \dfrac{\kappa - ru}{\lambda\mathbb{E}(Y^2)} y, & \text{if } \alpha m \leq u \leq u_s, \\ 0, & \text{if } u > u_s. \end{cases}$$

Recall that $u_s = \kappa/r$. Then, define $\mathbf{b}$ and $\mathbf{s}$ on $\mathbb{R}$ by

$$\mathbf{b}(u) = ru - \kappa + \eta\lambda\mathbb{E}\big(YR^*(u,Y)\big) - \frac{\eta}{2}\lambda\mathbb{E}\big((R^*(u,Y))^2\big),$$

and

$$\mathbf{s}(u) = \sqrt{\lambda\mathbb{E}\big((R^*(u,Y))^2\big)}.$$

One can show that $\mathbf{b}(u) = 0$ for $u < \alpha m$. Next, define the scale function p on $\mathbb{R}$ by

$$p(u) = \int_{\alpha m}^{u} \exp\left(-2\int_{\alpha m}^{y} \frac{\mathbf{b}(z)}{\mathbf{s}^2(z)}\,\mathrm{d}z\right)\mathrm{d}y, \tag{53}$$

and define the function v on $\mathbb{R} \times \mathbb{R}^+$ by

$$v(u,m) = \int_{\alpha m}^{u} p'(y)\int_{\alpha m}^{y} \frac{2\,\mathrm{d}z}{p'(z)\mathbf{s}^2(z)}\,\mathrm{d}y = \int_{\alpha m}^{u} \frac{2(p(u) - p(y))}{p'(y)\mathbf{s}^2(y)}\,\mathrm{d}y. \tag{54}$$

We want to show that $v(-\infty, m) = v(u_s, m) = \infty$. First, from $\mathbf{b}(u) = 0$ for $u < \alpha m$, it follows that $p(-\infty) = \int_{\alpha m}^{-\infty} 1\,\mathrm{d}y = -\infty$. Thus, the expression in (5.74) on page 348 of Karatzas & Shreve (1991) implies that $v(-\infty, m) = \infty$. Next, note that if optimally controlled surplus lies between αm and u_s, it follows the dynamics

$$\mathrm{d}\hat{U}_t^* = (\kappa - r\hat{U}_t^*)\left\{\left(1 - \frac{2(\kappa - r\hat{U}_t^*)}{\eta\lambda\mathbb{E}(Y^2)}\right)\mathrm{d}t + \frac{2}{\eta\sqrt{\lambda\mathbb{E}(Y^2)}}\,\mathrm{d}B_t\right\}. \tag{55}$$

Let $\delta = \eta^2\lambda\mathbb{E}(Y^2)/2r$, then according to (53) and (55), for $x \in (\alpha m, u_s)$,

$$\begin{aligned} p(u_s) - p(x) &= \int_x^{u_s} \exp\left\{\int_{\alpha m}^{y}\left(\eta - \frac{\delta}{u_s - z}\right)\mathrm{d}z\right\}\mathrm{d}y \\ &= \frac{\mathrm{e}^{-\eta\alpha m}}{(u_s - \alpha m)^\delta}\int_x^{u_s} \mathrm{e}^{\eta y}(u_s - y)^\delta\,\mathrm{d}y \end{aligned}$$

$$\geq \frac{e^{-\eta(\alpha m - x)}}{(u_s - \alpha m)^\delta} \frac{(u_s - x)^{\delta+1}}{\delta + 1},$$

and

$$\frac{2}{p'(x)\mathbf{s}^2(x)} = e^{\eta(\alpha m - x)} \frac{\delta(u_s - \alpha m)^\delta}{r(u_s - x)^{\delta+2}}.$$

Therefore,

$$\begin{aligned} v(u_s, m) &= \int_{\alpha m}^{u_s} \frac{2(p(u_s) - p(x))}{p'(x)\mathbf{s}^2(x)} \, dx \\ &\geq \frac{\delta}{r(\delta + 1)} \int_{\alpha m}^{u_s} \frac{dx}{u_s - x} = \infty. \end{aligned} \tag{56}$$

It follows from Feller's test for explosions (Theorem 5.5.29 on page 348 of Karatzas & Shreve 1991) that $\mathbb{P}^{u,m}(\tau_s^* < \tau_\alpha^*) = 0$ for $u \in (\alpha m, \min(m, u_s))$. ■

Remark 3.7: Intuitively, as the surplus gets closer to the safe level u_s, the insurer retains more of its insurance risk, transferring less to the reinsurer, as we observe in the expression for R^* in (39) and (40). Indeed, both the drift and the volatility of the optimally controlled surplus process approach 0 as the surplus approaches u_s. Thus, our intuition tells us that the safe level is not reachable, and Proposition 3.4 confirms our intuition.

Let $\tau = \tau_\alpha^* \wedge \tau_s^*$ denote the first hitting time of αm or u_s when the initial surplus u lies in $(\alpha m, \min(m, u_s))$. From Proposition 5.5.32 on page 350 of Karatzas & Shreve (1991) and from (56), we deduce that $0 < P(\tau < \infty) < 1$. Furthermore, in combination with Proposition 3.4, we deduce that either drawdown occurs with probability $\phi(u, m) = P(\tau < \infty)$ or the optimal controlled surplus value lies strictly between αm and u_s, for all time, with probability of $1 - \phi(u, m)$. Bayraktar & Zhang (2015) and Angoshtari et al. (2016b) reach similar conclusions in their work.

4. Numerical examples

Under condition (47), we present Examples 4.1–4.3 to illustrate our results when the reinsurance premium is calculated according to the expected-value premium principle and the variance premium principle. Then, in Example 4.4, we show the effect of the relative size of the risk-loading parameters θ and η when the reinsurance premium is calculated according to the mean-variance premium principle, again assuming that the premium for full reinsurance remains constant as we vary θ and η.

In the first example, we assume the claim size random variable Y is uniformly distributed. From Proposition 3.3, we deduce that, for either quota-share or excess-of-loss reinsurance, the variance principle gives a smaller reinsurance premium than the expected-value principle. The uniform distribution is not covered by Theorem 3.2 because ess sup $Y < \infty$; however, we can modify the proof of Theorem 3.2 to account for Y's bounded support.

Example 4.1: Assume the claim size random variable Y is uniformly distributed in the interval $[0, 2]$; then, $\mathbb{E}Y = 1$ and $\mathbb{E}(Y^2) = 4/3$. We set $(\theta, \eta) = (0.4, 0)$ for the expected-value principle and $(\theta, \eta) = (0, 0.6)$ for the variance premium principle. Thus, because we also set $\lambda = 3$, the premium rate for full reinsurance equals 4.2. Also, set $c = 3.3$ and $r = 0.05$, which implies that the safe level $u_s = 18$. Finally, set $\alpha = 0.1$ and $m = 40$, which implies that the drawdown level equals 4. Note that $m > u_s$; thus, minimizing the probability of drawdown reduces to one of minimizing the probability of ruin with ruin level 4.

Table 1. Values of $d^*(u)$, $q^*(u)$, $\phi(u,40)$, and the reinsurance premium rate.

u	1	3	5	7	9	11	13	15	17	18
$d^*(u)$	2.0000	1.7753	1.4189	1.1292	0.8787	0.6548	0.4505	0.2614	0.0845	0
Reins prem	0	0.0530	0.3546	0.7963	1.3202	1.9001	2.5210	3.1739	3.8525	4.2
$\phi(u,40)$	1.0000	1.0000	0.6977	0.3032	0.1078	0.0285	0.0046	0.0003	0.0000	0
$q^*(u)$	0.7083	0.6250	0.5417	0.4583	0.3750	0.2917	0.2083	0.1250	0.0417	0
Reins prem	0.9771	1.2938	1.6271	1.9771	2.3438	2.7271	3.1271	3.5438	3.9771	4.2
$\phi(u,40)$	1.0000	1.0000	0.5472	0.1236	0.0168	0.0011	0.0000	0.0000	0.0000	0

Under the expected-value premium principle, the optimal deductible $d^*(u)$ in excess-of-loss reinsurance is given by

$$d^*(u) = \left(3 - \sqrt{\frac{u}{2}}\right) \wedge 2, \tag{57}$$

with corresponding reinsurance premium rate, for $u \geq 2$,

$$(1+\theta)\lambda\mathbb{E}\big(\big(Y - d^*(u)\big)_+\big) = 1.05\left(\sqrt{\frac{u}{2}} - 1\right)^2, \tag{58}$$

and minimum probability of drawdown

$$\phi(u,m) = 1 - \frac{\int_4^u \mathrm{e}^{0.8\sqrt{2v}}\big(3\sqrt{2} - \sqrt{v}\big)^{4.8}\,\mathrm{d}v}{\int_4^{18} \mathrm{e}^{0.8\sqrt{2v}}\big(3\sqrt{2} - \sqrt{v}\big)^{4.8}\,\mathrm{d}v}. \tag{59}$$

Also, under the variance premium principle, the optimal quota-share reinsurance has retained proportion $q^*(u)$ given by

$$q^*(u) = \frac{18-u}{24}, \tag{60}$$

with corresponding reinsurance premium rate

$$\big(1 - q^*(u)\big)\lambda\mathbb{E}Y + \frac{\eta}{2}\big(1 - q^*(u)\big)^2\lambda\mathbb{E}(Y^2) = \frac{6+u}{24}\left(3 + 1.2\frac{6+u}{24}\right), \tag{61}$$

and minimum probability of drawdown

$$\phi(u,m) = 1 - \frac{\int_4^u \mathrm{e}^{\eta v}(18-v)^{24\eta}\,\mathrm{d}v}{\int_4^{18} \mathrm{e}^{\eta v}(18-v)^{24\eta}\,\mathrm{d}v}, \tag{62}$$

in which $\eta = 0.6$.

In Table 1, we give some numerical results to compare the two premium principles. Even though $u = 1$ and $u = 3$ lie below the drawdown level, we can compute the optimal reinsurance strategies at those surplus levels because the optimal strategy is independent of the drawdown level. In other words, the drawdown level 4 affects the probability of drawdown but not the optimal reinsurance strategy.

Note that the optimal retention strategy R^* decreases as u increases, which we expect from Corollary 3.2(i). It is not difficult to show that, by using $d^*(u)$ and $q^*(u)$ in (57) and (60), respectively, the optimal amount retained under the expected-value premium principle is greater than the one under the variance premium principle at all levels of surplus $u < u_s$. This result is expected since the variance premium principle is cheaper than the expected-value principle under condition (47). Furthermore, note that the minimum probability of drawdown is greater under the expected-value premium principle than under the variance premium. Indeed, when the company retains a greater share of each claim, drawdown is certainly more likely to occur.

Recall, from Equation (44) in Corollary 3.3 that, under the expected-value premium principle, the optimal deductible is approximately equal to $d^*(u) \approx 2/\theta \cdot (\kappa - ru)/\lambda$ when u is close to u_s. In this example, when $u = 17$, this approximation yields $1/12 = 0.083\overline{3}$, which is approximately equal to $d^*(17) = 0.0845$, for a relative error of 1.4%. When $u = 15$, the approximation yields 0.2500 as compared with $d^*(15) = 0.2733$, for a relative error of 8.5%. As expected, the approximation is worse the farther u lies from u_s.

In the next example, we assume the claim size random variable Y is exponentially distributed. From Proposition 3.3, we deduce that, for quota-share reinsurance, the variance principle gives a smaller reinsurance premium than the expected-value principle; however, for excess-of-loss reinsurance, the two premium principles result in the same reinsurance premium.

Example 4.2: Assume the claim size random variable Y is exponentially distributed with mean 1; then, $E(Y^2) = 2$. We set $(\theta, \eta) = (0.4, 0)$ for the expected-value principle and $(\theta, \eta) = (0, 0.4)$ for the variance premium principle. Thus, because we also set $\lambda = 3$, the premium rate for full reinsurance equals 4.2. Also, set $c = 3.3$ and $r = 0.05$, which implies that the safe level $u_s = 18$, as in Example 4.1. Finally, set $\alpha = 0.1$ and $m = 40$, which implies that the drawdown level equals 4 and $m > u_s$.

Under the expected-value premium principle, the optimal deductible $d^*(u)$ in excess-of-loss reinsurance uniquely solves

$$\frac{1}{d}\left(1 - \mathrm{e}^{-d}\right) = \frac{6+u}{24}, \tag{63}$$

with corresponding reinsurance premium rate

$$(1+\theta)\lambda\mathbb{E}\left(\left(Y - d^*(u)\right)_+\right) = 4.2\,\mathrm{e}^{-d^*(u)}, \tag{64}$$

and minimum probability of drawdown

$$\phi(u,m) = 1 - \frac{\int_4^u \exp\left\{-0.4\int_4^v \frac{1}{d^*(u)}\,\mathrm{d}w\right\}\mathrm{d}v}{\int_4^{18} \exp\left\{-0.4\int_4^v \frac{1}{d^*(u)}\,\mathrm{d}w\right\}\mathrm{d}v}. \tag{65}$$

Also, under the variance premium principle, the optimal quota-share reinsurance has retained proportion $q^*(u)$ given by

$$q^*(u) = \frac{18-u}{24}, \tag{66}$$

with corresponding reinsurance premium rate

$$\left(1 - q^*(u)\right)\lambda\mathbb{E}Y + \frac{\eta}{2}\left(1 - q^*(u)\right)^2\lambda\mathbb{E}(Y^2) = \frac{6+u}{24}\left(3 + 1.2\frac{6+u}{24}\right), \tag{67}$$

and minimum probability of drawdown

$$\phi(u,m) = 1 - \frac{\int_4^u \mathrm{e}^{\eta v}(18-v)^{24\eta}\,\mathrm{d}v}{\int_4^{18} \mathrm{e}^{\eta v}(18-v)^{24\eta}\,\mathrm{d}v}, \tag{68}$$

in which $\eta = 0.4$.

In Table 2, we give some numerical results to compare the two premium principles. Note that, except for values of u close to u_s under the expected value premium principle, the probability of

Table 2. Values of $d^*(u)$, $q^*(u)$, $\phi(u,40)$, and the reinsurance premium rate.

u	1	3	5	7	9	11	13	15	17	18
$d^*(u)$	3.3024	2.4325	1.8328	1.3832	1.0272	0.7344	0.4870	0.2733	0.0857	0
Reins prem	0.1545	0.3688	0.6718	1.0533	1.5037	2.0151	2.5809	3.1957	3.8549	4.2
$\phi(u,40)$	1.0000	1.0000	0.7341	0.3540	0.1401	0.0413	0.0074	0.0005	0.0000	0
$q^*(u)$	0.7083	0.6250	0.5417	0.4583	0.3750	0.2917	0.2083	0.1250	0.0417	0
Reins prem	0.9771	1.2938	1.6271	1.9771	2.3438	2.7271	3.1271	3.5438	3.9771	4.2
$\phi(u,40)$	1.0000	1.0000	0.6444	0.2204	0.0532	0.0075	0.0004	0.0000	0.0000	0

drawdown is greater under the exponential than under the uniform; compare with Table 1. Also, as in Example 4.1, the minimum probability of drawdown under the expected-value premium principle is greater than the one under the variance premium principle. Furthermore, we note that when Y is small enough, say $Y \le 2$, the optimal amount retained is greater under the expected-value premium principle than under the variance premium principle at all levels of surplus $u < u_s$. However, when Y is large enough, say $Y \ge 5$, the insurer retains less under the expected-value premium principle than under the variance premium principle at all levels of surplus $u < u_s$.

As in Example 4.1, the approximation for $d^*(u)$ in (44) fits well at $u = 17$, that is, $d^*(17) = 0.0857$ is close to the approximation from (44), namely, $0.083\bar{3}$, for a relative error of 2.8%.

In the third example, we assume the claim size random variable Y is Pareto distributed. From Proposition 3.3, we deduce that, for quota-share reinsurance, the variance principle gives a smaller reinsurance premium than the expected-value principle; however, for excess-of-loss reinsurance, the expected-value principle gives a smaller reinsurance premium than the variance principle.

Example 4.3: Assume the claim size random variable Y is Pareto distributed with mean and second moment equal to 1 and 3, respectively. We set $(\theta, \eta) = (0.4, 0)$ for the expected-value principle and $(\theta, \eta) = (0, 4/15)$ for the variance premium principle. Thus, because we also set $\lambda = 3$, the premium rate for full reinsurance equals 4.2. Also, set $c = 3.3$ and $r = 0.05$, which implies that the safe level $u_s = 18$, as in Examples 4.1 and 4.2. Finally, set $\alpha = 0.1$ and $m = 40$, which implies that the drawdown level equals 4 and $m > u_s$.

Under the expected-value premium principle, the optimal deductible $d^*(u)$ in excess-of-loss reinsurance is given by

$$d^*(u) = \frac{-0.3u + \sqrt{1.08u + 9.72}}{2(0.3 + 0.05u)} \wedge Y, \tag{69}$$

with corresponding reinsurance premium rate

$$(1+\theta)\lambda\mathbb{E}\big((Y - d^*(u))_+\big) = 113.4 \left(\frac{0.6 + 0.1u}{1.8 + \sqrt{1.08u + 9.72}}\right)^3, \tag{70}$$

and minimum probability of drawdown

$$\phi(u,m) = 1 - \frac{\int_4^u \exp\left\{-0.4 \int_4^v \frac{1}{d^*(u)}\,\mathrm{d}w\right\} \mathrm{d}v}{\int_4^{18} \exp\left\{-0.4 \int_4^v \frac{1}{d^*(u)}\,\mathrm{d}w\right\} \mathrm{d}v}. \tag{71}$$

Also, under the variance premium principle, the optimal quota-share reinsurance has retained proportion $q^*(u)$ given by

$$q^*(u) = \frac{18 - u}{24}, \tag{72}$$

Table 3. Values of $d^*(u)$, $q^*(u)$, $\phi(u, 40)$, and the reinsurance premium rate.

u	1	3	5	7	9	11	13	15	17	18
$d^*(u)$	4.2662	3.0000	2.1713	1.5822	1.1394	0.7927	0.5129	0.2815	0.0865	0
Reins prem	0.2956	0.5250	0.8200	1.1786	1.5988	2.0786	2.6160	3.2092	3.8565	4.2
$\phi(u, 40)$	1.0000	1.0000	0.7540	0.3851	0.1620	0.0509	0.0098	0.0007	0.0000	0
$q^*(u)$	0.7083	0.6250	0.5417	0.4583	0.3750	0.2917	0.2083	0.1250	0.0417	0
Reins prem	0.9771	1.2938	1.6271	1.9771	2.3438	2.7271	3.1271	3.5438	3.9771	4.2
$\phi(u, 40)$	1.0000	1.0000	0.7207	0.3268	0.1162	0.0285	0.0037	0.0001	0.0000	0

with corresponding reinsurance premium rate

$$\left(1 - q^*(u)\right)\lambda \mathbb{E}Y + \frac{\eta}{2}\left(1 - q^*(u)\right)^2 \lambda \mathbb{E}(Y^2) = \frac{6+u}{24}\left(3 + 1.2\frac{6+u}{24}\right), \tag{73}$$

and minimum probability of drawdown

$$\phi(u, m) = 1 - \frac{\int_4^u e^{\eta v}(18 - v)^{24\eta}\, dv}{\int_4^{18} e^{\eta v}(18 - v)^{24\eta}\, dv}, \tag{74}$$

in which $\eta = 4/15$.

Table 3 shows that the minimum probability of drawdown under the expected-value premium principle is larger than the one under the variance premium principle, as in the previous two examples. Also, the probability of drawdown is greater under the Pareto than under the uniform and the exponential; compare with Tables 1 and 2. Furthermore, we note that when Y is large enough, say $Y = 7$, the optimal amount retained under the variance premium principle is larger than under the expected-value premium principle at all levels of surplus $u < u_s$.

As in Examples 4.1 and 4.2, the approximation for $d^*(u)$ in (44) fits well at u $= 17$, that is, $d^*(17) = 0.0865$ is close to the approximation from (44), namely, $0.083\bar{3}$, for a relative error of 3.7%. Note that the relative error increases with increasing tail size over these three examples, which is not surprising.

Remark 4.1: In Examples 4.1–4.3, we have $\theta = 0.4$ and $\mathbb{E}Y = 1$; therefore, because condition (47) holds, the value of $(\eta/2)\mathbb{E}(Y^2) = 0.4$ in all three examples. Because the expression for $q^*(u)$ in (46) is determined by the values of u_s and $(\eta/2)\mathbb{E}(Y^2)$, the function $q^*(u)$ and corresponding reinsurance premium are equal across the three examples. However, the minimum probability of drawdown, when $\theta = 0$, varies from one distribution to the next because the value of η varies; note the dependence on η in the expressions in (62), (68), and (74).

In the next example, we assume that the claim size random variable Y is uniformly distributed, as in Example 4.1. We also assume that, when the insurer transfers all its claim risk to the reinsurer, that is, $R = 0$, the reinsurance premium under the mean-variance premium principle remains constant for various combinations of (θ, η). Hence, the safe level for the insurer is fixed.

Example 4.4: Assume the claim size random variable Y is uniformly distributed in the interval $[0, 2]$; then, $\mathbb{E}Y = 1$ and $\mathbb{E}(Y^2) = 4/3$. We set $\lambda = 3$, $c = 3.3$, and $r = 0.05$. Assume the relationship $1.5\theta + \eta = 0.6$ between $\theta \in [0, 0.4]$ and $\eta \in [0, 0.6]$, which implies that the safe level remains fixed at $u_s = 18$ as θ and η vary. Finally, set $\alpha = 0.1$ and $m = 40$, which implies that the drawdown level equals 4, as in the previous examples.

Note that when $\theta = 0$ or $\eta = 0$, we obtain the two special cases considered in Example 4.1, where we observed that the optimal amount retained and the corresponding minimum probability of drawdown are greater under the expected-value premium principle than under the variance premium principle at all levels of surplus $u < u_s$.

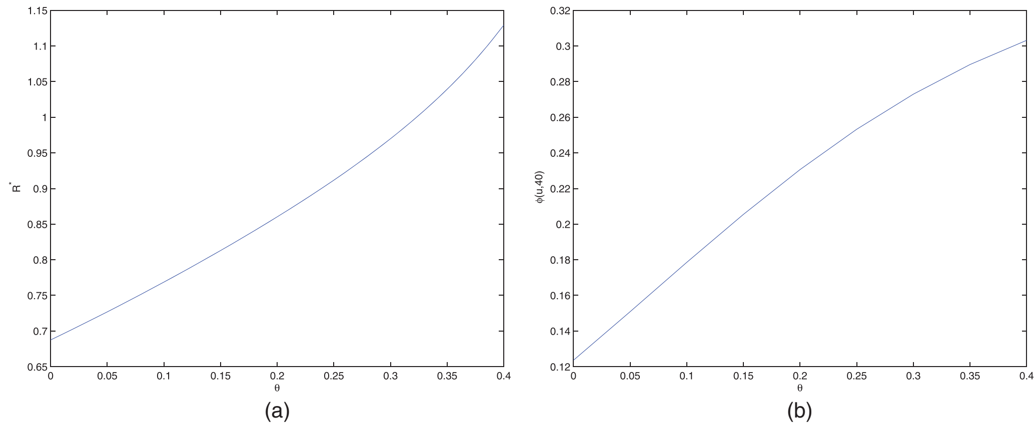

Figure 1. The effect of θ on $R^*(7, 1.5)$ and $\phi(7, 40)$.

In Figure 1(a), we plot the optimal amount retained when the surplus equals $u = 7$ and the claim size equals $Y = 1.5$, that is, $R^* = R^*(7, 1.5)$, as we vary $\theta \in [0, 0.4]$. In Figure 1(b), we plot the minimum probability of drawdown when the surplus equals $u = 7$, that is, $\phi(u, 40) = \phi(7, 40)$, as we vary $\theta \in [0, 0.4]$. We see from Figure 1 that the optimal retention strategy $R^*(7, 1.5)$ and the corresponding minimum probability of drawdown $\phi(7, 40)$ both increase with respect to $\theta \in [0, 0.4]$. Therefore, because we keep $1.5\theta + \eta$ fixed, $R^*(7, 1.5)$ and $\phi(7, 40)$ both decrease with respect to η. We expect these results because, when θ increases towards 0.4, η decreases to 0, and the premium principle tends to the expected-value premium principle. Thus, the optimal amount retained and the corresponding minimum probability of drawdown are smaller under the mean-variance premium principle (with $\theta > 0$ and $\eta > 0$) than under the expected-value premium principle ($\eta = 0$), but greater than under the variance premium principle ($\theta = 0$).

Remark 4.2: We conjecture that the last observation in Example 4.4 holds for more general distributions of Y. Specifically, we hypothesize that the optimal amount retained and the corresponding minimum probability of drawdown are smaller under the mean-variance premium principle (with $\theta > 0$ and $\eta > 0$) than under the expected-value premium principle ($\eta = 0$), but greater than under the variance premium principle ($\theta = 0$). We invite the interested reader to prove this hypothesis.

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Appendix. $\tau_{\alpha b} < \infty$ with probability 1

The proof of the following lemma is inspired by the proof of Lemma 3.1 in Luo et al. (2019); also, see Lemma 6.1 in Taksar & Markussen (2003).

Lemma A.1: *Fix an admissible retention strategy R, and define* $\tau_b = \inf\{t \geq 0 : \hat{U}_t^R \geq b\}$ *and* $\tau_{\alpha b} = \tau_\alpha \wedge \tau_b$.[5] *Then,* $\mathbb{P}^{u,m}(\tau_{\alpha b} < \infty) = 1$.

[5] These stopping times depend on R via $\hat{U}^R$, but for simplicity of notation, we suppress the superscript R on τ_α, τ_b, and $\tau_{\alpha b}$.

Proof: Define the process $Y = \{Y_t\}_{t \geq 0}$ by $Y_t = \mathrm{e}^{-\delta \hat{U}_t}$, in which $\hat{U}_t = \hat{U}_t^R$. In the expression for Y_t, δ is a positive constant to be chosen later in the proof. By applying Itô's formula to Y, we obtain

$$\begin{aligned} \mathrm{d}Y_t &= -\delta\, \mathrm{e}^{-\delta \hat{U}_t}\, \mathrm{d}\hat{U}_t + \frac{1}{2}\delta^2\, \mathrm{e}^{-\delta \hat{U}_t}\, d < \hat{U}, \hat{U} >_t \\ &= \delta\, \mathrm{e}^{-\delta \hat{U}_t} \left(\mathcal{M}_t\, \mathrm{d}t - \sqrt{\lambda \mathbb{E}\big(R_t^2\big)}\, \mathrm{d}B_t \right), \end{aligned}$$

in which

$$\mathcal{M}_t = \kappa - r\hat{U}_t - \theta\lambda \mathbb{E}R_t - \eta\lambda\mathbb{E}(YR_t) + \frac{\delta + \eta}{2}\lambda\mathbb{E}\big(R_t^2\big).$$

We wish to show that, for δ large enough, $\mathcal{M}_t \geq (\kappa - rb)/2 > 0$ for all $0 \leq t \leq \tau_{\alpha b}$ with probability 1. First, notice, for $u \leq b < u_s = \kappa/r$, we have $\kappa - ru \geq \kappa - rb > 0$. Next, find $\delta > 0$ such that

$$-\theta\lambda\mathbb{E}R_t - \eta\lambda\mathbb{E}(YR_t) + \frac{\delta + \eta}{2}\lambda\mathbb{E}\big(R_t^2\big) \geq -\frac{\kappa - rb}{2},$$

or equivalently,

$$\frac{\delta + \eta}{2} \geq \frac{\theta\mathbb{E}R + \eta\mathbb{E}(YR) - \frac{\kappa - rb}{2\lambda}}{\mathbb{E}\big(R^2\big)}, \tag{A1}$$

for all retention functions R such that $0 \leq R(y) \leq y$. In other words, we wish to show that the right side of inequality (A1) has a finite maximum over such R. By comparing the right side of inequality (A1) with f in (23), this problem is essentially identical to the one we solved in Lemma 3.1. By relying on the proof of that lemma, we deduce that the right side of (A1) is maximized by $\tilde{R}$ in which

$$\tilde{R}(y) = \frac{\theta + \eta y}{\tilde{\gamma}} \wedge y, \tag{A2}$$

in which $\tilde{\gamma} \in (\eta, \infty)$ uniquely solves

$$\theta\mathbb{E}R + \eta\mathbb{E}(YR) - \frac{\gamma}{2}\mathbb{E}\big(R^2\big) = \frac{\kappa - rb}{2\lambda}, \tag{A3}$$

with $R = \tilde{R}$ given in (A2). By substituting $\tilde{R}$ into the right side of (A1), we obtain the following inequality for δ:

$$\frac{\delta + \eta}{2} \geq \frac{\tilde{\gamma}}{2}.$$

This inequality holds for any δ greater than or equal to $\tilde{\gamma} - \eta$; thus, set $\delta = \tilde{\gamma} - \eta > 0$, and for this choice of δ, we have $\mathcal{M}_t \geq \frac{\kappa - rb}{2} > 0$ for all $0 \leq t \leq \tau_{\alpha b}$ with probability 1.

Assume $\hat{U}_0 = u \in (\alpha m, b)$; otherwise, $\tau_{\alpha b} = 0$. Apply Itô's formula to $\mathrm{e}^{-\delta \hat{U}_t}$ to obtain

$$\mathrm{e}^{-\delta \hat{U}_{\tau_{\alpha b} \wedge t}} - \mathrm{e}^{-\delta u} = \int_0^{\tau_{\alpha b} \wedge t} \delta\, \mathrm{e}^{-\delta \hat{U}_s} \mathcal{M}_s\, \mathrm{d}s - \int_0^{\tau_{\alpha b} \wedge t} \delta\, \mathrm{e}^{-\delta \hat{U}_s} \sqrt{\lambda\mathbb{E}\big(R_s^2\big)}\, \mathrm{d}B_s.$$

The integrand of the second integral is bounded, so the integral's expectation equals zero; thus, if we take the expectation of both sides, we get

$$\mathbb{E}^{u,m}\big(\mathrm{e}^{-\delta \hat{U}_{\tau_{\alpha b} \wedge t}}\big) - \mathrm{e}^{-\delta u} = \mathbb{E}^{u,m} \int_0^{\tau_{\alpha b} \wedge t} \delta\, \mathrm{e}^{-\delta \hat{U}_s} \mathcal{M}_s\, \mathrm{d}s.$$

Note that $\mathrm{e}^{-\delta b} \leq \mathrm{e}^{-\delta \hat{U}_{\tau_{\alpha b} \wedge t}} \leq \mathrm{e}^{-\delta \alpha m}$ for all $t \geq 0$ with probability 1, so we have the following sequence of (in)equalities

$$\begin{aligned} \mathrm{e}^{-\delta \alpha m} - \mathrm{e}^{-\delta u} &\geq \mathbb{E}^{u,m}\big(\mathrm{e}^{-\delta \hat{U}_{\tau_{\alpha b} \wedge t}}\big) - \mathrm{e}^{-\delta u} = \mathbb{E}^{u,m} \int_0^{\tau_{\alpha b} \wedge t} \delta\, \mathrm{e}^{-\delta \hat{U}_s} \mathcal{M}_s\, \mathrm{d}s \\ &\geq \delta\, \mathrm{e}^{-\delta b} \frac{\kappa - rb}{2} \mathbb{E}^{u,m} \int_0^{\tau_{\alpha b} \wedge t} 1\, \mathrm{d}s \\ &= \delta\, \mathrm{e}^{-\delta b} \frac{\kappa - rb}{2} \left(\mathbb{E}^{u,m} \int_0^{\tau_{\alpha b}} 1\, \mathrm{d}s \cdot \mathbf{1}_{\{\tau_{\alpha b} \leq t\}} + \mathbb{E}^{u,m} \int_0^{t} 1\, \mathrm{d}s \cdot \mathbf{1}_{\{\tau_{\alpha b} > t\}} \right) \\ &\geq \delta\, \mathrm{e}^{-\delta b} \frac{\kappa - rb}{2}\, t\, \mathbb{P}^{u,m}(\tau_{\alpha b} > t). \end{aligned}$$

By letting $t \to \infty$ in the last expression, we deduce that $\lim_{t \to \infty} \mathbb{P}^{u,m}(\tau_{\alpha b} > t) = 0$, or equivalently, $\mathbb{P}^{u,m}(\tau_{\alpha b} < \infty) = 1$. ■