Received: 1 February 2020 | Revised: 26 October 2020 | Accepted: 15 December 2020

DOI: 10.1002/oca.2702

RESEARCH ARTICLE

WILEY

Minimizing the probability of absolute ruin under the mean-variance premium principle

Xia Han[1] | **Zhibin Liang**[1] | **Kam C. Yuen**[2]

[1]School of Mathematical Sciences, Nanjing Normal University, Jiangsu, P.R. China

[2]Department of Statistics and Actuarial Science, The University of Hong Kong, Hong Kong, P.R. China

Correspondence
Zhibin Liang, School of Mathematical Sciences, Nanjing Normal University, Jiangsu 210023, P.R. China.
Email: liangzhibin111@hotmail.com

Funding information
National Natural Science Foundation of China, Grant/Award Number: 12071224; the Research Grants Council of the Hong Kong Special Administrative Region, China, Grant/Award Number: HKU17306220

Abstract

In this article, we assume that the insurer can purchase per-loss reinsurance and invest its surplus in a financial market consisting of a risk-free asset and a risky asset. It is also assumed that the investment amount of the risky asset is capped at a fixed level and that short-selling is prohibited. Our objective is to minimize the probability of absolute ruin, and the reinsurance premium is computed according to the mean-variance premium principle, that is, a combination of the expected-value and variance premium principles. By solving the corresponding Hamilton–Jacobi–Bellman equation, we derive explicit expressions for the S-shaped minimum absolute ruin function and its associated optimal reinsurance-investment strategy. We further study the same optimization problem for a slightly modified version of absolute ruin, and the corresponding optimal results are obtained as well. To gain insights into the optimal problems, we investigate the reason for the S-shaped value function and discover that the constraint on investment control can result in the kink of minimum absolute ruin function. Finally, some properties and numerical examples are presented to show the impact of model parameters on the optimal results.

KEYWORDS

absolute ruin, mean-variance premium principle, optimal reinsurance-investment strategy, per-loss reinsurance, S-shaped value function

1 | INTRODUCTION

Reinsurance and investment are two important business activities for an insurance company. Specifically, reinsurance can protect insurers against potentially large losses, while investment enables insurers to increase their profit. Optimization problems such as minimization of the probability of ruin, maximization of the expected utility of terminal wealth, and optimization under the mean-variance criterion have become a popular research topic in the actuarial literature. The technique of stochastic control theory and the corresponding Hamilton–Jacobi–Bellman (HJB) equation are widely used to cope with these problems.

Most of the literature considers problems under which the insurance company is restricted to buy either pure quota-share reinsurance (see Liang et al.,[1] Luo and Taksar,[2] and Promislow and Young[3]), or pure excess-of-loss reinsurance (see Bai et al.,[4] Gu et al.,[5] and Li et al.[6]), or a combination of the two (see Liang and Guo[7] and Zhang et al.[8]). In the literature, there also exist some research works studying the problem of optimal reinsurance without restricting the form of reinsurance. For example, under the criteria of minimizing the probability of ruin and maximizing the expected utility function of terminal wealth, Zhang et al.[9] investigated the optimal reinsurance-investment problem in which the insurer purchases a general reinsurance policy; and Liang and Young[10] derived the optimal investment and per-loss reinsurance strategy for an insurance company when the risk process follows a compound Poisson claim process; and Han et al.[11] found the general optimal reinsurance strategy that minimizes the probability of drawdown under the mean-variance premium principle, and observed that the optimal reinsurance strategy is identical to the one for minimizing the probability of ruin.

Due to its importance, the issue of absolute ruin problem has attracted much attention in the literature. The study of absolute ruin probability helps us to investigate the behavior of a company when it is in deficit. It is obvious that such an investigation cannot be done in the context of traditional ruin. In some articles, absolute ruin occurs when the premiums received are not sufficient to make the interest payments on the debt; see, for example, Dassios and Embrechts,[12] Cai,[13] Gerber and Yang,[14] and Zhou and Cai.[15] However, if there is a diffusion term in the surplus process, typically, when investment in a Black–Scholes risky asset is introduced, or the surplus process itself is modeled or perturbed by a Brownian motion, then the surplus process has a positive probability to bounce back from any negative level. In this spirit, Luo and Taksar[2] defined the absolute ruin as an event that *liminf* of the surplus process is negative infinity, and apply the HJB method to obtain the optimal proportional reinsurance and investment strategy that minimizes the probability of absolute ruin. It turns out that the value function is not always convex as the one in the case of traditional ruin. Liang and Long[16] investigated the same optimization problem under the assumption that the insurer's liabilities and capital gains in financial market are negatively correlated. They found that the solutions are S-shaped and the optimal strategy fails to be monotonic or continuous. Note that the results of Luo and Taksar[2] and Liang and Long[16] were derived under some restriction on the investment control. Furthermore, Bi and Zhang[17] relaxed the investment restriction in Luo and Taksar[2] and assumed that the claim process is correlated to the price process of the risky asset. They derived the convex minimum absolute ruin function explicitly and came to the conclusion that it is the restriction on investment that leads to the kink of minimal absolute ruin function.

Inspired by the above-mentioned works, we consider the optimal reinsurance-investment problem for an insurer who wishes to minimize the probability of absolute ruin defined in Luo and Taksa.[2] In our model set-up, the insurer can purchase per-loss reinsurance and invest its surplus in a financial market consisting of one risky asset and one risk-free asset, and the reinsurance premium is computed according to the mean-variance premium principle which combines the expected-value and variance premium principles. It is also assumed that short-selling is prohibited, and that borrowing is allowed but only to the extent that the level of the risky asset does not exceed a priori given amount. To make the optimization problem tractable, we further assume that the surplus process of the insurer follows the diffusion approximation of the classical Cramér–Lundberg model. By the technique of stochastic dynamic programming, we derive the corresponding HJB equation which has the same form as the one with the objective of minimizing the probability of traditional ruin. A verification theorem is then proved to show that a decreasing C^2 solution to the HJB equation coincides with the minimum absolute ruin function. Similar to Luo and Taksar,[2] the solution is no longer convex as the one in the classical case, but is S-shaped with a unique point of inflection. In addition, we investigate a similar optimization problem in which absolute ruin occurs if the surplus falls below a critical level (see Dassios and Embrechts[12]). It is interesting to find that this critical level exactly equals to the above-mentioned point of inflection, and thus the value function for this modified version of absolute ruin can be derived directly by means of our original analysis under the *liminf* definition of absolute ruin. We also find that the optimal strategies for these two different absolute ruin problems are the same if the surplus falls into some common regions. Moreover, we study the optimal results when the investment strategy is considered without constraint and show that the restriction on investment can result in the kink of minimal absolute ruin function, which is similar to the conclusion in Bi and Zhang.[17]

The rest of the article is organized as follows. In Section 2, we present the model and optimization problem. Explicit expressions for the optimal reinsurance-investment strategy and the corresponding minimum absolute ruin function are derived in Section 3. In Section 4, we present some properties of the optimal strategy and carry out some numerical analysis to show the impact of model parameters. Finally, we conclude the article in Section 5.

2 | MODEL AND PROBLEM FORMULATION

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with filtration $\{\mathcal{F}_t\}_{t\geq 0}$ containing all the objects defined in the following. We first introduce the classical Cramér–Lundberg risk model

$$X_t = u + ct - \sum_{i=1}^{N_t} Y_i, \tag{1}$$

where $u \geq 0$ is the initial surplus, and $c > 0$ is the premium rate. Moreover, $N = \{N_t\}_{t\geq 0}$ is a homogeneous Poisson process with intensity $\lambda > 0$, Y_i represents the size of the ith claim, and the positive claim sizes $Y_1, Y_2, \ldots$ are independent and identically distributed random variables and independent of N. Let Y be a generic random variable which has the same distribution as Y_i. Then we denote the common cumulative distribution function of $Y_i, i = 1, 2, \ldots,$ by $F_Y(y)$ with $F_Y(0) = 0$ and $0 < F_Y(y) < 1$ for $y > 0$. It is assumed that $\mathbb{E}(Y) < \infty$ and $\mathbb{E}(Y^2) < \infty$.

Suppose that the insurer can reinsure its claims with per-loss reinsurance via a continuously payable premium computed according to the mean-variance premium principle. Let $R_t(y)$ denote the retained claim at time t, as a function of the (possible) claim $Y = y$ at that time, thus, $y - R_t(y)$ is the amount of each claim transferred to the reinsurer. Then, the mean-variance premium rate at time t associated with R_t is given by

$$(1+\theta)\lambda\mathbb{E}(Y - R_t) + \frac{\eta}{2}\,\lambda\mathbb{E}((Y - R_t)^2), \tag{2}$$

where θ and η are the nonnegative risk loading parameters. If $\theta = 0$, then (2) reduces to the variance premium principle; similarly, if $\eta = 0$, then (2) reduces to the expected-value premium principle. A retention strategy $R = \{R_t\}_{t\geq 0}$ is said to be *admissible* if it (i) is adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$, (ii) is a nondecreasing function with respect to y, and (iii) satisfies $0 \leq R_t(y) \leq y$, for all $t \geq 0$ and $y \geq 0$.

Let $U = \{U_t\}_{t\geq 0}$ be the surplus process associated with a reinsurance strategy $R = \{R_t\}_{t\geq 0}$, that is, U_t is the surplus of the insurer at time t under R. Furthermore, we assume that the insurer can invest its surplus in a risky asset (stock or mutual fund) and a risk-free asset (bond or bank account) with rate $r > 0$. Specifically, the process of the risky asset follows a geometric Brownian motion given by

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

where $\mu > r$, $\sigma > 0$, and $W = \{W_t\}_{t\geq 0}$ is a standard Brownian motion. Let π_t denote the amount invested in the risky asset at time $t \geq 0$, and the rest of the surplus $(U_t - \pi_t)$ is invested in the risk-free asset. An investment strategy $\pi = \{\pi_t\}_{t\geq 0}$ is said to be *admissible* if it (i) is adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$, (ii) satisfies $\int_0^t \pi_s^2 ds < \infty$ with probability one for all $t \geq 0$, and (iii) holds that $0 \leq \pi_t \leq A$ almost surely for a priori given number $A > 0$ for all $t \geq 0$. Denote the set of admissible pairs of strategies (R, π) by $\mathcal{D}$. Thus, given a pair of controls $\nu = \{R_t, \pi_t\}_{t\geq 0} \in \mathcal{D}$, the controlled surplus process has the following dynamics

$$dU_t = \left(rU_t + c + (\mu - r)\pi_t - (1+\theta)\lambda\mathbb{E}(Y - R_t) - \frac{\eta}{2}\,\lambda\mathbb{E}((Y - R_t)^2)\right)dt + \sigma\pi_t dW_t - R_t dN_t. \tag{3}$$

To avoid mathematical trivialities, we assume that

$$\lambda\mathbb{E}(Y) < c < (1+\theta)\lambda\mathbb{E}(Y) + \frac{\eta}{2}\,\lambda\mathbb{E}(Y^2),$$

that is, the insurer's premium income is greater than the expected value of the claims but less than the premium for full reinsurance. Let κ be the positive difference

$$\kappa = (1+\theta)\lambda\mathbb{E}(Y) + \frac{\eta}{2}\,\lambda\mathbb{E}(Y^2) - c. \tag{4}$$

We solve this optimization problem by approximating the jump process in (3) by a diffusion (see Bai et al.,[4] Grandell,[18] and Liang and Yuen[19]). Specifically,

$$R_t\, dN_t \approx \lambda\mathbb{E}(R_t)\, dt - \sqrt{\lambda\mathbb{E}(R_t^2)}\, dB_t,$$

where $B = \{B_t\}_{t\geq 0}$ is another standard Brownian motion which is independent of W. The resulting process $\hat{U} = \{\hat{U}_t\}_{t\geq 0}$, then, evolves according to the dynamics

$$d\hat{U}_t = \left(r\hat{U}_t - \kappa + (\mu - r)\pi_t + \theta\lambda\mathbb{E}(R_t) + \eta\lambda\mathbb{E}(YR_t) - \frac{\eta}{2}\,\lambda\mathbb{E}(R_t^2)\right)dt + \sigma\pi_t dW_t + \sqrt{\lambda\mathbb{E}(R_t^2)}\, dB_t, \tag{5}$$

with initial surplus $\hat{U}_0 = u$.

In the literature, some results suggest that an insurer can apply the resulting per-loss reinsurance strategy to the claims of the original compound Poisson model even though we use the diffusion approximation to obtain the optimal retention strategy. For example, Liang and Young[10] presented numerical comparisons between the optimal retention strategies under the compound Poisson risk model and its diffusion approximation when reinsurance is priced according to the expected-value premium principle with the optimal strategy taking the form of excess-of-loss reinsurance. They showed that the optimal deductibles in both cases are close when the surplus is not too small, even when λ is small. Besides, Liang et al.[20] also investigated the connection between the results of the compound Poisson risk model and its diffusion approximation. They proved that, under an appropriate scaling of the classical risk process, the minimum probability of ruin converges to the one under the diffusion approximation.

In this article, we define *absolute ruin* as the event where the *liminf* of the surplus is equal to negative infinity (see Luo and Taskar[2] and Liang and Long[16]). Our objective is to minimize the probability of absolute ruin by selecting the optimal reinsurance-investment control policy ν. Thus, the event of absolute ruin under policy ν is defined as

$$\Theta^{\nu} = \{\omega \in \Omega : \liminf_{t\to\infty} \hat{U}_t = -\infty\}.$$

The minimum probability of absolute ruin ϕ is then given by

$$\phi(u) = \inf_{\nu\in D} \mathbb{P}_u(\Theta^{\nu}), \tag{6}$$

where $\mathbb{P}_u$ denotes the probability conditional on $\hat{U}_0 = u$. Therefore, the resulting optimization problem is an infinite-time horizon stochastic problem. Note that if the surplus $\hat{U}_t$ in (5) is negative, the company does not become bankrupt; rather it is allowed to borrow money to continue its business. We also point out that if the value of the surplus is greater than or equal to

$$u_s = \frac{\kappa}{r}, \tag{7}$$

with κ defined in (4), then the insurer can buy full reinsurance and invest all the surplus in the risk-free asset to earn interest rate r, and thus the surplus will never drop below its current value, that is, absolute ruin cannot occur in this case. For this reason, we call u_s the *safe level*. Therefore, it remains for us to determine the minimum probability of absolute ruin ϕ on the domain $(-\infty, u_s]$ in the following context.

3 | MINIMIZING THE PROBABILITY OF ABSOLUTE RUIN

In this section, for the risk model (5), we compute the optimal reinsurance strategy to minimize the probability of absolute ruin on region $(-\infty, u_s]$. In Section 3.1, we prove a verification theorem for finding ϕ. Then, in Section 3.2, we derive the explicit expressions for the optimal reinsurance and investment strategy and the corresponding minimum probability of absolute ruin. Finally, in Section 3.3, when absolute ruin is defined as in Dassios and Embrechts,[12] the same optimization problem is studied and the corresponding optimal results are obtained as well. Moreover, inspired by Bi and Zhang,[17] we compare the optimal results with those derived in the case that the restriction on the investment strategy in (5) is not imposed. Some interesting points are observed through the comparison.

3.1 | Verification theorem

For a given admissible strategy $\nu = (R, \pi)$ and any C^2 function on $(-\infty, u_s]$, define the differential operator $\mathcal{A}^{\nu}$ on appropriately differentiable functions as

$$\mathcal{A}^{\nu}h(u) = \left(ru - \kappa + (\mu - r)\pi + \theta\lambda\mathbb{E}(R) + \eta\lambda\mathbb{E}(YR) - \frac{\eta}{2}\,\lambda\mathbb{E}(R^2)\right) h_u + \frac{1}{2}\left(\sigma^2\pi^2 + \lambda\mathbb{E}(R^2)\right) h_{uu}. \tag{8}$$

This together with Lemmas A and B in the Appendix give the following verification theorem which can be proved by mimicking the corresponding proof given in Luo and Taksar.[2] So, we omit the details here.

Theorem 1. *Let V be a decreasing C^2 solution with bounded first derivative to the HJB equation $\inf_{v\in D}\mathcal{A}^v V(u)=0$ on $(-\infty,u_s]$, and satisfy the boundary conditions $V(-\infty)=1$ and $V(u_s)=0$. Then the value function ϕ defined in (6) coincides with V. Furthermore, let v^* be such that $\mathcal{A}^{v^*}V(u)=0$ for all $-\infty<u\le u_s$. Then the policy $v_t^*=(R_t^*,\pi_t^*)$ defined in feedback form via $R_t^*=R_t^*(\hat{U}_t^*,Y)$ and $\pi_t^*=\pi_t^*(\hat{U}_t^*)$ is the optimal reinsurance-investment strategy. Here, $\hat{U}_t^*$ denotes the optimally controlled surplus process under the optimal policy v^*.*

Remark 1. Lemma B shows that absolute ruin can be redefined as the event where the controlled process tends to negative infinity. By letting $M\to-\infty$ and $N\to u_s$ in Lemma A, we can see that the surplus process diverges to either safe level or negative infinity with probability 1. This shows the ergodicity of the controlled process.

3.2 Probability of absolute ruin

Consider the following control regions:

$$\begin{aligned}
\mathcal{O}_{V_1}&=\{-\infty<u\le u_s : V''(u)>0, \pi_t\le A\},\\
\mathcal{O}_{V_2}&=\{-\infty<u\le u_s : V''(u)>0, \pi_t> A\},\\
\mathcal{O}_{V_3}&=\{-\infty<u\le u_s : V''(u)\le 0\}.
\end{aligned}\tag{9}$$

Define

$$\begin{aligned}
\mathcal{L}(u,R,\pi)&=(ru-\kappa+(\mu-r)\pi)\,V'(u)+\frac{1}{2}\sigma^2\pi^2V''(u)\\
&\quad+\left(\theta\lambda\mathbb{E}\left(R\right)+\eta\lambda\mathbb{E}\left(YR\right)-\frac{\eta}{2}\,\lambda\mathbb{E}\left(R^2\right)\right)V'(u)+\frac{1}{2}\lambda\mathbb{E}\left(R^2\right)V''(u).
\end{aligned}\tag{10}$$

By using the cumulative distribution function of Y, we can rewrite $\mathcal{L}$ as

$$\begin{aligned}
\mathcal{L}(u,R,\pi)&=(ru-\kappa+(\mu-r)\pi)V'(u)+\frac{1}{2}\sigma^2\pi^2V''(u)\\
&\quad+\int_0^\infty\left\{\left(\theta\lambda R(y)+\eta\lambda yR(y)-\frac{\eta}{2}\lambda R^2(y)\right)V'(u)+\frac{1}{2}\lambda R^2(y)V''(u)\right\}dF_Y(y).
\end{aligned}\tag{11}$$

From this integral representation of $\mathcal{L}$, we can minimize $\mathcal{L}$ by minimizing the integrand for $0\le R(y)\le y$. Differentiating (11) with respect to R and π and setting the derivatives zero, we obtain

$$\hat{R}(u,y)=\frac{\theta+\eta y}{\beta(u)}\wedge y,\quad \hat{\pi}(u)=-\frac{u-r}{\sigma^2}\frac{1}{\eta-\beta(u)},\tag{12}$$

with $\beta(u)=\eta-\frac{V''(u)}{V'(u)}$. To simplify our analysis, we define the following functions

$$g_1(\beta)=\mathbb{E}R=\int_0^{\frac{\theta}{\beta-\eta}}S_Y(y)dy+\frac{\eta}{\beta}\int_{\frac{\theta}{\beta-\eta}}^\infty S_Y(y)dy,\tag{13}$$

$$g_2(\beta)=\mathbb{E}(YR)=2\int_0^{\frac{\theta}{\beta-\eta}}yS_Y(y)dy+\frac{1}{\beta}\int_{\frac{\theta}{\beta-\eta}}^\infty(\theta+2\eta y)S_Y(y)dy,\tag{14}$$

and

$$g_3(\beta)=\mathbb{E}(R^2)=2\int_0^{\frac{\theta}{\beta-\eta}}yS_Y(y)dy+\frac{2\eta}{\beta^2}\int_{\frac{\theta}{\beta-\eta}}^\infty(\theta+\eta y)S_Y(y)dy,\tag{15}$$

where $S_Y=1-F_Y$. In the following lemmas, we study the strategy (12) for each region.

Lemma 1. *For region $\mathcal{O}_{V_1}$, the extreme minimum point of $\mathcal{L}$ is given by*

$$R^*(u,y) = \frac{\theta + \eta y}{\beta_1^*(u)} \wedge y, \quad \pi^*(u) = -\frac{u-r}{\sigma^2}\frac{1}{\eta - \beta_1^*(u)}, \tag{16}$$

where $\beta_1^* > \eta$ uniquely solves $G_1(u,\beta) = 0$ with

$$G_1(u,\beta) = ru - \kappa + \theta\lambda g_1(\beta) + \eta\lambda g_2(\beta) - \frac{\beta}{2}\lambda\, g_3(\beta) - \frac{1}{2}\frac{(\mu-r)^2}{\sigma^2}\frac{1}{\eta-\beta}. \tag{17}$$

Under this strategy, the corresponding HJB equation becomes

$$(\beta_1^*(u) - \eta)V'(u) + V''(u) = 0.$$

Proof. Note that $\beta(u) = \eta - \frac{V''(u)}{V'(u)} > \eta$ in region $\mathcal{O}_{V_1}$. Thus, we have $\hat{R}(u) > 0$ and $\hat{\pi}(u) > 0$. Let $(R^*, \pi^*) = (\hat{R}, \hat{\pi})$, that is, $\hat{\pi}(u) \le A$. Inserting (12) into (10) yields

$$\mathcal{L}(u,\beta) = V'(u)G_1(u,\beta), \tag{18}$$

where we slightly abuse the notation of $\mathcal{L}$ by replacing its arguments (R,π) with β. We want to show that $\mathcal{L}$ has a unique zero at $\beta_1^*(u) > \eta$. First, we have

$$\lim_{\beta\to\eta^+} ru - \kappa + \theta\lambda g_1(\beta) + \eta\lambda g_2(\beta) - \frac{\beta}{2}\lambda\, g_3(\beta) = ru + c - \lambda\mathbb{E}(Y),$$

and

$$\lim_{\beta\to\eta^+} -\frac{1}{2}\frac{(\mu-r)^2}{\sigma^2}\frac{1}{\eta-\beta} = +\infty.$$

Thus, $G_1(u,\eta^+) \to \infty$ for any $u > -\infty$. Besides, it is not difficult to see that

$$\lim_{\beta\to\infty} G_1(u,\beta) = ru - \kappa \le 0.$$

Finally,

$$\begin{aligned}
G_1'(u,\beta) &= \theta\lambda g_1'(\beta) + \eta\lambda g_2'(\beta) - \frac{1}{2}\lambda\, g_3(\beta) - \frac{\beta}{2}\lambda\, g_3'(\beta) - \frac{1}{2}\frac{(\mu-r)^2}{\sigma^2}\frac{1}{(\eta-\beta)^2} \\
&= \frac{\beta}{2}\lambda\, g_3'(\beta) - \frac{1}{2}\lambda\, g_3(\beta) - \frac{\beta}{2}\lambda\, g_3'(\beta) - \frac{1}{2}\frac{(\mu-r)^2}{\sigma^2}\frac{1}{(\eta-\beta)^2} \\
&= -\frac{1}{2}\lambda\, g_3(\beta) - \frac{1}{2}\frac{(\mu-r)^2}{\sigma^2}\frac{1}{(\eta-\beta)^2} < 0.
\end{aligned}$$

The second line follows from the fact that $\theta g_1'(\beta) + \eta g_2'(\beta) = \frac{\beta}{2} g_3'(\beta)$. Thus, G_1 has a unique zero at $\beta_1^*(u) > \eta$, which implies that $\mathcal{L}$ in (18) indeed has a unique zero at $\beta_1^*(u) > \eta$. Here, we would like to point out that $\beta_1^*(u)$ depends on the value of the surplus u via the term $ru - \kappa$, which measures how far the surplus is below the safe level $u_s = \kappa/r$. ∎

In the following proposition, we study the property of the function $\beta_1^*(u)$.

Proposition 1. *For any $u \in \mathcal{O}_{V_1}$, $\beta_1^*(u)$ is an increasing function with respect to u. When u goes to u_s, we have*

$$\beta_1^*(u) \approx \frac{\lambda(\theta^2 + 2\theta\eta\mathbb{E}(Y) + \eta^2\mathbb{E}(Y^2)) + (\frac{\mu-r}{\sigma})^2}{2(\kappa - ru)}. \tag{19}$$

Proof. Note that $\beta_1^*(u)$ satisfies the equation $G_1(u, \beta_1^*(u)) = 0$. Taking the derivative with respect to u on both sides and simplifying the expression, we obtain

$$\frac{r}{(\beta_1^*(u))'} = \frac{1}{2}\lambda\, g_3(\beta_1^*(u)) + \frac{1}{2}\frac{(\mu-r)^2}{\sigma^2}\frac{1}{(\eta-\beta_1^*(u))^2}.$$

It is not difficult to see that $\beta_1^*(u)$ is an increasing function with respect to u because the right-hand side of the equality is positive and $r > 0$. Besides, we have $\beta_1^*(u) \to \infty$ as $u \to u_s$. Then it follows that

$$\lim_{\beta\to\infty}\beta\left(\theta g_1(\beta) + \eta g_2(\beta) - \frac{\beta}{2}\, g_3(\beta)\right) = \frac{1}{2}\theta^2 + \theta\eta\mathbb{E}(Y) + \frac{1}{2}\eta^2\mathbb{E}(Y^2),$$

from which we can derive the approximation given in (19) directly. ■

Remark 2. Since $\beta_1^*(u)$ is a strictly increasing function with respect to u, and $\beta_1^*(u)$ approaches ∞ when u tends to u_s, there exists a unique $u_1 < u_s$ such that $\beta_1^*(u_1) = \eta + \frac{\mu-r}{\sigma^2 A}$. It is not difficult to verify that $\pi^* \leq A$ when $u \geq u_1$, and $\pi^* > A$ when $u < u_1$. Therefore, we deduce that $\mathcal{O}_{V_1} = [u_1, u_s]$.

Lemma 2. *For region $\mathcal{O}_{V_2}$, the extreme minimum point of $\mathcal{L}$ is given by*

$$R^*(u,y) = \frac{\theta+\eta y}{\beta_2^*(u)} \wedge y, \quad \pi^*(u) = A, \tag{20}$$

where $\beta_2^* > \eta$ uniquely solves $G_2(u, \beta) = 0$ with

$$G_2(u,\beta) = ru - \kappa + (\mu-r)A + \frac{1}{2}\sigma^2A^2(\eta - \beta(u)) + \theta\lambda g_1(\beta) + \eta\lambda g_2(\beta) - \frac{\beta}{2}\lambda\, g_3(\beta). \tag{21}$$

Under this strategy, the corresponding HJB equation becomes

$$(\beta_2^*(u) - \eta)V'(u) + V''(u) = 0.$$

Proof. In region $\mathcal{O}_{V_2}$, we have $V''(u) > 0$ and $\pi_t > A$. It is easy to verify that $\pi_t = A$ minimizes $\mathcal{L}(u, R, \pi)$ over $0 \leq \pi \leq A$. Putting $\pi^* = A$ into (10) and differentiating $\mathcal{L}$ with respect to R yield

$$\overline{R}(u,y) = \frac{\theta+\eta y}{\beta(u)} \wedge y, \tag{22}$$

which is in the same form as $\hat{R}(u, y)$. Since $\overline{R} > 0$, inserting $(R, \pi) = (\overline{R}, A)$ into (10) gives

$$\mathcal{L}(u,\beta) = V'(u)G_2(u,\beta). \tag{23}$$

We want to show that $\mathcal{L}$ has a unique zero at $\beta_2^*(u) > \eta$. By noting that

$$\lim_{\beta\to\eta^+} G_2(u,\beta) = ru + c - \lambda\mathbb{E}(Y) + (\mu-r)A.$$

and $\lim_{\beta\to\infty} G_2(u, \beta) = -\infty$, we have

$$\begin{aligned}
G_2'(u,\beta) &= \theta\lambda g_1'(\beta) + \eta\lambda g_2'(\beta) - \frac{1}{2}\lambda\, g_3(\beta) - \frac{\beta}{2}\lambda\, g_3'(\beta) - \frac{1}{2}\sigma^2A^2 \\
&= \frac{\beta}{2}\lambda\, g_3'(\beta) - \frac{1}{2}\lambda\, g_3(\beta) - \frac{\beta}{2}\lambda\, g_3'(\beta) - \frac{1}{2}\sigma^2A^2 \\
&= -\frac{1}{2}\lambda\, g_3(\beta) - \frac{1}{2}\sigma^2A^2 < 0.
\end{aligned}$$

For notational convenience, we denote

$$u_2 = \frac{-(\mu-r)A - c + \lambda\mathbb{E}(Y)}{r}. \tag{24}$$

It turns out that when $u > u_2$, G_2 in (21) has a unique zero at $\beta_2^*(u) > \eta$, which implies that $\mathcal{L}$ in (23) has a unique zero at $\beta_2^*(u) > \eta$. We can see that β_2^* depends on the value of the surplus u via the term $ru + c - \lambda\mathbb{E}(Y) + (\mu - r)A$, which measures how far the surplus is above the value of u_2. ■

We now present the property of the function $\beta_2^*(u)$ in the following proposition.

Proposition 2. *For any $u \in \mathcal{O}_{V_2}$, $\beta_2^*(u)$ is an increasing function with respect to u. When u goes to u_2, we have*

$$\beta_2^*(u) \approx \frac{2(ru - \kappa + (\mu - r)A + \frac{1}{2}\sigma^2A^2\eta + \theta\lambda\mathbb{E}(Y) + \eta\lambda\mathbb{E}(Y^2))}{\lambda\mathbb{E}(Y^2) + \sigma^2A^2}. \tag{25}$$

Proof. Note that $\beta_2^*(u)$ satisfies the equation $G_2(u, \beta_2^*(u)) = 0$. Then taking the derivative with respect to u on both sides and simplifying the expression yield

$$\frac{r}{(\beta_2^*(u))'} = \frac{1}{2}\lambda\, g_3(\beta_2^*(u)) + \frac{1}{2}\sigma^2A^2.$$

Similarly, since the right-hand side of the equality is positive and $r > 0$, $\beta_2^*(u)$ is an increasing function with respect to u. Besides, Lemma 2 shows that $\beta_2^*(u) \to \eta$ as $u \to u_2$. As a result, the approximation of $\beta_2^*(u)$ given in (25) can be derived readily. ■

Remark 3. Because $\beta_2^*(u)$ is a strictly increasing function with respect to u and the equalities $\beta_1^*(u_1) = \beta_2^*(u_1) = \eta + \frac{\mu - r}{\sigma^2A}$ hold, we have $u_2 < u_1$. Therefore, we deduce that $\mathcal{O}_{V_2} = (u_2, u_1)$. In addition, we should point out that u_2 is negative due to the fact that $c > \lambda\mathbb{E}(Y)$.

Lemma 3. *For region $\mathcal{O}_{V_3}$, the extreme minimum point of $\mathcal{L}$ is given by*

$$R^*(u, y) = y, \quad \pi^*(u) = A. \tag{26}$$

Under this strategy, the corresponding HJB equation becomes

$$(ru + c + (\mu - r)A - \lambda\mathbb{E}(Y))V'(u) + \frac{1}{2}(\sigma^2A^2 + \lambda\mathbb{E}(Y^2))V''(u) = 0. \tag{27}$$

Proof. In region $\mathcal{O}_{V_3}$, we have $\hat{\pi}(u) < 0$ because $V''(u) < 0$. It is easy to verify that $\pi^* = A$ minimizes the expression (10). We now try to find the optimal reinsurance strategy R^*. Rewrite $\mathcal{L}$ as

$$\begin{aligned}\mathcal{L}(u, R, \pi) = &\,(ru - \kappa + (\mu - r)A)V'(u) + \frac{1}{2}\sigma^2A^2V''(u)\\ &+ \int_0^\infty \left\{(\theta\lambda R(y) + \eta\lambda yR(y))\,V'(u) + \frac{1}{2}\lambda R^2(y)(V''(u) - \eta V'(u))\right\} dF_Y(y).\end{aligned} \tag{28}$$

First, if the inequality $V''(u) - \eta V'(u) > 0$ holds, then $R^* = y$ minimizes the expression (28) since $\mathcal{L}$ is a quadratic function and $\overline{R}(u) > y$. Second, if $V''(u) - \eta V'(u) < 0$, we have $R^* = y$ because of $\overline{R}(u) < 0$. Finally, if $V''(u) - \eta V'(u) = 0$, $\pi^* = A$ and $R^* = y$ also minimize the function $\mathcal{L}$. ■

Remark 4. One can observe from (27) that $V''(u) \leq 0$ for all $u \leq u_2$. Thus, $\mathcal{O}_{V_3} = (-\infty, u_2]$.

In the following lemma, we show that there indeed exists large enough ϑ such that $C_t \geq \frac{\kappa - rN}{2} > 0$ for all $0 \leq t \leq \tau$ with probability 1. We discover that the derivation of the above optimal strategies facilitates the proof of the following lemma greatly.

Lemma 4. *There exists ϑ given by*

$$\vartheta = \begin{cases} \tilde{\beta}_1 - \eta, & 2u_1 - u_s \leq N \leq u_s,\\ \tilde{\beta}_2 - \eta, & 2u_2 - u_s < N < 2u_1 - u_s,\\ \tilde{\beta}_3 - \eta, & -\infty < N \leq 2u_2 - u_s,\end{cases} \tag{29}$$

such that $C_t \geq \frac{\kappa - rN}{2}$, where C_t is defined in (A2) of the Appendix; $\tilde{\beta}_i \in [\eta, \infty)$ $(i=1,2)$ uniquely solves $G_i\left(\frac{N}{2} + \frac{\kappa}{2r}, \beta\right) = 0$ with G_i defined in (17) and (21), respectively; and $\tilde{\beta}_3$ is given by

$$\tilde{\beta}_3 = \eta + \frac{2\left((\mu - r)A - \lambda\mathbb{E}(Y) + c + \frac{rN+\kappa}{2}\right)}{\lambda\mathbb{E}(Y^2) + \sigma^2 A^2}.$$

Proof. Recall that $\tau = \tau_M \wedge \tau_N$. Since $\hat{U}_t \leq N < u_s = \kappa/r$ for all $0 \leq t \leq \tau$, we have $\kappa - r\hat{U}_t \geq \kappa - rN > 0$. We then find $\vartheta > 0$ such that

$$-(\mu - r)\pi_t - \theta\lambda\mathbb{E}(R_t) - \eta\lambda\mathbb{E}(YR_t) + \frac{\eta + \vartheta}{2}\,\lambda\mathbb{E}(R_t^2) + \frac{1}{2}\vartheta\sigma^2\pi_t^2 \geq \frac{rN - \kappa}{2},$$

or equivalently,

$$\frac{1}{2}\vartheta \geq \frac{(\mu - r)\pi_t + \theta\lambda\mathbb{E}(R_t) + \eta\lambda\mathbb{E}(YR_t) - \frac{\eta}{2}\,\lambda\mathbb{E}(R_t^2) + \frac{rN-\kappa}{2}}{\lambda\mathbb{E}(R_t^2) + \sigma^2\pi_t^2}, \tag{30}$$

for all admissible strategies $\nu = (R, \pi)$. In other words, we want to show that the right side of the inequality (30) has a finite maximum over such ν. By comparing the right side of the inequality with $\mathcal{L}$ in (10), the problem is essentially identical to the ones in Lemmas 1–3. Making use of the proofs of these lemmas, we deduce that for any $N \in (-\infty, u_s)$, the right side of the inequality (30) is maximized by

$$(\tilde{R}, \tilde{\pi}) = \begin{cases} (y, A), & -\infty < N \leq 2u_2 - u_s, \\ \left(\frac{\theta+\eta y}{\tilde{\beta}_2} \wedge y, A\right), & 2u_2 - u_s < N \leq 2u_1 - u_s, \\ \left(\frac{\theta+\eta y}{\tilde{\beta}_1} \wedge y, -\frac{u-r}{\sigma^2}\frac{1}{\eta-\tilde{\beta}_1}\right), & 2u_1 - u_s < N \leq u_s, \end{cases}$$

where $\tilde{\beta}_i \in (\eta, \infty)$ $(i=1,2)$ uniquely solves $G_i(\frac{N}{2} + \frac{\kappa}{2r}, \beta) = 0$. By substituting $(\tilde{R}, \tilde{\pi})$ into the right side of (30) and using ϑ of (29), we have $C_t \geq \frac{\kappa - rN}{2} > 0$ for all $0 \leq t \leq \tau$ with probability 1. ■

Write the functions

$$F_1(u) = \int_{-\infty}^{u} \exp\left\{-\int_{-\infty}^{y} (\beta_3^*(w) - \eta)dw\right\} dy, \tag{31}$$

$$F_2(u) = \int_{u_2}^{u} \exp\left\{-\left(\int_{-\infty}^{u_2} (\beta_3^*(w) - \eta) + \int_{u_2}^{y} (\beta_2^*(w) - \eta)\right) dw\right\} dy, \tag{32}$$

and

$$F_3(u) = \int_{u_1}^{u} \exp\left\{-\left(\int_{-\infty}^{u_2} (\beta_3^*(w) - \eta) + \int_{u_2}^{u_1} (\beta_2^*(w) - \eta) + \int_{u_1}^{y} (\beta_1^*(w) - \eta)\right) dw\right\} dy. \tag{33}$$

In the following theorem, based on the analysis of Lemmas 1–3, we compute our candidate minimum probability of absolute ruin. In addition, it follows from the verification theorem in Section 3.1 that the resulting function is indeed the minimum probability of absolute ruin on $(-\infty, u_s]$.

Theorem 2. *When $-\infty < u \leq u_s$, the minimum probability of absolute ruin $\phi(u)$ for the surplus process (5) is given by*

$$\phi(u) = \begin{cases} 1 - \frac{F_1(u)}{F_1(u_2)+F_2(u_1)+F_3(u_s)}, & -\infty < u \leq u_2, \\ 1 - \frac{F_1(u_2)+F_2(u)}{F_1(u_2)+F_2(u_1)+F_3(u_s)}, & u_2 < u \leq u_1, \\ 1 - \frac{F_1(u_2)+F_2(u_1)+F_3(u)}{F_1(u_2)+F_2(u_1)+F_3(u_s)}, & u_1 < u \leq u_s, \end{cases} \tag{34}$$

where F_i $(i=1,2,3)$ *are given by (31), (32) and (33), respectively;* β_i^* $(i=1,2)$ *uniquely solve* $G_i(u,\beta)=0$ *with* G_i *given in (17) and (21), respectively;* β_3^* *is given by*

$$\beta_3^*(u)=\eta+\frac{2[ru+c+(\mu-r)A-\lambda\mathbb{E}(Y)]}{\sigma^2A^2+\lambda\mathbb{E}(Y)^2};$$

u_1 *is the surplus value such that* $\beta_1^*(u_1)=\eta+\frac{\mu-r}{\sigma^2A}$; *and* u_2 *is given in (24). The corresponding optimal reinsurance-investment strategy is given by* $(R^*,\pi^*)=(R^*(\hat{U}_t^*,Y),\pi^*(\hat{U}_t^*))$, *where*

$$(R^*(u,y),\pi^*(u))=\begin{cases}(y,A), & -\infty<u\le u_2,\\ \left(\frac{\theta+\eta y}{\beta_2^*(u)}\wedge y,A\right), & u_2<u\le u_1,\\ \left(\frac{\theta+\eta y}{\beta_1^*(u)}\wedge y,\frac{\mu-r}{\sigma^2(\beta_1^*(u)-\eta)}\right), & u_1<u\le u_s,\end{cases} \tag{35}$$

$\hat{U}_t^*$ *is the optimally controlled surplus at time t, and Y is the possible claim size at that time.*

Proof. First, we calculate the function V under the reinsurance-investment strategy (35). Under this strategy, V solves following boundary-value problem

$$\begin{cases}(\beta_1^*(u)-\eta)V'(u)+V''(u)=0, & u_1<u\le u_s,\\ (\beta_2^*(u)-\eta)V'(u)+V''(u)=0, & u_2<u\le u_1,\\ (\beta_3^*(u)-\eta)V'(u)+V''(u)=0, & -\infty<u\le u_2,\end{cases} \tag{36}$$

with boundary conditions

$$V(-\infty)=1,\quad V(u_s)=0.$$

Let V be the right-hand side of (34). Then it is clear that V solves this boundary-value problem. Furthermore, it is straightforward to show that V satisfies the conditions of Theorem 1. Thus, the probability of absolute ruin $\phi(u)$ and the corresponding reinsurance-investment strategy are indeed given by (34) and (35), respectively. ∎

Remark 5. The explicit solutions in Theorem 2 give insights into how an insurance company reacts when it is in deficit. It is interesting to note that, the minimum absolute ruin function ϕ is S-shaped with u_2 being the point of inflection. This does not appear in the traditional ruin probability minimization problem. Besides, we can see from (35) that the optimal feedback controls are monotone functions of the surplus level, and become more conservative as the surplus grows. Specifically, when the surplus level increases, the insurer prefers to invest less amount into the risky asset and purchase less reinsurance. We note here that the monotonicity of the optimal controls is also seen in the case of traditional ruin minimization. In particular, by setting $\eta=0$ or $\theta=0$ in (2), the mean-variance premium principle reduces to the expected-value or variance premium principle, respectively. We observe from (35) that the optimal reinsurance strategy is in the form of pure excess-of-loss reinsurance strategy under the expected-value principle, and that, under the variance premium principle, the optimal reinsurance strategy is in the form of the pure quota-share reinsurance. Both of these results are expected based on the work of Hipp and Taksar.[21]

3.3 | Some special cases

In this section, we compare the optimal results with those in the case where absolute ruin is defined as the first time when the drift coefficient in (2) turns negative, or in other words, the premiums received by the insurers are not sufficient to cover its debt. This modified version of absolute ruin was studied by many authors including Dassios and Embrechts[12] and Cai.[13] Specifically, we define

$$\tau_0=\inf\left\{t\ge 0:\ r\hat{U}_t-\kappa+(\mu-r)\pi_t+\theta\lambda\mathbb{E}(R_t)+\eta\lambda\mathbb{E}(YR_t)-\frac{\eta}{2}\,\lambda\mathbb{E}(R_t^2)<0\right\}.$$

Then the value function associated with this control problem is denoted by

$$\psi(u) = \inf_{v\in D} \mathbb{P}_u\{\tau_0 < \infty\}. \tag{37}$$

Note that if the insurer does not take any reinsurance and invests amount A into the risky asset, it turns out that the drift coefficient of the controlled process equals $r\hat{U}_t + (\mu - r)A + c - \lambda\mathbb{E}(Y)$, which is the maximum value of the drift coefficient for all admissible reinsurance policies. Define a critical level u_0 by

$$u_0 = \frac{-(\mu - r)A - c + \lambda\mathbb{E}(Y)}{r}. \tag{38}$$

It is not difficult to see that once the current surplus drops below u_0, we cannot find an admissible strategy to keep the drift coefficient positive, and hence absolute ruin occurs. Thus, the ultimate time of absolute ruin associated with the optimal strategy can be written as

$$\tau_0 = \inf\left\{t \geq 0 : \hat{U}_t < u_0\right\}.$$

Remark 6. It is interesting to find that u_0 defined in (38) is exactly equal to the value of u_2 (the point of inflection) in (24). Thus, we use the notation u_2 uniformly in the following context. Besides, we would like to point out that the optimal results for this kind of absolute ruin defined in (37) can be derived by using the analysis similar to that in Section 3.2. Therefore, we give the optimal strategy and the corresponding value function directly in Theorem 3.

Write the functions

$$\overline{F}_1(u) = \int_{u_2}^{u} \exp\left\{-\int_{u_2}^{y} (\beta_2^*(w) - \eta)dw\right\} dy, \tag{39}$$

and

$$\overline{F}_2(u) = \int_{u_1}^{u} \exp\left\{-\left(\int_{u_2}^{u_1} (\beta_2^*(w) - \eta) + \int_{u_1}^{y} (\beta_1^*(w) - \eta)\right) dw\right\} dy. \tag{40}$$

Theorem 3. *When $u_2 \leq u \leq u_s$, the minimum probability of absolute ruin $\psi(u)$ for the surplus process (5) is given by*

$$\psi(u) = \begin{cases} 1 - \frac{\overline{F}_1(u)}{\overline{F}_1(u_1)+\overline{F}_2(u_s)}, & u_2 \leq u \leq u_1, \\ 1 - \frac{\overline{F}_1(u_1)+\overline{F}_2(u)}{\overline{F}_1(u_1)+\overline{F}_2(u_s)}, & u_1 < u \leq u_s, \end{cases} \tag{41}$$

where $\overline{F}_i$ $(i=1,2)$ are given in (39) and (40), respectively; β_i^* $(i=1,2)$ uniquely solve $G_i(u,\beta) = 0$ with G_i given in (17) and (21), respectively; u_1 is the surplus value such that $\beta_1^*(u_1) = \eta + \frac{\mu-r}{\sigma^2 A}$; and u_2 is given in (24). The corresponding optimal reinsurance-investment strategy is given by $(R^*, \pi^*) = (R^*(\hat{U}_t^*, Y), \pi^*(\hat{U}_t^*))$, where

$$(R^*(u,y), \pi^*(u)) = \begin{cases} \left(\frac{\theta+\eta y}{\beta_2^*(u)} \wedge y, A\right), & u_2 \leq u \leq u_1, \\ \left(\frac{\theta+\eta y}{\beta_1^*(u)} \wedge y, -\frac{\mu-r}{\sigma^2(\eta-\beta_1^*(u))}\right), & u_1 < u \leq u_s, \end{cases} \tag{42}$$

$\hat{U}_t^*$ is the optimally controlled surplus at time t, and Y is the possible claim size at that time.

Remark 7. We discover that the optimal reinsurance-investment strategy in (42) is identical to the one given in (35) before the surplus drops below u_2. In fact, we can see from Hipp and Taksar[21] that, because the differential equations for these two different absolute ruin problems remain the same and the optimal strategy is the one that maximizes the ratio of the drift of a diffusion divided by its volatility squared, the different boundary conditions do not affect the optimal strategy when the surplus belongs to the same region. We can also deduce that the optimal strategy minimizes the traditional ruin probability before the surplus reaches the zero ruin level. Besides, we observe that the minimum absolute ruin function

ψ in Theorem 3 is convex, but not S-shaped as ϕ in Theorem 2. This can be explained by the fact that the surplus drift coefficient is nonnegative under the optimal strategy (42) for any $u \in [u_2, u_s]$, and hence ensures that the second derivative ψ'' stays positive.

As was mentioned before, by relaxing the investment restriction in Luo and Taksar[2] and assuming that the claim process is correlated to the price process of risky asset, Bi and Zhang[17] studied the optimal proportional reinsurance and investment strategy to minimize the probability of absolute ruin. They discovered that the minimum absolute ruin function is not S-shaped, but always convex. Motivated by their work, we want to compare the optimal results with those without the restriction on investment in our risk model. In the absence of constraint on risky asset, the control regions in (9) become

$$\begin{aligned}\overline{\mathcal{O}}_{V_1} &= \{-\infty < u < u_s : V''(u) > 0\},\\ \overline{\mathcal{O}}_{V_2} &= \{-\infty < u < u_s : V''(u) \le 0\}.\end{aligned} \tag{43}$$

Following the steps in the proof of Lemma 1, we find that the optimal strategy (R^*, π^*) in (26) holds for all $u \in (-\infty, u_s]$, and hence $\overline{\mathcal{O}}_{V_2} = \emptyset$. The minimum probability of absolute ruin without investment restriction $\tilde{\phi}$ can be derived directly.

Theorem 4. *When $-\infty < u \le u_s$, the minimum probability of absolute ruin $\phi(u)$ for the surplus process (5) is given by*

$$\tilde{\phi}(u) = 1 - \frac{\int_{-\infty}^{u} \exp\left\{-\int_{-\infty}^{y}(\beta_1^*(w) - \eta)dw\right\} dy}{\int_{-\infty}^{u_s} \exp\left\{-\int_{-\infty}^{y}(\beta_1^*(w) - \eta)dw\right\} dy}, \tag{44}$$

where β_1^* uniquely solves $G_1(u, \beta) = 0$ with G_1 given in (17). The corresponding optimal reinsurance-investment strategy is given by $(R^*, \pi^*) = (R^*(\hat{U}_t^*, Y), \pi^*(\hat{U}_t^*))$, where

$$(R^*(u), \pi^*(u)) = \left(\frac{\theta + \eta y}{\beta_1^*(u)} \wedge y, -\frac{u - r}{\sigma^2(\eta - \beta_1^*(u))}\right), \tag{45}$$

$\hat{U}_t^*$ is the optimally controlled surplus at time t, and Y is the possible claim size at that time.

Remark 8. We can see from Theorem 4 that the expression for the minimum absolute ruin function without restriction on risky investment becomes much simpler, and that the second derivative of $\tilde{\phi}$ is always positive. This observation coincides with the one derived in Bi and Zhang,[17] which can be explained from behavioral finance point of view. The insurers always make decisions based on the gains and losses in wealth. For our risk model, the investors are risk averse when the surplus is above the point of inflection u_2; in contrast, they are risk seeking when the surplus is below u_2. To alleviate the occurrence of absolute ruin, the insurers would rather buy no reinsurance and short a large amount of the risky asset, or borrow a lot from the bank to invest into the risky asset. As a result, the minimum absolute ruin function will not remain at a high level of risk for a long time, which makes the minimum absolute ruin function convex when the surplus is below u_2. Therefore, the restriction on investment is one of the reasons that lead to the kink of minimum absolute ruin function.

4 | SOME PROPERTIES AND NUMERICAL EXAMPLES

In this section, we first study some properties of the optimal strategy in Corollary 1. To do so, we make use of (R^*, π^*) given in (35), but in the interest of space, we omit the proof. When we write "increase" or "decrease," we mean in the weak or nonstrict, sense.

Corollary 1. *The optimal reinsurance-investment strategy satisfies the following properties:*

(i) $R^*(u, y)$ and $\pi^*(u)$ are decreasing functions with respect to u, and tend to 0 as the surplus u approaches u_s. When u goes to u_s, we have

$$R^*(u, y) \approx \frac{2(\kappa - ru)(\theta + \eta y)}{\lambda(\theta^2 + 2\theta\eta\mathbb{E}(Y) + \eta^2\mathbb{E}(Y^2)) + (\frac{\mu - r}{\sigma})^2},$$

and

$$\pi^*(u) \approx \frac{2(\kappa - ru)(\mu - r)}{\lambda\sigma^2(\theta^2 + 2\theta\eta\mathbb{E}(Y) + \eta^2\mathbb{E}(Y^2)) + (\mu - r)^2}.$$

(ii) *$R^*(u,y)$ and $\pi^*(u)$ increase with respect to λ.*
(iii) *$\pi^*(u)$ decreases as σ increases, but $R^*(u,y)$ increases as σ increases.*
(iii) *$R^*(u,y)$ and $y - R^*(u,y)$ increase as the claim size y increases.*
(iv) *$R^*(u,y)$ increases if Y increases in first-stochastic dominance. Specifically, if the claim size random variables are such that $Y_1 \leq_{\text{FSD}} Y_2$, that is, $S_{Y_1}(y) \leq S_{Y_2}(y)$ for all $y \geq 0$, then the optimal retention corresponding to Y_1 is less than the optimal retention corresponding to Y_2.*

Remark 9. We know that if λ increases, the reinsurance premium becomes more expensive, and the insurer purchases less reinsurance. Even though a larger value of λ means more claims on average, the expensive reinsurance premium has a larger effect on the insurer's optimal decision. When the reinsurance premium keeps increasing, to avoid absolute ruin, the insurer might optimally invest a larger amount in the risky asset to increase its profit. Note that the safe level also increases as λ increases. Furthermore, it is also expected that $\pi^*(u)$ decreases as σ increases, and that $R^*(u,y)$ increases as σ increases. A greater value of σ implies a greater risk of the risky asset. To reduce the risk, the insurer would like to invest less amount into the risky asset. Meanwhile, if the insurer is exposed to less investment risk, decrease in the purchase of reinsurance can help the insurer increase the profit. Finally, if the claim size random variable becomes larger as measured by first-stochastic dominance, reinsurance become more expensive, and hence the insurer tends to purchase less reinsurance even though the insurance risk is larger.

Since the forms of G_i $(i = 1, 2)$ given in (17) and (21) are complex, the monotonicity of the optimal strategy with respect to θ and η is not clear. Thus, in Example 1, we investigate the effect of θ and η on the optimal reinsurance-investment strategy. Note that the equation $G_1(u, \beta) = 0$ can be rewritten as

$$\int_0^{\frac{\theta}{\beta-\eta}} (1 + (\beta - \eta)y)S_Y(y)dy + \int_{\frac{\theta}{\beta-\eta}}^{\infty} \left(1 + \frac{\beta - \eta}{\beta}(\theta + \eta y)\right) S_Y(y)dy - \frac{(\mu - r)^2}{2\lambda\sigma^2(\beta - \eta)} = \frac{ru + c}{\lambda},$$

which is in a more explicit form. Similarly, $G_2(u, \beta) = 0$ can also be rewritten as

$$\int_0^{\frac{\theta}{\beta-\eta}} (1 + (\beta - \eta)y)S_Y(y)dy + \int_{\frac{\theta}{\beta-\eta}}^{\infty} \left(1 + \frac{\beta - \eta}{\beta}(\theta + \eta y)\right) S_Y(y)dy - \frac{(\mu - r)A + \sigma^2A^2(\eta - \beta)}{2\lambda} = \frac{ru + c}{\lambda}.$$

Besides, even though we see that the optimal policy derived in Theorem 2 minimizes the probability that the surplus drops below 0 or the critical negative level u_2, the relationship between the value functions is uncertain. Thus, in Example 2, we numerically present the monotonicity of the optimal results with respect to u, and compare the value functions for different cases derived in Section 3.3. Since Luo and Taksar[2] also considered the minimum probability of absolute of ruin but assumed that the insurer is allowed to purchase proportional reinsurance and the premium is calculated by the expected-value principle. In Example 3, we compare our optimal results with those derived in theorem 3.2 of Luo and Taksar,[2] and show that the optimal strategies derived in our article indeed reduce the risk exposure and make the minimum absolute ruin probability smaller.

In the following context, we assume that the claim size random variable Y is uniformly distributed in the interval $[0, 2]$, then, $\mathbb{E}(Y) = 1$ and $\mathbb{E}(Y^2) = 4/3$.

Example 1. We set $Y = 1.5$, $A = 2$, $\mu = 0.5$, $\sigma = 1$, $r = 0.05$, $c = 3.3$ and $\lambda = 3$. We also set $\eta = 0.3$ and $\theta \in [0, 0.4]$ for Figures 1 and 2; and $\theta = 0.2$ and $\eta \in [0, 0.6]$ for Figures 3 and 4. Consequently, the smallest safe level u_s equals 6.

In this example, we investigate the impact of the parameters θ and η on the optimal strategy according to the mean-variance premium principle. Note that $u_2 = -24$ with the given parameters. Then we have $u \in \mathcal{O}_{V_1}$. Figure 1 shows that $\beta_1^*(u)$ is a decreasing function with respect to θ. Thus, we see from the expression (35) that the optimal reinsurance and investment strategies increase as θ increases. This is clearly shown in Figure 2. In addition,

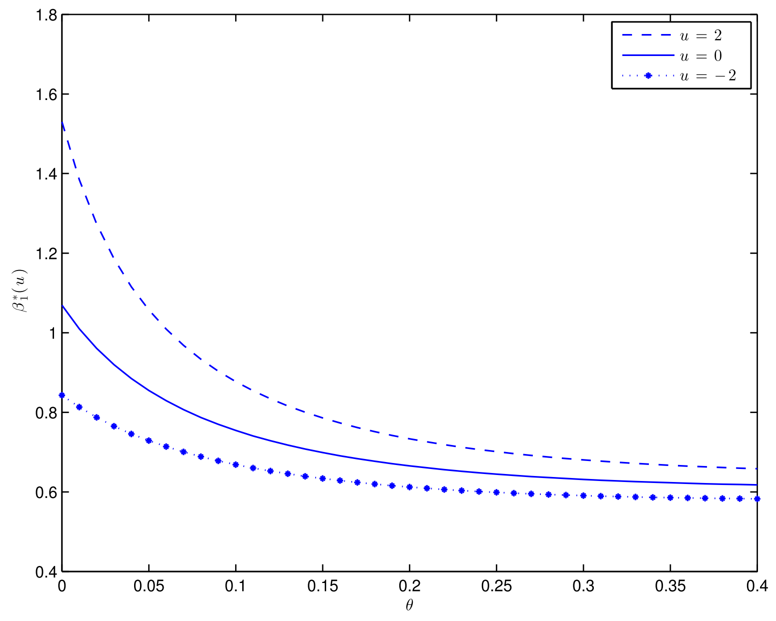

FIGURE 1 The effect of θ and u on $\beta_1^*(u)$ [Colour figure can be viewed at wileyonlinelibrary.com]

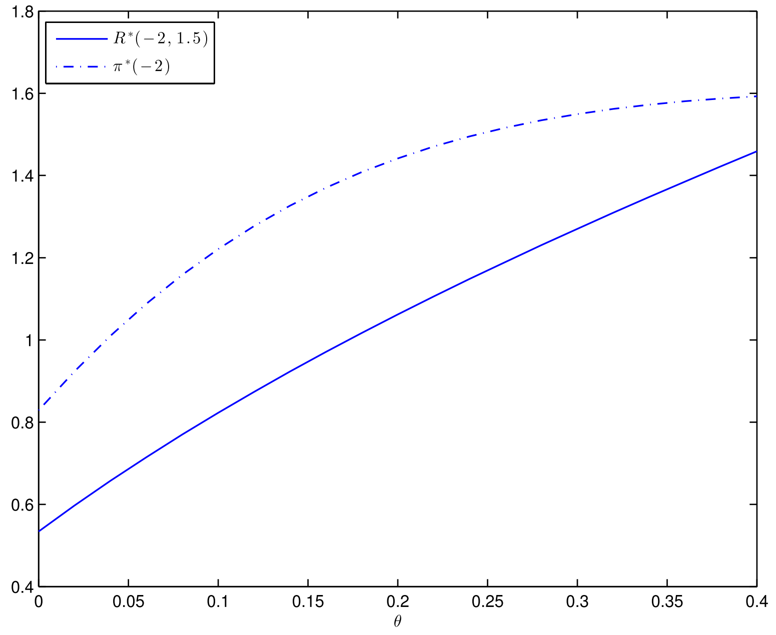

FIGURE 2 The effect of θ on optimal strategy [Colour figure can be viewed at wileyonlinelibrary.com]

it is easy to observe from Figure 4 that the optimal reinsurance and investment strategies increase as η increases, even though $\beta_1^*(u)$ in Figure 3 is not monotone with respect to η. These observations make sense since when θ and η increase, the reinsurance premium becomes more expensive, and hence the insurer prefers to purchase less reinsurance. However, if the reinsurance premium keeps increasing, to avoid absolute ruin, the insurer might optimally invest larger amount in the risky asset to increase its profit. Note that the safe level also increases as θ and η increase. Figures 1 and 3 also show that $\beta_1^*(u)$ is an increasing function with respect to u, which is expected from the proof of Proposition 1.

Example 2. We set $r = 0.05$, $c = 3.3$, $\lambda = 3$, $\theta = 0.2$ and $\eta = 0.3$. These values give the safe level $u_s = 18$. In addition, we set $Y = 1.5$, $A = 2$, $\mu = 0.5$ and $\sigma = 1$.

Figures 5 and 6 show that the optimal reinsurance-investment strategy (R^*, π^*) decreases as u increases. Meanwhile, the corresponding minimum absolute ruin probability $\phi(u)$ is a S-shaped decreasing function of u with u_2 being the point of inflection. These phenomena are kind of reasonable. When the value of the surplus increases toward u_s, the

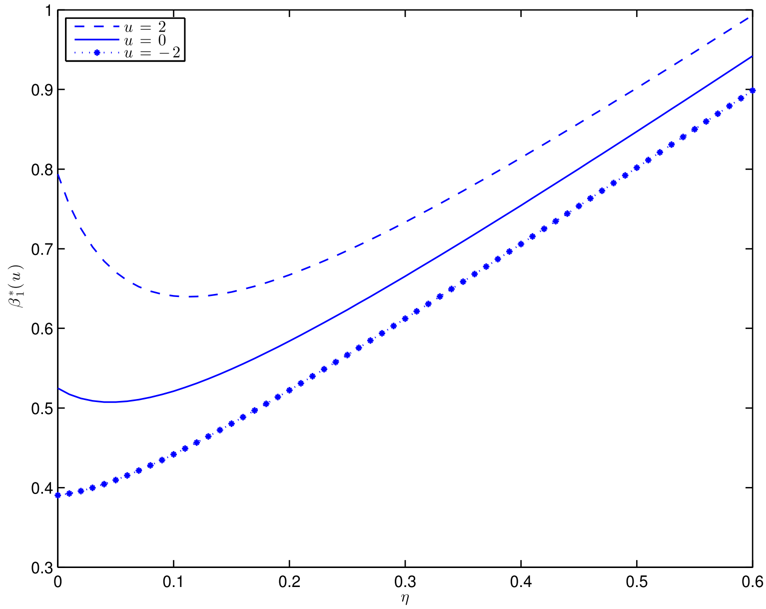

FIGURE 3 The effect of η and u on $\beta_1^*(u)$ [Colour figure can be viewed at wileyonlinelibrary.com]

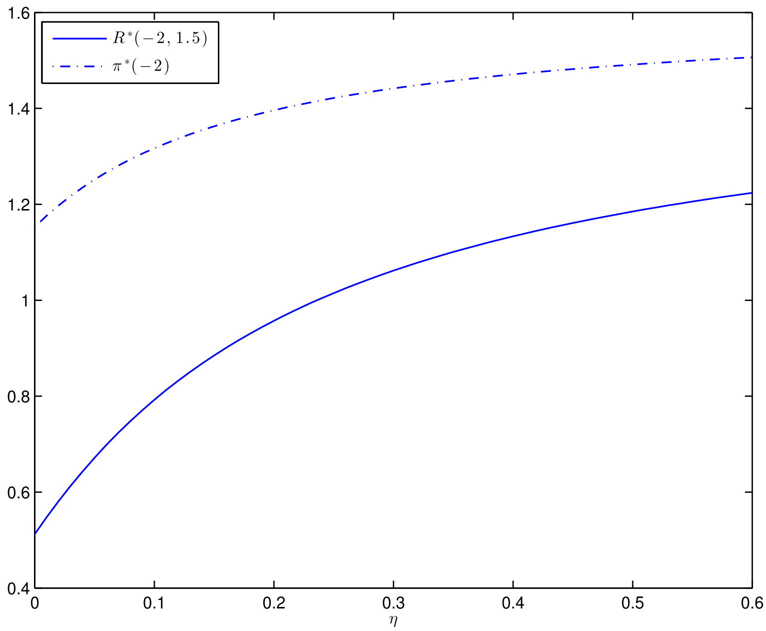

FIGURE 4 The effect of η on optimal strategy [Colour figure can be viewed at wileyonlinelibrary.com]

insurer can transfer all the risk to the reinsurer. As a result, the insurers' wealth will never decrease, and hence absolute ruin cannot happen. Besides, we observe from Figure 6 that the absolute ruin function ϕ is always larger than the one ($\tilde{\phi}$) where there is no constraint on the risky asset, but is always smaller than the one (ψ) where absolute ruin is defined in Dassios and Embrechts[12] and Cai.[13] When the risky asset has no constraint, the insurer has more choices (see Remark 8 for details) to avoid absolute ruin. However, absolute ruin is certainly more likely to occur when the ruin level increases. In particular, we would like to point out that both of the value functions in the two special cases, $\tilde{\phi}$ and ψ, are convex with respect to u. Moreover, recall from Corollary 1(i) that, when $u = 17$, the approximated value of (R^*, π^*) yields $(0.0624, 0.0432)$, which is close to $(R^*, \pi^*) = (0.0620, 0.0442)$ obtained using (35) with relative errors of -0.65% and 2.26%, respectively. When $u = 16$, the approximation yields $(0.1247, 0.0863)$ and $(R^*, \pi^*) = (0.1233, 0.0906)$, so the relative errors are -1.14% and 4.75%, respectively. As expected, the approximation is worse as u lies farther from the safe level.

Example 3. We set $r = 0.05$, $c = 3.3$, $\lambda = 3$, $\theta = 0.4$ and $\eta = 0$. These values give the safe level $u_s = 18$. In addition, we set $Y = 1.5$, $A = 2$, $\mu = 0.5$ and $\sigma = 1$.

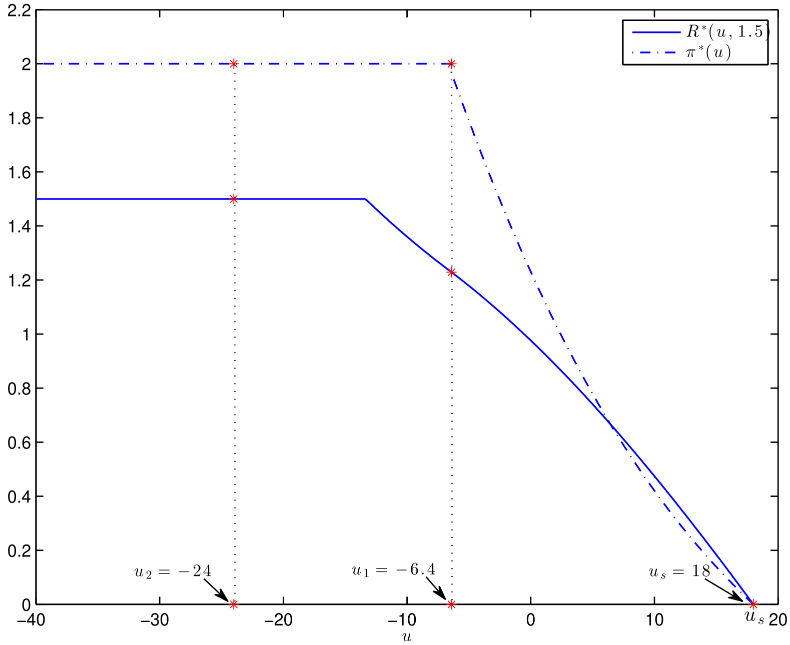

FIGURE 5 The effect of u on optimal strategy [Colour figure can be viewed at wileyonlinelibrary.com]

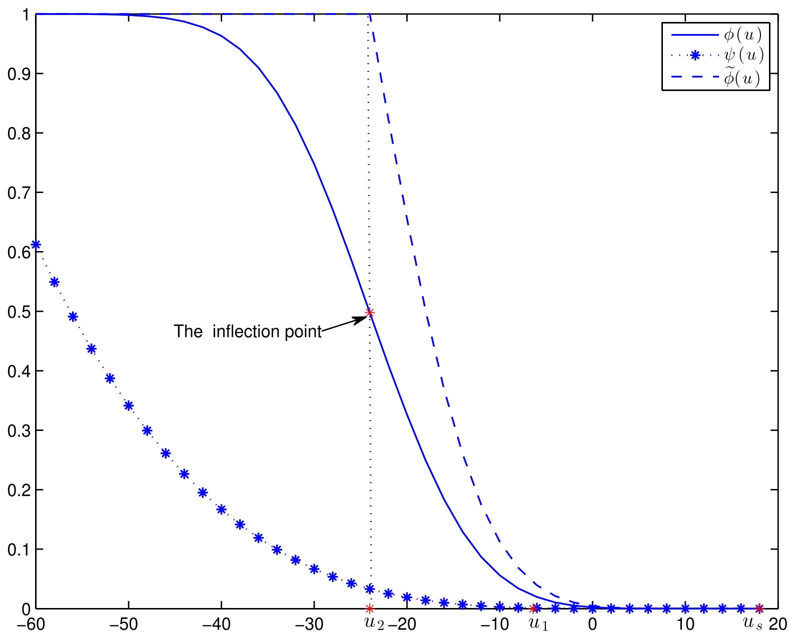

FIGURE 6 The effect of u on value functions [Colour figure can be viewed at wileyonlinelibrary.com]

In this example, we compare our optimal results with those derived in theorem 3.2 of Luo and Taksar[2] where the minimum probability of absolute ruin was studied in the case of proportional reinsurance under the expected-value principle. We use the notations (q^*, π_q^*) and V^q to represent the reinsurance-investment strategy and the associated value function obtained in their article. We also assume that the reinsurance premium principle in our risk model is using the expected-value principle by setting $\eta = 0$ in (5). In this special case, the optimal reinsurance strategy in (35) reduces to a pure excess-of-loss reinsurance strategy. We use the notations $(d^* \wedge Y, \pi_d^*)$ and ϕ^d to represent the reinsurance-investment strategy and the associated value function.

It is not difficult to see from Figure 7 that the values of $d^*(u) \wedge Y$ and $\pi_d^*(u)$ are always smaller than those of $q^*(u)Y$ and $\pi_q^*(u)$ as u increases. This observation implies that the insurer in our risk model prefers to transfer more risky exposure to the reinsurance company, and invest a smaller amount into the risky asset. Under the expected-value principle, we have shown that the optimal reinsurance strategy is in the form of pure excess-of-loss reinsurance strategy, so it is expected that the minimum probability of absolute ruin $\phi^d(u)$ is always smaller than $V^q(u)$ for any fixed u. This coincides with the

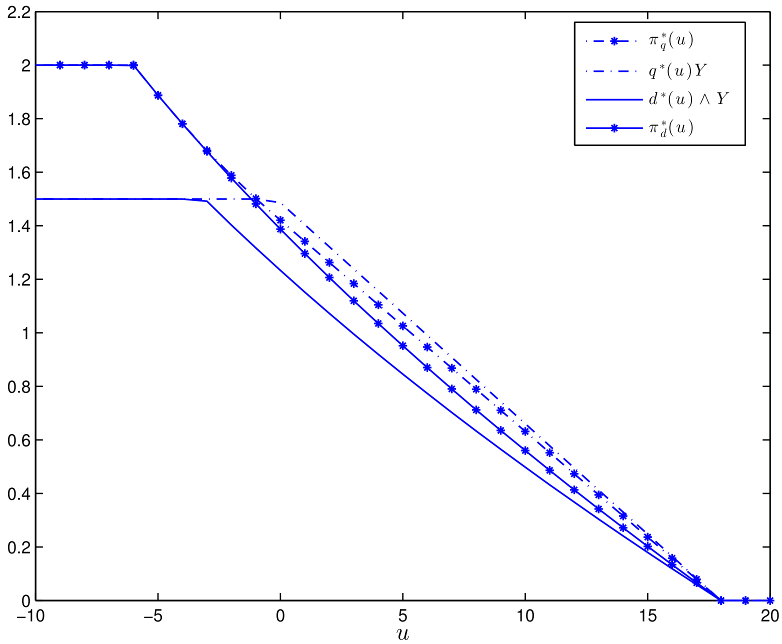

FIGURE 7 Optimal strategy under the expected-value principle [Colour figure can be viewed at wileyonlinelibrary.com]

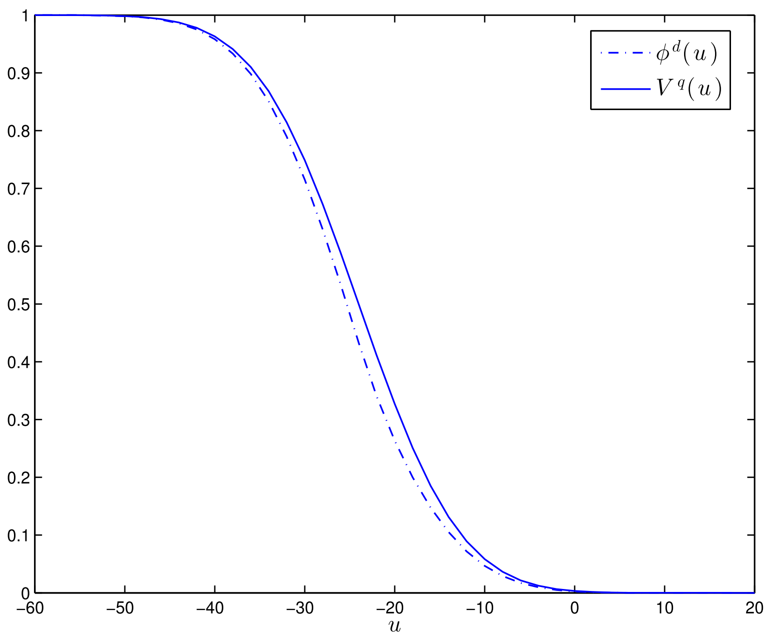

FIGURE 8 Value functions under the expected-value principle [Colour figure can be viewed at wileyonlinelibrary.com]

empirical evidence shown in Figure 8, which illustrates that the framework in our article is indeed more reasonable and practical than the one in Luo and Taksar.[2]

5 | CONCLUSION

In this article, we assume that the insurer can purchase per-loss reinsurance and invest its surplus in a financial market consisting of one risky asset and one risk-free asset. Under the criterion of minimizing the probability of absolute ruin, the optimization problem is fully solved, and the minimum absolute ruin function and its associated optimal reinsurance-investment strategy are derived explicitly. The solutions in this article give insights into how an insurance company reacts when it is in deficit, while the behavior of a company with negative surplus is not discussed under traditional ruin minimization. It is interesting to note that the minimum absolute ruin solution is S-shaped with a unique point of inflection but the traditional ruin functions are always convex. Besides, we study the same optimization problem when we use the absolute ruin definition of Dassios and Embrechts[12] to derive the optimal results. It turns out that the value function for this modified version of absolute ruin is convex everywhere. Moreover, similar to Bi and Zhang,[17] we come to the conclusion that the constraint on investment control can lead to the kink of minimum absolute ruin function.

Although the literature on optimal reinsurance is increasing rapidly, there are still many interesting problems that deserve to be investigated. It would be meaningful and practical to examine the minimum absolute ruin probability under the original compound Poisson process (without diffusion approximation), and compare the optimal results to those obtained in this article. Besides, we may introduce model uncertainty (ambiguity) into an insurer's controlled surplus process, and solve the optimal robust reinsurance policy which minimizes the probability of absolute ruin. It would also be interesting to consider the same optimization problems in which other restrictions on investment are imposed. For example, the borrowing rate is higher than the saving rate; or short-selling and borrowing are based on the fraction of wealth rather than a fixed amount.

ACKNOWLEDGMENTS

The authors would like to thank the anonymous referees for their careful reading and helpful comments on an earlier version of this article, which led to a considerable improvement of the presentation of the work. The research of Zhibin Liang and Xia Han was supported by National Natural Science Foundation of China (Grant No.12071224). The research of Kam Chuen Yuen was supported by a grant from the Research Grants Council of the Hong Kong Special Administrative Region, China (Project No. HKU17306220).

ORCID

Zhibin Liang https://orcid.org/0000-0002-3152-8081

REFERENCES

1. Liang Z, Bi J, Yuen KC, Zhang C. Optimal mean-variance reinsurance and investment in a jump-diffusion financial market with common shock dependence. *Math Meth Oper Res*. 2016;84(1):155-181.
2. Luo S, Taksar M. On absolute ruin minimization under a diffusion approximation model. *Insur Math Econom*. 2011;48(1):123-133.
3. Promislow SD, Young VR. Minimizing the probability of ruin when claims follow Brownian motion with drift. *North Am Actuar J*. 2005;9(3):110-128.
4. Bai L, Cai J, Zhou M. Optimal reinsurance policies for an insurer with a bivariate reserve risk process in a dynamic setting. *Insur Math Econom*. 2013;53(3):664-670.
5. Gu A, Guo X, Li Z, Zeng Y. Optimal control of excess-of-loss reinsurance and investment for insurers under a CEV model. *Insur Math Econom*. 2012;51(3):674-684.
6. Li D, Zeng Y, Yang H. Robust optimal excess-of-loss reinsurance and investment strategy for an insurer in a model with jumps. *Scand Actuar J*. 2018;2:145-171.
7. Liang Z, Guo J. Optimal combining quota-share and excess of loss reinsurance to maximize the expected utility. *J Appl Math Comput*. 2011;36(1–2):11-25.
8. Zhang X, Zhou M, Guo J. Optimal combinational quota-share and excess-of-loss reinsurance policies in a dynamic setting. *Appl Stoch Model Bus Ind*. 2007;23(1):63-71.
9. Zhang X, Meng H, Zeng Y. Optimal investment and reinsurance strategies for insurers with generalized mean-variance premium principle and no-short selling. *Insur Math Econom*. 2016;67:125-132.
10. Liang X, Virginia RY. Minimizing the probability of ruin: optimal per-loss reinsurance. *Insur Math Econom*. 2018;82:181-190.
11. Han X, Liang Z, Young VR. *Optimal Reinsurance Strategy to Minimize the Probability of Drawdown under a Mean-Variance Premium Principle, Working Paper*. Ann Arbor, MI: Department of Mathematics, University of Michigan; 2018.
12. Dassios A, Embrechts P. Martingales and insurance risk. *Stoch Model*. 1989;5(2):181-217.
13. Cai J. On the time value of absolute ruin with debit interest. *Adv Appl Probab*. 2009;39(2):343-359.
14. Gerber HU, Yang H. Absolute ruin probabilities in a jump diffusion risk model with investment. *North Am Actuar J*. 2007;11(3):159-169.
15. Zhou M, Cai J. Optimal dynamic risk control for insurers with state-dependent income. *J Appl Probab*. 2014;51(2):417-435.
16. Liang Z, Long M. Minimization of absolute ruin probability under negative correlation assumption. *Insur Math Econom*. 2015;65:247-258.
17. Bi X, Zhang S. Minimizing the risk of absolute ruin under a diffusion approximation model with reinsurance and investment. *J Syst Sci Complex*. 2015;28(1):144-155.
18. Grandell J. *Aspects of Risk Theory*. New York, NY: Springer; 1991.
19. Liang Z, Yuen KC. Optimal dynamic reinsurance with dependence risks: variance premium principle. *Scand Actuar J*. 2016;2016(1):18-36.

20. Liang X, Liang Z, Virginia RY. Optimal reinsurance under the mean-variance premium principle to minimize the probability of ruin. *Insur Math Econom*. 2020;92:128-146.
21. Hipp C, Taksar M. Optimal non-proportional reinsurance. *Insur Math Econom*. 2010;47(2):246-254.

How to cite this article: Han X, Liang Z, Yuen KC. Minimizing the probability of absolute ruin under the mean-variance premium principle. *Optim Control Appl Meth*. 2021;42:786–806. https://doi.org/10.1002/oca.2702

APPENDIX A. LEMMA

Let $\hat{U}_t$ be given in (5), and $-\infty < M < N < u_s$. Define $\tau_N = \inf\{t > 0 : \hat{U}_t = N\}$, $\tau_M = \inf\{t > 0 : \hat{U}_t = M\}$, and $\tau = \tau_M \wedge \tau_N$. Then, for any admissible strategy $\nu = \{R_t, \pi_t\}_{t\geq 0}$ and any initial surplus $u \in [M, N]$, it holds that $\mathbb{P}_u(\tau < \infty) = 1$.

Proof. Define the process $Y = \{Y_t\}_{t\geq 0}$ by $Y_t = e^{-\vartheta \hat{U}_t}$. In the expression for Y_t, ϑ is any constant to be chosen later in the proof. Applying Itô's formula to Y, we obtain

$$\begin{aligned} dY_t &= -\vartheta e^{-\vartheta \hat{U}_t} d\hat{U}_t + \frac{1}{2}\vartheta^2 e^{-\vartheta \hat{U}_t} d < \hat{U}, \hat{U} >_t \\ &= \vartheta e^{-\vartheta \hat{U}_t}\left(C_t dt - \sigma \pi_t \, dW_t - \sqrt{\lambda \mathbb{E}(R_t^2)} \, dB_t \right), \end{aligned} \tag{A1}$$

where

$$C_t = \kappa - r\hat{U}_t - (\mu - r)\pi_t - \theta\lambda\mathbb{E}(R_t) - \eta\lambda\mathbb{E}(YR_t) + \frac{\eta + \vartheta}{2}\,\lambda\mathbb{E}(R_t^2) + \frac{1}{2}\vartheta\sigma^2\pi_t^2. \tag{A2}$$

Suppose that there exists large enough ϑ such that $C_t \geq \frac{\kappa - rN}{2} > 0$ for all $0 \leq t \leq \tau$ with probability 1. Note that Lemma 4 shows that ϑ indeed exists and depends very much on the value of N. According to (A1), we have

$$e^{-\vartheta \hat{U}_{\tau\wedge t}} - e^{-\vartheta u} = \int_0^{\tau\wedge t} \vartheta e^{-\vartheta \hat{U}_s} C_s \, ds - \int_0^{\tau\wedge t} \vartheta e^{-\vartheta \hat{U}_s} \sigma \pi_s \, dW_s - \int_0^{\tau\wedge t} \vartheta e^{-\vartheta \hat{U}_s} \sqrt{\lambda \mathbb{E}(R_s^2)} \, dB_s.$$

The integrands of the second and third integrals are bounded, so both integrals have zero expectation. Thus, if we take expectation on both sides, we get

$$\mathbb{E}_u(e^{-\vartheta \hat{U}_{\tau\wedge t}}) - e^{-\vartheta u} = \mathbb{E}_u\left(\int_0^{\tau\wedge t} \vartheta e^{-\vartheta \hat{U}_s} C_s \, ds \right), \tag{A3}$$

where $\mathbb{E}_u$ denotes the expectation conditional on $\hat{U}_0 = u$. We proceed to prove the lemma in two different cases. On one hand, if $\vartheta \geq 0$, the inequalities $e^{-\vartheta N} \leq e^{-\vartheta \hat{U}_{\tau\wedge t}} \leq e^{-\vartheta M}$ hold for all $t \geq 0$ with probability 1, so we have the following sequence of (in)equalities

$$\begin{aligned} e^{-\vartheta M} - e^{-\vartheta u} &\geq \mathbb{E}_u(e^{-\vartheta \hat{U}_{\tau\wedge t}}) - e^{-\vartheta u} = \mathbb{E}_u\left(\int_0^{\tau\wedge t} \vartheta e^{-\vartheta \hat{U}_s} C_s \, ds \right) \\ &\geq \vartheta e^{-\vartheta N} \frac{\kappa - rN}{2} \mathbb{E}_u\left(\int_0^{\tau\wedge t} 1 ds \right) \\ &= \vartheta e^{-\vartheta N} \frac{\kappa - rN}{2} \left(\mathbb{E}_u\left(\int_0^{\tau} 1 ds \cdot \mathbf{1}_{\{\tau \leq t\}} \right) + \mathbb{E}_u\left(\int_0^{t} 1 ds \cdot \mathbf{1}_{\{\tau > t\}} \right) \right) \\ &\geq \vartheta e^{-\vartheta N} \frac{\kappa - rN}{2} \, t \, \mathbb{P}_u(\tau > t). \end{aligned}$$

Letting $t \to \infty$ in the last expression, we have $\lim_{t\to\infty}\mathbb{P}_u(\tau > t) = 0$, or equivalently, $\lim_{t\to\infty}\mathbb{P}_u(\tau < t) = 1$. On the other hand, if $\vartheta < 0$, the inequalities $e^{-\vartheta M} \leq e^{-\vartheta \hat{U}_{\tau\wedge t}} \leq e^{-\vartheta N}$ hold for all $t \geq 0$ with probability 1. Similar to the above steps, one can deduce that $\lim_{t\to\infty}\mathbb{P}_u(\tau < \infty) = 1$. ∎

APPENDIX B. LEMMA

Let $\hat{U}_t$ be given in (5). For any admissible strategy $\nu = \{R_t, \pi_t\}_{t\geq 0}$, if $\liminf_{t\to\infty}\hat{U}_t = -\infty$, then $\lim_{t\to\infty}\hat{U}_t = -\infty$; and if $\limsup_{t\to\infty}\hat{U}_t = u_s$, then $\lim_{t\to\infty}\hat{U}_t = u_s$.

Proof. Note that if $\hat{U}_0 = u < -D_0 = -r^{-1}(1 + \kappa + (\mu - r)A + \theta\lambda\mathbb{E}(Y) + \frac{3\eta}{2}\lambda\mathbb{E}(Y^2))$, we have

$$ru - \kappa + (\mu - r)\pi_t + \theta\lambda\mathbb{E}(R_t) + \eta\lambda\mathbb{E}(YR_t) - \frac{\eta}{2}\lambda\mathbb{E}(R_t^2) < -1,$$

for all $t \geq 0$. Let $D/2 > D_0$. Then take $M < -D$, $N = -D/2$ and assume that $\hat{U}_0 = -D \in (M, N)$. It follows from Lemma A that $\tau = \tau_M \wedge \tau_N < \infty$ with probability 1. Define the process $Z = \{Z_t\}_{t\geq 0}$ by $Z_t = e^{\delta\hat{U}_t}$. In the expression for Z_t, δ is any constant to be determined later. Applying Itô's formula to Z (just replacing $-\vartheta$ in (A3) by δ on both sides) and letting $t \to \infty$ yield

$$\mathbb{E}_{-D}(e^{\delta\hat{U}_\tau}) - e^{-\delta D} = \mathbb{E}_{-D}\left(\int_0^\tau \delta e^{\delta\hat{U}_s}\tilde{C}_s\, ds\right),$$

where

$$\tilde{C}_t = r\hat{U}_t - \kappa + (\mu - r)\pi_t + \theta\lambda\mathbb{E}(R_t) + \eta\lambda\mathbb{E}(YR_t) - \frac{\eta - \delta}{2}\lambda\mathbb{E}(R_t^2) + \frac{1}{2}\delta\sigma^2\pi_t^2, \tag{B1}$$

and $\mathbb{E}_{-D}$ denotes the expectation conditional on $\hat{U}_0 = -D$. Here, choosing any $0 < \delta < \frac{2}{\lambda\mathbb{E}(Y^2)+\sigma^2A^2}$, one can show that the value of $\tilde{C}_t$ in (B1) is negative for the chosen M and N. Therefore, we have

$$e^{-\delta D} > \mathbb{E}_{-D}(e^{\delta\hat{U}_\tau}) = e^{-\delta D/2}\mathbb{P}_{-D}(\tau_N < \tau_M) + e^{\delta M}\mathbb{P}_{-D}(\tau_N \geq \tau_M).$$

Letting $M \to -\infty$ gives

$$e^{-\delta D} > e^{-\delta D/2}\mathbb{P}_{-D}(\tau_N < \infty).$$

Thus, we obtain

$$\lim_{D\to\infty}\mathbb{P}_{-D}(\tau_N < \infty) < \lim_{D\to\infty} e^{-\delta D/2} \to 0. \tag{B2}$$

For a given initial surplus $\hat{U}_0 > -D$, we define

$$\tau^1 = \inf\{t > 0 : \hat{U}_t = -D\}, \quad \tau^2 = \inf\{t > \tau^1 : \hat{U}_t = -D/2\}.$$

Then, for any $-D/2 < K$, it can be shown that

$$\mathbb{P}(\liminf_{t\to\infty}\hat{U}_t = -\infty \cap \limsup_{t\to\infty}\hat{U}_t \geq K) \leq \mathbb{P}(\tau^1 < \infty, \tau^2 < \infty) \leq \mathbb{P}(\tau^2 < \infty | \tau^1 < \infty). \tag{B3}$$

From (B2) and the strong Markov property of $\{\hat{U}_t\}_{t\geq 0}$, we obtain

$$\lim_{D\to\infty}\mathbb{P}(\tau_2 < \infty | \tau_1 < \infty) = 0.$$

Thus, we have

$$\mathbb{P}(\liminf_{t\to\infty} \hat{U}_t = -\infty \cap \limsup_{t\to\infty} \hat{U}_t \geq K) = 0,$$

for any $K > -\infty$. Likewise, we can prove that

$$\mathbb{P}(\limsup_{t\to\infty} \hat{U}_t = u_s \cap \liminf_{t\to\infty} \hat{U}_t \leq K) = 0,$$

for any $K < u_s$. ■