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Optimal insurance with mean-deviation measures

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ABSTRACT

This paper studies an optimal insurance contracting problem in which the preferences of the decision maker are given by the sum of the expected loss and a convex, increasing function of a deviation measure. As for the deviation measure, our focus is on convex signed Choquet integrals (such as the Gini coefficient and a convex distortion risk measure minus the expected value) and on the standard deviation. We find that if the expected value premium principle is used, then stop-loss indemnities are optimal, and we provide a precise characterization of the corresponding deductible. Moreover, if the premium principle is based on Value-at-Risk or Expected Shortfall, then a particular layer-type indemnity is optimal, in which there is coverage for small losses up to a limit, and additionally for losses beyond another deductible. The structure of these optimal indemnities remains unchanged if there is a limit on the insurance premium budget. If the unconstrained solution is not feasible, then the deductible is increased to make the budget constraint binding. We provide several examples of these results based on the Gini coefficient and the standard deviation.

1. Introduction

Optimal insurance contract theory has gained substantial academic interest in recent years. In this theory, a decision maker (DM) or policyholder optimizes an objective function based on his/her terminal wealth, and insurance is priced using a well-defined premium principle. Early contributions to this problem studied expected utility (Arrow, 1963) or a mean-variance function (Borch, 1960) as objectives for the DM. More recent papers study more sophisticated objectives based on regulations or decision-theoretic frameworks that have gained popularity in behavioral economics. To list a few examples, researchers have considered distortion risk measures (Cui et al., 2013; Assa, 2015), expectiles (Cai and Weng, 2016), rank-dependent utilities (Ghossoub, 2019; Xu et al., 2019; Liang et al., 2022), regret-based objectives (Chi and Zhuang, 2022) and objectives with narrow framing (Zheng, 2020; Chi et al., 2022; Liang et al., 2023). In this paper, our focus is on an objective that is new in the context of optimal insurance contract theory: mean-deviation measures.

The class of generalized deviation measures was introduced by Rockafellar et al. (2006) via a set of four axioms. It is characterized based on a modified set of axioms compared to Artzner et al. (1999); in particular, the translation invariance property of a risk measure ρ is modified from $\rho(X+c)=\rho(X)+c$ (Artzner et al., 1999), which is also called cash additivity, to the property: $\rho(X+c)=\rho(X)$, for all random variables X and $c\in\mathbb{R}$. This allows for a natural separation between the actuarial value of a loss (expected loss) and the risk of a loss (measured via deviation measures). Canonical examples of deviation measures are the Gini coefficient and the standard deviation. The Gini coefficient ranges from 0 to 1, where 0 represents no risk and 1 represents the case in which all losses are concentrated in one state of the world (maximum dispersion in a distribution). It measures the extent to which the distribution deviates from a constant, and can be used to measure the risk of a random variable. The standard deviation is a more classical way to measure risk, and its use as risk measure is very popular for Gaussian distributions.

We study a special class of preferences, which can be seen as a generalization of mean-variance optimization in Markowitz (1952). In mean-variance optimization, an individual seeks to find a balance between expected return (mean) and risk (measured by the variance). A key advantage of mean-variance optimization is its simplicity, and this mean-variance structure allows us to explicitly reflect an individual's tolerance towards risk. In this paper, we keep such simple trade-off structure, and replace the variance as a function to measure risk by a more general deviation

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measure. Deviation measures preserve two key properties of variance: they are non-negative and translation invariant. In this way, a deterministic loss is measured as zero deviation. Deviation measures are consequently used to measure the “risk” within a mean-risk trade-off. While variance is not a special case of a deviation measure as it is not positively homogeneous and sub-additive, the standard deviation is. The objective that we study is called mean-deviation measures, and it considers the sum of a convex function of a deviation measure plus the expectation. By taking the square-function as convex function, the original mean-variance objective as in Markowitz (1952) is recovered as special case. Some specific mean-deviation measures have been extensively studied in the literature on portfolio selection problems, where the objective is to minimize the risk of a portfolio subject to a desired expected return, or to maximize the return among all portfolios with the risk not exceeding some threshold; see for example Sharpe (1964), Rockafellar et al. (2007) and Rockafellar and Uryasev (2013). Mean-deviation measures are also studied in the context of risk measures, see for example the mean-semideviation in Ogryczak and Ruszczyński (2001), mean-distortion risk measure mixtures in De Giorgi and Post (2008) and Cheung and Lo (2017) and mean-Expected Shortfall mixtures in Embrechts et al. (2021). For a general study on properties of mean-deviation risk measures, we refer to Han et al. (2023). It is shown by Rockafellar et al. (2006) that, under some bounded conditions, the generalized deviation measures are associated one-to-one with coherent risk measures (Artzner et al., 1999) via an additive relationship. The additive structure is for us only a special case, as our focus is on preferences given by the sum of the expected loss and a convex function a deviation measure. This structure allows for preferences that cannot be represented by a coherent risk measure, such as the mean-variance objective.

This paper contributes to the rapidly growing actuarial literature on optimal (re)insurance under risk measures. For an overview of this literature, we refer to Cai and Chi (2020). Suppose the insurance premium is given by the well-known expected value premium principle. Then, in the context of expected utility, stop-loss indemnities are well-known to be optimal (Arrow, 1963). With coherent distortion risk measures for the insurer, the stop-loss indemnity is also optimal (Lo, 2017), and the same holds true if the objective of the insurer is replaced with a mean-variance objective (Borch, 1960). This paper shows that the optimality of stop-loss indemnities holds true in a very general setting of mean-deviation measures, in which the deviation measure used is a convex signed Choquet integral or the standard deviation. We note that the coherent distortion risk measures and the mean-variance objective are special cases of such mean-deviation measures. The optimality of stop-loss indemnities still holds true even if the insurance premium is constrained by a constant budget. These results provide further evidence of the desirability of stop-loss insurance indemnities. In practice, stop-loss insurance is provided in public health insurance in the Netherlands, where participants need to pay their healthcare costs in a year up to a given deductible. Besides, we believe our setting retains a practical functional form that allows for a simple trade-off structure between the expected loss and the deviation measure. If the premium principle is based on Value-at-Risk or Expected Shortfall, we show in this paper that the optimal indemnity is generally a dual truncated stop-loss indemnity. In such indemnity function, there is coverage for small losses up to a limit, and additionally for losses beyond another deductible.

This paper is structured as follows. In Section 2, we formulate the precise problem that we study in this paper. Section 3 presents our main results with the expected value premium principle. Section 4 examines two special cases with a distortion premium principle. Section 5 examines the impact of a premium budget constraint, and Section 6 concludes. Appendix A provides some background axioms of risk measures that are referred to in this paper, and Appendix B provides insights into the monotonicity property of mean-deviation risk measures. Furthermore, Appendix C includes some supplementary special cases corresponding to Section 3, and Appendix D provides a proof that was omitted from Section 5.

2. Problem formulation

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and $\mathcal{L}^p, p \in [1, \infty)$ be the set of all random variables with finite p-th moment and $\mathcal{L}^\infty$ be the set of essentially bounded random variables. Each random variable represents a random risk that is realized at a well-defined future period. Throughout the paper, “increasing” and “decreasing” are in the non-strict (weak) sense, and all functionals we encounter are law-invariant (see Appendix A for the definition). Let $\mathcal{X}$ be a convex cone of random variables. For any $Z \in \mathcal{X}$, the cumulative distribution function associated with Z is denoted by F_Z. For any subset of I that is clear from the context, we define $\inf \emptyset = \text{ess-sup}\{x : x \in I\}$ and $\sup \emptyset = \text{ess-inf}\{x : x \in I\}$.

In decision-making, deviation measures are introduced and studied systematically for their application to risk management in areas like portfolio optimization and engineering. Roughly speaking, deviation measures evaluate the degree of non-constancy in a random variable, i.e., the extent to which outcomes may deviate from expectations. One example of such measures is the standard deviation (SD), which can be considered as a special case. Deviation measures do not necessarily need to be symmetric with respect to upward and downward risk. Fix $p \in [1, \infty]$. A mapping $D : L^p \to \mathbb{R}$, is called a *generalized deviation measure* (see, e.g., Rockafellar et al., 2006) if it satisfies

(D1) (Translation invariance) $D(Z + c) = D(Z)$ for all $Z \in L^p$ and $c \in \mathbb{R}$.
(D2) (Non-negativity) $D(Z) \geqslant 0$ for all $Z \in L^p$, with $D(Z) > 0$ for non-constant $Z \in L^p$.
(D3) (Positive homogeneity) $D(\lambda Z) = \lambda D(Z)$ for all $Z \in L^p$ and all $\lambda \geqslant 0$.
(D4) (Sub-additivity) $D(Y + Z) \leqslant D(Y) + D(Z)$ for all $Y, Z \in L^p$.

We can see that the combination of (D3) with (D4) implies convexity, thus D is a convex functional (see Appendix A for the definition). The set of generalized deviation measures includes, for instance, SD, semideviation, Expected Shortfall (ES) deviation and range-based deviation; see Examples 1 and 2 of Rockafellar et al. (2006) and Section 4.1 of Grechuk et al. (2012). Note that variance does not belong to the generalized deviation measures since it is not positive homogeneous. For more discussions and interpretations of these properties, we refer to Rockafellar et al. (2006). The continuity of D on L^p is defined respect to L^p-norm. We denote $\mathcal{D}^p$ as the set of continuous generalized deviation measures.

Deviation measures are not risk measures in the sense of Artzner et al. (1999), but the connection between deviation measures and risk measures is strong. It is shown in Theorem 2 of Rockafellar et al. (2006) that under some bounded conditions, the generalized deviation measures correspond one-to-one with coherent risk measures ρ with the relations that $D(Z) = \rho(Z) - \mathbb{E}[Z]$ or $\rho(Z) = D(Z) + \mathbb{E}[Z]$ for any $Z \in \mathcal{X}$. Note that the additive structure $\rho = D + \mathbb{E}$ is only as a special form of the combination of mean and deviation.

In the following definition, we state the mean-deviation (MD) preferences studied in this paper.

Definition 1. Fix $p \in [1, \infty]$, and let $D \in \mathcal{D}^p$. A mapping $\mathrm{MD}_g^D : L^p \to \mathbb{R}$ is defined by

$$\mathrm{MD}_g^D(Z) = g(D(Z)) + \mathbb{E}[Z], \tag{1}$$

where $g : [0, \infty) \to \mathbb{R}$ is a twice differentiable, strictly increasing, and convex function with $g(0) = 0$. We denote the set of functions g by $\mathcal{G}$.

The mapping MD_g^D in Definition 1 is not necessarily monotonic, as defined as property (A2) in Appendix A. Thus, MD_g^D is generally not a monetary risk measure. Theorem 1 in Han et al. (2023) showed that MD_g^D demonstrates monotonicity only under the condition that $g \in \mathcal{G}$ also satisfies $1/K$-Lipschitz continuity.[1] Here,

$$K =: \sup_{Z \in L^p \setminus C} \frac{D(Z)}{\text{ess-sup} Z - \mathbb{E}[Z]} < \infty, \tag{2}$$

where $C \subset L^p$ is the set of constants, almost surely. However, we do not impose this requirement on g in Definition 1 to accommodate classic objectives such as mean-variance or mean standard deviation. Nevertheless, it is worth noting that we can make MD_g^D a monotonic risk measure by imposing specific constraints on g. For example, in Section 3, numerical examples are provided where g is assumed to be a quadratic or exponential function, which ensures the monotonicity of MD_g^D. For a more detailed discussion of the monotonicity properties of MD_g^D, we refer to Appendix B.

In the following, we aim to study the optimal insurance problems under MD_g^D on $\mathcal{X} = L^p$ for some fixed $p \in [1, \infty]$ such that MD_g^D is finite. Note that MD_g^D is a convex risk measure since the expectation is linear and D is convex. Also, since g is strictly increasing, MD_g^D yields an aversion towards the deviation of Z, as measured by $D(Z)$.

Suppose that a DM faces a random loss $X \in \mathcal{X}_+$, where $\mathcal{X}_+ = \{\mathcal{X} \in \mathcal{X}, X \geqslant 0\}$. Consistent with the literature on optimal (re)insurance (Cai and Tan, 2007; Cai et al., 2008; Liu et al., 2020), the survival function $S_X(x)$ of X is assumed to be continuous and strictly decreasing on $(0, M] \setminus \{\infty\}$ with a possible jump at 0, where M is the essential supremum of X (may be finite or infinite). For simplicity, we denote $\alpha_0 = S_X(0)$. Note that $\alpha_0 = 1$ when the distribution function of X is continuous at 0. Under an insurance contract, the insurer agrees to cover a part of the loss X and requires a premium in return. The function $I : [0, M] \setminus \{\infty\} \to [0, M] \setminus \{\infty\}$ is commonly described as the indemnity or ceded loss function, and $R(x) := x - I(x)$ is known as the retained loss function. To prevent potential ex post moral hazard, where the DM might be incentivized to manipulate the size of the loss, we impose the incentive compatibility condition on the indemnity functions. We consider insurance contract $I \in \mathcal{I}$, where

$$\mathcal{I} := \{I : [0, M] \setminus \{\infty\} \to [0, M] \setminus \{\infty\} \mid I(0) = 0 \text{ and } 0 \leqslant I(x) - I(y) \leqslant x - y, \text{ for all } 0 \leqslant y \leqslant x\}. \tag{3}$$

Obviously, for any $I \in \mathcal{I}, I(x)$ and $x - I(x)$ are increasing in x. The assumption that $I \in \mathcal{I}$ is common in the literature; see, e.g., Assa (2015) and the review paper by Cai and Chi (2020).

Any $I \in \mathcal{I}$ is 1-Lipschitz continuous. Given that a Lipschitz-continuous function is absolutely continuous, it is almost everywhere differentiable and its derivative is essentially bounded by its Lipschitz constant. Therefore, function I can be written as the integral of its derivative, and $\mathcal{I}$ can be represented as

$$\mathcal{I} = \left\{ I : [0, M] \setminus \{\infty\} \to [0, M] \setminus \{\infty\} \mid I(x) = \int_0^x q(t)\mathrm{d}t,\ 0 \leqslant q \leqslant 1 \right\}. \tag{4}$$

We introduce the space of marginal indemnification functions as

$$\mathcal{Q} = \left\{ q : [0, M] \setminus \{\infty\} \to \mathbb{R}_+ \mid 0 \leqslant q \leqslant 1 \right\}.$$

For any indemnification function $I \in \mathcal{I}$, the associated marginal indemnification is a function $q \in \mathcal{Q}$ such that $I(x) = \int_0^x q(t)dt,\ x \in [0, M] \setminus \{\infty\}$.

For a given $I \in \mathcal{I}$, the insurer prices indemnity functions using $\Pi(I(X))$, then the risk exposure of the DM after purchasing insurance is given by

$$T_I(X) = X - I(X) + \Pi(I(X)).$$

We assume that the DM would like to use MD_g^D to measure the risk and aims to solve the following problem

$$\min_{I \in \mathcal{I}} \mathrm{MD}_g^D(T_I). \tag{5}$$

If Π is $\|\cdot\|_p$-continuous, the problem (5) admits an optimal solution $I^* \in \mathcal{I}$. To be more precise, take a sequence $\left\{I_n\right\}_{n=1}^{\infty} \subset \mathcal{I}$ such that

$$\lim_{n \to \infty} g(D(X - I_n(X))) + \mathbb{E}[X] + \Pi[I_n(X)] = \inf_{I \in \mathcal{I}} \{g(D(X - I(X))) + \Pi(I(X))\}.$$

Since there exists a subsequence $\left\{I_{n_k}\right\}_{k=1}^{\infty}$ that uniformly converges to $I^* \in \mathcal{I}$, we know that $I_{n_k}(X) \to I^*(X)$ in L^p as $k \to \infty$. Since D and Π are $\|\cdot\|_p$-continuous, and g is continuous, then I^* is a minimizer for (5). Note that continuity is a technical condition commonly satisfied by most risk measures. For instance, VaR is continuous on L^∞ whereas ES is continuous on L^1. Below, we consider a set $\mathcal{X}$ such that both D and Π are continuous.

3. Results under expected value premium principle

In this section, we assume that the insurer prices indemnity functions using a premium principle defined by the expected value premium principle:

$$\Pi(I(X)) = (1 + \theta)\mathbb{E}[I(X)], \tag{6}$$

where $\theta > 0$ is the safety loading parameter.

[1] For $\lambda > 0$, a real function g is *λ-Lipschitz* if $|g(x) - g(y)| \leqslant \lambda|x - y|$ for x, y in the domain of g.

3.1. Optimal solutions with convex signed Choquet integrals

To find an explicit solution of (5), we focus on a subset of generalized deviation measures $\mathcal{D}^p$ by assuming that D is a convex signed Choquet integral. Denote by

$$\widetilde{\mathcal{H}}_c = \{h : h \text{ maps } [0,1] \text{ to } \mathbb{R}, h \text{ is of bounded variation}, \ h(0)=0 \text{ and } h(1)=c\}$$

with $c \geqslant 0$. Let

$$\rho_h^c(X) = \int_0^\infty h(S_X(x))\mathrm{d}x,$$

where $h \in \widetilde{\mathcal{H}}_c$. The function h is called the distortion function of ρ_h^c. For $X \in \mathcal{X}$ with its distribution function given by F, the Value at Risk (VaR) of X at level $p \in (0,1]$ is defined, for $x \in \mathbb{R}$, as

$$\mathrm{VaR}_p(X) = F_X^{-1}(p) = \inf\{x \in \mathbb{R} : F(x) \geqslant p\}, \tag{7}$$

which is the left-quantile of X. It is useful to note that if h is left-continuous, ρ_h^c admits a quantile representation as follows

$$\rho_h^c(X) = \int_0^1 \mathrm{VaR}_{1-p}(X)\mathrm{d}h(p); \tag{8}$$

see Lemma 1 of Wang et al. (2020b). Also, by Theorem 1 of Wang et al. (2020b), we know that if h is concave, then ρ_h^c is convex and comonotonic additive (see Appendix A for the definitions). In the following, we use $\mathcal{H}_c$ to denote the subset of $\widetilde{\mathcal{H}}_c$ where h is also concave. For $h \in \mathcal{H}_c$, ρ_h^c is finite on L^p for $p \in [1,\infty]$ if and only if $\|h'\|_q < \infty$, where $\|h'\|_q = (\int_0^1 |h'(t)|^q \mathrm{d}t)^{1/q}$ and $q = (1-1/p)^{-1}$, and ρ_h^c is always finite on L^∞; see Lemma 2.1 of Liu et al. (2020).

There has been an extensive literature on a subclass of signed Choquet integrals, in which $h \in \widetilde{\mathcal{H}}_1$ is increasing; we call this class of functionals distortion risk measures (DRM). Furthermore, the signed Choquet integrals are also used as measures of distributional variability, where $h \in \widetilde{\mathcal{H}}_0$. In this case, h is not monotone. Note that ρ_h^0 with $h \in \mathcal{H}_0$ satisfies all the four properties of (D1)-(D4), and thus belong to the class of the generalized deviation measures. In particular, by Theorem 1 of Wang et al. (2020b), if a generalized deviation measure D is comonotonic additive, then D can only be a signed Choquet integral. When $h \in \mathcal{H}_0$, we refer to Appendix A for more specific examples.

Thus, when D is a signed Choquet integral, our objective in (5) can be rewritten as

$$\min_{I\in\mathcal{I}} \mathrm{MD}_g^D(T_I) = \min_{I\in\mathcal{I}} \left\{ g(D_h(X) - D_h(I(X))) + \mathbb{E}[X] + \theta\mathbb{E}[I(X)] \right\}, \tag{9}$$

where

$$D_h(X) := \rho_h^0(X) = \int_0^\infty h(S_X(x))\mathrm{d}x, \tag{10}$$

with $h \in \mathcal{H}_0$.[2] This is a direct consequence of $D_h(X - I(X)) = D_h(X) - D_h(I(X))$, which is due to comonotonic additivity of D_h.

Theorem 1. *Suppose that D is given by (10) and Π is given by (6). The following statements hold:*

(i) For every $I \in \mathcal{I}$, we can construct a stop-loss insurance treaty $I_d(x) = (x-d)_+$ for some $0 \leqslant d \leqslant M$ such that $\mathrm{MD}_g^D(T_{I_d}) \leqslant \mathrm{MD}_g^D(T_I)$. Further, I_{d^*} with

$$d^* = \sup\left\{ x : \ g'\left(\int_0^x h(S_X(t))\mathrm{d}t\right) h(S_X(x)) - \theta S_X(x) \leqslant 0, \ \text{and } 0 \leqslant x < M \right\}, \tag{11}$$

is a solution to problem (5).

(ii) If $h''(0) < 0$, the optimal solution to problem (5) is unique on $[0,M]$, i.e., we have $I_{d^*} = \arg\min_{I\in\mathcal{I}} \mathrm{MD}_g^D(T_I)$.

Proof. To show (i), we first fix $D_h(X - I(X)) = s \in [0, D_h(X)]$ and solve (5) subject to this constraint. That is, we want to solve

$$\min_{I\in\mathcal{I}} f(I) := g(s) + \theta\mathbb{E}[I(X)] + \mathbb{E}[X] + \lambda(D_h(X - I(X))) - s), \tag{12}$$

with $\lambda \geqslant 0$ being the Karush-Kuhn-Tucker (KKT) multiplier. By (4) and (8), we have

$$f(I) = \theta\int_0^1 \mathrm{VaR}_{1-t}(I(X))\mathrm{d}t + \lambda\int_0^1 \mathrm{VaR}_{1-t}(X - I(X))\mathrm{d}h(t) + g(s) + \mathbb{E}[X] - \lambda s$$

---

[2] We remark that all convex signed Choquet integrals on L^p are L^p-continuous; see Corollary 7.10 in Rüschendorf (2013) for the L^p-continuity of the finite-valued convex risk measures on L^p.

$$
\begin{aligned}
&= \theta \int_0^1 I(\mathrm{VaR}_{1-t}(X))\mathrm{d}t + \lambda \int_0^1 (\mathrm{VaR}_{1-t}(X) - I(\mathrm{VaR}_{1-t}(X)))\mathrm{d}h(t) + g(s) + \mathbb{E}[X] - \lambda s \\
&= \theta \int_0^M S_X(x) q(x)\mathrm{d}x + \lambda \int_0^M h(S_X(x))(1-q(x))\mathrm{d}x + g(s) + \mathbb{E}[X] - \lambda s \\
&= \int_0^M (\theta S_X(x) - \lambda h(S_X(x)))q(x)\mathrm{d}x + \lambda D_h(X) + g(s) + \mathbb{E}[X] - \lambda s.
\end{aligned}
$$

The second equality follows from comonotonic additivity of VaR and $f(\mathrm{VaR}_t(X)) = \mathrm{VaR}_t(f(X))$ for any increasing function f and $t \in (0,1)$, and the third equality follows from a change of variable and integration by parts. Define

$$\underline{d}_\lambda = \sup\{x:\ \theta S_X(x) - \lambda h(S_X(x)) > 0,\ \text{and}\ 0 \leqslant x < M\},$$

and

$$\overline{d}_\lambda = \sup\{x:\ \theta S_X(x) - \lambda h(S_X(x)) \geqslant 0,\ \text{and}\ 0 \leqslant x < M\}.$$

It is obvious that $\underline{d}_\lambda \leqslant \overline{d}_\lambda$ for any fixed $\lambda \in [0,\infty)$. Define $H(x) = \theta S_X(x) - \lambda h(S_X(x))$. It is clear that $H(0) = \theta\alpha_0 - \lambda h(\alpha_0)$, $\lim_{x\to M} H(x) = 0$, and $H'(x) = (\theta - \lambda h'(S_X(x)))S'_X(x)$. Since h is a concave function with $h(0) = h(1) = 0$, if $\lambda < \theta/h'(0)$ (i.e., $H'(M) < 0$), we have $q(x) = 0$ and $\underline{d}_\lambda = \overline{d}_\lambda = M$. In this case, we have $H(0) > 0$ and $I(x) = 0$. Otherwise, if $\lambda \geqslant \theta/h'(0)$, it is clear that the following q will minimize (12)

$$
q(x) = \begin{cases} 0, & \text{if } \theta S_X(x) - \lambda h(S_X(x)) > 0 \ \ (i.e.,\ x < \underline{d}_\lambda), \\ 1, & \text{if } \theta S_X(x) - \lambda h(S_X(x)) < 0 \ \ (i.e.,\ x > \overline{d}_\lambda), \\ c(x), & \text{otherwise,} \end{cases} \tag{13}
$$

where $c(x)$ could be any $[0,1]$-valued function. In particular, if $H(0) < 0$, $\underline{d}_\lambda = \overline{d}_\lambda = 0$. Thus, we can select the function c to be of the form $c(x) = 1_{\{x>d\}}$ for some $d \in [\underline{d}_\lambda, \overline{d}_\lambda]$. Then, $I(x) = I_d(X) := \int_0^x q(t)\mathrm{d}t = (x-d)_+$. Now, λ is such that $D_h(X - I_{\underline{d}_\lambda}(X)) \geqslant s$ and $D_h(X - I_{\overline{d}_\lambda}(X)) \leqslant s$, and since $D_h(X - I_d(X))$ is increasing in d, there exists $d \in [\underline{d}_\lambda, \overline{d}_\lambda]$ such that

$$s = D_h(X - I_d(X)) = \int_0^d h(S_X(x))\mathrm{d}x.$$

That is, for every s, there exists an $I_d(x) = (x-d)_+$ that does better than any $I \in \mathcal{I}$.

We next show that for any I^* that solves (5), there exists an $I_d(x) = (x-d)_+$ such that $\mathrm{MD}_g^D(T_{I^*}) = \mathrm{MD}_g^D(T_{I_d})$. We fix I^* that solves (5), and define $s = D_h(X - I^*(X))$. By the above steps, for any given s, we can always construct an insurance treaty $I_d(x) = (x-d)_+$ for some $0 \leqslant d \leqslant M$ such that $\mathrm{MD}_g^D(T_{I_d}) \leqslant \mathrm{MD}_g^D(T_{I^*})$. Since I^* is optimal, then we have $\mathrm{MD}_g^D(T_{I^*}) = \mathrm{MD}_g^D(T_{I_d})$. Hence, there exists an optimal indemnity that is of a stop-loss form.

Finally, we aim to find the optimal d for problem (5) by assuming that the insurance contract is given by I_d for some $d \in [0,M]$, that is,

$$\min_{d\in[0,M]} F(d) := g\left(\int_0^d h(S_X(x))\mathrm{d}x\right) + \theta \int_d^M S_X(x)\mathrm{d}x + \mathbb{E}[X]. \tag{14}$$

To find the optimal d, with the first-order condition, we use

$$F'(d) = g'\left(\int_0^d h(S_X(x))\mathrm{d}x\right) h(S_X(d)) - \theta S_X(d).$$

It is clear that $F'(M) = 0$. Moreover,

$$F''(d) = g''\left(\int_0^d h(S_X(x))\mathrm{d}x\right) h^2(S_X(d)) + g'\left(\int_0^d h(S_X(x))\mathrm{d}x\right) h'(S_X(d))S'_X(d) - \theta S'_X(d).$$

Since g is convex, $S_X(d)$ decreases in d and h is concave, F'' has at most one intersection with the x-axis. Let

$$d^* = \sup\left\{x:\ g'\left(\int_0^x h(S_X(t))\mathrm{d}t\right) h(S_X(x)) - \theta S_X(x) \leqslant 0,\ \text{and}\ 0 \leqslant x < M\right\},$$

then d^* is the optimal solution to (14). This concludes the proof of (i).

To show (ii), if $h''(0) < 0$, then it holds for any concave function with $h(0) = 0$ that $h(s)/s$ is strictly decreasing, and since S_X is strictly decreasing on $[0,M]$, therefore it holds that the set $\{x \in [0,M] \mid \theta S_X(x) - \lambda h(S_X(x)) = 0\}$ has Lebesgue measure zero. In other words, if $h''(0+) < 0$, then $\underline{d}_\lambda = \overline{d}_\lambda$. Then, the necessary condition for optimality of the insurance contract I_{d^*} with (11) becomes a sufficient condition. It implies that d^* is a saddle point of the function $f(d)$ on $[0, M)$ or $d^* = M$, i.e. $I_{d^*} = \arg\min_{I\in\mathcal{I}} f(I)$. □

Remark 1. In the proof of Theorem 1(i), we can also employ the convex ordering approach to show the optimal insurance treaty is in a stop-loss form. Specifically, for any admissible ceded loss function $I \in \mathcal{I}$, we can construct an insurance treaty $I_d(x) = (x-d)_+$ for some $0 \leqslant d \leqslant M$ such that $\mathbb{E}[I(X)] = \mathbb{E}[(X-d)_+]$; the existence of such d is demonstrated in the proof of Theorem 2. According to Lemma 1, we have $I \leqslant_{cx} I_d$. Since $h \in \mathcal{H}_0$ and $D(X - I(X)) = D(X) - D(I(X))$, as per Theorem 2 of Wang et al. (2020a), we obtain $D(I_d(X)) \geqslant D(I(X))$, implying that $\mathrm{MD}_g^D(T_{I_d}) \leqslant \mathrm{MD}_g^D(T_I)$. In other words, there exists an $I_d(x) = (x-d)_+$ that outperforms any $I \in \mathcal{I}$.

Moreover, the problem in (9) can be linked to a constrained DRM-based problem (see, e.g., Lo, 2017). Instead of fixing $D_h(X - I(X)) = s$ in the proof of Theorem 1, if one fixes $\mathbb{E}[X - I(X) + \Pi(X)] = L$, then solving (9) is equivalent to solving

$$\min_{I\in\mathcal{I}} g\left(D_h(X) - D_h(I(X))\right),$$
$$\text{s.t. } \mathbb{E}[X - I(X) + \Pi(I(X))] = L.$$

Furthermore, for any $h \in \mathcal{H}_0$, we have $I_h = I_{h_+} - I_{h_-}$, where $h_+ \in \mathcal{H}_c$ and $h_- \in \mathcal{H}_c$ are increasing functions such that $h = h_+ - h_-$ via the Jordan decomposition (see Wang et al., 2020b). Consequently, due to the monotonicity of g, the above problem can be further reduced to[3]

$$\min_{I\in\mathcal{I}} \rho_{h_-}(I(X)) - \rho_{h_+}(I(X)),$$
$$\text{s.t. } \mathbb{E}[X - I(X) + \Pi(I(X))] = L,$$

which is a budget-constrained DRM-based minimization problem; see (2.4) in Lo (2017). Thus, following the analysis in Lo (2017), we can also determine that the optimal insurance treaty is in a stop-loss form. Thus, the optimality of a stop-loss contract can be shown in two alternative ways, and the first part of our proof of Theorem 1(i) provides an alternative way to demonstrate this. Next, we still need to derive the optimal d^* and the uniqueness of the solution if $h''(0) < 0$, and this is also shown in the proof of Theorem 1.

In the following corollary, we consider $g(x) = \alpha x + \beta x^2$. Then we have $g'(x) = a + 2\beta x$ and $g''(x) = 2\beta$. Since $g \in \mathcal{G}$, we assume that $\alpha \geqslant 0$ and $\beta \geqslant 0$, and at least one of the inequalities holds strictly.

Corollary 1. *Suppose that D is given by (10) with $h''(0) < 0$. Let $g(x) = \alpha x + \beta x^2$ with $\alpha \geqslant 0$ and $\beta \geqslant 0$. Then we have $I_{d^*}(x) = (x - d^*)_+$, where*

$$d^* = \sup\left\{ x : h(S_X(x))\left(\alpha + 2\beta\int_0^x h(S_X(t))\mathrm{d}t\right) - \theta S_X(x) \leqslant 0,\ \textit{and}\ 0 \leqslant x < M \right\}.$$

In particular, if $\beta = 0$, we have

$$d^* = \sup\{x : \alpha h(S_X(x)) - \theta S_X(x) \leqslant 0,\ \textit{and}\ 0 \leqslant x < M\}.$$

We remark that d^* in Corollary 1 decreases as α and β increase, but increases as θ increases. In fact, larger values of α and β mean that the DM is more concerned with the variability of the risk exposure. Thus, it is to be expected that the DM is willing to buy more insurance when more weight is given to the deviation. Specifically, we have $d^* \to M$ as $\alpha \to 0$ and $\beta \to 0$, which implies the DM would like to buy no insurance. In this situation, the DM is risk neutral since $\mathrm{MD}_g^D = \mathbb{E}$. Here, we observe that the quadratic function g can be understood as the DM considering or penalizing the second-order changes in the deviation. Furthermore, as the value of θ increases, the insurer sets a relatively higher insurance premium, which consequently leads the DM to reduce the amount of insurance purchased.

In the following, we show one special example by assuming that $D_h(X)$ in Corollary 1 is the Gini deviation. Let $X \in L^1$ and X_1, X_2, X are i.i.d.,

$$D_h(X) = \mathrm{Gini}(X) := \frac{1}{2}\mathbb{E}\left[|X_1 - X_2|\right]. \tag{15}$$

The Gini deviation is a signed Choquet integral with a concave distortion function h given by $h(t) = t - t^2, t \in [0,1]$. This is due to its alternative form (see, e.g., Denneberg, 1990)

$$\mathrm{Gini}(X) = \int_0^1 F_X^{-1}(t)(2t-1)\mathrm{d}t.$$

Since $h''(0) = -1 < 0$ for Gini, by Theorem 1 (ii), I_{d^*} is the unique optimal solution. Moreover, we find $K = 1$ for Gini (see Proposition S.1 in Appendix B), where K is defined in (2). As discussed below Definition 1, to make MD_g^D monotonic on the relevant domain, we must ensure that $g'(D(X - I(X))) = \alpha + 2\beta(D(X) - D(I(X))) \leqslant 1$ for all $I \in \mathcal{I}$. Thus, since $D(I(X)) \geqslant 0$ by the non-negativity property (D2), we may define

$$\mathcal{A} = \{(\alpha,\beta) : \alpha \geqslant 0,\ \beta \geqslant 0,\ (\alpha,\beta) \neq (0,0),\ \text{ and } \alpha + 2\beta D(X) \leqslant 1\}. \tag{16}$$

In the parameter setting below, we will restrict the values of α and β to satisfy (16). Moreover, we consider the performance when g is an exponential function or when D_h is in the form of mean median difference. Since the findings are qualitatively similar, we include these examples in Appendix C.

[3] While the decomposition may not be unique, each decomposition yields identical outcomes after the quantile reformulation within a single integral.

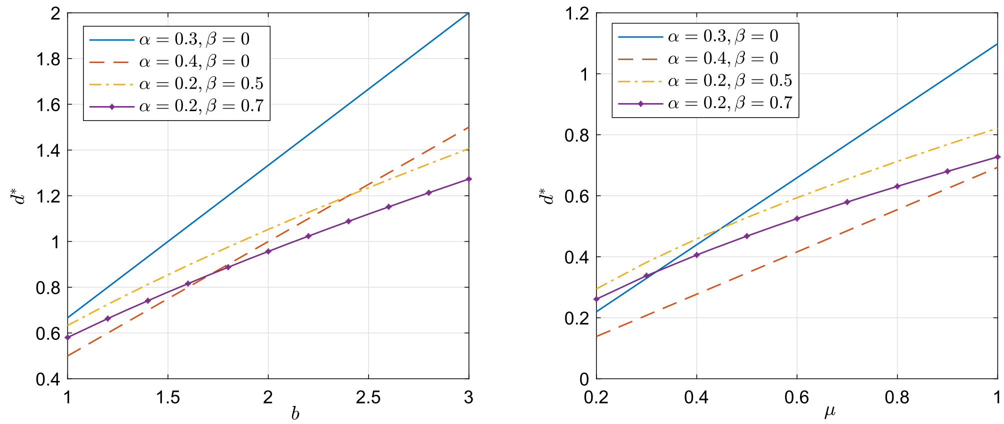

Fig. 1. Optimal deductible d^* as a function of b for the uniform distribution with $a = 0$ (left figure) and as a function of $\mu := 1/\lambda$ for the exponential distribution (right figure) with $D = \text{Gini}$ and $g(x) = \alpha x + \beta x^2$.

Example 1. Let $D = \text{Gini}$ and $g(x) = \alpha x + \beta x^2$ with $(\alpha, \beta) \in \mathcal{A}$. If $\beta = 0$, then

$$\theta S_X(x) - \alpha(S_X(x) - S_X^2(x)) = S_X(x)(\theta - \alpha + \alpha S_X(x)).$$

Thus, we can see that if $\theta < \alpha$, then $d^* = S_X^{-1}(\frac{\alpha-\theta}{\alpha})$; otherwise, $d^* = M$. For the case of $\beta \neq 0$, we have

$$\theta S_X(x) - h(S_X(x))\left(\alpha + 2\beta \int_0^x h(S_X(t))\mathrm{d}t\right) = S_X(x)\left((\theta - \alpha + \alpha S_X(x)) - 2\beta(1 - S_X(x)) \int_0^x (S_X(t) - S_X^2(t))\mathrm{d}t\right).$$

If $X \sim U[a, b]$,[4] then $\text{Gini}(X) = (b - a)/6$. Take $\theta = 0.2$, we can compute d^* numerically by

$$d^* = \sup\left\{x : \alpha\frac{x-a}{b-a} - 2\beta\frac{x-a}{(b-a)^3}\left(\frac{x^3-a^3}{3} - \frac{(a+b)(x^2-a^2)}{2} + ab(x-a)\right) - \theta \leqslant 0, \text{ and } 0 \leqslant x < b\right\}.$$

If $X \sim \exp(\lambda)$ with any $\lambda > 0$, then $\text{Gini}(X) = 1/(2\lambda)$. Again, we can compute d^* numerically

$$d^* = \sup\left\{x : \alpha - \alpha e^{-\lambda x} + \frac{\beta}{\lambda}(1 - e^{-\lambda x})^3 - \theta \leqslant 0, \text{ and } x \geqslant 0\right\}.$$

In Fig. 1, we display the optimal deductible d^* as a function of b for the uniform distribution and as a function of $\mu := 1/\lambda$ for the exponential distribution. We find that increasing the expected loss leads to a strict increase in the deductible. This pattern is linear when the function g is linear ($\beta = 0$), and concave when the function g is strictly convex ($\beta > 0$). Also, the optimal deductible d^* is decreasing in both α and β. The reason is that both parameters lead to a larger weight of the deviation in the optimization, and a more deviation-averse DM prefers a lower deductible and thus more insurance coverage. For the uniform distribution and linear function g, we note that the expected loss before insurance is $b/2$, and the deductible is approximately $b/2$ ($\alpha = 0.4$) or $2b/3$ ($\alpha = 0.3$). Thus the deductible is paid in full by the DM with a probability of around 0.5 or $1/3$, respectively. For the exponential distribution, the deductible generally around 0.7μ ($\alpha = 0.4$) or 1.1μ ($\alpha = 0.3$). Note that here the deductible is paid in full by the DM with a probability of around $\exp(-0.7) \approx 0.50$ or $\exp(-1.1) \approx 1/3$, respectively. This is similar as for the uniform distribution. Similar observations hold true for the convex functions g (with $\beta > 0$).

3.2. Standard deviation based measures

As mentioned in Section 2, SD is a generalized deviation measure, but variance does not satisfy (D3). Also, neither SD nor variance are convex signed Choquet integrals, so we cannot use Theorem 1 for SD. In particular, SD can be written as $\text{SD}(X) = \sup\{\int_0^1 \text{VaR}_t(X)\mathrm{d}h(t) : h \in \mathcal{H}_0, \|h'\|_2^2 \leqslant 1\}, X \in L^\infty$; see Example 2.1 of Wang et al. (2020b) for a simple proof of this representation.

Since SD and variance are commonly used deviation measures, we also want to solve (9) with $D = \text{SD}$:

$$\min_{I \in \mathcal{I}} \{g(\text{SD}(X - I(X))) + \mathbb{E}[X] + \theta\mathbb{E}[I(X)]\}. \tag{17}$$

In particular, if $g(x) = \gamma x^2$ for $\gamma > 0$, it is the mean-variance criterion. The following lemma is well-known (see, e.g., Property 3.4.19 in Denuit et al. (2005) and Lemma A.2 in Chi (2012)).

Lemma 1. *Provided that the random variables Y_1 and Y_2 have finite expectations, if they satisfy*

$$\mathbb{E}[Y_1] = \mathbb{E}[Y_2], \quad F_{Y_1}(t) \leqslant F_{Y_2}(t), \quad t < t_0, \quad S_{Y_1}(t) \leqslant S_{Y_2}(t), \quad t \geqslant t_0$$

for some $t_0 \in \mathbb{R}$, then $Y_1 \leqslant_{cx} Y_2$, i.e.

[4] When $a > 0$, the uniform distribution is not covered by Theorem 1 because ess-inf X can be larger than 0; however, we can modify the proof of Theorem 1 to account for X with any bounded and non-negative support.

$$\mathbb{E}\left[G\left(Y_1\right)\right] \leqslant \mathbb{E}\left[G\left(Y_2\right)\right]$$

for any convex function $G(x)$ provided the expectations exist.

Denote by

$$w_1(d) = \int_0^d S_X(x)\mathrm{d}x, \quad \text{and} \quad w_2(d) = 2\int_0^d xS_X(x)\mathrm{d}x.$$

Theorem 2. *For problem (17), we can construct a stop-loss insurance treaty $I_d(x) = (x-d)_+$ for some $0 \leqslant d \leqslant M$ such that $\mathrm{MD}_g^D(T_{I_d}) \leqslant \mathrm{MD}_g^D(T_I)$ for any admissible ceded loss function $I \in \mathcal{I}$. Further, we have $I_{d^*}(x) = (x-d^*)_+$ with*

$$d^* = \sup\left\{ x : g'\left(\sqrt{w_2(x) - w_1^2(x)}\right)\sqrt{\frac{(x-w_1(x))^2}{w_2(x)-w_1^2(x)}} - \theta \leqslant 0, \ \textit{and}\ 0 \leqslant x < M \right\}.$$

Proof. For any admissible ceded loss function $I \in \mathcal{I}$, we can construct an insurance treaty $I_d(x) = (x-d)_+$ for some $0 \leqslant d \leqslant M$ such that $\mathbb{E}[I(X)] = \mathbb{E}[(X-d)_+]$. Since $k(d) := \mathbb{E}[(X-d)_+]$ is a decreasing function in d, and $k(0) = \mathbb{E}[X]$ and $k(M) = 0$ with $0 \leqslant \mathbb{E}[I(X)] \leqslant \mathbb{E}[X]$, the existence of d can be verified. Furthermore, by taking $t_0 = d$ in Lemma 1, we have $\mathbb{E}[(X \wedge d)^2] \leqslant \mathbb{E}[(X - I(X))^2]$. Thus, we have $\mathrm{SD}(X - I_d(X)) \leqslant \mathrm{SD}(X - I(X))$, which implies that $\mathrm{MD}_g^D(T_{I_d}) \leqslant \mathrm{MD}_g^D(T_I)$. Therefore, we have

$$g(\mathrm{SD}(X \wedge d)) + \mathbb{E}[X] + \theta\mathbb{E}[(X-d)_+] = g\left(\left(w_2(d) - w_1^2(d)\right)^{1/2}\right) + \mathbb{E}[X] + \theta\int_d^M S_X(x)\mathrm{d}x.$$

Let

$$f(d) = g(\sqrt{w(d)}) + \mathbb{E}[X] + \theta\int_d^M S_X(x)\mathrm{d}x,$$

where $w(d) = w_2(d) - w_1^2(d)$. It is clear that

$$w'(d) = 2dS_X(d) - 2S_X(d)\int_0^d S_X(x)\mathrm{d}x = 2S_X(d)(d - w_1(d)) \geqslant 0.$$

Then we have

$$\begin{aligned}
f'(d) &= \frac{1}{2\sqrt{w(d)}}g'(\sqrt{w(d)})w'(d) - \theta S_X(d) \\
&= S_X(d)\left(\frac{g'(\sqrt{w(d)})}{\sqrt{w(d)}}(d - w_1(d)) - \theta\right) \\
&= S_X(d)\left(g'(\sqrt{w(d)})\sqrt{\frac{(d-w_1(d))^2}{w(d)}} - \theta\right).
\end{aligned}$$

Let $F(d) = g'(\sqrt{w(d)})\sqrt{\phi(d)} - \theta$, where $\phi(d) = \frac{(d-w_1(d))^2}{w_2(d)-w_1(d)}$. Note that

$$\begin{aligned}
\phi'(d) &= \frac{d-w_1(d)}{(w_2(d)-w_1^2(d))^2}\left(2F_X(d)(w_2(d)-w_1^2(d)) - (d-w_1(d))(w_2'(d) - 2w_1(d)w_1'(d))\right) \\
&= \frac{d-w_1(d)}{(w_2(d)-w_1^2(d))^2}\left(2F_X(d)(w_2(d)-w_1^2(d)) - 2S_X(d)(d-w_1(d))^2\right) \\
&\geqslant \frac{2(d-w_1(d))}{(w_2(d)-w_1^2(d))^2}\left(F_X(d)w_2(d) - (w_1(d) - S_X(d)d)^2\right) \\
&= \frac{2(d-w_1(d))}{(w_2(d)-w_1^2(d))^2}\left(F_X(d)S_X(d)d^2 + F_X(d)\int_0^d x^2\mathrm{d}F_X(x) - \left(\int_0^d x\mathrm{d}F_X(x)\right)^2\right) \\
&= \frac{2(d-w_1(d))}{(w_2(d)-w_1^2(d))^2}\left(S_X(d)F_X(d)d^2 - S_X(d)\left(\int_0^d x\mathrm{d}F_X(x)\right)^2 + F_X(d)\int_0^d x^2\mathrm{d}F_X(x) - F_X(d)\left(\int_0^d x\mathrm{d}F_X(x)\right)^2\right) \geqslant 0.
\end{aligned}$$

Together with $\lim_{d\to 0}\phi(d) = 0$, it follows that $\phi(d) \geqslant 0$ for $d \in [0, M]$. Also, we know that $g'(\sqrt{w(d)})$ increases in d as w increases in d and g is convex. Moreover, it is not difficult to verify that $F(0) = -\theta$ and

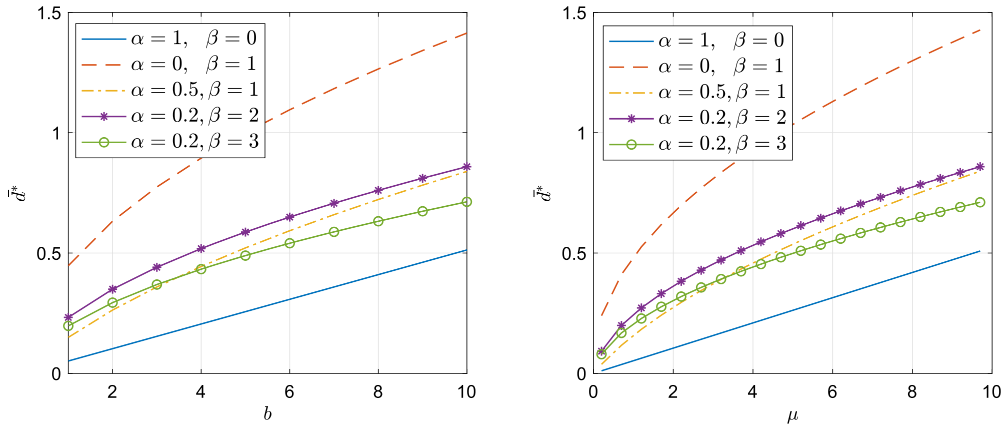

Fig. 2. Optimal deductible $\overline{d}^*$ as a function of b for the uniform distribution (left figure) and as a function of $\mu := 1/\lambda$ for the exponential distribution (right figure) with $D = \mathrm{SD}$ and $g(x) = \alpha x + \beta x^2$.

$$F(M) = g'(\mathrm{SD}(X))\frac{M - \mathbb{E}[X]}{\mathrm{SD}(X)} - \theta.$$

Therefore, if $F(M) < 0$, then f is a decreasing function of d and thus $d^* = M$. On the other hand, if $F(M) \geqslant 0$, f first decreases and then increases in d, and thus

$$d^* = \sup\left\{ x : g'\left(\sqrt{w_2(x) - w_1^2(x)}\right)\sqrt{\frac{(x - w_1(x))^2}{w_2(x) - w_1^2(x)}} - \theta \leqslant 0, \text{ and } 0 \leqslant x < M \right\}. \quad \square$$

Remark 2. For the same reasoning outlined in Remark 1, if one fixes $\mathbb{E}[X - I(X) + \Pi(X)] = L$, then solving (17) is equivalent to solving the following optimization problem:

$$\begin{aligned} &\min_{I \in \mathcal{I}} \mathrm{Var}(X - I(X)), \\ &\text{s.t. } \mathbb{E}[X - I(X) + \Pi(I(X))] = L, \end{aligned}$$

which is a well-established problem investigated in Borch (1960), and the optimal insurance treaty takes a stop-loss form. Consequently, the optimization problem can be simplified to determine the optimal d^*, as demonstrated in the proof of Theorem 2.

Furthermore, problem (17) is intricately linked to the work of Chi (2012), which addressed the optimal structure of reinsurance indemnities when the DM aims to minimize VaR or ES of the total risk exposure. In their framework, the reinsurance premium is determined through a variance-related principle, represented as $\Pi(X) = \mathbb{E}[X] + g(\mathrm{Var}(X))$. In contrast, we adopt this as our primary objective. Furthermore, we extend this objective to include a budget constraint in Section 5. The insight remains consistent: layer reinsurance generally emerges as an optimal strategy.

Note that, for the quadratic function $g(x) = \alpha x + \beta x^2$, if $\alpha > 0$ and $\beta = 0$, then MD_g^D corresponds to mean-SD; and if $\alpha = 0$ and $\beta > 0$, MD_g^D embodies mean-variance, a pivotal objective investigated in Li and Young (2021) and Liang et al. (2023). We cannot guarantee that MD_g^D will be monotonic when $D = \mathrm{SD}$ because SD does not satisfy (2) for any $p \in [1, \infty]$; see Proposition S.1 in Appendix B.

Corollary 2. *If $g(x) = \alpha x + \beta x^2$ with $\alpha \geqslant 0$ and $\beta \geqslant 0$, we have $I_{d^*}(x) = (x - d^*)_+$ with*

$$d^* = \sup\left\{ x : \alpha\sqrt{\frac{(x - w_1(x))^2}{w_2(x) - w_1^2(x)}} + 2\beta(x - w_1(x)) - \theta \leqslant 0, \textit{ and } 0 \leqslant x < M \right\}.$$

Example 2. Let $D = \mathrm{SD}$ and $g(x) = \alpha x + \beta x^2$ with $\alpha \geqslant 0$ and $\beta \geqslant 0$. For $X \sim U[0, b]$, we have $w_1(x) = (2bx - x^2)/(2b)$ and $w_2(x) = x^2(3b - 2x)/(3b)$. By setting $\theta = 0.2$, we can compute d^* numerically by

$$d^* = \sup\left\{ x : \alpha\left(\frac{3x}{4b - 3x}\right)^{1/2} + \frac{\beta x^2}{b} - \theta \leqslant 0, \text{ and } 0 \leqslant x < b \right\}.$$

For $X \sim \exp(\lambda)$ with any $\lambda > 0$, we have $w_1(x) = (1 - e^{-\lambda x})/\lambda$ and $w_2(x) = \frac{2}{\lambda^2}(1 - e^{-\lambda x}) - \frac{2}{\lambda}xe^{-\lambda x}$. By setting $\theta = 0.2$, we can compute d^* numerically by

$$d^* = \sup\left\{ x : \alpha\left(\frac{(\lambda x - 1 + e^{-\lambda x})^2}{1 - e^{-2\lambda x} - 2\lambda x e^{-\lambda x}}\right)^{1/2} + 2\beta(x - \frac{1 - e^{-\lambda x}}{\lambda}) - \theta \leqslant 0, \text{ and } x \geqslant 0 \right\}.$$

In Fig. 2, we display the optimal deductible d^* as a function of b for the uniform distribution and as a function of μ for the exponential distribution. Similar to Fig. 1, we find that increasing the expected loss leads to a strict increase in the deductible. Again, this graph is linear when the function g is linear ($\beta = 0$), and concave when the function g is strictly convex ($\beta > 0$). We do find that the size of the deductible is substantially smaller than Fig. 1, which is an indication that SD and variance make the DM more risk averse.

4. Results for two distortion premium principles

For $h \in \widetilde{\mathcal{H}}_1$ being increasing, the distortion premium principle Π_h is given by

$$\Pi_h(I(X)) := \int_0^\infty h(S_{I(X)}(x))\mathrm{d}x = \int_0^\infty h(S_X(x))q(x)\mathrm{d}x, \tag{18}$$

where q is defined in (4), and the second equality above is shown in the proof of Theorem 1. When the distortion function h is concave, the amount $\int_0^\infty h(S_X(x))\mathrm{d}x - \mathbb{E}[X]$ is non-negative and can be interpreted as the risk loading that is added to the expected loss.

In this section, suppose that $D = D_{h_1}$ with $h_1 \in \mathcal{H}_0$, we aim to solve

$$\min_{I\in\mathcal{I}} \mathrm{MD}_g^D(T_I) = \min_{I\in\mathcal{I}} \left\{ g(D_{h_1}(X - I(X))) + \mathbb{E}[X - I(X)] + \Pi_{h_2}(I(X)) \right\}, \tag{19}$$

where $h_2 \in \widetilde{\mathcal{H}}_1$ is increasing. As we know, VaR and ES are special distortion risk measures, where the ES at level $p \in (0,1)$ is the functional $\mathrm{ES}_p : L^1 \to \mathbb{R}$ defined by

$$\mathrm{ES}_p(Z) = \frac{1}{1-p}\int_p^1 \mathrm{VaR}_s(Z)\mathrm{d}s,$$

where VaR is defined in (7), and $\mathrm{ES}_1(Z) =$ ess-sup $(Z) = \mathrm{VaR}_1(Z)$ which may be infinite. In particular, we have $h(t) = \mathbb{1}_{\{t>1-p\}}$ for VaR_p and $h(t) = \frac{t}{1-p} \wedge 1$ for ES_p. The explicit solutions are derived when the DM uses VaR and ES as the premium principles. For notational convenience, we write $x_p := \mathrm{VaR}_p(X)$ for some $p \in (0,1)$.

4.1. Value-at-risk

We give the optimal results for $\Pi = \mathrm{VaR}_p$ for $p \in (0,1)$ in the following proposition.

Proposition 1. *Suppose that D is given by (10), and $h_2(t) = \mathbb{1}_{\{t>1-p\}}$ with $p \in (0,1)$, i.e., $\Pi_{h_2}(X) = \mathrm{VaR}_p(X)$. The unique solution to problem (19) is given by*

$$I_{d^*,x_p}(x) = x \wedge d^* + (x - x_p)_+,$$

with

$$d^* = \sup\left\{ x : \ 1 - S_X(x) - g'\left(\int_x^{x_p} h_1(S_X(x))\mathrm{d}x\right) h_1(S_X(x)) \leqslant 0, \ \text{ and } \ d_0^* \leqslant x \leqslant x_p \right\}. \tag{20}$$

Proof. The proof is similar to the one of Theorem 1, so we only provide the major steps that highlight the differences. We first fix $D_h(X - I(X)) = s \in [0, D_h(X)]$ and solve problem (19) subject to this constraint. That is, we want to solve

$$\min_{I\in\mathcal{I}} f(I) := g(s) + \mathbb{E}[X] - \mathbb{E}[I(X)] + \Pi_{h_2}(I(X)) + \lambda(D_{h_1}(X - I(X)) - s)$$

with $\lambda \geqslant 0$ being the KKT multiplier. As shown in Theorem 1, $f(I)$ can be written as

$$f(I) = \int_0^M (h_2(S_X(x)) - S_X(x) - \lambda h_1(S_X(x)))q(x)\mathrm{d}x + \lambda D_{h_1}(X) + g(s) + \mathbb{E}[X] - \lambda s, \tag{21}$$

and it is clear that the following q will minimize (21):

$$q(x) = \begin{cases} 0, & \text{if } h_2(S_X(x)) - S_X(x) - \lambda h_1(S_X(x)) > 0, \\ 1, & \text{if } h_2(S_X(x)) - S_X(x) - \lambda h_1(S_X(x)) < 0, \\ c, & \text{otherwise}, \end{cases} \tag{22}$$

where c could be any $[0,1]$-valued function on $h_2(S_X(x)) - S_X(x) - \lambda h_1(S_X(x)) = 0$. Define

$$H(x) = h_2(S_X(x)) - S_X(x) - \lambda h_1(S_X(x)).$$

Let $h_2(t) = \mathbb{1}_{\{t>1-p\}}$ with $p \in (0,1)$.

(i) For $t < 1-p$, or equivalently, $x_p < x \leqslant M$, we always have $H(x) = -S_X(x) - \lambda h_1(S_X(x)) \leqslant 0$, which implies $q(x) = 1$ for $x_p < x < M$.

(ii) For $t \geqslant 1-p$, or equivalently, $x \leqslant x_p$, we have $H(x) = 1 - S_X(x) - \lambda h_1(S_X(x))$. Moreover, h_1 is concave with $h_1(0) = h_1(1) = 0$, $H'(x) = -S'_X(x)(1 + \lambda h'_1(S_X(x))$, $H(x_p) = p - \lambda h(1-p)$ and $H(0) = 1 - \alpha_0 - \lambda h_1(\alpha_0)$. The function H has one discontinuity at x_p, and is differentiable on $(0, x_p)$. On $[0, x_p)$, if $H(x) > 0$ and $H'(x) \leqslant 0$, then $h_1(S_X(x)) < (1 - S_X(x))/\lambda$ and $h'_1(S_X(x)) \leqslant -1/\lambda$. Combining this with concavity of h_1 and $h_1(1) = 0$ yields a contradiction. Thus, if $H(x) > 0$, it must hold that $H'(x) > 0$, making H increasing on $(0, x_p)$, and thus there is no zero of H on $[0, x_p)$. If $H(0) \leqslant 0$, then H may first decrease and then increase on $[0, x_p)$. Hence, there is at most one zero d_λ on $(0, x_p)$.

Define

$$d_\lambda = \sup\{x:\ 1-S_X(x)-\lambda h_1(S_X(x))\leqslant 0,\ \text{ and }\ 0\leqslant x\leqslant x_p\},$$

then we have $I(x)=\int_0^x q(t)\mathrm{d}t = x\wedge d_\lambda + (x-x_p)_+$. That is, for every s, there exists an $I_{d,x_p}(x)=x\wedge d+(x-x_p)_+$ that does better than any $I\in\mathcal{I}$.

Next, we aim to find the optimal d for problem (19) when the insurance contract is given by I_{d,x_p} for some $0\leqslant d\leqslant x_p$, that is,

$$\min_{0\leqslant d\leqslant x_p} F(d) := \int_0^d (1-S_X(x))\mathrm{d}x - \int_{x_p}^M S_X(x)\mathrm{d}x + \mathbb{E}[X] + g\left(\int_d^{x_p} h_1(S_X(x))\mathrm{d}x\right). \tag{23}$$

It is straightforward to show that

$$F'(d) = -g'\left(\int_d^{x_p} h_1(S_X(x))\mathrm{d}x\right) h_1(S_X(d)) + (1-S_X(d)),$$

and

$$F''(d) = g''\left(\int_d^{x_p} h_1(S_X(x))\mathrm{d}x\right) h_1^2(S_X(d)) - g'\left(\int_d^{x_p} h_1(S_X(x))\mathrm{d}x\right) h_1'(S_X(d))S_X'(d) - S_X'(d).$$

Since g is convex, $S_X(x)$ decreases in x and h is concave with $h_1(0)=h_1(1)=0$, F'' has at most one intersection with the x-axis, then d^* in (20) is the unique optimal solution to (23). □

So, if insurance premium is based on the VaR, the optimal indemnity is a dual truncated stop-loss indemnity. To be precise, the optimal indemnity provides full coverage for small losses up to a limit, and additionally for losses beyond another deductible that is based on $\mathrm{VaR}_p(X)$. This implies that the retained loss after insurer is bounded: $X - I^*(X)\leqslant \mathrm{VaR}_p(X)-d^*$. We remark that the optimal solution for $\Pi=\mathrm{VaR}_p$ with $p\in(0,1)$ is unique. This is because S_X is strictly decreasing on $[0,M]$, therefore it holds that the set $\{x\in[0,M]|\ -S_X(x)-\lambda h(S_X(x))=0\}$ has Lebesgue measure zero.

Remark 3. From the proof of Proposition 1, we find that if

$$F'(d)|_{d=0} = -g'\left(\int_0^{x_p} h_1(S_X(x))\mathrm{d}x\right) h_1(\alpha_0) + (1-\alpha_0)\geqslant 0,$$

and

$$F''(d)|_{d=0} = g''\left(\int_0^{x_p} h_1(S_X(x))\mathrm{d}x\right) h_1^2(\alpha_0) - g'\left(\int_0^{x_p} h_1(S_X(x))\mathrm{d}x\right) h_1'(\alpha_0)S_X'(0) - S_X'(0)\geqslant 0,$$

then $F'(d)\geqslant 0$ for all $d\in[0,x_p]$. In this case, we have $d^*=0$, and the optimal solution takes the form of stop-loss. In particular, if $\alpha_0=1$, i.e., X is continuous, then we have $F'(d)|_{d=0}=0$ due to $h(1)=0$, and

$$F''(d) = -g'\left(\int_d^{x_p} h_1(S_X(x))\mathrm{d}x\right) h_1'(1)S_X'(0) - S_X'(0).$$

To ensure $F''(d)\geqslant 0$, together with the convexity of g, we have

$$g'\left(\int_0^{x_p} h_1(S_X(x))\mathrm{d}x\right) h_1'(1)\geqslant -1. \tag{24}$$

Since $h_1'(1)\leqslant 0$ and g' is increasing in x_p, we can conclude that if p is sufficiently small such that (24) holds, then the optimal solution is in the form of a stop-loss contract.

We once again focus on $D=\text{Gini}$ to illustrate the behavior of d^* when the premium is based on the VaR. Since the behaviors under exponential distribution and uniform distribution are similar, we only give the results of uniform distribution. Moreover, we have that $h'(1)=-1$ and $K=1$ for Gini, and $g'\leqslant 1$ if MD_g^D is monotonic, by (24), the optimal solution is in the form of stop-loss $I=(x-x_p)_+$. Consequently, in the following example, we refrain from imposing constraints on MD_g^D to adhere to monotonicity, thus encompassing a broader range of scenarios.

Example 3. Let $D=\text{Gini}$ and $g(x)=\alpha x+\beta x^2$ with $\alpha\geqslant 0$ and $\beta\geqslant 0$. If $X\sim U[0,b]$, we have $x_p=pb$. Then d^* in (20) becomes

$$d^* = \sup\left\{x:\ \frac{x}{b} - \frac{bx-x^2}{b^2}\left(\alpha + \frac{\beta}{b^2}\left(bx_p^2 - \frac{2}{3}x_p^3 - bx^2 + \frac{2}{3}x^3\right)\right)\leqslant 0,\ \text{ and }\ 0\leqslant x\leqslant x_p\right\}.$$

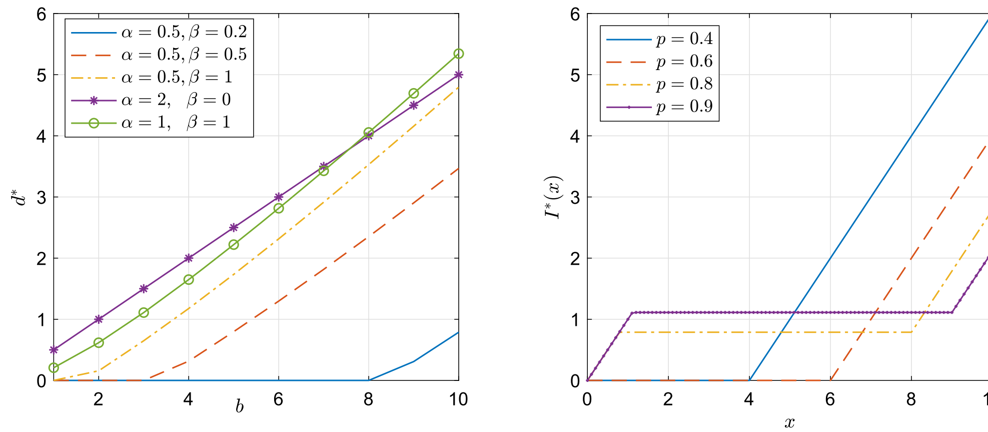

Fig. 3. Optimal threshold d^* with $p=0.8$ (left figure) and optimal indemnity function I^* with $b=10,\ \alpha=0.5,\ \beta=0.2$ (right figure) with $D=\text{Gini}$ and $g(x)=\alpha x+\beta x^2$.

In Fig. 3, we display the threshold d^* as a function of b (left figure) and the optimal indemnities as a function of quantile p. Overall, we can see that the threshold d^* is increasing in b, and strictly increasing whenever the threshold is strictly positive. Also, we can see that a larger threshold d is associated with larger values of α and β, because a larger weight on the Gini-deviation means that the DM prefers to purchase more insurance.

4.2. Expected shortfall

We next give the optimal results for $\Pi=\text{ES}_p$ for $p\in(0,1)$. The following proposition shows that the optimal indemnity has a similar structure as for the case with $\Pi=\text{VaR}_p$ (see Proposition 1), but with a more complex selection of the deductible parameter (denoted as d_2^* below) beyond which the indemnity provides full marginal coverage.

Proposition 2. *Suppose that D is given by (10), and $h_2(t)=\frac{t}{1-p}\wedge 1$ with $p\in(0,1)$, i.e., $\Pi_{h_2}(X)=\text{ES}_p(X)$. The following statements hold:*

(i) For every $I\in\mathcal{I}$, we can construct a dual truncated stop-loss insurance treaty $I_{d_1,d_2}(x)=x\wedge d_1+(x-d_2)_+$ for some $0\leqslant d_1\leqslant x_p<d_2\leqslant M$ such that $\text{MD}_g^D(T_{I_{d_1,d_2}})\leqslant\text{MD}_g^D(T_I)$. Further, an optimal solution to problem (19) is given by

$$I_{d_1^*,d_2^*}(x)=x\wedge d_1^*+(x-d_2^*)_+,$$

where d_1^* and d_2^* can be derived by solving

$$d_1^*=\sup\left\{x:\ 1-S_X(x)-g'\left(\int_x^{d_2^*}h_1(S_X(t))\mathrm{d}t\right)h_1(S_X(x))\leqslant 0,\ \text{and}\ 0\leqslant x\leqslant x_p\right\},\tag{25}$$

and

$$d_2^*=\sup\left\{x:\ g'\left(\int_{d_1^*}^{x}h_1(S_X(x))\mathrm{d}x\right)h_1(S_X(x))-\frac{p}{1-p}S_X(x)\leqslant 0,\ \text{and}\ x_p<x<M\right\}.\tag{26}$$

(ii) If $h_1''(0)<0$, the optimal solution to problem (19) is unique on $[0,M]$.

Proof. The steps are similar as in Proposition 1, and q in (22) minimizes (21) when $h_2(t)=\frac{t}{1-p}\wedge 1$ since it holds for a general $h_2\in\widetilde{H}_1$. Again, let $H(x)=h_2(S_X(x))-S_X(x)-\lambda h_1(S_X(x))$.

(i) For $t\geqslant 1-p$, or equivalently, $x\leqslant x_p$, the analysis is similar to the case of VaR.
(ii) For $t<1-p$, or equivalently, $x>x_p$, we have $H(x)=\frac{p}{1-p}S_X(x)-\lambda h_1(S_X(x))$ and thus $H(x_p)=p-\lambda h_1(1-p)$. When $H(x_p)>0$, if $H'(M)=(\frac{p}{1-p}-\lambda h_1'(0))S_X'(M)>0$, then there exists a unique $d_{1\lambda}$ such that $H(x)>0$ for $x_p<x<d_{2\lambda}$, and $H(x)<0$ for $d_{2\lambda}<x<M$; if $H'(M)\leqslant 0$, then $H(x)\geqslant 0$ for any $x\in(x_p,M]$. When $H(x_p)\leqslant 0$, then we have $H_1(x)\leqslant 0$ for any $x_p<x<M$.

Define

$$d_{1\lambda}=\sup\left\{x:\ 1-S_X(x)-\lambda h_1(S_X(x))\leqslant 0,\ \text{and}\ 0\leqslant x\leqslant x_p\right\},$$
$$\underline{d}_{2\lambda}=\sup\left\{x:\ \frac{p}{1-p}S_X(x)-\lambda h_1(S_X(x))>0,\ \text{and}\ x_p<x<M\right\},$$

and

$$\overline{d}_{2\lambda} = \sup\left\{ x : \frac{p}{1-p} S_X(x) - \lambda h_1(S_X(x)) \geqslant 0, \text{ and } x_p < x < M \right\}.$$

It is clear that $0 \leqslant d_{1\lambda} \leqslant x_p < \underline{d}_{2\lambda} \leqslant \overline{d}_{2\lambda}$. Thus, similar to Theorem 1, we can select the function c to be of the form $c(x) = 1_{\{x > d_{2\lambda}\}}$ for some $d_{2\lambda} \in [\underline{d}_{2\lambda}, \overline{d}_{2\lambda}]$, and $I(x) = I_{d_{1\lambda}, d_{2\lambda}}(x) := \int_0^x q(t)\mathrm{d}t = x \wedge d_{1\lambda} + (x - d_{2\lambda})_+$. Now, λ is such that $D_h(X - I_{d_{1\lambda}, \underline{d}_{2\lambda}}(X)) \geqslant s$ and $D_h(X - I_{d_{1\lambda}, \overline{d}_{2\lambda}}(X)) \leqslant s$, and since $D_h(X - I_{d_{1\lambda}, d_{2\lambda}}(X))$ is increasing in $d_{2\lambda}$, there exists $d_{2\lambda} \in [\underline{d}_{2\lambda}, \overline{d}_{2\lambda}]$ such that

$$s = D_h(X - I_{d_{1\lambda}, d_{2\lambda}}(X)) = \int_{d_{1\lambda}}^{d_{2\lambda}} h(S_X(x))\mathrm{d}x.$$

That is, for every s, there exists an $I_{d_1, d_2}(x) = x \wedge d_1 + (x - d_2)_+$ that does better than any $I \in \mathcal{I}$.

Next, we show that for any I^* that solves (19), there exists an $I_{d_1, d_2}(x) = x \wedge d_1 + (x - d_2)_+$ such that $\mathrm{MD}_g^D(T_{I^*}) = \mathrm{MD}_g^D(T_{I_{d_1, d_2}})$. We fix that $s = D_h(X - I^*(X))$. By the above steps, for any given s, we can always construct an insurance treaty $I_{d_1, d_2}(x) = x \wedge d_1 + (x - d_2)_+$ for some $0 \leqslant d_1 \leqslant x_p < d_2 \leqslant M$ such that $\mathrm{MD}_g^D(T_{I_{d_1, d_2}}) \leqslant \mathrm{MD}_g^D(T_{I^*})$. Since I^* is optimal, then we have $\mathrm{MD}_g^D(T_{I^*}) = \mathrm{MD}_g^D(T_{I_{d_1, d_2}})$.

Finally, we aim to find the optimal d_1 and d_2 for the problem (19), that is,

$$\min_{0 \leqslant d_1 \leqslant x_p < d_2 \leqslant M} F(d_1, d_2) := \int_0^{d_1} (1 - S_X(x))\mathrm{d}x + \int_{d_2}^{M} \left(\frac{p}{1-p} S_X(x) \right) \mathrm{d}x + g\left(\int_{d_1}^{d_2} h_1(S_X(x))\mathrm{d}x \right) + \mathbb{E}[X].$$

To use the first-order condition, we get

$$\frac{\partial F(d_1, d_2)}{\partial d_1} = -g'\left(+ \int_{d_1}^{d_2} h_1(S_X(x))\mathrm{d}x \right) h_1(S_X(d_1)) + 1 - S_X(d_1),$$

and

$$\frac{\partial F(d_1, d_2)}{\partial d_2} = g'\left(\int_{d_1}^{d_2} h_1(S_X(x))\mathrm{d}x \right) h_1(S_X(d_2)) - \frac{p}{1-p} S_X(d_2).$$

Moreover,

$$\frac{\partial F^2(d_1, d_2)}{\partial d_1^2} = -g'\left(\int_{d_1}^{d_2} h_1(S_X(x))\mathrm{d}x \right) h_1'(S_X(d_1)) S_X'(d_1) - S_X'(d_1) + g''\left(\int_{d_1}^{d_2} h_1(S_X(x))\mathrm{d}x \right) h_1^2(S_X(d_2)),$$

and

$$\frac{\partial F^2(d_1, d_2)}{\partial d_2^2} = g'\left(\int_{d_1}^{d_2} h_1(S_X(x))\mathrm{d}x \right) h_1'(S_X(d_1)) S_X'(d_1) - \frac{p}{1-p} S_X'(d_1) + g''\left(\int_{d_1}^{d_2} h_1(S_X(x))\mathrm{d}x \right) h_1^2(S_X(d_1)).$$

Since g is convex, $S_X(x)$ decreases in x and h is concave with $h_1(0) = h_1(1) = 0$, we can check that $\frac{\partial F^2(d_1, d_2)}{\partial d_1^2}$ and $\frac{\partial F^2(d_1, d_2)}{\partial d_2^2}$ have at most one intersection point with the x-axis. Then, d_1^* and d_2^* are solved by (25) and (26). This concludes the proof of (i). The proof of (ii) is similar to the one for Theorem 1 (ii). □

Remark 4. Note that if

$$\left. \frac{\partial F(d_1, d_2)}{\partial d_1} \right|_{d_1 = 0} = -g'\left(\int_0^{d_2} h_1(S_X(x))\mathrm{d}x \right) h_1(\alpha_0) + (1 - \alpha_0) \geqslant 0,$$

and

$$\left. \frac{\partial F^2(d_1, d_2)}{\partial d_1^2} \right|_{d_1 = 0} = g''\left(\int_0^{d_2} h_1(S_X(x))\mathrm{d}x \right) h_1^2(\alpha_0) - g'\left(\int_0^{d_2} h_1(S_X(x))\mathrm{d}x \right) h_1'(\alpha_0) S_X'(0) - S_X'(0) \geqslant 0,$$

then we will have $\frac{\partial F(d_1, d_2)}{\partial d_1} \geqslant 0$ for all $d_1 \geqslant 0$. In this case, we have $d_1^* = 0$, and the optimal solution takes the form of stop-loss. In particular, if $\alpha_0 = 1$, i.e., X is continuous, then we have $\left. \frac{\partial F(d_1, d_2)}{\partial d_1} \right|_{d_1 = 0} = 0$ due to $h(1) = 0$, and

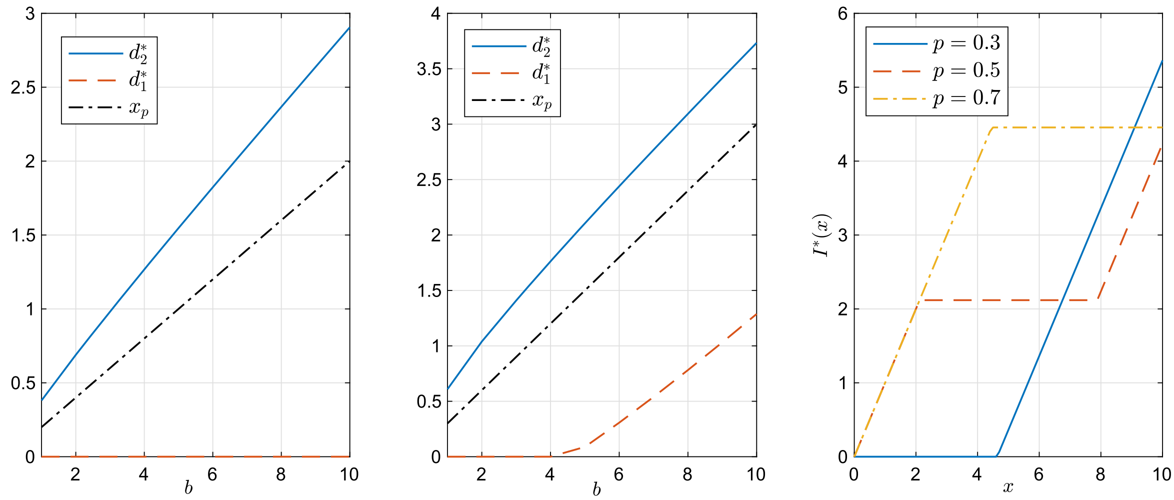

Fig. 4. Optimal parameters d_1^* and d_2^* corresponding to Example 4 for the cases $p = 0.2, \alpha = 0.5, \beta = 0.3$ (left figure), and $p = 0.3, \alpha = 0.7, \beta = 0.5$ (middle figure) with $D = \text{Gini}$ and $g(x) = \alpha x + \beta x^2$. The right figure shows the optimal indemnity for three choices of the parameter p, with $b = 10,\ \alpha = 0.5,\ \beta = 0.3$.

$$\left.\frac{\partial F^2(d_1, d_2)}{\partial d_1^2}\right|_{d_1=0} = -g'\left(\int_0^{d_2} h_1(S_X(x))\mathrm{d}x\right) h_1'(1)S_X'(0) - S_X'(0).$$

To ensure $\left.\frac{\partial F^2(d_1,d_2)}{\partial d_1^2}\right|_{d_1=0} \geqslant 0$, together with the convexity of g, we have

$$g'\left(\int_0^{d_2} h_1(S_X(x))\mathrm{d}x\right) h_1'(1) \geqslant -1. \tag{27}$$

By (26), we know that d_2^* increases in p. Since $h_1'(1) \leqslant 0$ and g' is increasing in d_2, we can conclude that if p is sufficiently small such that (27) holds, then the optimal solution is in the form of a stop-loss contract.

We next illustrate the optimal indemnity function for premium calculation using ES under $D = \text{Gini}$. In line with the VaR case it holds that the optimal solution consistently appears as a stop-loss with $I(x) = (x - d_2^*)_+$ due to the properties of the derivative of the Gini deviation and the condition $g' \leqslant 1$ as shown in (27), assuming MD_g^D is monotonic. Here, in line with Example 3, we do not restrict the monotonicity of MD_g^D to show more cases.

Example 4. Let $D = \text{Gini}$ and $g(x) = \alpha x + \beta x^2$ with $\alpha \geqslant 0$ and $\beta \geqslant 0$. Take $\theta = 0.2$. If $X \sim U[0, b]$, we have $x_p = pb$. Then d_1^* and d_2^* in (25) and (26) become

$$d_1^* = \sup\left\{x : \ \frac{x}{b} - \frac{bx - x^2}{b^2}\left(\alpha + \frac{\beta}{b^2}\left(bd_1^2 - \frac{2}{3}d_1^3 - bx^2 + \frac{2}{3}x^3\right)\right) \leqslant 0, \text{ and } 0 \leqslant x \leqslant x_p\right\},$$

and

$$d_2^* = \sup\left\{x : \ \frac{x}{b}\left(\alpha + \frac{\beta}{b^2}\left(bx^2 - \frac{2}{3}x^3 - bd_2^2 + \frac{2}{3}d_2^3\right)\right) - \frac{p}{(1-p)} \leqslant 0, \text{ and } x_p < x < b\right\}.$$

In Fig. 4, we display these two thresholds as a function of b for two sets of parameters. We can see that $d_1^* = 0$ for all $b \in [0, 10]$ in the left figure, which suggests that the optimal indemnity is of a stop-loss form. For larger values of b, this observation does not hold true in the middle figure. In both figures, the parameters d_1^* and d_2^* are increasing in b, and strictly increasing whenever d_1^* is strictly positive. Moreover, the right figure shows three optimal indemnity functions for three different choices of p. Interestingly, we can see that for larger values of p, the second parameter d_2^* is larger, and thus the indemnity functions provide less coverage in the right tail.

5. The budget constraint problem

In this section, we assume that the insurer faces a fixed budget to purchase insurance. This yields the following constraint:

$$\Pi(I(X)) \leqslant \overline{\Pi}, \qquad \text{for some budget threshold } \overline{\Pi} > 0. \tag{28}$$

We refer to the minimization problem (5) subject to (28) as the budget constraint problem. For simplicity, we focus in this section only on the cases under which we showed uniqueness of the optimal solution in Sections 3 and 4.

Assume that an unconstrained optimal solution I^* has premium equal to $\Pi_0 = \Pi(I^*(X))$. To avoid redundant arguments, we assume $\overline{\Pi} < \Pi_0$, that is, $\overline{\Pi}$ is no larger than the minimal premium for optimal solutions without budget constraint. This means that the optimal solution to the unconstrained problem is no longer feasible in the constrained problem.

Proposition 3. *When Π is calculated by the expected value premium principle in (6) or the distortion premium principle in (18), the constraint (28) is binding to (5) for $\overline{\Pi} < \Pi_0$.*

Proof. Suppose (5) with (28) admits a solution $\widetilde{I}$ for which the constraint (28) is slack. Note that

$$\mathrm{MD}_g^D(X - I^*(X) + \Pi(I^*(X))) < \mathrm{MD}_g^D(X - \widetilde{I}(X) + \Pi(\widetilde{I}(X))).$$

There exists $\lambda \in (0,1)$ such that $\Pi(I(X)) = \lambda\Pi(\widetilde{I}(X)) + (1-\lambda)\Pi(I^*(X)) = \overline{\Pi}$, where $I = \lambda\widetilde{I} + (1-\lambda)I^*$ due to the fact that both the expected premium principle and the distortion premium principles are comonotonic additive. Since MD_g^D is convex, we have

$$\begin{aligned}\mathrm{MD}_g^D(X - I(X) + \Pi(I(X))) &= \mathrm{MD}_g^D(\lambda(X - \widetilde{I}(X) + \Pi(\widetilde{I}(X))) + (1-\lambda)(X - I^*(X) + \Pi(I^*(X))))\\ &\leqslant \lambda\mathrm{MD}_g^D(X - \widetilde{I}(X) + \Pi(\widetilde{I}(X))) + (1-\lambda)\mathrm{MD}_g^D(X - I^*(X) + \Pi(I^*(X)))\\ &< \mathrm{MD}_g^D(X - \widetilde{I}(X) + \Pi(\widetilde{I}(X))),\end{aligned}$$

which contradicts the optimality of $\widetilde{I}$. Thus, the constraint (28) should be binding to (5). □

Theorem 3. *Suppose $\overline{\Pi} < \Pi_0$, Π is calculated by the expected value premium principle in (6), and one of the following holds:*

- *$D = D_h$ with $h''(0) < 0$ as given by (10), or*
- *$D = \mathrm{SD}$.*

Then, the optimal indemnity $\widetilde{I}_{d^*} \in \mathcal{I}$ for (5) with constraint (28) is given by

$$\widetilde{I}_{d^*}(x) = (x - \widetilde{d}^*)_+,$$

where $\widetilde{d}^*$ is the solution to $\Pi((X - \widetilde{d}^*)_+) = \overline{\Pi}$.

Proof. Case 1: $D = D_h$. We fix $D_h(X - I(X)) = s \in [0, D_h(X)]$ and solve (5) subject to constraint (28). We translate the constrained minimization problem to a non-constrained problem by using the Lagrangian multiplier method. Consider the following minimization problem

$$\min_{I\in\mathcal{I}} \widetilde{f}(I) := g(s) + \theta\mathbb{E}[I(X)] + \mathbb{E}[X] + \lambda_1(D_h(X - I(X)) - s) + \lambda_2((1+\theta)\mathbb{E}[I(X)] - \overline{\Pi})$$

with $\lambda_1 \geqslant 0$ and $\lambda_2 \geqslant 0$ being the KKT multipliers. By similar arguments as in the proof of Theorem 1, we can write

$$\widetilde{f}(I) = \int_0^M (\theta S_X(x) + \lambda_2(1+\theta)S_X(x) - \lambda_1 h(S_X(x)))q(x)\mathrm{d}x + \lambda_1 D_h(X) + g(s) + \mathbb{E}[X] - \lambda_1 s - \lambda_2\overline{\Pi}. \tag{29}$$

Let

$$H(x) = \theta S_X(x) + \lambda_2(1+\theta)S_X(x) - \lambda_1 h(S_X(x)).$$

For any $\lambda_1, \lambda_2 \in [0,\infty)$, we have $H(0) = \theta\alpha_0 + \lambda_2(1+\theta)\alpha_0 - \lambda_1 h(\alpha_0)$, $\lim_{x\to M} H(x) = 0$, and

$$H'(x) = (\theta + \lambda_2(1+\theta) - \lambda_1 h'(S_X(x)))S'_X(x).$$

Since h is a concave function with $h(0) = h(1) = 0$, if $\theta + \lambda_2(1+\theta) - \lambda_1 h'(0) < 0$, there exists $d_{\lambda_1,\lambda_2} \geqslant 0$ such that $H(x) \leqslant 0$ for $x \in [d_{\lambda_1,\lambda_2}, M)$ and $H(x) > 0$ for $x \in [0, d_{\lambda_1,\lambda_2})$. Thus, if $\theta + \lambda_2(1+\theta) - \lambda_1 h'(0) < 0$, then the following $\widetilde{q}$ will minimize (29)

$$\widetilde{q}(x) = \begin{cases} 0, & \text{if } H(x) > 0 \ \ (i.e.,\ x < d_{\lambda_1,\lambda_2}),\\ 1, & \text{if } H(x) < 0 \ \ (i.e.,\ x > d_{\lambda_1,\lambda_2}),\\ c, & \qquad\qquad otherwise,\end{cases}$$

where c could be any $[0,1]$-valued constant on $H(x) = 0$ (i.e., $x = d_{\lambda_1,\lambda_2}$). On the other hand, if $\theta + \lambda_2(1+\theta) - \lambda_1 h'(0) \geqslant 0$, $H'(x) \leqslant 0$ for all $x \geqslant 0$, which implies $H(x) \geqslant 0$ for all $x \geqslant 0$. In this case, $d_{\lambda_1,\lambda_2} = M$. Then we have $I(x) = I_{d_{\lambda_1,\lambda_2}}(X) := \int_0^x q(t)\mathrm{d}t = (x - d_{\lambda_1,\lambda_2})_+$.

Next, we aim to find the optimal d for problem (5) subject to (28) when the insurance contract is given by I_d for some $d \in [0, M]$, that is,

$$\min_{d\in[0,M]} \widetilde{F}(d) = \int_d^M (\theta S_X(x) + \lambda_2(1+\theta)S_X(x)\mathrm{d}x + g\left(\int_0^d h(S_X(x))\mathrm{d}x\right) + \mathbb{E}[X] - \lambda_2\overline{\Pi}. \tag{30}$$

To use the first-order condition, we get

$$\widetilde{F}'(d) = g'\left(\int_0^d h(S_X(x))\mathrm{d}x\right)h(S_X(d)) - (\theta S_X(d) + \lambda_2(1+\theta)S_X(d)).$$

Assume that there exists a constant $\lambda_2^* \geqslant 0$ such that $d_{\lambda_2^*}$ solves problem (30) for $\lambda_2 = \lambda_2^*$ and $\int_{d_{\lambda_2^*}}^M (1+\theta)S_X(x)\mathrm{d}x = \overline{\Pi}$. Then, we can show $\widetilde{d}^* = \widetilde{d}_{\lambda_2^*}$ solves problem (5) subject to the constraint (28). We denote the optimal value of problem (5) with constraint (28) by $V(\overline{\Pi})$. Then, it follows that

$$V(\overline{\Pi}) = \sup_{\substack{d\in[0,M] \\ \int_d^M (1+\theta)S_X(x)\mathrm{d}x \leqslant \overline{\Pi}}} \mathrm{MD}_g^D(T_I) \leqslant \sup_{\substack{d\in[0,M] \\ \int_d^M (1+\theta)S_X(x)\mathrm{d}x \leqslant \overline{\Pi}}} \left\{ \mathrm{MD}_g^D(T_I) - \lambda_2^* \left(\int_d^M (1+\theta)S_X(x)\mathrm{d}x - \overline{\Pi} \right) \right\}$$
$$\leqslant \sup_{d\in[0,M]} \left\{ \mathrm{MD}_g^D(T_I) - \lambda_2^* \left(\int_d^M (1+\theta)S_X(x)\mathrm{d}x - \overline{\Pi} \right) \right\} = \mathrm{MD}_g^D(T_{I_{\widetilde{d}_{\lambda_2^*}}}) \leqslant V(\overline{\Pi}).$$

The last inequality is because $I_{\widetilde{d}_{\lambda_2^*}}$ is feasible to problem (5) without the constraint. Hence, $\widetilde{d}^* = \widetilde{d}_{\lambda_2^*}$ solves problem (5) subject to (28). Thus, we have $\Pi((X-\widetilde{d}^*)_+) = \overline{\Pi}$. In this case, λ_2 can be solved by

$$\lambda_2^* = \inf\left\{ \lambda_2 : \; g'\left(\int_0^{\widetilde{d}^*} h(S_X(x))\mathrm{d}x \right) h(S_X(x)) - \theta S_X(x) + \lambda_2(1+\theta)S_X(x) \leqslant 0, \text{ and } \lambda_2 \geqslant 0 \right\}.$$

Case 2: $D = \mathrm{SD}$. With the budget constraint, we first consider the following minimization problem

$$\begin{aligned} \inf_{0\leqslant d<M} \widetilde{f}(d) &:= g(\mathrm{SD}(X\wedge d)) + \mathbb{E}[X] + \theta\mathbb{E}[(X-d)_+] + \lambda((1+\theta)\mathbb{E}[(X-d)_+] - \overline{\Pi}) \\ &= g\left(\sqrt{w_2(d) - w_1^2(d)}\right) + \mathbb{E}[X] + (\theta + \lambda(1+\theta)) \int_d^M S_X(x)\mathrm{d}x - \lambda\overline{\Pi}. \end{aligned} \tag{31}$$

We only need to replace θ in Theorem 2 with $\theta + \lambda(1+\theta)$. By the first order condition, we have

$$\begin{aligned} \widetilde{f}'(d) &= \frac{1}{2\sqrt{w(d)}} g'\left(\sqrt{w(d)}\right) w'(d) - (\theta + \lambda(1+\theta))S_X(d) \\ &= S_X(d)\left(g'(\sqrt{w(d)})\sqrt{\frac{(d-w_1(d))^2}{w(d)}} - (\theta + \lambda(1+\theta)) \right). \end{aligned}$$

Again, assume that there exists a constant $\lambda^* \geqslant 0$ such that d_{λ^*} solves problem (31) for $\lambda = \lambda^*$ and $\int_{d_{\lambda^*}}^M (1+\theta)S_X(x)\mathrm{d}x = \overline{\Pi}$. Then, we aim to show $\widetilde{d}^* = \widetilde{d}_{\lambda^*}$ solves problem (5) with constraint (28). The process is similar to the first part, and we have $\Pi((X-\widetilde{d}^*)_+) = \overline{\Pi}$. In this case, λ can be solved by

$$\lambda^* = \inf\left\{ \lambda : g'(\sqrt{w(d)})\sqrt{\frac{(d-w_1(d))^2}{w(d)}} - (\theta + \lambda(1+\theta)) \leqslant 0, \text{ and } \lambda \geqslant 0 \right\},$$

which yields the result. □

Recall Examples 1 and 2 in Section 3 where $D = \mathrm{Gini}$ or $D = \mathrm{SD}$. In the next example, we further assume that DM has a budget $\overline{\Pi}$ for purchasing insurance.

Example 5. Let $g(x) = \alpha x + \beta x^2$ with $\alpha = 0.2$ and $\beta = 0.7$ for $D = \mathrm{Gini}$. Based on Example 1, we can compute that $d^* = 1.27$ for $X \sim U[0,3]$. Since $\theta = 0.2$, we have $\Pi(I_{d^*}) = (1+\theta)\mathbb{E}[I_{d^*}(X)] = 0.60$, and thus we assume that $\overline{\Pi} < 0.60$ in left panel of Fig. 5. Similarly, we can compute $d^* = 0.73$ for $X \sim \exp(1)$ and $\Pi(I_{d^*}) = (1+\theta)\mathbb{E}[I_{d^*}(X)] = 0.58$; thus we assume that $\overline{\Pi} < 0.58$ in right panel of Fig. 5.

Let $g(x) = \alpha x + \beta x^2$ with $\alpha = 0.5$ and $\beta = 1$ for $D = \mathrm{SD}$, based on Example 2, we can compute $d^* = 0.84$ for $X \sim U[0,10]$. Since $\theta = 0.2$, we have $\Pi(I_{d^*}) = 5.03$, and thus we assume that $\overline{\Pi} < 5.03$ in left panel of Fig. 6. Similarly, we can compute $d^* = 0.54$ for $X \sim \exp(0.2)$ and $\Pi(I_{d^*}) = 5.39$, and thus we assume that $\overline{\Pi} < 5.39$ in right panel of Fig. 6.

We can see from Fig. 5 that the optimal deductible increases as the constraint $\overline{\Pi}$ increases, which implies that the DM chooses to retain more claims if the premium budget is relatively small. In particular, when the budget is relatively larger, say $\overline{\Pi} > 0.60$ in left panel of Fig. 5 and $\overline{\Pi} > 0.58$ in right panel of Fig. 5, the constraint is not binding. Thus, the optimal results are identical to those without constraint in Fig. 1. The same illustrations also apply to Fig. 6.

We next present the optimal indemnity function with a budget constraint when the premium is calculated by VaR or ES. We will show that the optimal indemnity remains a dual truncated stop-loss indemnity when we add the budget constraint, but the corresponding parameters are modified. Since the proof is similar to Propositions 1-2 and Theorem 3, we only present the major steps that highlight the differences. Also, the proof is relatively lengthy, so we put it in Appendix D. Let

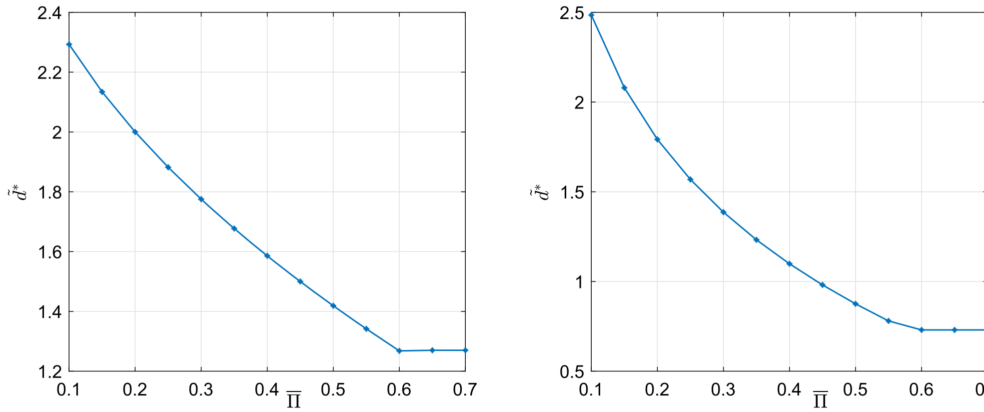

Fig. 5. Optimal deductible $\widetilde{d}^*$ for the uniform distribution (left figure) and exponential distribution (right figure) with $D = \text{Gini}$ and $g(x) = 0.2x + 0.7x^2$.

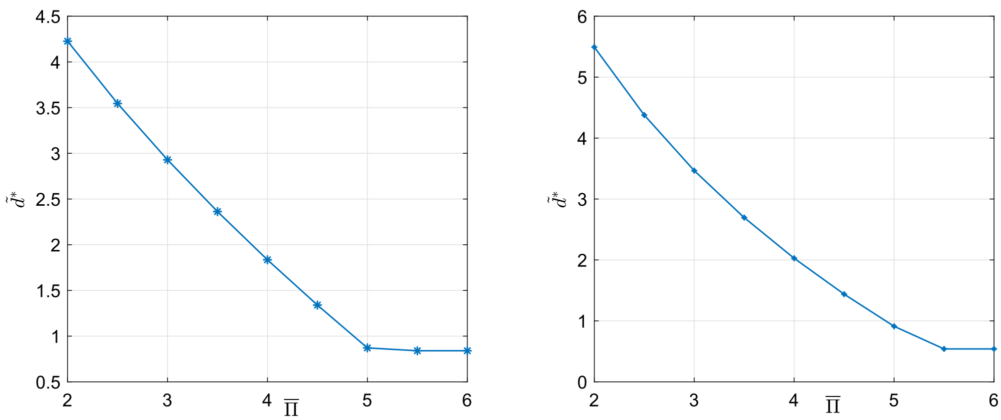

Fig. 6. Optimal deductible $\widetilde{d}^*$ for the uniform distribution (left figure) and the exponential distribution (right figure) with $D = \text{SD}$ and $g(x) = 0.5x + x^2$.

$$\mathcal{L}(a,b,c) = g'\left(\int_0^a h_1(S_X(x))\mathrm{d}x + \int_b^c h_1(S_X(x))\mathrm{d}x\right). \tag{32}$$

Theorem 4. *Suppose $\overline{\Pi} < \Pi_0$ and $D = D_{h_1}$ with $h_1 \in \mathcal{H}_0$ and $h_1''(0) < 0$, when $\Pi = \text{VaR}_p$ or ES_p for some $p \in [0,1)$, the optimal indemnity $\widetilde{I}_{\widetilde{d}_0^*,\widetilde{d}_1^*,\widetilde{d}_2^*} \in \mathcal{I}$ for (5) with constraint $\Pi(\widetilde{I}_{\widetilde{d}_0^*,\widetilde{d}_1^*,\widetilde{d}_2^*}(X)) \leqslant \overline{\Pi}$ is given by*

$$\widetilde{I}_{\widetilde{d}_0^*,\widetilde{d}_1^*,\widetilde{d}_2^*}(x) = (x - \widetilde{d}_0^*)_+ \wedge (\widetilde{d}_1^* - \widetilde{d}_0^*) + (x - \widetilde{d}_2^*)_+,$$

where $\widetilde{d}_0^*$, $\widetilde{d}_1^*$ and $\widetilde{d}_2^*$ and can be derived by solving $\widetilde{d}_1^*$, $\widetilde{d}_2^*$ and $\widetilde{d}_3^*$ can be derived by solving

$$\begin{aligned}
\widetilde{d}_0^* &= \inf\{x:\ (1+\lambda_2 - S_X(x)) - \mathcal{L}(x, \widetilde{d}_1^*, \widetilde{d}_2^*)h_1(S_X(x)) \leqslant 0,\ \textit{and}\ 0 \leqslant x \leqslant x_p\},\\
\widetilde{d}_1^* &= \sup\{x:\ (1+\lambda_2 - S_X(x)) - \mathcal{L}(\widetilde{d}_1^*, x, \widetilde{d}_2^*)h_1(S_X(x)) \leqslant 0,\ \textit{and}\ \widetilde{d}_1^* \leqslant x \leqslant x_p\},\\
\widetilde{d}_2^* &= \begin{cases} x_p & \textit{if}\ \Pi = \text{VaR}_p,\\ \sup\left\{x:\ \frac{p+\lambda_2}{1-p}S_X(x) - \mathcal{L}(\widetilde{d}_0^*, \widetilde{d}_1^*, x)h_1(S_X(x)) \leqslant 0,\ \textit{and}\ x_p < x < M\right\} & \textit{if}\ \Pi = \text{ES}_p,\end{cases}
\end{aligned} \tag{33}$$

where λ_2 is determined by

$$\lambda_2 = \inf\left\{\lambda_2 : \Pi(\widetilde{I}_{\widetilde{d}_0^*,\widetilde{d}_1^*,\widetilde{d}_2^*}(X)) - \overline{\Pi} \leqslant 0,\ \textit{and}\ \lambda_2 \geqslant 0\right\}.$$

Note that for the VaR, the parameter $\widetilde{d}_2^*$ does not change after we add the budget constraint. The reason is that increasing this parameter beyond x_p reduces the coverage, but not the corresponding premium. Also note that if Π is large enough, it will hold that $\lambda_2 = 0$, and then we recover the structure of the indemnity function in the unconstrained case in Propositions 1-2.

6. Conclusion

This paper contributes to the field of optimal insurance contract theory by introducing and analyzing the use of mean-deviation measures as an objective for decision-makers. The findings highlight the desirability of stop-loss insurance indemnities and provide valuable insights into the optimal design of insurance contracts under different premium principles. Further research can build upon these results by exploring additional deviation measures and their implications for insurance contract optimization.

We conclude this paper with several possible extensions. First, future research could explore the use of other deviation measures. Our focus in this paper is on convex signed Choquet integrals and the standard deviation. Second, the paper focuses on the case when the premium principle is either based on expected value, Value-at-Risk, or Expected Shortfall. Future research could investigate other premium principles and their implications on optimal insurance contract design. Finally, the paper only considers a single policyholder that is used to determine the premium charged by the insurer. Future research could examine the implications of multiple policyholders on optimal insurance contract design and explore the use of game theory in this context.

CRediT authorship contribution statement

Tim J. Boonen: Writing – review & editing, Writing – original draft. **Xia Han:** Writing – review & editing, Writing – original draft.

Declaration of competing interest

No potential conflict of interest was reported by the authors.

Data availability

No data was used for the research described in the article.

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Appendix A. Some background on risk measures

In this appendix we collect some common terminology and results on risk measures, which are briefly mentioned in the text of the paper, but not essential to the presentation of our main results. All random variables are tacitly assumed to be in the space $\mathcal{X}$.

We next list some properties of risk measures. To do so, we first define *comonotonicity*. A random vector $(Z_1,\dots,Z_n)$ is comonotonic if there exists a random variable Z and increasing functions $f_1,\dots,f_n$ on $\mathbb{R}$ such that $Z_i = f_i(Z)$ a.s. for every $i=1,\dots,n$. We define the following properties for a mapping $\rho:\mathcal{X}\to\mathbb{R}$:

(A1) (Law invariant) $\rho(Y)=\rho(Z)$ for all $Y,Z\in\mathcal{X}$ if Y and Z follow the same distribution,
(A2) (Cash invariance) $\rho(Y+c)=\rho(Y)+c$ for all $c\in\mathbb{R}$,
(A3) (Monotonicity) $\rho(Y)\leqslant\rho(Z)$ for all $Y,Z\in\mathcal{X}$ with $Y\leqslant Z$,
(A4) (Convexity) $\rho(\lambda Y+(1-\lambda)Z)\leqslant\lambda\rho(Y)+(1-\lambda)\rho(Z)$ for all $Y,Z\in\mathcal{X}$ and $\lambda\in[0,1]$,
(A5) (Comonotonic additive) $\rho(Y+Z)=\rho(Y)+\rho(Z)$ whenever Y and Z are comonotonic.

Here, (A1) states that the risk value depends on the loss via its distribution. Using the standard terminology in Föllmer and Schied (2016), a risk measure is a *monetary risk measure* if it satisfies (A2) and (A3), it is a *convex risk measure* if it is monetary and further satisfies (A4), and it is a *coherent risk measure* if it is monetary and further satisfies (D3) and (D4). Clearly, (D3) together with (D4) implies (A4). Thus, convex risk measures are more general than the coherent risk measures.

Below, we list some classic convex signed Choquet integrals with $h\in\mathcal{H}_0$, which are formulated on their respective effective domains. In fact, with some bounded assumptions of ρ_h^c defined in (8), there exists a one-to-one correspondence between deviation measures and distortion risk measures with the relation $\rho_h^0(X)=\rho_h^1(X)-\mathbb{E}[X]$.

(i) The Gini deviation with $h(t)=t-t^2$:

$$\frac{1}{2}\mathbb{E}\left[\left|X_1-X_2\right|\right],\quad X\in L^1, X_1, X_2, X \text{ are iid}.$$

(ii) The mean median difference with $h(t)=\min\{t,1-t\}$:

$$\min_{x\in\mathbb{R}}\mathbb{E}[|X-x|],\quad X\in L^1.$$

(iii) The range with $h(t)=\mathbb{1}_{\{0<t<1\}}$:

$$\text{ess}-\text{sup}(X)-\text{ess}-\text{inf}(X),\quad X\in L^\infty.$$

(iv) The inter-ES range with $h(t)=\frac{t}{1-\alpha}\wedge 1+\frac{\alpha-t}{1-\alpha}\wedge 0$:

$$\mathrm{ES}_\alpha(X)+\mathrm{ES}_\alpha(-X),\quad \alpha\in(0,1),\ X\in L^1.$$

(v) The ES deviation with $h(t)=\frac{\alpha t}{1-\alpha}\wedge(1-t)$:

$$\mathrm{ES}_\alpha(X)-\mathbb{E}(X),\quad \alpha\in(0,1),\ X\in L^1.$$

Appendix B. Monotonicity of MD_g^D

The mapping MD_g^D in Definition 1 is not necessarily monotonic, as defined as property (A2) in Appendix A. Thus, MD_g^D is generally not a monetary risk measure. In fact, MD_g^D satisfies weak monotonicity which implies that $\mathrm{MD}_g^D(c_1) \leqslant \mathrm{MD}_g^D(c_2)$ if $c_1 \leqslant c_2$ for any $c_1, c_2 \in \mathbb{R}$. Han et al. (2023) characterized recently the mean-deviation measures which are monotonic in the general mean-deviation model defined below.

Definition S.1 *(Mean-deviation model).* Fix $p \in [1, \infty]$. A *mean-deviation model* is a continuous functional $U : L^p \to (-\infty, \infty]$ defined by

$$U(X) = V(\mathbb{E}[X], D(X)), \tag{S.1}$$

where $V : \mathbb{H} \to (-\infty, \infty]$ with $\mathbb{H} = \{(x, y) \in \mathbb{R} \times [0, \infty)\}$ such that (i) $V(m, d)$ is strictly increasing in m for every d; (ii) $V(m, d)$ is strictly increasing in d for every m; (ii) $V(m, 0) = m$ for every m (normalization).

For $p \in [1, \infty]$, we define

$$\overline{\mathcal{D}}^p = \left\{ D \in \mathcal{D}^p : \sup_{X \in L^p \setminus C} \frac{D(X)}{\text{ess-sup}X - \mathbb{E}[X]} = 1 \right\}.$$

Theorem S.1 *(Theorem 1 of Han et al. (2023)). Fix $p \in [1, \infty]$. Suppose that $U : L^p \to (-\infty, \infty]$ is a mean-deviation model in (S.1) with $D \in \mathcal{D}^p$. The following statements are equivalent.*

(i) *U is a monetary risk measure.*
(ii) *For some $\lambda > 0$, $\lambda D \in \overline{\mathcal{D}}^p$ and $U = g \circ D + \mathbb{E}$ where $g : [0, \infty) \to \mathbb{R}$ is a non-constant increasing and λ-Lipschitz function satisfying $g(0) = 0$.*

Note that we use Gini in (15), the mean median difference (MMD) in (S.2) or SD for numerical examples throughout the paper. In the next proposition, we demonstrate that K, as defined in (2), equals 1 for both Gini and MMD. Therefore, in order to ensure that MD_g^D is monotonic, the function g should be 1-Lipschitz. Additionally, we show that SD does not satisfy (2) for any $p \in [1, \infty]$, and thus we cannot ensure MD_g^D to be monotonic when D is SD.

Proposition S.1. *For $D = \text{Gini}$ or $D = \text{MMD}$, we have $K = 1$; and for $D = \text{SD}$, we have $K = \infty$.*

Proof. Let $X \in L^1$ and X_1, X_2, X are i.i.d. For $D = \text{Gini}$, we have

$$\begin{aligned}
\sup_{X \in L^1 \setminus C} \frac{\text{Gini}(X)}{\text{ess-sup}X - \mathbb{E}[X]} &= \frac{1}{2} \sup_{X \in L^1 \setminus C} \frac{\mathbb{E}\left[|X_1 - X_2|\right]}{\text{ess-sup}X - \mathbb{E}[X]} \\
&= \frac{1}{2} \sup_{X \in L^1 \setminus C} \frac{\mathbb{E}\left[|X_1 - \text{ess-sup}X + \text{ess-sup}X - X_2|\right]}{\text{ess-sup}X - \mathbb{E}[X]} \\
&\leqslant \frac{1}{2} \sup_{X \in L^1 \setminus C} \frac{\mathbb{E}[|\text{ess-sup}X - X_1|] + \mathbb{E}|\text{ess-sup}X - X_2|]}{\text{ess-sup}X - \mathbb{E}[X]} = 1.
\end{aligned}$$

Furthermore, let $\{X_n\}_{n=1}^{\infty}$ be a sequence with $\mathbb{P}\left[X_n = n\right] = 1 - 1/n$ and $\mathbb{P}\left[X_n = 0\right] = 1/n$. Then $\text{ess-sup}X_n - E[X_n] = 1$, and $\lim_{n \to \infty} \text{Gini}(X_n) = 1$, so $K = 1$.

For $D = \text{MMD}$, we have

$$\begin{aligned}
\sup_{X \in L^1 \setminus C} \frac{\text{MMD}(X)}{\text{ess-sup}X - \mathbb{E}[X]} &= \sup_{X \in L^1 \setminus C} \frac{\min_{x \in \mathbb{R}} \mathbb{E}[|X - x|]}{\text{ess-sup}X - \mathbb{E}[X]} \\
&\leqslant \sup_{X \in L^1 \setminus C} \frac{\mathbb{E}[|X - \text{ess-sup}X|]}{\text{ess-sup}X - \mathbb{E}[X]} = 1.
\end{aligned}$$

Also, for a sequence $\{X_n\}_{n=1}^{\infty}$ with $\mathbb{P}\left[X_n = n\right] = 1 - 1/n$ and $\mathbb{P}\left[X_n = 0\right] = 1/n$, $\lim_{n \to \infty} \text{MMD}(X_n) = 1$, implying that $K = 1$ for $D = \text{MMD}$.

For $D = \text{SD}$, we use a proof by contradiction. Suppose there exists $K < \infty$ for $D = \text{SD}$. Let X be the random variable with $\mathbb{P}[X = 0] = (K + 2)^{-2}$ and $\mathbb{P}\left[X = (K + 2)^2\right] = 1 - (K + 2)^{-2}$. Then $\text{ess-sup}X - E[X] = 1$ and $\text{SD}(X) = (K + 2)\sqrt{K^2 + 4K + 3 - 1/(K + 2)^2} > K$, which leads to a contradiction. □

Appendix C. Special cases corresponding to Section 3

We assume that $D_h(X)$ is the mean median difference:

$$D_h(X) = \text{MMD}(X) := \min_{x \in \mathbb{R}} \mathbb{E}[|X - x|] = \mathbb{E}\left[\left|X - F_X^{-1}\left(\frac{1}{2}\right)\right|\right], \quad X \in L^1. \tag{S.2}$$

The MMD is a signed Choquet integral with a concave distortion function h given by $h(t) = \min\{t, 1 - t)\}, t \in [0, 1]$. We recall that $\mathcal{A}$ is defined in (16).

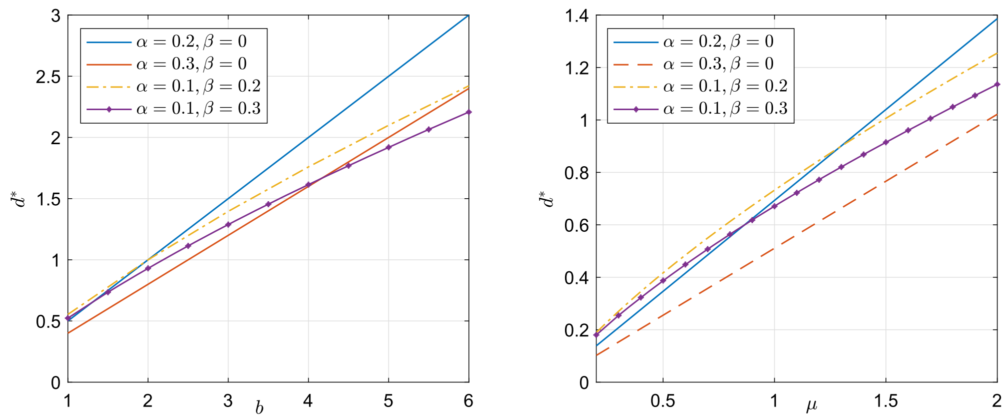

Fig. S.1. Optimal deductible d^* for the uniform distribution with $a = 0$ (left panel) and exponential distribution with μ (right panel) with $D = \text{MMD}$ and $g(x) = \alpha x + \beta x^2$. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

Example S.1. Let $D = \text{MMD}$ and $g(x) = \alpha x + \beta x^2$ with $(\alpha, \beta) \in \mathcal{A}$. If $\beta = 0$, then

$$\theta S_X(x) - \frac{\alpha}{2}\min(S_X(x), 1 - S_X(x)) = \max\left(\theta S_X(x) - \frac{\alpha}{2}S_X(x), \theta S_X(x) + \frac{\alpha}{2}S_X(x) - \frac{\alpha}{2}\right).$$

If $\theta < \alpha/2$, then $d^* = F_X^{-1}(\frac{2\theta}{2\theta+\alpha})$; otherwise, $d^* = M$. For the case of $\beta \neq 0$, we have

$$\theta S_X(x) - h(S_X(x))\left(\alpha + 2\beta\int_0^x h(S_X(t))\mathrm{d}t\right) = \theta S_X(x) - \min(S_X(x), 1 - S_X(x))\left(\alpha + 2\beta\int_0^x \min(S_X(t), 1 - S_X(t))\mathrm{d}t\right).$$

Assume that $X \sim U[a, b]$, then $\text{MMD}(X) = (b - a)/4$. By taking $\theta = 0.2$, we can compute d^* numerically by

$$d^* = \sup\left\{x : \min\left(1, \frac{x-a}{b-x}\right)\left(\alpha(b-a) + 2\beta\int_a^x \min(b-t, t-a)\mathrm{d}t\right) - \theta(b-a) \leqslant 0, \text{ and } 0 \leqslant x < b\right\}.$$

We assume that $X \sim \exp(\lambda)$ with any $\lambda > 0$, then $\text{MMD}(X) = \ln(2)/\lambda$. Again, we can compute d^* numerically

$$d^* = \sup\left\{x : \min(1, e^{\lambda x} - 1)\left(\alpha + 2\beta\int_0^x \min(e^{-\lambda t}, 1 - e^{-\lambda t})\mathrm{d}t\right) - \theta \leqslant 0, \text{ and } x \geqslant 0\right\}.$$

The optimal deductible d^* as a function of parameter b for the uniform distribution and parameter μ for the exponential distribution is displayed in Fig. S.1. This figure shows similar patterns as those in Fig. 1.

In the next two examples, we further assume that $g(x) = e^{\beta x} - 1$ with $\beta > 0$, as it is a common choice for an increasing and convex function with $g(0) = 0$. Again, to ensure that MD_g^D is monotonic on the relevant domain, we must have $g'(D(X - I(X))) = \beta\exp\{\beta(D(X) - D(I(X))\} \leqslant 1$ for all $I \in \mathcal{I}$. Define

$$\mathcal{B} = \{\beta : \beta > 0, \text{ and } \beta D(X) + \ln(\beta) \leqslant 0\}.$$

Example S.2. Let $D = \text{Gini}$ and $g(x) = e^{\beta x} - 1$ with $\beta \in \mathcal{B}$. Then we have

$$\theta S_X(x) - \beta h(S_X(x))\exp\left\{\beta\int_0^x h(S_X(t))\mathrm{d}t\right\} = S_X(x)\left(\theta - \beta(1 - S_X(x))\exp\left\{\beta\int_0^x (S_X(t) - S_X^2(t))\mathrm{d}t\right\}\right).$$

If $X \sim U[a, b]$, then $\text{Gini}(X) = (b - a)/6$. Take $\theta = 0.2$, we can compute d^* numerically by

$$d^* = \sup\left\{x : \beta\frac{x-a}{b-a}\exp\left\{-\frac{\beta}{(b-a)^2}\left(\frac{x^3 - a^3}{3} - \frac{(a+b)(x^2 - a^2)}{2} + ab(x-a)\right)\right\} - \theta \leqslant 0, \text{ and } 0 \leqslant x < b\right\}.$$

If $X \sim \exp(\lambda)$ with $\lambda > 0$, then $\text{Gini}(X) = 1/(2\lambda)$. Again, we can compute d^* numerically

$$d^* = \sup\left\{x : \beta(1 - e^{-\lambda x})\exp\left\{\frac{1}{2\lambda}(1 - e^{-\lambda x})^2\right\} - \theta \leqslant 0, \text{ and } x \geqslant 0\right\}.$$

In Fig. S.2, we once again examine the optimal deductible d^* as it changes with b for the uniform distribution, and with μ for the exponential distribution. We observe that as the expected loss increases, the deductible also rises. When β is small, the DM tends to opt for a larger deductible, while larger values of β indicate heightened aversion to risk deviations, leading to a preference for a smaller deductible. The shape of the optimal

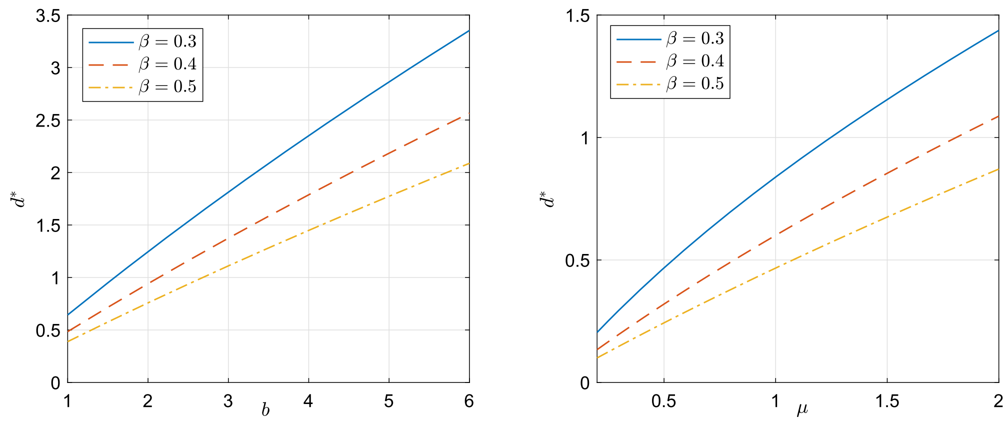

Fig. S.2. Optimal deductible d^* as a function of b for the uniform distribution with $a=0$ (left figure) and as a function of μ for the exponential distribution (right figure) with $D=\text{Gini}$ and $g(x)=e^{\beta x}-1$.

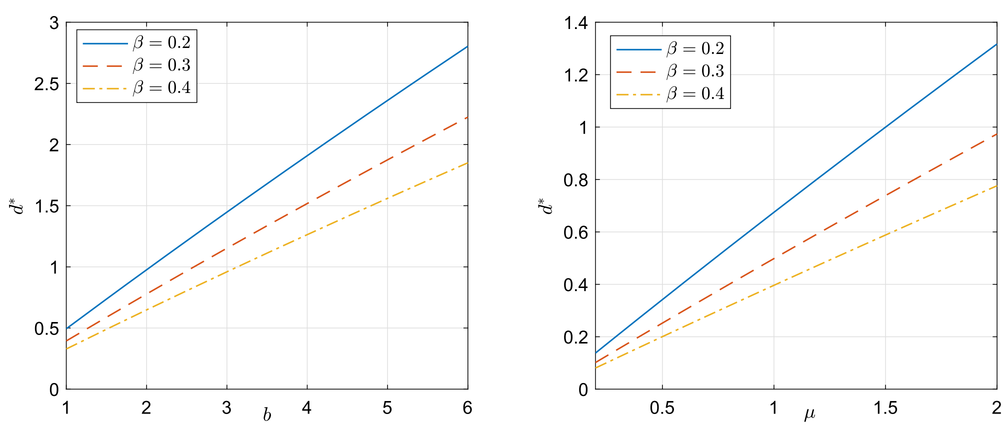

Fig. S.3. Optimal deductible d^* for the uniform distribution with $a=0$ (left panel) and exponential distribution with μ (right panel) with $D=\text{MMD}$ and $g(x)=e^{\beta x}-1$.

deductible as a function of the underlying distribution parameter appears qualitatively similar to Fig. 1, when the function g is non-linear and quadratic.

Example S.3. Let $D=\text{MMD}$ and $g(x)=e^{\beta x}-1$ with $\beta\in\mathcal{B}$. Thus, we have

$$\theta S_X(x)-\beta h(S_X(x))\exp\left\{\beta\int_0^x h(S_X(t))\mathrm{d}t\right\}=\theta S_X(x)-\beta\min(S_X(x),1-S_X(x))\exp\left\{\beta\int_0^x \min(S_X(t),1-S_X(t))\mathrm{d}t\right\}.$$

Assume that $X\sim U[a,b]$, then $\text{MMD}(X)=(b-a)/4$. By taking $\theta=0.2$, we can compute d^* numerically by

$$d^*=\sup\left\{x:\beta\min\left(1,\frac{x-a}{b-x}\right)\exp\left\{\frac{\beta}{b-a}\int_a^x\min(b-t,t-a)\mathrm{d}t\right\}-\theta\leqslant 0,\ \text{and } 0\leqslant x<b\right\}.$$

We assume that $X\sim\exp(\lambda)$ with $\lambda>0$, then $\text{MMD}(X)=\ln(2)/\lambda$. Again, we can compute d^* numerically

$$d^*=\sup\left\{x:\beta\min(1,e^{\lambda x}-1)\exp\left\{\beta\int_0^x\min(e^{-\lambda t},1-e^{-\lambda t})\mathrm{d}t\right\}-\theta\leqslant 0,\ \text{and } x\geqslant 0\right\}.$$

The optimal deductible d^* is reported in Fig. S.3. The patterns have a similar explanation as in Fig. S.2.

Next, we give one more example corresponding to Section 3.2. We assume that $g(x)=e^{\beta x}-1$ with $\beta>0$.

Example S.4. Let $D=\text{SD}$ and $g(x)=e^{\beta x}-1$ with $\beta>0$. For $X\sim U[0,b]$, we have $w_1(x)=(2bx-x^2)/(2b)$ and $w_2(x)=x^2(3b-2x)/(3b)$. By setting $\theta=0.2$, we can compute d^* numerically by

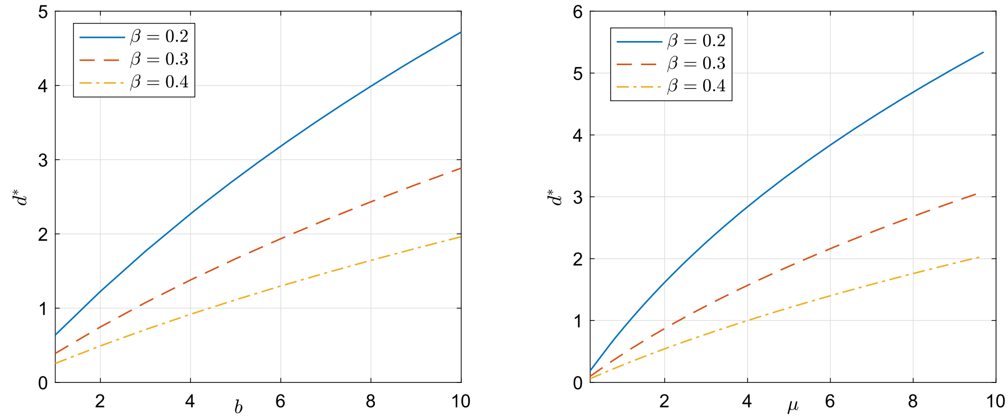

Fig. S.4. Optimal deductible d^* as a function of b for the uniform distribution (left figure) and as a function of μ for the exponential distribution (right figure) with $D = \mathrm{SD}$ and $g(x) = e^{\beta x} - 1$.

$$d^* = \sup\left\{x : \beta\exp\left\{\beta\left(\frac{4bx^3 - 3x^4}{12b^2}\right)^{1/2}\right\}\left(\frac{3x}{4b-3x}\right)^{1/2} - \theta \leqslant 0, \text{ and } 0 \leqslant x < b\right\}.$$

For $X \sim \exp(\lambda)$ with any $\lambda > 0$, we have $w_1(x) = (1 - e^{-\lambda x})/\lambda$ and $w_2(x) = \frac{2}{\lambda^2}(1 - e^{-\lambda x}) - \frac{2}{\lambda}xe^{-\lambda x}$. By setting $\theta = 0.2$, we can compute d^* numerically by

$$d^* = \sup\left\{x : \beta\exp\left\{\frac{\beta}{\lambda}\left(1 - 2\lambda xe^{-\lambda x} - e^{-2\lambda x}\right)^{1/2}\right\}\frac{\lambda x - 1 + e^{-\lambda x}}{(1 - 2\lambda xe^{-\lambda x} - e^{-2\lambda x})^{1/2}} - \theta \leqslant 0, \text{ and } x \geqslant 0\right\}.$$

The optimal deductible d^* is reported in Fig. S.4. Again, the shape of the optimal deductible as a function of the underlying distribution parameter is similar to the case with a non-linear and quadratic function g (cf. Fig. 2).

Appendix D. Proof of Theorem 4

Proof. Case 1: $\Pi = \mathrm{VaR}_p$. Along similar lines in the proof of Proposition 2 and Theorem 3, we consider the following minimization problem

$$\min_{I\in\mathcal{I}} \widetilde{f}(I) := \int_0^M ((1+\lambda_2)h_2(S_X(x)) - S_X(x) - \lambda_1 h_1(S_X(x)))q(x)\mathrm{d}x + \lambda_1 D_{h_1}(X) + g(s) + \mathbb{E}[X] - \lambda_1 s - \lambda_2\overline{\Pi}, \tag{S.3}$$

where $\lambda_1 \geqslant 0$ and $\lambda_2 \geqslant 0$ are the KKT multipliers, and the following q will minimize (S.3)

$$q(x) = \begin{cases} 0, & \text{if } (1+\lambda_2)h_2(S_X(x)) - S_X(x) - \lambda_1 h_1(S_X(x)) > 0, \\ 1, & \text{if } (1+\lambda_2)h_2(S_X(x)) - S_X(x) - \lambda_1 h_1(S_X(x)) < 0, \\ c, & \qquad\qquad\qquad\qquad\qquad\qquad\qquad \textit{otherwise}, \end{cases} \tag{S.4}$$

where c could be any $[0,1]$-valued constant on $(1+\lambda_2)h_2(S_X(x)) - S_X(x) - \lambda_1 h_1(S_X(x)) = 0$. Define

$$H(x) = (1+\lambda_2)h_2(S_X(x)) - S_X(x) - \lambda_1 h_1(S_X(x)).$$

Let $h_2(t) = \mathbb{1}_{\{t>1-p\}}$ with $p \in (0,1)$.

(i) For $t < 1-p$, or equivalently, $x_p < x \leqslant M$, we always have $H(x) = -S_X(x) - \lambda_1 h_1(S_X(x)) < 0$ for $x_p < x < M$, which implies $q(x) = 1$.
(ii) For $t \geqslant 1-p$, or equivalently, $x \leqslant x_p$, we have $H(x) = 1 + \lambda_2 - S_X(x) - \lambda_1 h_1(S_X(x))$. Since h_1 is concave with $h_1(0) = h_1(1) = 0$, $H'(x) = -S'_X(x)(1 + \lambda_1 h'_1(S_X(x))$, $H(x_p) = \lambda_2 + p - \lambda_1 h(1-p)$ and $H(0) = \lambda_2 + 1 - \alpha_0 - \lambda_1 h_1(\alpha_0)$, there at most exists two zeros $d_{i,\lambda_1,\lambda_2} \leqslant x_p$ $(i = 1,2)$ such that $H(d_{i,\lambda_1,\lambda_2}) = 0$.

Define

$$d_{0,\lambda_1,\lambda_2} = \inf\{x :\ 1 + \lambda_2 - S_X(x) - \lambda_1 h_1(S_X(x)) \leqslant 0, \text{ and } \ 0 \leqslant x \leqslant x_p\},$$

and

$$d_{1,\lambda_1,\lambda_2} = \sup\{x :\ 1 + \lambda_2 - S_X(x) - \lambda_1 h_1(S_X(x)) \leqslant 0, \text{ and } \ d_{0,\lambda_1,\lambda_2} \leqslant x \leqslant x_p\}.$$

Then we have $I(x) = I_{d_{0,\lambda_1,\lambda_2}, d_{1,\lambda_1,\lambda_2}, x_p}(X) := \int_0^x q(t)\mathrm{d}t = (x - d_{0,\lambda_1,\lambda_2})_+ \wedge (d_{1,\lambda_1,\lambda_2} - d_{0,\lambda_1,\lambda_2}) + (x - x_p)_+$. That is, for every s, there exists an $I_{d_0,d_1,x_p}(x) = (x - d_0)_+ \wedge (d_1 - d_0) + (x - x_p)_+$ that does better than any $I \in \mathcal{I}$.

Next, we aim to find the optimal d_0 and d_1 for problem (S.3) subject to (28) when the insurance contract is given by I_{d_0,d_1,x_p} for some $d_0, d_1 \in [0, x_p]$, that is,

$$\min_{d_0,d_1\in[0,x_p]} \widetilde{F}(d_0,d_1) := \int_{d_0}^{d_1} (1+\lambda_2 - S_X(x))\mathrm{d}x - \int_{x_p}^{M} S_X(x)\mathrm{d}x + g\left(\int_0^{d_0} h_1(S_X(x))\mathrm{d}x + \int_{d_1}^{x_p} h_1(S_X(x))\mathrm{d}x\right) + \mathbb{E}[X] - \lambda_2\overline{\Pi}. \tag{S.5}$$

Assume that there exists a constant $\lambda_2^*\geqslant 0$ such that $\widetilde{d}_{0\lambda_2^*}$ and $\widetilde{d}_{1\lambda_2^*}$ solve problem (S.5) for $\lambda_2=\lambda_2^*$ and $\mathrm{VaR}_p(I_{\widetilde{d}_{0\lambda_2^*},\widetilde{d}_{1\lambda_2^*},x_p})=\overline{\Pi}$. Then, we aim to show $\widetilde{d}_{0\lambda_2^*}$ and $\widetilde{d}_{1\lambda_2^*}$ solve problem (5) with constraint (28). We denote the optimal value of problem (5) with constraint (28) by $V(\overline{\Pi})$. Then, it follows that

$$\begin{aligned}
V(\overline{\Pi}) = \inf_{\substack{d_0,d_1\in[0,x_p]\\ \mathrm{VaR}_p(I_{d_0,d_1,x_p})\leqslant\overline{\Pi}}} \mathrm{MD}_g^D(T_I) &\leqslant \inf_{\substack{d_0,d_1\in[0,x_p]\\ \mathrm{VaR}_p(I_{d_0,d_1,x_p})\leqslant\overline{\Pi}}} \left\{\mathrm{MD}_g^D(T_I) - \lambda_2^*\left(\mathrm{VaR}_p(I_{d_0,d_1,x_p})-\overline{\Pi}\right)\right\} \\
&\leqslant \inf_{d_0,d_1\in[0,x_p]} \left\{\mathrm{MD}_g^D(T_I) - \lambda_2^*\left(\mathrm{VaR}_p(I_{d_0,d_1,x_p})-\overline{\Pi}\right)\right\} = \mathrm{MD}_g^D(T_{I_{\widetilde{d}_{0\lambda_2^*},\widetilde{d}_{1\lambda_2^*}}}) \leqslant V(\overline{\Pi}).
\end{aligned}$$

The last inequality is because $I_{\widetilde{d}_{0\lambda_2^*},\widetilde{d}_{1\lambda_2^*}}$ is feasible to problem (5) without the constraint. Hence, $(\widetilde{d}_0^*,\widetilde{d}_1^*)=(\widetilde{d}_{0\lambda_2^*},\widetilde{d}_{1\lambda_2^*})$ solves problem (S.5). To use the first-order condition, we get

$$\frac{\partial\widetilde{F}(d_0,d_1)}{\partial d_0} = g'\left(\int_0^{d_0} h_1(S_X(x))\mathrm{d}x + \int_{d_1}^{x_p} h_1(S_X(x))\mathrm{d}x\right) h_1(S_X(d_0)) - (1+\lambda_2 - S_X(d_0)),$$

and

$$\frac{\partial\widetilde{F}(d_0,d_1)}{\partial d_1} = -g'\left(\int_0^{d_0} h_1(S_X(x))\mathrm{d}x + \int_{d_1}^{x_p} h_1(S_X(x))\mathrm{d}x\right) h_1(S_X(d_1)) + (1+\lambda_2 - S_X(d_1)).$$

Then, we can solve $\widetilde{d}_0^*$, $\widetilde{d}_1^*$ and λ_2^* by (33), and

$$\lambda_2^* = \inf\left\{\lambda_2 : \mathrm{VaR}_p(I_{\widetilde{d}_0^*,\widetilde{d}_1^*,x_p}) - \overline{\Pi}\leqslant 0, \text{ and } \lambda_2\geqslant 0\right\}. \tag{S.6}$$

Case 2: $\Pi=\mathrm{ES}_p$. Let $h_2(t)=\frac{t}{1-p}\wedge 1$ with $p\in(0,1)$.

(i) For $t\geqslant 1-p$, or equivalently, $x\leqslant x_p$, the analysis is similar to the case of VaR.
(ii) For $t<1-p$, or equivalently, $x>x_p$, we always have $H(x)=\frac{p+\lambda_2}{1-p}S_X(x)-\lambda_1 h_1(S_X(x))$. When $H(x_p)=p+\lambda_2-\lambda_1 h_1(1-p)>0$, if $H'(M)=\frac{p+\lambda_2}{1-p}-\lambda_1 h_1'(0)<0$, then there exists a unique $d_{2,\lambda_1,\lambda_2}$ such that $H_1(x)>0$ for $x_p<x<d_{2,\lambda_1,\lambda_2}$, and $H_1(x)<0$ for $d_{2,\lambda_1,\lambda_2}<x<M$; if $H'(0)>0$, then $H(x)\geqslant 0$ for any $x\in[x_p,M]$. When $H(x_p)=p-\lambda h_1(1-p)<0$, then we have $H_1(x)\leqslant 0$ for any $x>x_p$.

Define

$$d_{2,\lambda_1,\lambda_2} = \sup\left\{x : \frac{p+\lambda_2}{1-p}S_X(x)-\lambda_1 h_1(S_X(x))\geqslant 0, \text{ and } x_p<x<M\right\}. \tag{S.7}$$

It is clear that $d_{0,\lambda_1,\lambda_2}\leqslant d_{1,\lambda_1,\lambda_2}<x_p\leqslant d_{2,\lambda_1,\lambda_2}$. Then problem (S.3) can be minimized by $I(x)=I_{d_{0,\lambda_1,\lambda_2},d_{1,\lambda_1,\lambda_2},d_{2,\lambda_1,\lambda_2}}(X) := \int_0^x q(t)\mathrm{d}t = (x-d_{0,\lambda_1,\lambda_2})_+\wedge(d_{1,\lambda_1,\lambda_2}-d_{0,\lambda_1,\lambda_2})+(x-d_{2,\lambda_1,\lambda_2})_+$. That is, for every s, there exists an $I_{d_0,d_1,d_2}(x)=(x-d_0)_+\wedge(d_1-d_0)+(x-d_2)_+$ that does better than any $I\in\mathcal{I}$. Next, we aim to find the optimal d_0, d_1 and d_2 for problem (S.3) subject to (28) when the insurance contract is given by I_{d_0,d_1,d_2} for some $d_0,d_1\in[0,x_p]$ and $d_2\in(x_p,M]$, that is,

$$\min_{d_0,d_1\in[0,x_p],d_2\in(x_p,M]} \widetilde{F}(d_0,d_1,d_2) := \int_{d_0}^{d_1}(1+\lambda_2-S_X(x))\mathrm{d}x + \int_{d_2}^{M}\frac{p+\lambda_2}{1-p}S_X(x)\mathrm{d}x + g\left(\int_0^{d_0} h_1(S_X(x))\mathrm{d}x + \int_{d_1}^{d_2} h_1(S_X(x))\mathrm{d}x\right) + \mathbb{E}[X]-\lambda_2\overline{\Pi}. \tag{S.8}$$

To use the first-order condition, we get

$$\frac{\partial\widetilde{F}(d_0,d_1,d_2)}{\partial d_0} = g'\left(\int_0^{d_0} h_1(S_X(x))\mathrm{d}x + \int_{d_1}^{d_2} h_1(S_X(x))\mathrm{d}x\right) h_1(S_X(d_0)) - (1+\lambda_2-S_X(d_0)),$$

$$\frac{\partial\widetilde{F}(d_0,d_1,d_2)}{\partial d_1} = -g'\left(\int_0^{d_0} h_1(S_X(x))\mathrm{d}x + \int_{d_1}^{d_2} h_1(S_X(x))\mathrm{d}x\right) h_1(S_X(d_1)) + (1+\lambda_2-S_X(d_1)),$$

and

$$\frac{\partial \widetilde{F}(d_0,d_1,d_2)}{\partial d_2} = g'\left(\int_0^{d_0} h_1(S_X(x))\mathrm{d}x + \int_{d_1}^{d_2} h_1(S_X(x))\mathrm{d}x\right) h_1(S_X(d_2)) - \frac{p+\lambda_2}{1-p} S_X(d_2).$$

Assume that there exists a constant $\lambda_2^* \geqslant 0$ such that $\widetilde{d}_{0\lambda_2^*}$, $\widetilde{d}_{1\lambda_2^*}$ and $\widetilde{d}_{2\lambda_2^*}$ solve problem (S.8) for $\lambda_2 = \lambda_2^*$ and $\Pi(I_{\widetilde{d}_{0\lambda_2^*},\widetilde{d}_{1\lambda_2^*},\widetilde{d}_{2\lambda_2^*}}) = \overline{\Pi}$. In this case, $\widetilde{d}_0^*$, $\widetilde{d}_1^*$, $\widetilde{d}_2^*$ and λ_2^* can be solved by (33) and

$$\lambda_2^* = \inf\left\{\lambda_2 : \mathrm{VaR}_p(I_{\widetilde{d}_0^*,\widetilde{d}_1^*,\widetilde{d}_2^*}) - \overline{\Pi} \leqslant 0, \text{ and } \lambda_2 \geqslant 0\right\}. \quad \square$$

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