Insurance: Mathematics and Economics 113 (2023) 185–197

Contents lists available at ScienceDirect

Insurance: Mathematics and Economics

journal homepage: www.elsevier.com/locate/ime

Diversification quotients based on VaR and ES

Xia Han [a], Liyuan Lin [b,*], Ruodu Wang [b]

[a] School of Mathematical Sciences and LPMC, Nankai University, China
[b] Department of Statistics and Actuarial Science, University of Waterloo, Canada

ARTICLE INFO

Article history:
Received January 2023
Received in revised form August 2023
Accepted 23 August 2023
Available online 1 September 2023

JEL classification:
C44
G11

Keywords:
Value-at-Risk
Expected Shortfall
Diversification quotient
Elliptical models
Regular varying models

ABSTRACT

The diversification quotient (DQ) is recently introduced for quantifying the degree of diversification of a stochastic portfolio model. It has an axiomatic foundation and can be defined through a parametric class of risk measures. Since the Value-at-Risk (VaR) and the Expected Shortfall (ES) are the most prominent risk measures widely used in both banking and insurance, we investigate DQ constructed from VaR and ES in this paper. In particular, for the popular models of elliptical and multivariate regular varying (MRV) distributions, explicit formulas are available. The portfolio optimization problems for the elliptical and MRV models are also studied. Our results further reveal favorable features of DQ, both theoretically and practically, compared to traditional diversification indices based on a single risk measure.

1. Introduction

In order to mitigate risks in portfolios of financial investment quantitatively, a common approach is to compute a quantitative index of the portfolio model, based on e.g., the volatility, variance, an expected utility or a risk measure, following the seminal idea of Markowitz (1952) on portfolio diversification. In the literature, one of the most prominent examples of the diversification index based on a general risk measure is defined by Tasche (2007) which is referred as diversification ratio (DR). Choueifaty and Coignard (2008) investigated the theoretical and empirical properties of DR in portfolio construction and compared the behavior of the resulting portfolio to common, simple strategies. See Embrechts et al. (2015) and Koumou and Dionne (2022) for theories of DR and other diversification indices. Bürgi et al. (2008) defined a closely related notion of DR which is called the diversification gain and explored various methods of modeling dependence and their influence on diversification gain.

Different from the traditional diversification indices such as DR in the above literature, Han et al. (2022) proposed six axioms – non-negativity, location invariance, scale invariance, rationality, normalization and continuity – which jointly characterize a new diversification index, called the diversification quotient (DQ), whose definition is based on a class of risk measures decreasing in an index α. All commonly used risk measures belong to a monotonic parametric family, and this includes VaR, ES, expectiles, mean-variance, and entropic risk measures. They argued that DQ has many appealing features both theoretically and practically, while these properties, in particular the six axioms above, are not shared by DR based on VaR, ES, or any other commonly used risk measure. Moreover, portfolio optimization of DQs based on VaR and ES can be computed very efficiently, and thus DQ can be easily applied to real data.

Most properties of DQ are studied by Han et al. (2022) for a general class of risk measures. In this paper, we focus on specific risk measures, in particular, the Value-at-Risk (VaR) and the Expected Shortfall (ES). Even though VaR has been criticized because of its lack of subadditivity and ES requires the loss to have a finite mean, VaR and ES are still the two most common classes of risk measures in practice, widely employed in global banking and insurance regulatory frameworks; see Basel III/IV (BCBS (2019)) and Solvency II (EIOPA (2011)). More theoretical properties and discussions of VaR and ES can be found in, e.g., Artzner et al. (1999), Embrechts et al. (2014, 2018), Emmer et al. (2015) and the references therein. We pay particular attention to two popular models in finance and insurance, namely, elliptical and multivariate regular variation (MRV) distributions. Elliptical distributions, including

* Corresponding author.
E-mail addresses: xiahan@nankai.edu.cn (X. Han), l89lin@uwaterloo.ca (L. Lin), wang@uwaterloo.ca (R. Wang).

https://doi.org/10.1016/j.insmatheco.2023.08.006
0167-6687/

normal and t-distributions as special cases, are the most standard tools for quantitative risk management (McNeil et al. (2015)). They have been studied for DR with convenient properties; see Cui et al. (2022) and the references therein. The MRV model is widely used in Extreme Value Theory for investigating the portfolio diversification; see, e.g., Mainik and Rüschendorf (2010), Mainik and Embrechts (2013) and Bignozzi et al. (2016).

This paper is an extension of Han et al. (2022) in which an axiomatic framework of diversification indices is proposed and general properties of DQ are studied. As a new concept of diversification index, studying properties such as explicit formulas and limiting behavior of DQ under specific risk measures and special risk models will help us to better understand and use DQ in risk management applications. In addition, the advantages of DQ and the connection between DQ and DR are clearer under the elliptical and MRV models, revealing many attractive features of choosing DQ instead of DR to quantify diversification risk, especially for tail heaviness and common shocks.

The paper is organized as follows. In Section 2, the definition of DQ and some preliminaries on risk measures are collected. In Section 3, we study general properties for DQs based on VaR and ES. Since DQs based on VaR and ES have natural ranges of $[0,n]$ and $[0,1]$, respectively, some special dependence structures of the portfolio that correspond to the special values of 0, 1, and n are constructed with clear interpretation for values in between (Theorem 1). In Section 4, we focus on DQ for large portfolios. By the Law of Large Numbers, we show that DQs based VaR and ES for a portfolio with independent components tend to 0 as the number of assets in the portfolio increases to infinity (Theorem 2). The limits for DQs based on VaR and ES for portfolios with exchangeable components do not necessarily tend to 0. We show that the upper bound for the limit decreases in the bivariate correlation coefficient. (Proposition 1). In Section 5, DQ is applied to elliptical models; explicit formulas and the limiting behavior of DQs based on VaR and ES are available (Proposition 2 and Theorem 3). Moreover, we present several numerical results for the two most important elliptical distributions used in finance and insurance, namely the multivariate normal distribution and the multivariate t-distribution, and show that DQ can properly capture tail heaviness. As a popular tool for modeling heavy-tailed phenomena, MRV models for DQ are studied in Section 6. Furthermore, we generalize the results to the optimal portfolio selection problem in Section 7. Under elliptical models, the optimization problem can boil down to a well-studied problem (see e.g., Choueifaty and Coignard (2008)) and a limiting result in MRV models is also derived (Theorem 4 and Proposition 5). We conclude the paper in Section 8.

2. Diversification quotients

Throughout this paper, $(\Omega,\mathcal{F},\mathbb{P})$ is an atomless probability. The atomless assumption in our context is very weak and it is widely used in statistics and risk management; see Delbaen (2002) and Section A.3 of Föllmer and Schied (2016) for details of atomless probability spaces. Almost surely equal random variables are treated as identical. A risk measure ϕ is a mapping from $\mathcal{X}$ to $\mathbb{R}$, where $\mathcal{X}$ is a convex cone of random variables on $(\Omega,\mathcal{F},\mathbb{P})$ representing losses faced by a financial institution or an investor, and $\mathcal{X}$ is assumed to include all constants (i.e., degenerate random variables). For $p\in(0,\infty)$, denote by $L^p=L^p(\Omega,\mathcal{F},\mathbb{P})$ the set of all random variables X with $\mathbb{E}[|X|^p]<\infty$ where $\mathbb{E}$ is the expectation under $\mathbb{P}$. Furthermore, $L^\infty=L^\infty(\Omega,\mathcal{F},\mathbb{P})$ is the space of all essentially bounded random variables, and $L^0=L^0(\Omega,\mathcal{F},\mathbb{P})$ is the space of all random variables. Write $X\sim F$ if the random variable X has the distribution function F under $\mathbb{P}$, and $X\overset{\mathrm{d}}{=}Y$ if two random variables X and Y have the same distribution. We always write $\mathbf{X}=(X_1,\dots,X_n)$ and $\mathbf{0}$ for the n-vector of zeros. Further, denote by $[n]=\{1,\dots,n\}$, $\mathbb{R}_+=[0,\infty)$ and $\overline{\mathbb{R}}=[-\infty,\infty]$. Terms such as increasing or decreasing functions are in the non-strict sense. For $X\in\mathcal{X}$, ess-sup(X) and ess-inf(X) are the essential supremum and the essential infimum of X, respectively.

A *diversification index* D is a mapping from $\mathcal{X}^n$ to $\overline{\mathbb{R}}$, which is used to quantify the magnitude of diversification of a risk vector $\mathbf{X}\in\mathcal{X}^n$ representing portfolio losses. Our convention is that a smaller value of $D(\mathbf{X})$ represents a stronger diversification. Measuring diversification is closely related to risk measures. Some standard properties of a risk measure $\phi:\mathcal{X}\to\mathbb{R}$ are collected below.

[M] Monotonicity: $\phi(X)\leqslant\phi(Y)$ for all $X,Y\in\mathcal{X}$ with $X\leqslant Y$.
[CA] Constant additivity: $\phi(X+c)=\phi(X)+c$ for all $c\in\mathbb{R}$ and $X\in\mathcal{X}$.
[PH] Positive homogeneity: $\phi(\lambda X)=\lambda\phi(X)$ for all $\lambda\in(0,\infty)$ and $X\in\mathcal{X}$.
[SA] Subadditivity: $\phi(X+Y)\leqslant\phi(X)+\phi(Y)$ for all $X,Y\in\mathcal{X}$.

The two popular classes of risk measures in banking and insurance practice are VaR and ES. The VaR at level $\alpha\in[0,1)$ is defined as

$$\mathrm{VaR}_\alpha(X)=\inf\{x\in\mathbb{R}:\mathbb{P}(X\leqslant x)\geqslant 1-\alpha\},\quad X\in L^0,$$

and the ES (also called CVaR, TVaR or AVaR) at level $\alpha\in(0,1)$ is defined as

$$\mathrm{ES}_\alpha(X)=\frac{1}{\alpha}\int_0^\alpha \mathrm{VaR}_\beta(X)\mathrm{d}\beta,\quad X\in L^1,$$

and $\mathrm{ES}_0(X)=\text{ess-sup}(X)=\mathrm{VaR}_0(X)$ which may be ∞. The probability level α above is typically very small, e.g., 0.01 or 0.025 in BCBS (2019); note that we use the "small α" convention as in Han et al. (2022). Both VaR and ES satisfy the properties [M], [CA] and [PH], while ES also satisfies the property [SA].

To measure diversification quantitatively, a new index, called diversification quotient (DQ), is introduced as follows.

Definition 1 *(Han et al. (2022)).* Let $\rho=(\rho_\alpha)_{\alpha\in I}$ be a class of risk measures indexed by $\alpha\in I=(0,\bar\alpha)$ with $\bar\alpha\in(0,\infty]$ such that ρ_α is decreasing in α. For $\mathbf{X}\in\mathcal{X}^n$, the diversification quotient based on the class ρ at level $\alpha\in I$ is defined by

$$\mathrm{DQ}_\alpha^\rho(\mathbf{X})=\frac{\alpha^*}{\alpha},$$
$$\text{where }\alpha^*=\inf\left\{\beta\in I:\rho_\beta\left(\sum_{i=1}^n X_i\right)\leqslant\sum_{i=1}^n\rho_\alpha(X_i)\right\}$$

with the convention $\inf(\varnothing)=\bar\alpha$.

Remark 1. The value of DQ^ρ_α depends on how the class $\rho=(\rho_\alpha)_{\alpha\in I}$ is parametrized. For instance, one could, hypothetically, use a different parametrization $\mathrm{VaR}'_\alpha=\mathrm{VaR}_{\alpha^2}$ for the class VaR, although there is no real reason to do so. The value of $\mathrm{DQ}^{\mathrm{VaR}'}_\alpha$ is generally different from $\mathrm{DQ}^{\mathrm{VaR}}_{\alpha^2}$, but they generate the same order; that is, $\mathrm{DQ}^{\mathrm{VaR}'}_\alpha(\mathbf{X})\leqslant\mathrm{DQ}^{\mathrm{VaR}'}_\alpha(\mathbf{Y})$ if and only if $\mathrm{DQ}^{\mathrm{VaR}}_{\alpha^2}(\mathbf{X})\leqslant\mathrm{DQ}^{\mathrm{VaR}}_{\alpha^2}(\mathbf{Y})$, which can be checked by definition. Therefore, different parametrizations do not affect the application of DQ in portfolio optimization.

Han et al. (2022, Theorem 1) characterized a subclass of DQ via six axioms: non-negativity, location invariance, scale invariance, rationality, normalization and continuity; such DQs are defined on the class of risk measures satisfying [M], [CA] and [PH].

DQ is defined based on a monotonic parametric class of risk measures. All commonly used risk measures belong to a monotonic parametric family; for instance, this includes VaR, ES, expectiles, mean-variance, and entropic risk measures; see Föllmer and Schied (2016) for a general treatment of risk measures.

In finance and insurance, the risk measures VaR and ES play prominent roles, as they are specified in regulatory documents such as BCBS (2019) and EIOPA (2011). We will focus on VaR or ES as the risk measures assessing diversification by DQ in this paper. In particular, both VaR and ES satisfy the properties [M], [CA] and [PH], and hence $\mathrm{DQ}^{\mathrm{VaR}}_\alpha$ and $\mathrm{DQ}^{\mathrm{ES}}_\alpha$ satisfy the six above axioms.

Another popular diversification index is the diversification ratio (see e.g., Tasche (2007) and Embrechts et al. (2015)), defined as

$$\mathrm{DR}^{\phi}(\mathbf{X}) = \frac{\phi\left(\sum_{i=1}^n X_i\right)}{\sum_{i=1}^n \phi(X_i)}, \tag{1}$$

where ϕ is a suitably chosen risk measure, such as VaR_α, ES_α, variance (var), or standard deviation (SD). Although DR generally does not satisfy some of the six axioms, we will compare DQ and DR in several parts of the paper.

3. DQ based on VaR and ES

In this section, we will focus on the theoretical properties of $\mathrm{DQ}^{\mathrm{VaR}}_\alpha$ and $\mathrm{DQ}^{\mathrm{ES}}_\alpha$. For VaR and ES, the interval in Definition 1 has a natural range of $I=(0,1)$. Similarly to Han et al. (2022), we let $\mathcal{X}^n$ be $(L^0)^n$ when we discuss $\mathrm{DQ}^{\mathrm{VaR}}_\alpha$ and $(L^1)^n$ when we discuss $\mathrm{DQ}^{\mathrm{ES}}_\alpha$. To compute $\mathrm{DQ}^{\mathrm{ES}}_\alpha$, we first define the superquantile transform (Liu et al. (2021, Example 4)). The term "superquantile" is an alternative name for ES; see Rockafellar et al. (2014).

Definition 2. The *superquantile transform* of a distribution F with finite mean is a distribution $\widetilde{F}$ with quantile function $p \mapsto \mathrm{ES}_{1-p}(X)$ for $p \in (0,1)$, where $X \sim F$.

The following alternative formulas for DQs based on VaR and ES will be useful later. They are shown in Theorem 3 of Han et al. (2022). For a given $\alpha \in (0,1)$, $\mathrm{DQ}^{\mathrm{VaR}}_\alpha$ and $\mathrm{DQ}^{\mathrm{ES}}_\alpha$ can be computed by

$$\mathrm{DQ}^{\mathrm{VaR}}_\alpha(\mathbf{X}) = \frac{1 - F\left(\sum_{i=1}^n \mathrm{VaR}_\alpha(X_i)\right)}{\alpha} \quad \text{and} \quad \mathrm{DQ}^{\mathrm{ES}}_\alpha(\mathbf{X}) = \frac{1 - \widetilde{F}\left(\sum_{i=1}^n \mathrm{ES}_\alpha(X_i)\right)}{\alpha}, \tag{2}$$

where F is the distribution of $\sum_{i=1}^n X_i$ and $\widetilde{F}$ is the superquantile transform of F.

Remark 2. Let $S=\sum_{i=1}^n X_i$. If S has a continuous and strictly monotone quantile function, then (2) can be rewritten as

$$\mathrm{DQ}^{\mathrm{VaR}}_\alpha(\mathbf{X}) = \frac{1}{\alpha}\mathbb{P}\left(S > \sum_{i=1}^n \mathrm{VaR}_\alpha(X_i)\right), \quad \mathbf{X} \in \mathcal{X}^n,$$

and

$$\mathrm{DQ}^{\mathrm{ES}}_\alpha(\mathbf{X}) = \frac{1}{\alpha}\mathbb{Q}\left(S > \sum_{i=1}^n \mathrm{ES}_\alpha(X_i)\right), \quad \mathbf{X} \in \mathcal{X}^n,$$

for some probability measure $\mathbb{Q}$. To give a formula for $\mathbb{Q}$, let F be the distribution of S, and $\alpha_0 = 1 - F(\mathbb{E}[S])$. There exists an increasing and continuous function $g:(0,1)\to[0,1]$ such that $\mathrm{ES}_{g(\alpha)}(S) = \mathrm{VaR}_\alpha(S)$ for all $\alpha \in (0,\alpha_0)$ and $g(\alpha)=1$ for $\alpha \in [\alpha_0, 1)$. We can express $\mathbb{Q}$ by $\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P} = g'(1-F(S))$.

Remark 3. DQ based on ES admits another convenient formula in Han et al. (2022, Theorem 3). If $\mathbb{P}(\sum_{i=1}^n X_i > \sum_{i=1}^n \mathrm{ES}_\alpha(X_i)) > 0$, then

$$\mathrm{DQ}^{\mathrm{ES}}_\alpha(\mathbf{X}) = \frac{1}{\alpha}\min_{r\in(0,\infty)} \mathbb{E}\left[\left(r\sum_{i=1}^n (X_i - \mathrm{ES}_\alpha(X_i)) + 1\right)_+\right], \tag{3}$$

and otherwise $\mathrm{DQ}^{\mathrm{ES}}_\alpha(\mathbf{X}) = 0$. The main advantage of this formula of $\mathrm{DQ}^{\mathrm{ES}}_\alpha$ is computation and optimization. In particular, this formula allows us to write the portfolio optimization problem of $\mathrm{DQ}^{\mathrm{ES}}_\alpha$ as a convex program; this is shown in Proposition 5 of Han et al. (2022).

Next, we see that if $\alpha \in (0, 1/n)$, there are three special values of $\mathrm{DQ}^{\mathrm{VaR}}_\alpha$, which are 0, 1 and n, corresponding to different representative dependence structures. The last value of n is based on a useful inequality

$$\mathrm{VaR}_{n\alpha}\left(\sum_{i=1}^n X_i\right) \leqslant \sum_{i=1}^n \mathrm{VaR}_\alpha(X_i) \tag{4}$$

from Corollary 1 of Embrechts et al. (2018), and its sharpness is stated in Corollary 2 therein. For $\mathrm{DQ}^{\mathrm{ES}}_\alpha$, there are two special numbers, 0 and 1, because ES is a class of subadditive risk measures. As a natural question, we wonder for what types of dependence structures these special values are attained. Next, we address this question.

We first present the concept of risk concentration in Wang and Zitikis (2021) which will be useful to understand the dependence structures corresponding to special values of $\mathrm{DQ}^{\mathrm{VaR}}_\alpha$ and $\mathrm{DQ}^{\mathrm{ES}}_\alpha$.

Definition 3 *(Tail event and α-concentrated).* Let X be a random variable and $\alpha \in (0,1)$.

(i) A tail event of X is an event $A \in \mathcal{F}$ with $0 < \mathbb{P}(A) < 1$ such that $X(\omega) \geqslant X(\omega')$ holds for a.s. all $\omega \in A$ and $\omega' \in A^c$, where A^c stands for the complement of A.
(ii) A random vector $(X_1, \dots, X_n)$ is α-concentrated if its component share a common tail event of probability α.[1]

Theorem 4 of Wang and Zitikis (2021) gives that a random vector $(X_1,\dots,X_n)$ is α-concentrated for all $\alpha \in (0,1)$ if and only if it is comonotonic, and hence the dependence notion of α-concentration is weaker than comonotonicity. A random vector $(X_1,\dots,X_n)$ is *comonotonic* if there exists a random variable Z and increasing functions $f_1,\dots,f_n$ on $\mathbb{R}$ such that $X_i = f_i(Z)$ a.s. for every $i \in [n]$.

We first address the case that $\mathrm{DQ}^{\mathrm{VaR}}_\alpha(\mathbf{X}) = n$, which involves the dependence concepts of both risk concentration and mutual exclusivity (see Dhaene and Denuit (1999)). Thus, to arrive at the maximum value of $\mathrm{DQ}^{\mathrm{VaR}}_\alpha(\mathbf{X}) = n$, one requires a dependence structure that is a combination of positive and negative dependence. This phenomenon is common in problems in VaR aggregation; see Puccetti and Wang (2015) for extremal dependence concepts. For this purpose, we propose the *α-concentration-exclusion* (α-CE) model for $\alpha \in (0, 1/n)$, which is a random vector $\mathbf{X} \in \mathcal{X}^n$ satisfying four conditions:

(i) $\mathbb{P}(X_i > \mathrm{VaR}_\alpha(X_i)) = \alpha$;
(ii) $\mathbb{P}(X_i \geqslant \mathrm{VaR}_\alpha(X_i)) \geqslant n\alpha$;
(iii) $\{X_i > \mathrm{VaR}_\alpha(X_i)\}$, $i \in [n]$, are mutually exclusive;

[1] Wang and Zitikis (2021) used the "large α" convention, and hence our α-concentration corresponds to their $(1-\alpha)$-concentration.

(iv) $(X_1,\dots,X_n)$ are $(n\alpha)$-concentrated.

For a class ρ of risk measures ρ_α decreasing in α, we say that ρ is *non-flat from the left* at (α, X) if $\rho_\beta(X) > \rho_\alpha(X)$ for all $\beta \in (0,\alpha)$, and ρ is *left continuous* at (α, X) if $\alpha \mapsto \rho_\alpha(X)$ is left continuous.

Remark 4. For any given $X \in L^0$, if VaR is non-flat from the left at $(n\alpha, X)$, then there exists α-CE random vector $\mathbf{X} \in \mathcal{X}^n$ such that $\sum_{i=1}^n X_i = X$. For instance, let $A = \{X > \mathrm{VaR}_{n\alpha}(X)\}$. As VaR is non-flat from the left at $(n\alpha, X)$, we have $\mathbb{P}(A) = n\alpha$. Let $(A_1,\dots,A_n)$ be a partition of A with $\mathbb{P}(A_i) = \alpha$ for $i \in [n]$. Also, let $X_i = (X-m)\mathbf{1}_{A_i}$ for $i \in [n-1]$ and $X_n = (X-m)\mathbf{1}_{\{A_n \cup A^c\}} + m$ where $m = \mathrm{VaR}_{n\alpha}(X)$ is a constant. It follows that $\sum_{i=1}^n X_i = X$, and it is clear that $\mathbf{X} = (X_1,\dots,X_n)$ is an α-CE model; such a construction is essentially the one in Embrechts et al. (2018, Theorem 2). More generally, we give a sufficient condition for $\mathbf{X}$ to satisfy the α-CE model. A random vector (X, Y) is said to be counter-monotonic if $(X, -Y)$ is comonotonic. If each pair (X_i, X_j) is counter-monotonic for $i \neq j$, and for each $i \in [n]$, $\mathbb{P}(X_i > \mathrm{VaR}_\alpha(X_i)) = \alpha$ and $\mathrm{VaR}_\alpha(X_i) = \text{ess-inf}(X_i)$, then $\mathbf{X}$ follows an α-CE model. For recent results on pairwise counter-monotonicity, see Lauzier et al. (2023).

In the next result, we summarize several dependence structures that correspond to special values 0, 1 and n of $\mathrm{DQ}^{\mathrm{VaR}}_\alpha$ and the special values 0 and 1 of $\mathrm{DQ}^{\mathrm{ES}}_\alpha$.

Theorem 1. *For $\alpha \in (0,1)$ and $n \geqslant 2$, the following hold:*

(i) $\{\mathrm{DQ}^{\mathrm{VaR}}_\alpha(\mathbf{X}) \mid \mathbf{X} \in \mathcal{X}^n\} = [0, \min\{n, 1/\alpha\}]$ and $\{\mathrm{DQ}^{\mathrm{ES}}_\alpha(\mathbf{X}) \mid \mathbf{X} \in \mathcal{X}^n\} = [0,1]$.
(ii) For ρ being VaR or ES, $\mathrm{DQ}^\rho_\alpha(\mathbf{X}) = 0$ if and only if $\sum_{i=1}^n X_i \leqslant \sum_{i=1}^n \rho_\alpha(X_i)$ a.s. In case $\sum_{i=1}^n X_i$ is a constant, $\mathrm{DQ}^{\mathrm{VaR}}_\alpha(\mathbf{X}) = 0$ if $\alpha < 1/n$ and $\mathrm{DQ}^{\mathrm{ES}}_\alpha(\mathbf{X}) = 0$.
(iii) For ρ being VaR or ES, if $\mathbf{X}$ is α-concentrated, then $\mathrm{DQ}^\rho_\alpha(\mathbf{X}) \leqslant 1$. If, in addition, ρ is continuous and non-flat from the left at $(\alpha, \sum_{i=1}^n X_i)$, then $\mathrm{DQ}^\rho_\alpha(\mathbf{X}) = 1$.
(iv) If $\alpha < 1/n$ and $\mathbf{X}$ has an α-CE model, then $\mathrm{DQ}^{\mathrm{VaR}}_\alpha(\mathbf{X}) = n$ and $\mathrm{DQ}^{\mathrm{ES}}_{n\alpha}(\mathbf{X}) = 1$.

Proof. (i) We first prove the case of VaR. By Corollary 1 of Embrechts et al. (2018), we have

$$\mathrm{VaR}_{n\alpha}\left(\sum_{i=1}^n X_i\right) \leqslant \sum_{i=1}^n \mathrm{VaR}_\alpha(X_i),$$

which implies $\alpha^* \leqslant n\alpha$, and hence $\mathrm{DQ}^{\mathrm{VaR}}_\alpha(\mathbf{X}) \leqslant n$. By definition, $\alpha^* \in [0,1]$, and hence $0 \leqslant \mathrm{DQ}^{\mathrm{VaR}}_\alpha(\mathbf{X}) \leqslant 1/\alpha$. To summarize, $\{\mathrm{DQ}^{\mathrm{VaR}}_\alpha(\mathbf{X}) \mid \mathbf{X} \in \mathcal{X}^n\} \subseteq [0, \min\{n, 1/\alpha\}]$.

Next, we show that every point in the interval $[0, \min\{n, 1/\alpha\}]$ is attainable by $\mathrm{DQ}^{\mathrm{VaR}}_\alpha$. Take any $\mathbf{X} \in \mathcal{X}^n$ and let $a = \mathrm{DQ}^{\mathrm{VaR}}_\alpha(\mathbf{X})$. Since $\mathrm{DQ}^{\mathrm{VaR}}_\alpha$ satisfies [LI], we can replace each component X_i of $\mathbf{X}$ with $X_i - \mathrm{VaR}_\alpha(X_i)$ for $i \in [n]$. Hence, it is safe to assume that VaR_α of each component of $\mathbf{X}$ is 0. Let $\mathbf{Z} = \mathbf{X}\mathbb{1}_A$ where $A \in \mathcal{F}$ is independent of $\mathbf{X}$ and $\mathbb{P}(A) = p \in (0,1)$. Since the mapping $F \mapsto \mathrm{VaR}_\alpha(X)$ where $X \sim F$ has convex level sets (e.g., Gneiting (2011)), VaR_α of each component of $\mathbf{Z}$ is 0. By (2), we have

$$\mathrm{DQ}^{\mathrm{VaR}}_\alpha(\mathbf{Z}) = \frac{1}{\alpha}\mathbb{P}\left(\sum_{i=1}^n Z_i > 0\right) = \frac{p}{\alpha}\mathbb{P}\left(\sum_{i=1}^n X_i > 0\right) = p\,\mathrm{DQ}^{\mathrm{VaR}}_\alpha(\mathbf{X}).$$

Since $p \in (0,1)$ is arbitrary, any point in $[0,a]$ belongs to the range of $\mathrm{DQ}^{\mathrm{VaR}}_\alpha$. To complete the proof, it suffices to construct $\mathbf{X}$ such that $\mathrm{DQ}^{\mathrm{VaR}}_\alpha(\mathbf{X}) = \min\{n, 1/\alpha\}$.

In case $\alpha \geqslant 1/n$, let $\mathbf{X}$ follow an n-dimensional multinomial distribution with parameters $(1/n,\dots,1/n)$. It is clear that $\sum_{i=1}^n X_i = 1$. Since $\alpha \geqslant 1/n$, then $\mathrm{VaR}_\alpha(X_i) = 0$. In this case, by (2), $\mathrm{DQ}^{\mathrm{VaR}}_\alpha(\mathbf{X}) = 1/\alpha$. In case $\alpha < 1/n$, we can find $\mathbf{X}$ satisfying $\mathrm{DQ}^{\mathrm{VaR}}_\alpha(\mathbf{X}) = n$, which is constructed in part (iv) of the proof below.

Next, we prove the case of ES. Since ES satisfies [SA], the range of $\mathrm{DQ}^{\mathrm{ES}}_\alpha$ is contained in $[0,1]$. Take any $t \in [0,2]$, and let each of X_1 and X_2 follow a uniform distribution on $[-1,1]$ such that $X_1 + X_2$ is uniformly distributed on $[-t,t]$. The existence of such (X_1, X_2) is shown by Theorem 3.1 of Wang and Wang (2016). Let $X_i = 0$ for $i = 3,\dots,n$. We can easily compute $\mathrm{ES}_\alpha(X_1) = \mathrm{ES}_\alpha(X_2) = 1-\alpha$ and $\mathrm{ES}_\beta(X_1+X_2) = t(1-\beta)$. Hence,

$$\mathrm{DQ}^{\mathrm{ES}}_\alpha(X_1,\dots,X_n) = \frac{1}{\alpha}\inf\{\beta \in (0,1) : t(1-\beta) \leqslant 2-2\alpha\} = \frac{1}{\alpha}\left(1 - \frac{2-2\alpha}{t}\right)_+.$$

For letting t vary in $[0,2]$, we get that every point in $[0,1]$ is attained by $\mathrm{DQ}^{\mathrm{ES}}_\alpha$.

(ii) The first part follows directly from Theorem 2 (i) of Han et al. (2022). In particular, if $\sum_{i=1}^n X_i$ is a constant, we have $\mathrm{VaR}_0\left(\sum_{i=1}^n X_i\right) = \mathrm{VaR}_{n\alpha}\left(\sum_{i=1}^n X_i\right) \leqslant \sum_{i=1}^n \mathrm{VaR}_\alpha(X_i)$ for $\alpha < 1/n$, and $\mathrm{ES}_0\left(\sum_{i=1}^n X_i\right) = \mathrm{ES}_\alpha\left(\sum_{i=1}^n X_i\right) \leqslant \sum_{i=1}^n \mathrm{ES}_\alpha(X_i)$. Thus, we have $\mathrm{DQ}^{\mathrm{ES}}_\alpha(\mathbf{X}) = 0$ if $\alpha < 1/n$ and $\mathrm{DQ}^{\mathrm{ES}}_\alpha(\mathbf{X}) = 0$.

(iii) By Theorem 6 in Wang and Zitikis (2021), if $\mathbf{X}$ is α-concentrated, we have

$$\mathrm{VaR}_\alpha\left(\sum_{i=1}^n X_i\right) \leqslant \sum_{i=1}^n \mathrm{VaR}_\alpha(X_i),$$

which implies $\alpha^* \leqslant \alpha$ and then $\mathrm{DQ}^{\mathrm{VaR}}_\alpha(\mathbf{X}) \leqslant 1$. Further, as VaR is continuous and non-flat from the left at $(\alpha, \sum_{i=1}^n X_i)$, by Theorem 6 in Wang and Zitikis (2021), the inequality above is an equality. Thus, we have $\alpha^* = \alpha$, which leads to $\mathrm{DQ}^{\mathrm{VaR}}_\alpha(\mathbf{X}) = 1$. Moreover, from Theorem 5 of Wang and Zitikis (2021), we know that $\mathrm{ES}_\alpha\left(\sum_{i=1}^n X_i\right) = \sum_{i=1}^n \mathrm{ES}_\alpha(X_i)$ if $(X_1,\dots,X_n)$ is α-concentrated. Combining with the fact that $\mathrm{ES}_\alpha(\sum_{i=1}^n X_i)$ is non-flat from left at $(\alpha, \mathbf{X})$, we have $\mathrm{DQ}^{\mathrm{ES}}_\alpha(\mathbf{X}) = 1$.

(iv) As $X_1,\dots,X_n$ are $(n\alpha)$-concentrated, there exists an event B such that B is a tail event for all X_i and $\mathbb{P}(B) = n\alpha$. Let $B_i = \{X_i > \mathrm{VaR}_\alpha(X_i)\}$. By Lemma A.3 of Wang and Zitikis (2021), we have $\{X_i > \mathrm{VaR}_{n\alpha}(X_i)\} \subseteq B$. As $\mathrm{VaR}_\alpha(X_i) \geqslant \mathrm{VaR}_{n\alpha}(X_i)$, it gives $B_i \subseteq B$ for all $i \in [n]$. From $\mathbb{P}(X_i \geqslant \mathrm{VaR}_\alpha(X_i)) \geqslant n\alpha$, we know that $X_i(\omega) \geqslant \mathrm{VaR}_\alpha(X_i)$ for all $\omega \in B$. Further, as $B_1,\dots,B_n$ are mutually exclusive, we have $X_i(\omega) > \mathrm{VaR}_\alpha(X_i)$ and $X_j(\omega) = \mathrm{VaR}_\alpha(X_j)$ for all $\omega \in B_i$ and $j \neq i$. Hence, for all $\omega \in \bigcup_{i=1}^n B$, we have $\sum_{i=1}^n X_i(\omega) > \sum_{i=1}^n \mathrm{VaR}_\alpha(X_i)$ while $\sum_{i=1}^n X_i(\omega) \leqslant \sum_{i=1}^n \mathrm{VaR}_\alpha(X_i)$ for $\omega \in \left(\bigcup_{i=1}^n A_i\right)^c = \bigcap_{i=1}^n A_i^c$. Therefore, if $\alpha < 1/n$,

$$\mathbb{P}\left(\sum_{i=1}^n X_i > \sum_{i=1}^n \mathrm{VaR}_\alpha(X_i)\right) = \mathbb{P}\left(\bigcup_{i=1}^n B_i\right) = \sum_{i=1}^n \mathbb{P}(B_i) = n\alpha.$$

By (2), we have $\mathrm{DQ}^{\mathrm{VaR}}_\alpha(\mathbf{X}) = n$.

For the case of ES, as $X_1,\dots,X_n$ are $(n\alpha)$-concentrated, by Theorem 5 of Wang and Zitikis (2021), we have $\mathrm{ES}_{n\alpha}\left(\sum_{i=1}^n X_i\right) = \sum_{i=1}^n \mathrm{ES}_{n\alpha}(X_i)$. Together with the fact that $\beta \mapsto \mathrm{ES}_\beta\left(\sum_{i=1}^n X_i\right)$ is strictly decreasing at $\beta = n\alpha$, we get that $\mathrm{DQ}^{\mathrm{ES}}_{n\alpha}(\mathbf{X}) = 1$. □

Note that comonotonicity is stronger than α-concentration, and hence it is a sufficient condition for (iii) in Theorem 1 replacing α-concentration.

In summary, both $\mathrm{DQ}^{\mathrm{VaR}}_\alpha$ and $\mathrm{DQ}^{\mathrm{ES}}_\alpha$ take values on a bounded interval. In contrast, the diversification ratio $\mathrm{DR}^{\mathrm{VaR}_\alpha}$ is unbounded, and $\mathrm{DR}^{\mathrm{ES}_\alpha}$ is bounded above by 1 only when the ES of the total risk is non-negative. The continuous ranges of DQs also give more information on diversification. Moreover, similarly to the continuity axiom of preferences (e.g., Föllmer and Schied (2016)), a bounded interval can provide mathematical convenience for applications. The values of DQs are simple to interpret. To be specific, for $\mathrm{DQ}^{\mathrm{VaR}}_\alpha$, its value is 0 if there is a very good hedge in the sense of Theorem 1 (ii); its value is 1 if there is strong positive dependence such as comonotonicity, and its value is n if there is strong negative dependence conditional on the tail event. For $\mathrm{DQ}^{\mathrm{ES}}_\alpha$, its value is 0 if there is a very good hedge in the sense of Theorem 1 (ii) and its value is 1 if there is strong positive dependence such as comonotonicity or α-concentration.

4. Diversification for large portfolios

In this section, we will focus on the asymptotic behavior of DQ for large portfolios. First, since the independent portfolio is widely recognized as an effectively diversified portfolio, we anticipate that DQ for this type of portfolio would be close to zero as n tends to ∞.

Theorem 2. *Let $X_1, X_2, \dots$ be a sequence of uncorrelated random variables in L^2. Assume $\sup_{i\in\mathbb{N}} \mathrm{var}(X_i) < \infty$ and $\inf_{i\in\mathbb{N}}\{\rho_\alpha(X_i) - \mathbb{E}[X_i]\} > 0$. For $\alpha \in (0,1)$ and ρ being VaR or ES,*

$$\lim_{n\to\infty} \mathrm{DQ}^{\rho}_\alpha(X_1,\dots,X_n) = 0. \tag{5}$$

Proof. Let $\mathbf{X}_n = (X_1,\dots,X_n)$ and $S_n = \sum_{i=1}^n X_i$. As DQ^ρ_α is location invariant, we can assume that $\mathbb{E}[X_i] = 0$ for $i = 1, 2, \dots$. Hence, by the L^2-Law of Large Numbers in the form of Durrett (2019, Theorem 2.2.3), we have $S_n/n \xrightarrow{L^2} 0$. (In fact, L^1 convergence is sufficient to prove our result.)

We first prove the case of VaR. Note that $S_n/n \xrightarrow{L^2} 0$ implies $\lim_{n\to\infty} \mathbb{P}(S_n/n > x) = 0$ for all $x > 0$. Let $\varepsilon = \inf_{i\in\mathbb{N}}\{\rho_\alpha(X_i) - \mathbb{E}[X_i]\}$. As $\mathrm{VaR}_\alpha(X_i) > \varepsilon$, $i = 1, 2, \dots$, we have

$$\mathbb{P}\left(S_n > \sum_{i=1}^n \mathrm{VaR}_\alpha(X_i)\right) \leqslant \mathbb{P}\left(S_n/n > \varepsilon\right) \to 0.$$

Thus, $\lim_{n\to\infty} \mathbb{P}(S_n > \sum_{i=1}^n \mathrm{VaR}_\alpha(X_i)) = 0$. By (2), we have

$$\lim_{n\to\infty} \mathrm{DQ}^{\mathrm{VaR}}(\mathbf{X}_n) = \lim_{n\to\infty} \frac{1}{\alpha}\mathbb{P}\left(S_n > \sum_{i=1}^n \mathrm{VaR}_\alpha(X_i)\right) = 0.$$

Next, we prove the case of ES. As ES is a convex distortion risk measure, ES is L^1-continuous (see Rüschendorf (2013, Corollary 7.10)). Further, since $\mathrm{ES}_\beta(0) = 0$, we have $\mathrm{ES}_\beta(S_n/n) \to 0$ as $n \to \infty$ for all $\beta \in (0,1)$. As a result, for every $\beta \in (0,1)$, there exists N_β such that $\mathrm{ES}_\beta(S_n/n) < \varepsilon$ for all $n > N_\beta$. Therefore, we have

$$\alpha^* = \inf\left\{\beta \in (0,1) : \mathrm{ES}_\beta(S_n) \leqslant \sum_{i=1}^n \mathrm{ES}_\alpha(X_i)\right\}$$
$$\leqslant \inf\left\{\beta \in (0,1) : \mathrm{ES}_\beta(S_n/n) \leqslant \varepsilon)\right\} \to 0$$

as $n \to \infty$. Hence, we have $\mathrm{DQ}^{\mathrm{ES}}_\alpha(\mathbf{X}_n) = \alpha^*/\alpha \to 0$ as $n \to \infty$. □

Note that Theorem 2 does not imply that all independent portfolios are good hedges, because (5) holds under some assumptions. In case the components of the portfolio have very heavy tails, DQ based on VaR can be close to n even if the individual losses are iid, as we will see in Theorem 3 below.

Remark 5. In the special case that $X_1, X_2, \dots$ are iid, Theorem 2 implies that, if $\rho_\alpha(X_1) > \mathbb{E}[X_1]$, we have

$$\lim_{n\to\infty} \mathrm{DQ}^{\rho}_\alpha(X_1,\dots,X_n) = 0$$

for ρ being VaR or ES.

Next, we focus on portfolios with exchangeable components, which may represent a homogeneous subgroup of assets from a large asset pool. An infinite sequence of random variables $X_1, X_2, \dots$ is said to be exchangeable if $(X_1,\dots,X_n) \overset{\mathrm{d}}{=} (X_{\pi(1)},\dots,X_{\pi(n)})$ for all $n \geqslant 2$ and $\pi \in \mathfrak{S}_n$, where $\mathfrak{S}_n$ is the set of permutations of $[n]$. Exchangeability is closely related to iid sequence of random variables due to de Finetti's theorem, which says that any infinite exchangeable sequence is conditionally iid. However, for the exchangeable portfolio, the value of DQ does not necessarily converge to 0 as n goes to infinity. By the Birkhoff–Khinchin theorem (see Aleksandr and Khinchin (1949)), if $\mathbb{E}[|X_1|] < \infty$, we have $\sum_{i=1}^n X_i/n \to \mathbb{E}[X_1|\mathcal{G}]$ a.s. for some sub-σ-algebra $\mathcal{G} \subseteq \mathcal{F}$. By (2), we get

$$\mathrm{DQ}^{\mathrm{VaR}}_\alpha(X_1,\dots,X_n) \to \frac{1 - F\left(\mathrm{VaR}_\alpha(X_1)\right)}{\alpha} \quad \text{as} \quad n \to \infty,$$

and

$$\mathrm{DQ}^{\mathrm{ES}}_\alpha(X_1,\dots,X_n) \to \frac{1 - \widetilde{F}\left(\mathrm{ES}_\alpha(X_1)\right)}{\alpha} \quad \text{as} \quad n \to \infty,$$

where F is the distribution of $\mathbb{E}[X_1|\mathcal{G}]$ and $\widetilde{F}$ is the superquantile transform of F.

The above formulas depend on $\mathcal{G}$ which may not be explicit. In the next proposition, we derive an upper bound on the limit.

Proposition 1. *Let $X_1, X_2, \dots$ be a sequence of exchangeable random variables in L^2. Denote by $\mu = \mathbb{E}[X_1]$, $\sigma^2 = \mathrm{var}(X_1)$ and $r = \mathrm{corr}(X_1, X_2)$. For $\alpha \in (0,1)$ and ρ being VaR or ES, if $\rho_\alpha(X_1) > \mu$, then*

$$\lim_{n\to\infty} \mathrm{DQ}^{\rho}_\alpha(X_1,\dots,X_n) \leqslant \frac{1}{\alpha}\frac{r\sigma^2}{r\sigma^2 + (\rho_\alpha(X_1) - \mu)^2}. \tag{6}$$

Proof. Let $S_n = \sum_{i=1}^n X_i$. As $(X_1,\dots,X_n)$ is exchangeable, we have $\mathbb{E}[S_n] = n\mu$ and $\mathrm{var}(S_n) = (n + n(n-1)r)\sigma^2$. The mean and variance of S_n imply the bound

$$\rho_\beta(S_n) \leqslant n\mu + \sigma\sqrt{n + n(n-1)r}\sqrt{\frac{1-\beta}{\beta}}$$

for all $\beta \in (0,1)$; see Table 1 of Li et al. (2018). As a result, we have

$$\mathrm{DQ}^{\rho}_\alpha(X_1,\dots,X_n)$$
$$\leqslant \frac{1}{\alpha}\inf\left\{\beta \in (0,1) : n\mu + \sigma\sqrt{n + n(n-1)r}\sqrt{\frac{1-\beta}{\beta}} \leqslant n\rho_\alpha(X_1)\right\}$$
$$= \frac{1}{\alpha}\frac{\frac{1+(n-1)r}{n}\sigma^2}{\frac{1+(n-1)r}{n}\sigma^2 + (\rho_\alpha(X_1) - \mu)^2}.$$

Sending $n \to \infty$, we get the desired result. □

The upper bound (6) on $\lim_{n\to\infty} \mathrm{DQ}^{\rho}_\alpha(X_1,\dots,X_n)$ in Proposition 1 decreases as the correlation r between assets decreases. Intuitively, this means that less positive dependence leads to greater diversification. In particular, if $r \downarrow 0$, then $\lim_{n\to\infty} \mathrm{DQ}^{\rho}_\alpha(X_1,\dots,X_n) \to 0$. The upper bound (6) holds true also without exchangeability, as long as the average of the bivariate correlations of assets converges to r and all assets are identically distributed.

5. Elliptical models

The most commonly used classes of multivariate distributions are the elliptical models which include the multivariate normal and t-distributions as special cases. For a general treatment of elliptical models in risk management, see McNeil et al. (2015). In this section, we study DQs based on VaR and ES for elliptical models.

5.1. Explicit formulas for DQ

A random vector $\mathbf{X}$ is *elliptically distributed* if its characteristic function can be written as

$$\psi(\mathbf{t}) = \mathbb{E}\left[\exp\left(\mathrm{i}\mathbf{t}^\top \mathbf{X}\right)\right] = \exp\left(\mathrm{i}\mathbf{t}^\top \boldsymbol{\mu}\right)\tau\left(\mathbf{t}^\top \Sigma \mathbf{t}\right),$$

for some $\boldsymbol{\mu} \in \mathbb{R}^n$, positive semi-definite matrix $\Sigma \in \mathbb{R}^{n\times n}$, and $\tau : \mathbb{R}_+ \to \mathbb{R}$ called the characteristic generator. We denote this distribution by $\mathrm{E}_n(\boldsymbol{\mu}, \Sigma, \tau)$. We will assume that Σ is not a matrix of zeros. Each marginal distribution of an elliptical distribution is a one-dimensional elliptical distribution with the same characteristic generator. The most common examples of elliptical distributions are normal and t-distributions. An n-dimensional t-distribution $\mathrm{t}(\nu, \boldsymbol{\mu}, \Sigma)$ with $\nu > 0$ has density function f given by (if $|\Sigma| > 0$)

$$f(\mathbf{x}) = \frac{\Gamma((\nu+n)/2)}{\Gamma(\nu/2)\nu^{n/2}\pi^{n/2}|\Sigma|^{1/2}} \times \left(1 + \frac{1}{\nu}(\mathbf{x}-\boldsymbol{\mu})^\top \Sigma^{-1}(\mathbf{x}-\boldsymbol{\mu})\right)^{-(\nu+n)/2},$$

where Γ is the gamma function and $|\Sigma|$ is the determinant of the dispersion matrix Σ.

We remind the reader that for elliptical models, VaR and ES behave very similarly. For instance, VaR_α is subadditive for $\alpha \in (0, 1/2)$ in this setting; see (McNeil et al., 2015, Theorem 8.28). Moreover, for $\mathbf{X} \sim \mathrm{E}_n(\boldsymbol{\mu}, \Sigma, \tau)$ and $\mathbf{a} \in \mathbb{R}^n$, both $\mathrm{VaR}_\alpha(\mathbf{a}^\top \mathbf{X})$ and $\mathrm{ES}_\alpha(\mathbf{a}^\top \mathbf{X})$ have the form $y\sqrt{\mathbf{a}^\top \Sigma \mathbf{a}} + \mathbf{a}^\top \boldsymbol{\mu}$ for some constant y being $y_\alpha^{\mathrm{VaR}} := \mathrm{VaR}_\alpha(Y)$ or $y_\alpha^{\mathrm{ES}} := \mathrm{ES}_\alpha(Y)$ where $Y \sim \mathrm{E}_1(0, 1, \tau)$. As a consequence, the behavior of DQ based on VaR is similar to that based on ES, except for the case of infinite mean.

For a positive semi-definite matrix Σ, we write $\Sigma = (\sigma_{ij})_{n\times n}$, $\sigma_i^2 = \sigma_{ii}$, and $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_n)$, and define the constant

$$k_\Sigma = \frac{\sum_{i=1}^n \left(\mathbf{e}_i^\top \Sigma \mathbf{e}_i\right)^{1/2}}{\left(\mathbf{1}^\top \Sigma \mathbf{1}\right)^{1/2}} = \frac{\sum_{i=1}^n \sigma_i}{\left(\sum_{i,j}^n \sigma_{ij}\right)^{1/2}} \in [1, \infty), \tag{7}$$

where $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^n$ and $\mathbf{e}_1, \dots, \mathbf{e}_n$ are the column vectors of the $n \times n$ identity matrix I_n. Moreover, $k_\Sigma = 1$ if and only if $\Sigma = \boldsymbol{\sigma}\boldsymbol{\sigma}^\top$, which means that $\mathbf{X} \sim \mathrm{E}_n(\boldsymbol{\mu}, \Sigma, \tau)$ is comonotonic.

Explicit formulas and the limiting behavior of DQs based on VaR and ES for elliptical models are given by the following few results.

Proposition 2. *Suppose that $\mathbf{X} \sim \mathrm{E}_n(\boldsymbol{\mu}, \Sigma, \tau)$. We have, for $\alpha \in (0, 1)$,*

$$\mathrm{DQ}_\alpha^{\mathrm{VaR}}(\mathbf{X}) = \frac{1 - F(k_\Sigma \mathrm{VaR}_\alpha(Y))}{\alpha} \quad \text{and}$$

$$\mathrm{DQ}_\alpha^{\mathrm{ES}}(\mathbf{X}) = \frac{1 - \widetilde{F}(k_\Sigma \mathrm{ES}_\alpha(Y))}{\alpha},$$

where $Y \sim \mathrm{E}_1(0, 1, \tau)$ with distribution function F, and $\widetilde{F}$ is the superquantile transform of F in (2). Moreover,

(i) $\alpha \mapsto \mathrm{DQ}_\alpha^{\mathrm{VaR}}(\mathbf{X})$ takes value in $[0, 1]$ on $(0, 1/2]$ and it takes value in $[1, 2]$ on $(1/2, 1)$;
(ii) $k_\Sigma \mapsto \mathrm{DQ}_\alpha^{\mathrm{VaR}}(\mathbf{X})$ is decreasing for $\alpha \in (0, 1/2]$ and increasing for $\alpha \in (1/2, 1)$;
(iii) $k_\Sigma \mapsto \mathrm{DQ}_\alpha^{\mathrm{ES}}(\mathbf{X})$ is decreasing for $\alpha \in (0, 1)$.

Proof. We first consider the case of VaR. Since $\mathbf{X} \sim \mathrm{E}_n(\boldsymbol{\mu}, \Sigma, \tau)$, the linear structure of elliptical distributions gives $\sum_{i=1}^n X_i \sim \mathrm{E}_1(\mathbf{I}^\top \boldsymbol{\mu}, \mathbf{I}^\top \Sigma \mathbf{I}, \tau)$. That is, $\sum_{i=1}^n X_i \overset{\mathrm{d}}{=} \sum_{i=1}^n \mu_i + \|\mathbf{1}^\top A\|_2 Y$, where A is the Cholesky decomposition of Σ. Also, we have $\mathrm{VaR}_\alpha(X_i) = \mu_i + \|\mathbf{e}_i^\top A\|_2 \mathrm{VaR}_\alpha(Y)$. By (2),

$$\begin{aligned}
&\mathrm{DQ}_\alpha^{\mathrm{VaR}}(\mathbf{X}) \\
&= \frac{1}{\alpha}\mathbb{P}\left(\sum_{i=1}^n X_i > \sum_i^n \mu_i + \|\mathbf{e}_i^\top A\|_2 \mathrm{VaR}_\alpha(Y)\right) \\
&= \frac{1}{\alpha}\mathbb{P}\left(\sum_{i=1}^n \mu_i + \|\mathbf{1}^\top A\|_2 Y > \sum_i^n \mu_i + \|\mathbf{e}_i^\top A\|_2 \mathrm{VaR}_\alpha(Y)\right) \\
&= \frac{1 - F(k_\Sigma \mathrm{VaR}_\alpha(Y))}{\alpha}.
\end{aligned}$$

By replacing VaR with ES and $\sum_{i=1}^n X_i$ with $\mathrm{ES}_U(\sum_{i=1}^n X_i)$, we can get the first formula of $\mathrm{DQ}_\alpha^{\mathrm{ES}}(\mathbf{X})$.

(i) For $\alpha \in (0, 1/2]$, we have $\mathrm{VaR}_\alpha(Y) \leqslant k_\Sigma \mathrm{VaR}_\alpha(Y)$ and $1 - \alpha \leqslant F(k_\Sigma \mathrm{VaR}_\alpha(Y)) \leqslant 1$. Hence, $0 \leqslant \mathrm{DQ}_\alpha^{\mathrm{VaR}}(\mathbf{X}) \leqslant 1$. For $\alpha \in (1/2, 1)$, $\mathrm{VaR}_\alpha(Y) \geqslant k_\Sigma \mathrm{VaR}_\alpha(Y)$ and $\alpha \leqslant 1 - F(k_\Sigma \mathrm{VaR}_\alpha(Y)) \leqslant 1$. Hence, $1 \leqslant \mathrm{DQ}_\alpha^{\mathrm{VaR}}(\mathbf{X}) \leqslant 1/\alpha \leqslant 2$.
(ii) If $\alpha \in (0, 1/2]$, then $\mathrm{VaR}_\alpha(Y) \geqslant 0$, and thus $\mathrm{DQ}_\alpha^{\mathrm{VaR}}(\mathbf{X})$ decreases in k_Σ. If $\alpha \in (1/2, 1)$, then $\mathrm{VaR}_\alpha(Y) \leqslant 0$, and thus $\mathrm{DQ}_\alpha^{\mathrm{VaR}}(\mathbf{X})$ increases in k_Σ.
(iii) For $\alpha \in (0, 1)$, $\mathrm{ES}_\alpha(Y) \geqslant 0$. Hence, $\mathrm{DQ}_\alpha^{\mathrm{ES}}(\mathbf{X})$ increases in k_Σ. □

In the discussions below, we will assume $\alpha \in (0, 1/2)$, which is the most common setting in risk management. In Proposition 2, we see that, for $\alpha \in (0, 1/2)$, $\mathrm{DQ}_\alpha^{\mathrm{VaR}}(\mathbf{X}) \in [0, 1]$. This is in contrast to Theorem 1, where the range of $\mathrm{DQ}_\alpha^{\mathrm{VaR}}$ is $[0, n]$ instead of $[0, 1]$, when we do not restrict to elliptical models. This phenomenon should not be surprising, because, as we mentioned before, VaR_α for $\alpha \in (0, 1/2)$ is similar to ES_α for elliptical models, and $\mathrm{DQ}_\alpha^{\mathrm{ES}}$ has range $[0, 1]$.

In case $Y \sim \mathrm{E}_1(0, 1, \tau)$ has a positive density on $\mathbb{R}$, we can see from Proposition 2 that $\mathrm{DQ}_\alpha^{\mathrm{VaR}}(\mathbf{X}) = 1$ if and only if $k_\Sigma = 1$ (i.e., $\mathbf{X}$ is comonotonic) or $\mathrm{VaR}_\alpha(Y) = 0$ (i.e., $\alpha = 1/2$). Similarly, $\mathrm{DQ}_\alpha^{\mathrm{ES}}(\mathbf{X}) = 1$ if and only if $k_\Sigma = 1$.

In case the elliptical distribution is asymptotically uncorrelated, we will see that $\mathrm{DQ}_\alpha^{\mathrm{VaR}}(\mathbf{X}) \to 0$ and $\mathrm{DQ}_\alpha^{\mathrm{ES}}(\mathbf{X}) \to 0$ as $n \to \infty$. This is consistent with our intuition that, if the individual risks are asymptotically uncorrelated, then full diversification can be achieved asymptotically, thus the diversification index goes to 0. The value $\mathrm{AC}_\Sigma = \sum_{i,j}^n \sigma_{ij} / (\sum_{i=1}^n \sigma_i)^2 = 1/k_\Sigma^2$ will be called the average correlation (AC) of Σ.

Proposition 3. *Suppose that $\mathbf{X} \sim \mathrm{E}_n(\boldsymbol{\mu}, \Sigma, \tau)$.*

(i) Let $Y \sim \mathrm{E}_1(0, 1, \tau)$ and f be the density function of Y. We have

$$\lim_{\alpha \downarrow 0} \mathrm{DQ}_\alpha^{\mathrm{VaR}}(\mathbf{X}) = \lim_{x\to\infty} k_\Sigma \frac{f(k_\Sigma x)}{f(x)} \quad \text{if } \mathrm{VaR}_0(Y) = \infty \text{ and the limit exists}, \tag{8}$$

and $\lim_{\alpha\downarrow 0} \mathrm{DQ}_\alpha^{\mathrm{VaR}}(\mathbf{X}) = 0$ if $\mathrm{VaR}_0(Y) < \infty$.
(ii) If $\lim_{n\to\infty} \mathrm{AC}_\Sigma = 0$, then

$$\lim_{n\to\infty} \mathrm{DQ}_\alpha^{\mathrm{VaR}}(\mathbf{X}) = \lim_{n\to\infty} \mathrm{DQ}_\beta^{\mathrm{ES}}(\mathbf{X}) = 0$$

for $\alpha \in (0, 1/2)$ *and* $\beta \in (0, 1)$.

Proof. (i) If $\mathrm{VaR}_0(Y) < \infty$, then $\mathrm{VaR}_0(Y) \leqslant k_\Sigma \mathrm{VaR}_0(Y)$ as $k_\Sigma \geqslant 1$. Hence, $\mathrm{DQ}_0^{\mathrm{VaR}}(\mathbf{X}) = 0$. If $\mathrm{VaR}_0(Y) = \infty$, then $\mathrm{VaR}_0(Y) > k_\Sigma \mathrm{VaR}_\alpha(Y)$ for $\alpha > 0$. Therefore,

$$\lim_{\alpha \to 0} \mathrm{DQ}_\alpha^{\mathrm{VaR}}(\mathbf{X}) = \lim_{\alpha \to 0} \frac{1 - F(k_\Sigma \mathrm{VaR}_\alpha(Y)))}{\alpha} = \lim_{\alpha \to 0} k_\Sigma \frac{f(k_\Sigma \mathrm{VaR}_\alpha(Y)))}{f(\mathrm{VaR}_\alpha(Y)))} = \lim_{x \to \infty} k_\Sigma \frac{f(k_\Sigma x)}{f(x)},$$

and we get the desired result.

(ii) We only show the proof of $\mathrm{DQ}_\alpha^{\mathrm{VaR}}$ as the result for $\mathrm{DQ}_\beta^{\mathrm{ES}}$ can be obtained along the same analogy. By Proposition 2, it is clear that $\mathrm{AC}_\Sigma \to \mathrm{DQ}_\alpha^{\mathrm{VaR}}(\mathbf{X})$ is increasing for $\alpha \in (0, 1/2)$ and $\mathrm{AC}_\Sigma \to \mathrm{DQ}_\beta^{\mathrm{ES}}(\mathbf{X})$ is increasing for $\alpha \in (0, 1)$. Moreover, if AC_Σ goes to 0 as $n \to \infty$, we have $\lim_{n \to \infty} k_\Sigma = \infty$. Thus, we have $\mathrm{DQ}_\alpha^{\mathrm{VaR}}(\mathbf{X}) \to 0$ as $n \to \infty$ by Proposition 2. □

Explicit formulas of (8) for normal and t-distributions are provided in Section 5.2.

Remark 6. In general, we do not have a limiting result for $\mathrm{DQ}_\alpha^{\mathrm{ES}}$ in the form of Proposition 3 (i). If $\mathbf{X} \sim \mathrm{t}(\nu, \boldsymbol{\mu}, \Sigma)$ for $\nu > 1$, then $\mathrm{DQ}_\alpha^{\mathrm{ES}}$ has the same limit as $\mathrm{DQ}_\alpha^{\mathrm{VaR}}$ in (8) as $\alpha \downarrow 0$ because $\mathrm{VaR}_\alpha(Y)/\mathrm{ES}_\alpha(Y)$ has a constant limit $(\nu - 1)/\nu$ for a t-distributed Y by the Karamata theorem; see Theorem A.7 of McNeil et al. (2015).

From the results above, $\mathrm{DQ}_\alpha^{\mathrm{VaR}}(\mathbf{X})$ and $\mathrm{DQ}_\alpha^{\mathrm{ES}}(\mathbf{X})$ depend on both τ and α. In sharp contrast, DR of a centered elliptical distribution is always $1/k_\Sigma$, which ignores the shape of the distribution. More precisely, for $\mathbf{X} \sim \mathrm{E}_n(\mathbf{0}, \Sigma, \tau)$ and $\alpha \in (0, 1/2)$, we have

$$\mathrm{DR}^{\mathrm{VaR}_\alpha}(\mathbf{X}) = \frac{\mathrm{VaR}_\alpha(\sum_{i=1}^n X_i)}{\sum_{i=1}^n \mathrm{VaR}_\alpha(X_i)} = \frac{\left(\sum_{i,j}^n \sigma_{ij}\right)^{1/2} \mathrm{VaR}_\alpha(Y)}{\sum_{i=1}^n \sigma_i \mathrm{VaR}_\alpha(Y)} = \frac{1}{k_\Sigma}, \tag{9}$$

and similarly, $\mathrm{DR}^{\mathrm{ES}_\alpha}(\mathbf{X}) = 1/k_\Sigma$. Note that in this case, $\mathrm{DR}^{\mathrm{VaR}_\alpha}$ and $\mathrm{DR}^{\mathrm{ES}_\alpha}$ do not depend on τ, α or whether the risk measure is VaR or ES. Indeed, DR based on var or SD also has the same value $1/k_\Sigma$.

For $\mathbf{X} \sim \mathrm{E}_n(\boldsymbol{\mu}, \Sigma, \tau)$ with $\boldsymbol{\mu} \neq \mathbf{0}$, $\mathrm{DR}^{\mathrm{VaR}_\alpha}(\mathbf{X})$ and $\mathrm{DR}^{\mathrm{ES}_\alpha}(\mathbf{X})$ depend also on $\boldsymbol{\mu}$, which is arguably undesirable as it conflicts location invariance. Nevertheless, $\lim_{\alpha \downarrow 0} \mathrm{DR}^{\mathrm{VaR}_\alpha}(\mathbf{X}) = 1/k_\Sigma$ if $\mathrm{VaR}_0(Y) = \infty$ (i.e., the value taken by Y is unbounded from above), and this limit does not depend on $\boldsymbol{\mu}$. On the other hand, $\mathrm{DQ}_\alpha^{\mathrm{VaR}}(\mathbf{X})$ has a limit in (8) which depends on both k_Σ and τ. The above observations suggest that DQ is more comprehensive than DR by utilizing the information on the shape of the distribution.

A similar result to Proposition 3 (ii) holds for DR of centered elliptical distributions. More precisely, if $\alpha \in (0, 1/2)$, $\boldsymbol{\mu} = \mathbf{0}$, and $\lim_{n \to \infty} \mathrm{AC}_\Sigma = 0$, then we have $\lim_{n \to \infty} \mathrm{DR}^{\mathrm{VaR}_\alpha}(\mathbf{X}) = 0$ by (9), and similarly, $\lim_{n \to \infty} \mathrm{DR}^{\mathrm{ES}_\alpha}(\mathbf{X}) = 0$. These limits do not hold if $\boldsymbol{\mu} \neq \mathbf{0}$.

5.2. Normal and t-distributions

Next, we take a close look at the two most important elliptical distributions used in finance and insurance, namely the multivariate normal distribution and the multivariate t-distribution. The explicit formulas for DQ for these distributions are available through the explicit formulas of VaR and ES; see Examples 2.14 and 2.15 of McNeil et al. (2015).

Han et al. (2022) proposed three simple models where the components of portfolio vectors follow the iid normal model, iid t-model and the common shock t-model, respectively, and showed that the diversification is the strongest according to DQ for the iid normal model and the iid t-model has a smaller DQ than the common shock t-model. In contrast, DR reports a similar value for all three models; see their Section 5.2 for details. Therefore, DQ has the nice feature that it can capture heavy tails and common shocks.

We present some formulas and numerical results for correlated normal and t-models. We focus our discussions mainly on $\mathrm{DQ}_\alpha^{\mathrm{VaR}}$ as the case of $\mathrm{DQ}_\alpha^{\mathrm{ES}}$ is similar. We first compute the limit of DQ as $\alpha \downarrow 0$ according to (8). By direct calculation,

$$\lim_{\alpha \downarrow 0} \mathrm{DQ}_\alpha^{\mathrm{VaR}}(\mathbf{X}) = \mathbb{1}_{\{k_\Sigma = 1\}} \quad \text{if } \mathbf{X} \sim \mathrm{N}(\boldsymbol{\mu}, \Sigma); \tag{10}$$

$$\lim_{\alpha \downarrow 0} \mathrm{DQ}_\alpha^{\mathrm{VaR}}(\mathbf{X}) = k_\Sigma^{-\nu} \quad \text{if } \mathbf{X} \sim \mathrm{t}(\nu, \boldsymbol{\mu}, \Sigma). \tag{11}$$

The above two values properly reflect the fact that the normal distribution is tail independent unless $k_\Sigma = 1$ (i.e., comonotonic), whereas the t-distribution is tail dependent; see Examples 7.38 and 7.39 of McNeil et al. (2015). DQ is able to capture this phenomenon well, by providing, for α close to 0, $\mathrm{DQ}_\alpha^{\mathrm{VaR}} \approx 0$ (strong diversification) for normal distribution and $\mathrm{DQ}_\alpha^{\mathrm{VaR}} \approx k_\Sigma^{-\nu}$ (moderate diversification for common choices of Σ and ν; see Fig. 3) for a t-distribution. On the other hand, DR of centered normal and t-distributions is always $1/k_\Sigma$, which fails to distinguish the tail of the t-distribution from that of the normal distribution (see (9)).

For numerical illustrations, we consider two specific dispersion matrices, parameterized by $r \in [0, 1]$ and $n \in \mathbb{N}$,

$$\Sigma_1 = (\sigma_{ij})_{n \times n}, \quad \text{where } \sigma_{ii} = 1 \text{ and } \sigma_{ij} = r \text{ for } i \neq j, \text{ and}$$

$$\Sigma_2 = (\sigma_{ij})_{n \times n}, \quad \text{where } \sigma_{ii} = 1 \text{ and } \sigma_{ij} = r^{|j-i|} \text{ for } i \neq j.$$

Note that Σ_1 represents an equicorrelated model and Σ_2 represents an autoregressive model AR(1). For $r = 0$, $r = 1$ or $n = 2$, these two dispersion matrices are identical. We take four models $\mathbf{X}_i \sim \mathrm{N}(\boldsymbol{\mu}, \Sigma_i)$ and $\mathbf{Y}_i \sim \mathrm{t}(\nu, \boldsymbol{\mu}, \Sigma_i)$, $i = 1, 2$, and we will let r, ν, α, n vary. Note that the location $\boldsymbol{\mu}$ does not matter in computing DQ, and we can simply take $\boldsymbol{\mu} = \mathbf{0}$. The default parameters are set as $r = 0.3$, $n = 4$, $\nu = 3$ and $\alpha = 0.05$ if not explained otherwise.

DQ for the t-models as the parameter of degrees of freedom ν varies Fig. 1 presents the values of DQ for the t-models with varying ν, where $\nu \in (0, 10]$ for VaR and $\nu \in (1, 10]$ for ES. We observe a monotonic relation that $\mathrm{DQ}_\alpha^{\mathrm{VaR}}$ and $\mathrm{DQ}_\alpha^{\mathrm{ES}}$ are decreasing in ν. In particular, if ν is close to 0, we see that $\mathrm{DQ}_\alpha^{\mathrm{VaR}} \approx 1$ which means there is almost no diversification effect for such super heavy-tailed models. On the other hand, DR completely ignores ν and always reports the same value. Note that the values of DQ and DR are not directly comparable as they are not on the same scale.

DQ for elliptical models as the correlation parameter r varies In Fig. 2, we report how DQ changes over $r \in [0, 1]$ in the four models. Intuitively, for r close to 1 which corresponds to comonotonicity, DQ is close to 1 in all models since there is no or very weak diversification in this case. More interestingly, for r close to 0, there is very strong diversification for the normal models, meaning $\mathrm{DQ}_\alpha^{\mathrm{VaR}} \approx 0$ and $\mathrm{DQ}_\alpha^{\mathrm{ES}} \approx 0$, whereas for the t-models, $\mathrm{DQ}_\alpha^{\mathrm{VaR}}$ and $\mathrm{DQ}_\alpha^{\mathrm{ES}}$ are clearly away from 0. Note that the components of a t-distribution are tail dependent even for zero or negative correlation (see Example 7.39 of McNeil et al. (2015)). Hence, DQ is able to capture dependence created by the common factor in the t-model, in addition to its correlation structure.

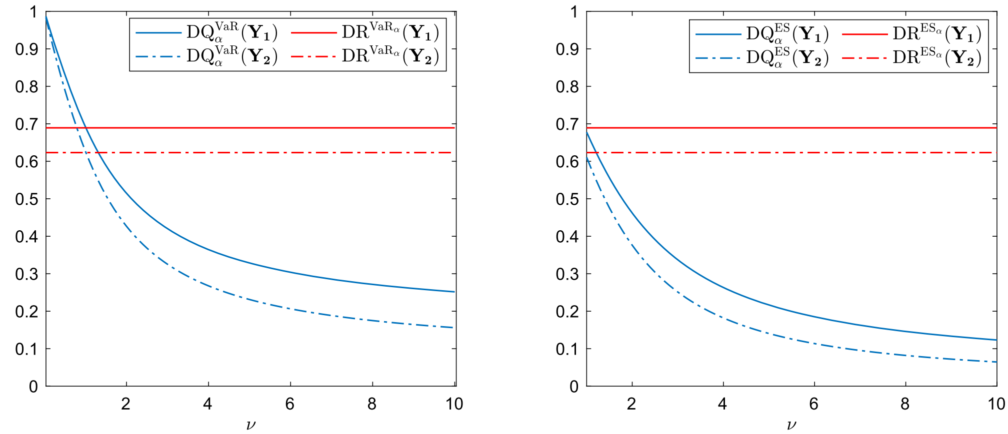

Fig. 1. DQ and DR based on VaR for $\nu \in (0, 10]$ and ES for $\nu \in (1, 10]$ with fixed $\alpha = 0.05$, $r = 0.3$ and $n = 4$.

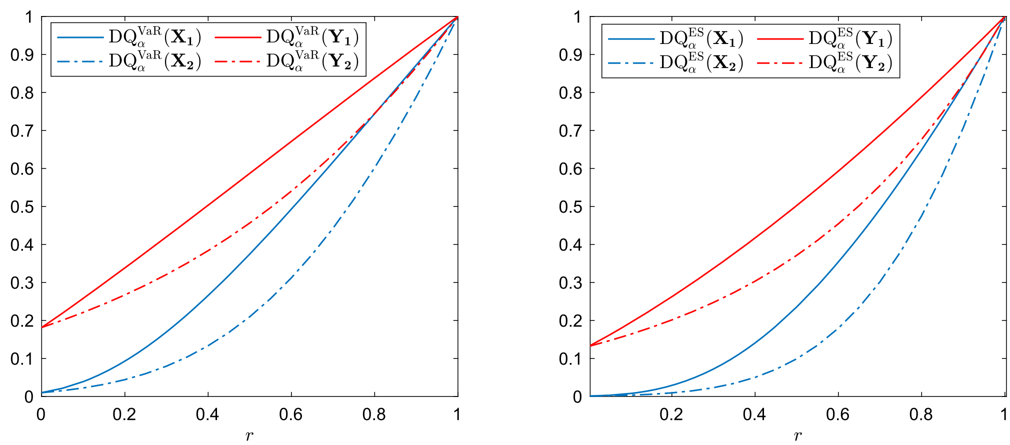

Fig. 2. DQ based on VaR and ES for $r \in [0, 1]$ with fixed $\alpha = 0.05$, $\nu = 3$, and $n = 4$.

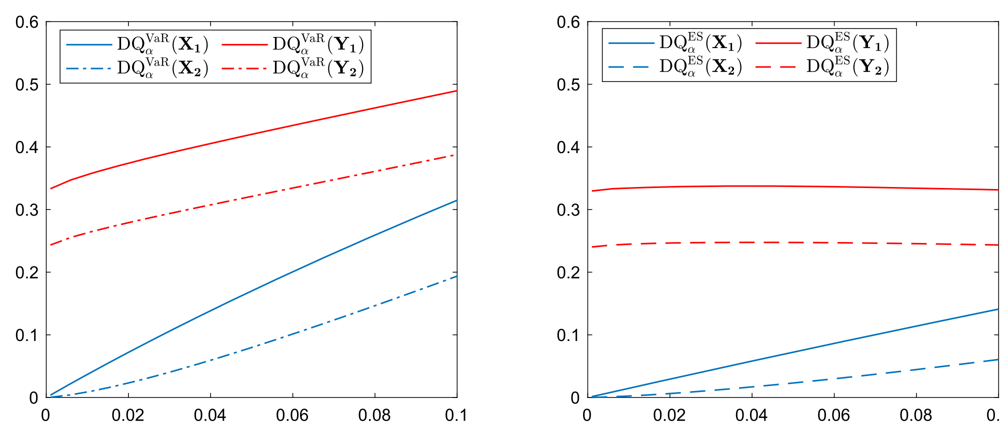

Fig. 3. DQ based on VaR and ES for $\alpha \in (0, 0.1)$ with fixed $\nu = 3$, $r = 0.3$ and $n = 4$.

DQ for varying α and its limit In Fig. 3, we report $\mathrm{DQ}_\alpha^{\mathrm{VaR}}$ and $\mathrm{DQ}_\alpha^{\mathrm{ES}}$ for $\alpha \in (0, 1)$ in the four models with correlation matrices specified in Section 5.2. We can see from Fig. 3 that DQ can be non-monotonic with respect to α (see the curves of $\mathrm{DQ}_\alpha^{\mathrm{ES}}$ for $\mathbf{X}_i \sim \mathrm{t}(\nu, \boldsymbol{\mu}, \Sigma_i)$). In addition, we can compute $k_{\Sigma_1} = 1.4510$ and $k_{\Sigma_2} = 1.6046$. Hence, it can be anticipated from Proposition 2 that, since DQ is decreasing in k_Σ, models with Σ_1 have larger DQ than the corresponding models with Σ_2. Moreover, as $\alpha \downarrow 0$, we can see that $\mathrm{DQ}_\alpha^{\mathrm{VaR}}$ converges to its corresponding limits in (10) and (11); also note that $\mathrm{DQ}_\alpha^{\mathrm{ES}}$ has the same limits as $\mathrm{DQ}_\alpha^{\mathrm{VaR}}$ for t-distributions as discussed in Remark 6.

DQ for elliptical models as the dimension n varies Fig. 4 is related to Section 5.2 and reports how DQ changes over $n \in [2, 100]$ in the four models. We choose $r = 0.5$ in this experiment for better visibility. As we can see, DQ decreases to 0 for models with the AR(1) dispersion Σ_2, and DQ converges to a non-zero constant for models with the equicorrelated dispersion Σ_1. This is consistent with Proposition 3 (ii) because $\mathrm{AC}_{\Sigma_1} \to r$ and $\mathrm{AC}_{\Sigma_2} \to 0$ as $n \to \infty$.

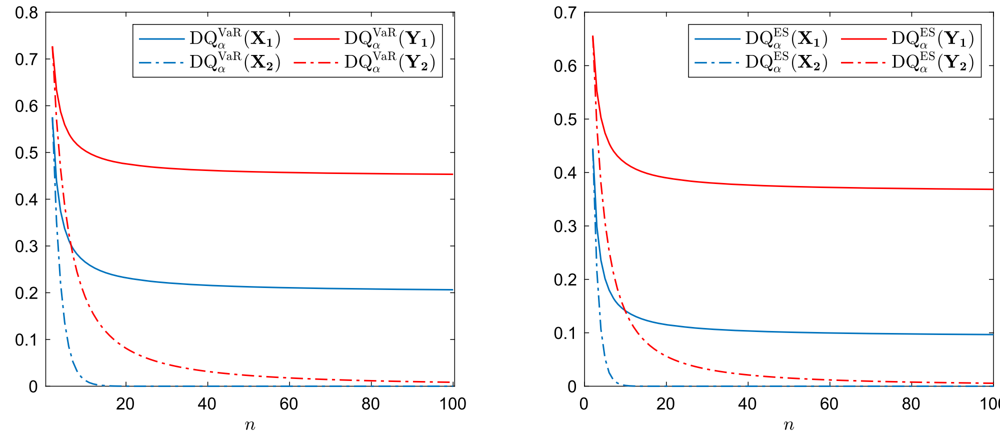

Fig. 4. DQs based on VaR and ES for $n \in [2, 100]$ with fixed $\alpha = 0.05$, $r = 0.5$ and $\nu = 3$.

Table 1
Values of DQs based on VaR at level $\alpha = 0.01$ and ES at level $c\alpha$, where $n = 4$ and $r = 0.3$.

	c	$c\alpha$	$\mathrm{DQ}_\alpha^{\mathrm{VaR}}$	$\mathrm{DQ}_{c\alpha}^{\mathrm{ES}}$
$\mathbf{X}_1 \sim \mathrm{N}(\boldsymbol{\mu}, \Sigma_1)$	2.58	0.0258	0.0369	0.0377
$\mathbf{X}_2 \sim \mathrm{N}(\boldsymbol{\mu}, \Sigma_2)$	2.58	0.0258	0.0024	0.0025
$\mathbf{Y}_1 \sim \mathrm{t}(3, \boldsymbol{\mu}, \Sigma_1)$	3.31	0.0331	0.3558	0.3373
$\mathbf{Y}_2 \sim \mathrm{t}(3, \boldsymbol{\mu}, \Sigma_2)$	3.31	0.0331	0.2094	0.1961

Cross-comparison between DQ based on VaR and ES One may be tempted to compare values of DQ based on VaR to those based on ES. Although we see from Fig. 3 that the curve $\mathrm{DQ}_\alpha^{\mathrm{VaR}}$ often dominates the curve $\mathrm{DQ}_\alpha^{\mathrm{ES}}$ for the same model, such a comparison is not meaningful, since VaR and ES are not meant to be compared at the same level α. For a fair comparison, one needs to associate a VaR level α to an ES level $c\alpha$ where $c \geqslant 1$ is PELVE of Li and Wang (2022) defined via $\mathrm{ES}_{c\alpha}(X) = \mathrm{VaR}_\alpha(X)$ for X being normally or t-distributed; note that the location and scale of X do not matter. The values of c, $\mathrm{DQ}_\alpha^{\mathrm{VaR}}$ and $\mathrm{DQ}_{c\alpha}^{\mathrm{ES}}$ for $\alpha = 0.01$ are summarized in Table 1. As we observe from Table 1, the values of DQs based on VaR and ES are quite close when the probability level is calibrated via PELVE. This is consistent with the afore-mentioned fact that VaR behaves similarly to ES in the setting of elliptical models.

6. Multivariate regularly varying models

Heavy-tailed distributions are known to exhibit complicated and even controversial phenomena in finance (see e.g., Ibragimov et al. (2011)), and they are typically modeled via multivariate regularly varying (MRV) models, important objects in Extreme Value Theory. Such models are particularly relevant for tail risk measures such as VaR and ES at high levels (McNeil et al. (2015)). In particular, MRV models have been applied to DR based on VaR (e.g., Mainik and Rüschendorf (2010) and Mainik and Embrechts (2013)).

Definition 4. A random vector $\mathbf{X} \in \mathcal{X}^n$ has an MRV model with some $\gamma > 0$ if there exists a Borel probability measure Ψ on the unit sphere $\mathbb{S}^n := \left\{\mathbf{s} \in \mathbb{R}^n : \|\mathbf{s}\| = 1\right\}$ such that for any $t > 0$ and any Borel set $S \subseteq \mathbb{S}^n$ with $\Psi(\partial S) = 0$,

$$\lim_{x\to\infty} \frac{\mathbb{P}(\|\mathbf{X}\| > tx,\ \mathbf{X}/\|\mathbf{X}\| \in S)}{\mathbb{P}(\|\mathbf{X}\| > x)} = t^{-\gamma}\Psi(S),$$

where $\|\cdot\|$ is the L_1-norm (one could use any other norm equivalent to the L_1-norm). We call γ the tail index of $\mathbf{X}$ and Ψ the spectral measure of $\mathbf{X}$. This is written as $\mathbf{X} \in \mathrm{MRV}_\gamma(\Psi)$.

The univariate regular variation with tail index γ is defined as

$$\text{for all } t > 0,\ \lim_{x\to\infty} \frac{1 - F_X(tx)}{1 - F_X(x)} = t^{-\gamma},$$

where F is the distribution function of X. We write $X \in \mathrm{RV}_\gamma$ for this property. As a consequence of $\mathbf{X} \in \mathrm{MRV}_\gamma(\Psi)$, $\|\mathbf{X}\|$ satisfies univariate regular variation with the same tail index γ.

Regular variation is one of the basic notions for describing heavy-tailed distributions and dependence in the tails. In what follows, we limit our discussion to $\mathbf{X} \in \mathrm{MRV}_\gamma(\Psi)$ under the non-degeneracy condition:

$$\Psi\left(\left\{\mathbf{s} \in \mathbb{S}^n : \mathbf{s} \in (0, \infty)^n\right\}\right) > 0.$$

Note that if $\mathbf{X} \in \mathrm{MRV}_\gamma(\Psi)$ satisfies non-degeneracy condition, we have $\mathbf{w}^\top\mathbf{X} \in \mathrm{RV}_\gamma$ (See Mainik and Embrechts (2013)). Since $\mathrm{VaR}_\alpha(X)/\mathrm{ES}_\alpha(X) \to (\gamma - 1)/\gamma$ as $\alpha \downarrow 0$ for $X \in \mathrm{RV}_\gamma$ with finite mean (see e.g., McNeil et al. (2015, p.154)), we only present the case of VaR.

Let $\mathbf{X} \in \mathrm{MRV}_\gamma(\Psi)$ be a random vector with identical marginals. If $X_1, \ldots, X_n$ have a finite mean, then VaR is asymptotically subadditive in the following sense (see e.g., Embrechts et al. (2009))

$$\mathrm{VaR}_\alpha\left(\sum_{i=1}^n X_i\right) \leqslant \sum_{i=1}^n \mathrm{VaR}_\alpha(X_i) \quad \text{for } \alpha \text{ close enough to } 0,$$

but the inequality is reversed if $X_1, \ldots, X_n$ do not have a finite mean. Next, in contrast to Proposition 2 and Remark 5, we will show that DQ based on VaR can be arbitrarily close to n even if the individual losses are iid.

Theorem 3. *Suppose that $\mathbf{X} \in \mathrm{MRV}_\gamma(\Psi)$ and $\mathbf{X}$ has positive joint density on the support of $\mathbf{X}$. Then,*

$$\lim_{\alpha\downarrow 0} \mathrm{DQ}_\alpha^{\mathrm{VaR}}(\mathbf{X}) = \eta_{\mathbf{1}}\left(\sum_{i=1}^n \eta_{\mathbf{e}_i}^{1/\gamma}\right)^{-\gamma}, \tag{12}$$

where $\eta_{\mathbf{x}} = \int_{\mathbb{S}^n}\left(\mathbf{x}^\top\mathbf{s}\right)_+^\gamma \Psi(\mathrm{d}\mathbf{s})$ for $\mathbf{x} \in \mathbb{R}^n$. Moreover, if $X_1, \ldots, X_n$ are iid random variables, then $\mathrm{DQ}_\alpha^{\mathrm{VaR}}(\mathbf{X}) \to n^{1-\gamma}$ as $\alpha \downarrow 0$.

Proof. A more general result of (12) and its proof are shown in Proposition 5, where the asymptotic behavior of $\mathrm{DQ}_\alpha^{\mathrm{VaR}}$ for weighted portfolios is investigated. Since DQ is scale-invariant, by taking $\mathbf{w} = (1/n, \ldots, 1/n)$ in Proposition 5, it gives

$$\lim_{\alpha\downarrow 0}\mathrm{DQ}_\alpha^{\mathrm{VaR}}(\mathbf{X})=\lim_{\alpha\downarrow 0}\mathrm{DQ}_\alpha^{\mathrm{VaR}}(1/nX_1,\dots,1/nX_n)=\frac{\eta_{\mathbf{w}}}{\left(\sum_{i=1}^n w_i\eta_{\mathbf{e}_i}^{1/\gamma}\right)^\gamma},$$

where $\eta_{\mathbf{w}}=n^{-\gamma}\int_{\mathbb{S}^n}(\mathbf{1}^\top s)_+^\gamma\,\Psi(\mathrm{d}\mathbf{s})=n^{-\gamma}\eta_{\mathbf{1}}$. As a result, we have

$$\lim_{\alpha\downarrow 0}\mathrm{DQ}_\alpha^{\mathrm{VaR}}(\mathbf{X})=\eta_{\mathbf{1}}\left(\sum_{i=1}^n\eta_{\mathbf{e}_i}^{1/\gamma}\right)^{-\gamma}.$$

If $X_1,\dots,X_n$ are iid non-negative random variables, by Example 3.1 of Embrechts et al. (2009), we have

$$\eta_{\mathbf{1}}^{1/\gamma}=\lim_{\alpha\downarrow 0}\frac{\mathrm{VaR}_\alpha\left(\sum_{i=1}^n X_i\right)}{\mathrm{VaR}_\alpha(X_1)}=n^{1/\gamma},$$

which implies that $\eta_{\mathbf{1}}=n$. Moreover,

$$(\eta_{\mathbf{e}_i})^{1/\gamma}=\lim_{\alpha\downarrow 0}\frac{\mathrm{VaR}_\alpha(X_i)}{\mathrm{VaR}_\alpha(X_1)}=1.$$

Hence, $\lim_{\alpha\downarrow 0}\mathrm{DQ}_\alpha^{\mathrm{VaR}}(\mathbf{X})=n^{1-\gamma}$. Further, if $\gamma\downarrow 0$, then $\mathrm{DQ}_\alpha^{\mathrm{VaR}}(\mathbf{X})\to n$. □

The α-CE model in Theorem 1 with $\mathrm{DQ}_\alpha^{\mathrm{VaR}}(\mathbf{X})=n$ is complicated and involves both positive and negative dependence. Theorem 3 suggests that $\mathrm{DQ}_\alpha^{\mathrm{VaR}}(\mathbf{X})\approx n$ can be obtained for some very heavy-tailed iid model with γ close to 0. Therefore, the upper bound n on $\mathrm{DQ}_\alpha^{\mathrm{VaR}}$ is relevant when analyzing very heavy-tailed risks such as catastrophe losses; we refer to Embrechts et al. (1997) for a general treatment of heavy-tailed risks in insurance and finance.

Remark 7. Suppose that $X_1,\dots,X_n$ are iid random variables with $X_1\in\mathrm{RV}_\gamma$ having positive density over its support. We have $\mathbf{X}=(X_1,\dots,X_n)\in\mathrm{MRV}_\gamma(\Psi)$ by Kulik and Soulier (2020, Example 2.1.4), and thus $\mathrm{DQ}_\alpha^{\mathrm{VaR}}(\mathbf{X})\to n^{1-\gamma}$ as $\alpha\downarrow 0$.

Remark 8. We note that the intersection between elliptical distributions and MRV distributions is non-empty. For $\mathbf{X}\sim\mathrm{E}_n(\mu,\Sigma,\tau)$, we have

$$\mathbf{X}\overset{\mathrm{d}}{=}\mu+RAU,$$

where $A\in\mathbb{R}^{n\times n}$ satisfying $AA^\top=\Sigma$, U is uniformly distributed on the Euclidean sphere $\mathbb{S}_2^d$ and R is a non-negative random variable that is independent of U. Theorem 4.3 of Hult and Lindskog (2002) showed that $\mathbf{X}$ has an MRV model if and only if $R\in\mathrm{RV}_\gamma$ for some $\gamma>0$. Assume that the elliptically distributed $\mathbf{X}$ is in $\mathrm{MRV}_\gamma(\Psi)$ with $\gamma>0$. As a result, we have $Y\sim E_1(0,1,\tau)\in\mathrm{RV}_\gamma$. Let f be the density of Y. Following Proposition 3 (i) and the fact that $\mathrm{VaR}_\alpha(Y)/\mathrm{ES}_\alpha(Y)\to(\gamma-1)/\gamma$ as $\alpha\downarrow 0$ for $Y\in\mathrm{RV}_\gamma$ with finite mean, we have

$$\lim_{\alpha\downarrow 0}\mathrm{DQ}_\alpha^{\mathrm{ES}}(\mathbf{X})=\lim_{\alpha\downarrow 0}\mathrm{DQ}_\alpha^{\mathrm{VaR}}(\mathbf{X})=\lim_{x\to\infty}k_\Sigma\frac{f(k_\Sigma x)}{f(x)}=k_\Sigma^{-\gamma}.$$

If $\mathbf{X}$ follows an elliptical distribution in the MRV class, then $\mathrm{DQ}_\alpha^{\mathrm{ES}}(\mathbf{X})$ has the same limit as $\mathrm{DQ}_\alpha^{\mathrm{VaR}}(\mathbf{X})$. For example, if $\mathbf{X}\sim t(\nu,\mu,\Sigma)$, we have $\mathbf{X}\in\mathrm{MRV}_\gamma(\Psi)$ with $\gamma=\nu$ as we have shown in (11) that $\lim_{\alpha\downarrow 0}\mathrm{DQ}_\alpha^{\mathrm{VaR}}(\mathbf{X})=\lim_{\alpha\downarrow 0}\mathrm{DQ}_\alpha^{\mathrm{ES}}(\mathbf{X})=k_\Sigma^{-\nu}$.

To end this section, we show that if there exists an asset with a strictly heavier tail than the other assets in the portfolio, then DQ based on VaR tends to 1 as $\alpha\downarrow 0$.

Proposition 4. *Suppose $X_i\in\mathrm{RV}_{\gamma_i}$ for $i\in[n]$ such that $\gamma_1<\min_{i=2,\dots,n}\gamma_i$. If $X_1,\dots,X_n$ have positive densities on their support, then $\lim_{\alpha\downarrow 0}\mathrm{DQ}_\alpha^{\mathrm{VaR}}(\mathbf{X})=1$.*

Proof. Since $\gamma_1<\min_{i=2,\dots,n}\gamma_i$, X_1 has a heavier tail than $X_2,\dots,X_n$. As a result, we have $\sum_{i=1}^n X_i\in\mathrm{RV}_{\gamma_1}$ regardless of the dependence between all random variables (See Kulik and Soulier (2020, Lemma 1.3.2)), that is,

$$\lim_{x\to\infty}\frac{\mathbb{P}\left(\sum_{i=1}^n X_i>x\right)}{\mathbb{P}(X_1>x)}=1.$$

Moreover, X_1 having a heavier tail than $X_2,\dots,X_n$ also implies that $\lim_{\alpha\downarrow 0}\mathrm{VaR}_\alpha(X_i)/\mathrm{VaR}_\alpha(X_1)=0$ for all $i=2,\dots,n$, and thus $\lim_{\alpha\downarrow 0}\sum_{i=1}^n\mathrm{VaR}_\alpha(X_i)/\mathrm{VaR}_\alpha(X_1)=1$. Therefore, we have

$$\begin{aligned}\lim_{\alpha\downarrow 0}\mathrm{DQ}_\alpha^{\mathrm{VaR}}(\mathbf{X})&=\lim_{\alpha\downarrow 0}\frac{\mathbb{P}\left(\sum_{i=1}^n X_i>\sum_{i=1}^n\mathrm{VaR}_\alpha(X_i)\right)}{\alpha}\\&=\lim_{\alpha\downarrow 0}\frac{\mathbb{P}\left(\sum_{i=1}^n X_i>\sum_{i=1}^n\mathrm{VaR}_\alpha(X_i)\right)}{\mathbb{P}(X_1>\mathrm{VaR}_\alpha(X_1))}\\&=\lim_{\alpha\downarrow 0}\frac{\mathbb{P}\left(\sum_{i=1}^n X_i>\sum_{i=1}^n\mathrm{VaR}_\alpha(X_i)\right)}{\mathbb{P}(X_1>\sum_{i=1}^n\mathrm{VaR}_\alpha(X_i))}\\&\quad\times\frac{\mathbb{P}(X_1>\sum_{i=1}^n\mathrm{VaR}_\alpha(X_i))}{\mathbb{P}(X_1>\mathrm{VaR}_\alpha(X_1))}\\&=\lim_{\alpha\downarrow 0}\left(\frac{\sum_{i=1}^n\mathrm{VaR}_\alpha(X_i)}{\mathrm{VaR}_\alpha(X_1)}\right)^{-\gamma_1}=1.\end{aligned}$$

Thus, we get the desired result. □

Proposition 4 illustrates the intuitive fact that, if the tail of one asset is strictly heavier than the others, then the portfolio has no diversification in the tail region, i.e., as $\alpha\downarrow 0$.

7. Optimization for the elliptical models and MRV models

We analyze portfolio diversification for a random vector $\mathbf{X}\in\mathcal{X}^n$ representing losses from n assets and a vector $\mathbf{w}=(w_1,\dots,w_n)\in\Delta_n$ of portfolio weights, where

$$\Delta_n:=\left\{\mathbf{x}\in[0,1]^n:x_1+\dots+x_n=1\right\}.$$

The total loss of the portfolio is $\mathbf{w}^\top\mathbf{X}$. We write $\mathbf{w}\odot\mathbf{X}=(w_1X_1,\dots,w_nX_n)$ which is the portfolio loss vector with the weight $\mathbf{w}$. For a portfolio selection problem, we need to treat $\mathrm{DQ}_\alpha^\rho(\mathbf{w}\odot\mathbf{X})$ as a function of the portfolio weight $\mathbf{w}$.

Han et al. (2022) studied the following optimization diversification problem

$$\min_{\mathbf{w}\in\Delta_n}\mathrm{DQ}_\alpha^{\mathrm{VaR}}(\mathbf{w}\odot\mathbf{X})\quad\text{and}\quad\min_{\mathbf{w}\in\Delta_n}\mathrm{DQ}_\alpha^{\mathrm{ES}}(\mathbf{w}\odot\mathbf{X});\tag{13}$$

for general $\mathbf{X}$. Moreover, efficient algorithms are obtained to optimize $\mathrm{DQ}_\alpha^{\mathrm{VaR}}$ and $\mathrm{DQ}_\alpha^{\mathrm{ES}}$ in real-data applications; see their Sections 6.2 and 7. In this section, we focus on the portfolio optimization problems for elliptical and MRV models.

For the elliptical models, the optimization of $\mathrm{DQ}_\alpha^{\mathrm{VaR}}$, $\mathrm{DQ}_\alpha^{\mathrm{ES}}$ boils down to maximizing $k_{\mathbf{w}\Sigma\mathbf{w}^\top}$ in (7) since DQ of $\mathbf{w}\odot\mathbf{X}$ is decreasing in $k_{\mathbf{w}\Sigma\mathbf{w}^\top}$. We assume that Σ is invertible, and write $\Sigma=(\sigma_{ij})_{n\times n}$, with diagonal entries $\sigma_{ii}=\sigma_i^2$, $i\in[n]$, and $\boldsymbol{\sigma}=(\sigma_1,\dots,\sigma_n)$. Note that

$$k_{\mathbf{w}\Sigma\mathbf{w}^\top}=\frac{\mathbf{w}^\top\boldsymbol{\sigma}}{\sqrt{\mathbf{w}^\top\Sigma\mathbf{w}}},$$

and we immediately give the optimizer of (13) for the elliptical models.

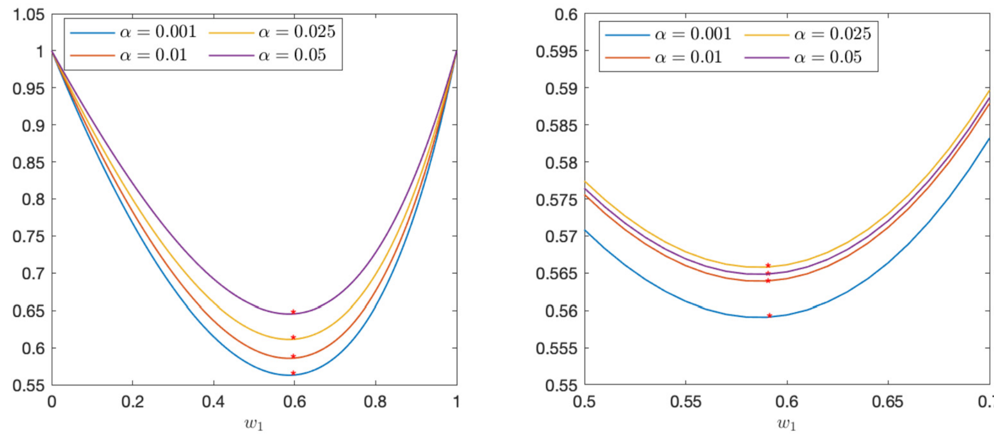

Fig. 5. Values of $\mathrm{DQ}_\alpha^{\mathrm{VaR}}(\mathbf{w}\odot\mathbf{X})$ and $\mathrm{DQ}_\alpha^{\mathrm{ES}}(\mathbf{w}\odot\mathbf{X})$ for $w_1\in[0,1]$.

Theorem 4. *Suppose that* $\mathbf{X}\sim\mathrm{E}_n(\boldsymbol{\mu},\Sigma,\tau)$, Σ *is invertible and* $\alpha\in(0,1/2)$, *then the vector*

$$\mathbf{w}^* = \mathop{\arg\max}_{\mathbf{w}\in\Delta_n}\frac{\mathbf{w}^\top\boldsymbol{\sigma}}{\sqrt{\mathbf{w}^\top\Sigma\mathbf{w}}} \tag{14}$$

minimizes (13)*, that is,*

$$\min_{\mathbf{w}\in\Delta_n}\mathrm{DQ}_\alpha^\rho(\mathbf{w}\odot\mathbf{X}) = \mathrm{DQ}_\alpha^\rho(\mathbf{w}^*\odot\mathbf{X}) \tag{15}$$

for ρ *being* VaR *or* ES.

The optimization problem (14) is well studied in the literature, and the existence and uniqueness of the solution can be verified if Σ is invertible, see, e.g. Choueifaty and Coignard (2008). Note that the optimizer for problem (15) does not depend on the tail probability level α. It is straightforward to see that

$$\mathop{\arg\min}_{\mathbf{w}\in\Delta_n}\mathrm{DR}^{\rho_\alpha}(\mathbf{w}\odot\mathbf{X}) = \mathop{\arg\max}_{\mathbf{w}\in\Delta_n}\frac{\mathbf{w}^\top\boldsymbol{\mu}+\mathbf{w}^\top\boldsymbol{\sigma}\rho_\alpha(Y)}{\mathbf{w}^\top\boldsymbol{\mu}+\sqrt{\mathbf{w}^\top\Sigma\mathbf{w}}\rho_\alpha(Y)}$$

for ρ being VaR or ES and $Y\sim\mathrm{E}_1(0,1,\tau)$. This optimizer is the same as that of (15) if $\boldsymbol{\mu}=\mathbf{0}$. This shows that for centered elliptical models, optimizing DQ and optimizing DR are equivalent problems, both of which are further equivalent to optimizing DR based on SD (assuming it exists). This is intuitive as for a fixed τ, centered elliptical distributions are parameterized by their dispersion matrices.

Example 1. Assume that $\mathbf{X}\sim\mathrm{t}(\nu,\boldsymbol{\mu},\Sigma)$ where $\nu=3$ and the dispersion matrix is given by

$$\Sigma=\begin{pmatrix}1 & 0.5\\ 0.5 & 2\end{pmatrix}.$$

Clearly, DQ does not depend on $\boldsymbol{\mu}$. We show the curves of $\mathrm{DQ}_\alpha^{\mathrm{VaR}}(\mathbf{w}\odot\mathbf{X})$ and $\mathrm{DQ}_\alpha^{\mathrm{ES}}(\mathbf{w}\odot\mathbf{X})$ against the weight w_1 with various values of $\alpha=0.001, 0.01, 0.025, 0.05$. It can be anticipated from (14) that although DQ depends on α, the optimizer does not. By solving (14), we get $w_1^*=0.5860$ and $w_2^*=0.4140$, which corresponds to the observations in Fig. 5. Recall that $\mathrm{DQ}_\alpha^{\mathrm{ES}}$ is quite flat when α varies in Fig. 3, and hence curves of $\mathrm{DQ}_\alpha^{\mathrm{ES}}(\mathbf{w}\odot\mathbf{X})$ look similar for different α.

Next, we turn to the MRV model. The following result gives the limit of DQ of the portfolio $\mathbf{w}\odot\mathbf{X}$ where $\mathbf{X}$ follows an MRV model. Due to the same reason stated in Section 6, we only present the case of VaR. In the proofs below, for any positive functions f and g, we write $f(x)\simeq g(x)$ as $x\to x_0$ to represent $\lim_{x\to x_0}f(x)/g(x)=1$.

Proposition 5. *Suppose that* $\mathbf{X}\in\mathrm{MRV}_\gamma(\Psi)$ *and* $\mathbf{X}$ *has positive joint density on the support of* $\mathbf{X}$. *Then, for* $\mathbf{w}\in\Delta_n$,

$$\lim_{\alpha\downarrow 0}\mathrm{DQ}_\alpha^{\mathrm{VaR}}(\mathbf{w}\odot\mathbf{X}) = f(\mathbf{w}),$$

where $f(\mathbf{w})=\eta_{\mathbf{w}}/\left(\sum_{i=1}^n w_i\eta_{\mathbf{e}_i}^{1/\gamma}\right)^\gamma$ *and* $\eta_{\mathbf{x}}=\int_{\mathbb{S}^n}(\mathbf{x}^\top\mathbf{s})_+^\gamma\,\Psi(\mathrm{d}\mathbf{s})$ *for* $\mathbf{x}\in\mathbb{R}^n$.

Proof. If $\mathbf{X}\in\mathrm{MRV}_\gamma(\Psi)$ with $\gamma\in(0,1)$, we have (Lemma 2.2 of Mainik and Embrechts (2013))

$$\lim_{\alpha\downarrow 0}\frac{\mathrm{VaR}_\alpha\left(\sum_{i=1}^n w_iX_i\right)}{\mathrm{VaR}_\alpha\left(\|\mathbf{X}\|_1\right)}=\eta_{\mathbf{w}}^{1/\gamma},$$

and

$$\lim_{\alpha\downarrow 0}\sum_{i=1}^n\frac{w_i\mathrm{VaR}_\alpha(X_i)}{\mathrm{VaR}_\alpha\left(\|\mathbf{X}\|_1\right)}=\sum_{i=1}^n w_i\eta_{\mathbf{e}_i}^{1/\gamma},$$

where $\|\mathbf{X}\|_1=\sum_{i=1}^n|X_i|$. As $\mathbf{X}$ has positive joint density, VaR_α is continuous for $\sum_{i=1}^n w_iX_i$. Then we have $\mathrm{VaR}_{\alpha^*}(\sum_{i=1}^n w_iX_i)=\sum_{i=1}^n w_i\mathrm{VaR}_\alpha(X_i)$. Thus, it follows that

$$\frac{\mathrm{VaR}_\alpha\left(\sum_{i=1}^n w_iX_i\right)}{\mathrm{VaR}_{\alpha^*}(\sum_{i=1}^n w_iX_i)}\to\frac{\eta_{\mathbf{w}}^{1/\gamma}}{\sum_{i=1}^n w_i\eta_{\mathbf{e}_i}^{1/\gamma}}\qquad\text{as }\alpha\downarrow 0.$$

Since $\sum_{i=1}^n w_iX_i\in\mathrm{RV}_\gamma$, for $c>0$,

$$\frac{\mathrm{VaR}_\alpha\left(\sum_{i=1}^n w_iX_i\right)}{\mathrm{VaR}_{c\alpha}\left(\sum_{i=1}^n w_iX_i\right)}\simeq\left(\frac{1}{c}\right)^{-1/\gamma}\qquad\text{as }\alpha\downarrow 0.$$

Let $c=\alpha^*/\alpha$, we have

$$\left(\frac{\alpha}{\alpha^*}\right)^{-1/\gamma}\to\frac{\eta_{\mathbf{w}}^{1/\gamma}}{\sum_{i=1}^n w_i\eta_{\mathbf{e}_i}^{1/\gamma}}.$$

Hence,

$$\mathrm{DQ}_\alpha^{\mathrm{VaR}}(\mathbf{w}\odot\mathbf{X})=\frac{\alpha^*}{\alpha}\to\frac{\eta_{\mathbf{w}}}{\left(\sum_{i=1}^n w_i\eta_{\mathbf{e}_i}^{1/\gamma}\right)^\gamma}.$$

The desired result is obtained. □

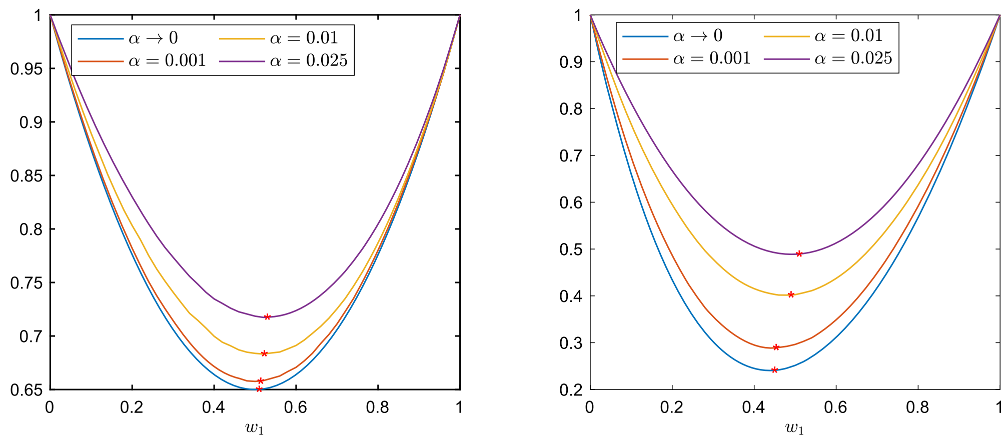

Fig. 6. Values of $\mathrm{DQ}^{\mathrm{VaR}}_{\alpha}(\mathbf{w} \odot \mathbf{X})$ with $\nu = 2$ (left) and $\nu = 4$ (right).

Proposition 5 allows us to approximately optimize $\mathrm{DQ}^{\mathrm{VaR}}_{\alpha}$ by minimizing $f(\mathbf{w})$. For $\mathbf{X} \in \mathrm{MRV}_{\gamma}(\Psi)$ with $\gamma > 1$, by assuming $\Psi\left(\{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}^\top \mathbf{x} = 0\}\right) = 0$ for any $\mathbf{a} \in \mathbb{R}^n$, which means that all components are relevant for the extremes of $\mathbf{X}$, the existence and uniqueness of $\mathbf{w}^* = \arg\min_{\mathbf{w} \in \Delta_n} f(\mathbf{w})$ are guaranteed. In fact, the existence of $\mathbf{w}^*$ is due to the continuity of $f(\mathbf{w})$ and the compactness of Δ_n. To show uniqueness, we can rewrite the above minimization problem as

$$\min_{\mathbf{w} \in \Delta_n} \eta_{\mathbf{w}} \quad \text{s.t.} \quad \sum_{i=1}^{d} w_i \eta_{\mathbf{e}_i}^{1/\gamma} = 1.$$

Note that the set of constraints is compact and $\eta_{\mathbf{w}}$ is strictly convex, and hence $\mathbf{w}^*$ is unique.

Example 2. Assume that Y_1 and Y_2 are iid following a standard t-distribution with $\nu > 1$ degrees of freedom. A random vector $\mathbf{X} = (X_1, X_2)$ is defined as

$$\mathbf{X} = A\mathbf{Y} \quad \text{with } A = \begin{pmatrix} 1 & 0 \\ r & \sqrt{1-r^2} \end{pmatrix}.$$

The random vectors $\mathbf{X}$ and $\mathbf{Y}$ are not elliptically distributed. Using the results in Mainik and Embrechts (2013), we have

$$\frac{\eta_{\mathbf{w}}}{\eta_{\mathbf{1}_1}} = (w_1 + w_2 r)^{\nu} + \left(w_2\sqrt{1-r^2}\right)^{\nu},$$

and

$$\frac{\eta_{\mathbf{w}}}{\eta_{\mathbf{1}_2}} = \frac{(w_1 + w_2 r)^{\nu} + \left(w_2\sqrt{1-r^2}\right)^{\nu}}{r^{\nu} + \sqrt{1-r^2}^{\nu}}.$$

Hence,

$$f(\mathbf{w}) = \left(w_1 \left((w_1 + w_2 r)^{\nu} + \left(w_2\sqrt{1-r^2}\right)^{\nu} \right)^{-\frac{1}{\nu}} + w_2 \left(\frac{(w_1 + w_2 r)^{\nu} + \left(w_2\sqrt{1-r^2}\right)^{\nu}}{r^{\nu} + \sqrt{1-r^2}^{\nu}} \right)^{-\frac{1}{\nu}} \right)^{-\nu}.$$

Take $r = 0.3$. We show the curves of $\mathrm{DQ}^{\mathrm{VaR}}_{\alpha}(\mathbf{w} \odot \mathbf{X})$ against w_1 for $\alpha = 0.001, 0.01, 0.025$ and $\nu = 2, 4$. Also, we use $f(\mathbf{w})$ to approximate $\mathrm{DQ}^{\mathrm{VaR}}_{\alpha}(\mathbf{w} \odot \mathbf{X})$ as α tends to 0. From Fig. 6, we can see that the optimizer w_1^* is converging to the one that maximizes $f(\mathbf{w})$ as α tends to 0.

Remark 9. Some negative dependence concepts yield small values of DQ. The joint mix dependence usually leads to a zero DQ as we see in Theorem 1 (ii). The negative dependence concept of Lee and Ahn (2014), weaker than joint mix dependence, does not necessarily lead to a small value of DQ. For instance, the portfolio vector $\mathbf{X} = (X, -\varepsilon X)$ is counter-monotonic for $\varepsilon > 0$, but its DQ can be close to 1 for small ε. In particular, we have $\mathrm{DQ}^{\mathrm{VaR}}_{\alpha}(\mathbf{X}) \approx 0.9333$ and $\mathrm{DQ}^{\mathrm{ES}}_{\alpha}(\mathbf{X}) \approx 0.9044$ for $\alpha = 0.05$ and $\varepsilon = 0.01$ when X follows a standard normal distribution.

8. Conclusion

The DQs based on VaR and ES are investigated in this paper, following the theory of DQ in Han et al. (2022). In particular, for elliptical and MRV models, these DQs have simple forms. Comparisons between DQ and DR illustrate some attractive features of DQ. These results enhance the theory and applications of DQ.

We summarize some features below. (i) In cases of VaR and ES, DQs have simple formulas, in a way comparable to DRs. (ii) DQs based on VaR and ES take values in bounded intervals and have natural ranges of $[0, n]$ and $[0, 1]$, respectively. The special values 0, 1 and n which correspond to special dependence structures can be constructed. (iii) DQs based on VaR and ES for elliptical distributions and MRV models have convenient expressions and it can capture heavy tails in an intuitive way. (iv) Portfolio optimization for elliptical models boils down to a well-studied problem in the literature. For centered elliptical models, optimizing DQ and optimizing DR are equivalent problems.

We discuss some future directions for the research of DQ. As a potential alternative to ES, expectiles (Bellini et al. (2014)) have received increasing attention in the recent literature; indeed, they are the only elicitable coherent risk measures (Ziegel (2016)). It would be interesting to formulate DQ based on expectiles and investigate its properties that are different from DQ based on ES or VaR. As another interesting class of risk measures, the optimized certainty equivalents (Ben-Tal and Teboulle (2007)) are introduced from decision-theoretic criterion based on utility functions. It would be useful to construct DQ based on utility functions or optimized certainty equivalents and analyze the decision-theoretic implications of DQ.

Declaration of competing interest

The authors declare that they have no conflict of interest.

Data availability

No data was used for the research described in the article.

Acknowledgements

The authors thank the editor and two anonymous referees for constructive comments. The authors also thank Fabio Bellini, Aleš Černý, An Chen, Jan Dhaene, Paul Embrechts, Paul Glasserman, Pablo Koch-Medina, Dimitrios Konstantinides, Henry Lam, Jean-Gabriel Lauzier, Fabio Macherroni, Massimo Marinacci, Cosimo Munari, Alexander Schied, Qihe Tang, Stan Uryasev, Peter Wakker, and Zuoquan Xu for helpful comments and discussions on various issues of DQ. Xia Han is supported by the National Natural Science Foundation of China (Grant No. 12301604, 12371471) and the Fundamental Research Funds for the Central Universities, Nankai University (Grant No. 63231138). Liyuan Lin is supported by the Hickman Scholarship from the Society of Actuaries. Ruodu Wang is supported by the Natural Sciences and Engineering Research Council of Canada (RGPIN-2018-03823, RGPAS-2018-522590) and the Sun Life Research Fellowship.

References

Aleksandr, I., Khinchin, A., 1949. Mathematical Foundations of Statistical Mechanics. Courier Corporation.
Artzner, P., Delbaen, F., Eber, J.-M., Heath, D., 1999. Coherent measures of risk. Mathematical Finance 9 (3), 203–228.
BCBS, 2019. Minimum Capital Requirements for Market Risk. February 2019. Basel Committee on Banking Supervision. Document d457. Bank for International Settlements, Basel.
Bellini, F., Klar, B., Müller, A., Rosazza Gianin, E., 2014. Generalized quantiles as risk measures. Insurance: Mathematics and Economics 54 (1), 41–48.
Ben-Tal, A., Teboulle, M., 2007. An old-new concept of convex risk measures: the optimized certainty equivalent. Mathematical Finance 17 (3), 449–476.
Bignozzi, V., Mao, T., Wang, B., Wang, R., 2016. Diversification limit of quantiles under dependence uncertainty. Extremes 19 (2), 143–170.
Bürgi, R., Dacorogna, M.M., Iles, R., 2008. Risk aggregation, dependence structure and diversification benefit. In: Stress Testing for Financial Institutions. Riskbooks, Incisive Media, London, pp. 265–306.
Choueifaty, Y., Coignard, Y., 2008. Toward maximum diversification. The Journal of Portfolio Management 35 (1), 40–51.
Cui, H., Tan, K.S., Yang, F., Zhou, C., 2022. Asymptotic analysis of portfolio diversification. Insurance: Mathematics and Economics 106, 302–325.
Delbaen, F., 2002. Coherent risk measures on general probability spaces. In: Sandmann, K., Schönbucher, P.J. (Eds.), Advances in Finance and Stochastics: Essays in Honour of Dieter Sondermann, pp. 1–37.
Dhaene, J., Denuit, M., 1999. The safest dependence structure among risks. Insurance: Mathematics and Economics 25 (1), 11–21.
Durrett, R., 2019. Probability: Theory and Examples, vol. 49. Cambridge University Press.
EIOPA, 2011. Equivalence assessment of the Swiss supervisory system in relation to articles 172, 227 and 260 of the Solvency II Directive. EIOPA-BoS-11-028.
Embrechts, P., Klüppelberg, C., Mikosch, T., 1997. Modelling Extremal Events for Insurance and Finance. Springer, Berlin.
Embrechts, P., Lambrigger, D., Wüthrich, M., 2009. Multivariate extremes and the aggregation of dependent risks: examples and counter-examples. Extremes 12 (2), 107–127.
Embrechts, P., Liu, H., Wang, R., 2018. Quantile-based risk sharing. Operations Research 66 (4), 936–949.
Embrechts, P., Puccetti, G., Rüschendorf, L., Wang, R., Beleraj, A., 2014. An academic response to Basel 3.5. Risks 2 (1), 25–48.
Embrechts, P., Wang, B., Wang, R., 2015. Aggregation-robustness and model uncertainty of regulatory risk measures. Finance and Stochastics 19 (4), 763–790.
Emmer, S., Kratz, M., Tasche, D., 2015. What is the best risk measure in practice? A comparison of standard measures. The Journal of Risk 18 (2), 31–60.
Föllmer, H., Schied, A., 2016. Stochastic Finance. An Introduction in Discrete Time, fourth edition. Walter de Gruyter, Berlin.
Gneiting, T., 2011. Making and evaluating point forecasts. Journal of the American Statistical Association 106 (494), 746–762.
Han, X., Lin, L., Wang, R., 2022. Diversification quotients: quantifying diversification via risk measures. arXiv:2206.13679.
Hult, H., Lindskog, F., 2002. Multivariate extremes, aggregation and dependence in elliptical distributions. Advances in Applied Probability 34 (3), 587–608.
Ibragimov, R., Jaffee, D., Walden, J., 2011. Diversification disasters. Journal of Financial Economics 99 (2), 333–348.
Koumou, G.B., Dionne, G., 2022. Coherent diversification measures in portfolio theory: an axiomatic foundation. Risks 10 (11), 205.
Kulik, R., Soulier, P., 2020. Heavy-Tailed Time Series. Springer, Berlin.
Lauzier, J.G., Lin, L., Wang, R., 2023. Pairwise counter-monotonicity. Insurance: Mathematics and Economics 111, 279–287.
Lee, W., Ahn, J.Y., 2014. On the multidimensional extension of countermonotonicity and its applications. Insurance: Mathematics and Economics 56, 68–79.
Li, H., Wang, R., 2022. PELVE: probability equivalent level of VaR and ES. Journal of Econometrics 234 (1), 353–370.
Li, L., Shao, H., Wang, R., Yang, J., 2018. Worst-case range value-at-risk with partial information. SIAM Journal on Financial Mathematics 9 (1), 190–218.
Liu, P., Schied, A., Wang, R., 2021. Distributional transforms, probability distortions, and their applications. Mathematics of Operations Research 46 (4), 1490–1512.
Mainik, G., Embrechts, P., 2013. Diversification in heavy-tailed portfolios: properties and pitfalls. Annals of Actuarial Science 7 (1), 26–45.
Mainik, G., Rüschendorf, L., 2010. On optimal portfolio diversification with respect to extreme risks. Finance and Stochastics 14 (4), 593–623.
Markowitz, H., 1952. Portfolio selection. The Journal of Finance 7 (1), 77–91.
McNeil, A.J., Frey, R., Embrechts, P., 2015. Quantitative Risk Management: Concepts, Techniques and Tools, revised edition. Princeton University Press, Princeton, NJ.
Puccetti, G., Wang, R., 2015. Extremal dependence concepts. Statistical Science 30 (4), 485–517.
Rockafellar, R.T., Royset, J.O., Miranda, S.I., 2014. Superquantile regression with applications to buffered reliability, uncertainty quantification, and conditional value-at-risk. European Journal of Operational Research 234 (1), 140–154.
Rüschendorf, L., 2013. Mathematical Risk Analysis. Dependence, Risk Bounds, Optimal Allocations and Portfolios. Springer, Heidelberg.
Tasche, D., 2007. Capital allocation to business units and sub-portfolios: the Euler principle. arXiv:0708.2542.
Wang, B., Wang, R., 2016. Joint mixability. Mathematics of Operations Research 41 (3), 808–826.
Wang, R., Zitikis, R., 2021. An axiomatic foundation for the Expected Shortfall. Management Science 67 (3), 1413–1429.
Ziegel, J.F., 2016. Coherence and elicitability. Mathematical Finance 26 (4), 901–918.