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Diversification Quotients: Quantifying Diversification via Risk Measures

Xia Han,[a] **Liyuan Lin,**[b] **Ruodu Wang**[c,*]

[a] School of Mathematical Sciences, and The Key Laboratory of Pure Mathematics and Combinatorics, Ministry of Education, Nankai University, Tianjin 300071, China; [b] Department of Econometrics and Business Statistics, Monash University, Clayton, Victoria 3168, Australia; [c] Department of Statistics and Actuarial Science, University of Waterloo, Waterloo, Ontario N2L3G1, Canada

*Corresponding author

Contact: xiahan@nankai.edu.cn, https://orcid.org/0009-0004-1201-9515 (XH); LiyuanLin@monash.edu, https://orcid.org/0000-0002-8845-2436 (LL); wang@uwaterloo.ca, https://orcid.org/0000-0003-3849-4555 (RW)

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Abstract. We establish the first axiomatic theory for diversification indices using six intuitive axioms: nonnegativity, location invariance, scale invariance, rationality, normalization, and continuity. The unique class of indices satisfying these axioms, called the diversification quotients (DQs), are defined based on a parametric family of risk measures. A further axiom of portfolio convexity pins down DQs based on coherent risk measures. The DQ has many attractive properties, and it can address several theoretical and practical limitations of existing indices. In particular, for the popular risk measures value at risk and expected shortfall, the corresponding DQ admits simple formulas, and it is efficient to optimize in portfolio selection. Moreover, it can properly capture tail heaviness and common shocks, which are neglected by traditional diversification indices. When illustrated with financial data, the DQ is intuitive to interpret, and its performance is competitive against other diversification indices.

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1. Introduction

Portfolio diversification refers to investment strategies that spread out among many assets, usually with the hope of reducing the volatility or risk of the resulting portfolio. A mathematical formalization of diversification in a portfolio selection context was made by Markowitz (1952), and some early literature on diversification includes Sharpe (1964), Samuelson (1967), Levy and Sarnat (1970), and Fama and Miller (1972), among others.

Although diversification is conceptually simple, the question of how to measure diversification *quantitatively* is never well settled. An intuitive but nonquantitative approach is to simply count the number of distinct stocks or industries of substantial weight in the portfolio; see, for example, Green and Hollifield (1992), Denis et al. (2002), and DeMiguel et al. (2009) in different contexts. This approach is heuristic, as it does not involve statistical or stochastic modeling. The second approach is to compute a quantitative index of the portfolio model based on, for example, volatility, variance (var), an expected utility, or a risk measure; this idea is certainly in the direction of Markowitz (1952). In addition, one may empirically address diversification by combining both approaches; see, for example, Tu and Zhou (2011) for the performance of different diversified portfolio strategies, D'Acunto et al. (2019) in the context of robo-advising, and Berger and Eeckhoudt (2021) for the perspective of risk aversion and ambiguity aversion. Green and Hollifield (1992) studied conditions under which the two approaches are roughly in line with each other.

In this paper, we take the second approach by assigning a quantifier, called a *diversification index*, to each modeled portfolio. Carrying the idea of Markowitz (1952), we start our journey with a simple index, the diversification ratio (DR) based on the standard deviation (SD). For a random vector $\mathbf{X} = (X_1, \dots, X_n)$ representing future random losses and profits of individual components in a portfolio in one period,[1] the DR based on the SD is defined as

$$\mathrm{DR}^{\mathrm{SD}}(\mathbf{X}) = \frac{\mathrm{SD}(\sum_{i=1}^n X_i)}{\sum_{i=1}^n \mathrm{SD}(X_i)}; \tag{1}$$

see Choueifaty and Coignard (2008). One can also replace SD with variance. Intuitively, with a smaller value indicating a stronger diversification, the index DR^{SD} quantifies the improvement of the portfolio SD over the sum of the SD of its components, and it has several convenient properties. Nevertheless, it is well-known that SD is a coarse, nonmonotone, and symmetric measurement of risk, making it unsuitable for many risk management applications, especially in the presence of heavy-tailed and skewed loss distributions; see Embrechts et al. (2002) for thorough discussions.

Risk measures, in particular, the value at risk (VaR) and expected shortfall (ES), are more flexible quantitative tools, widely used in both financial institutions' internal risk management and banking and insurance regulatory frameworks, such as Basel III/IV (Basel Committee on Banking Supervision 2019) and Solvency II (EIOPA 2011). ES has many nice theoretical properties and satisfies the four axioms of coherence (Artzner et al. 1999), whereas VaR is not subadditive in general, but it enjoys other practically useful properties; see Embrechts et al. (2014, 2018), Emmer et al. (2015), and the references therein for more discussions on the issues of VaR versus ES.

Some indices of diversification based on various risk measures have been proposed in the literature. For a given risk measure ϕ, an example of a diversification index is the DR in (1) with the SD replaced by ϕ; see Tasche (2007). For a review of diversification indices, see Koumou (2020). We find several demerits of DRs built on a general risk measure ϕ such as VaR or ES in Section 2. A natural question is whether we can design a suitable index based on risk measures to quantify the magnitude of diversification, which avoids the deficiencies of the DR. Answering this and related questions is the main purpose of this paper.

We take an axiomatic approach to find our desirable diversification indices. Axiomatic approaches for risk and decision indices have been prolific in economic and statistical decision theories; see, for example, the recent discussions of Gilboa et al. (2019) and the monographs of Gilboa (2009) and Wakker (2010). Closely related to diversification indices, risk measures (Artzner et al. 1999, Frittelli and Rosazza Gianin 2002, Föllmer and Schied 2016) and acceptability indices (Cherny and Madan 2009) also admit sound axiomatic foundation; the particular cases of VaR and ES are studied by Chambers (2009) and Wang and Zitikis (2021).

In Section 3, as our main contributions, we establish the first axiomatic foundation of diversification indices.[2] This axiomatic theory leads to the class of diversification quotients (DQs), the main object of this paper, which have an interpretation parallel to DRs. Six simple axioms—nonnegativity, location invariance, scale invariance, rationality, normalization, and continuity—are introduced and justified for their desirability in quantifying diversification. Their interpretations are self-evident, and they describe the basic requirements for a diversification index. In Theorem 1, these six axioms characterize DQs based on monetary and positive homogeneous risk measures. A seventh axiom of portfolio convexity, planting an intuitive ordering over portfolio weights in the index, further pins down DQs based on coherent risk measures in Theorem 2. Further, Proposition 1 gives conditions for which such DQs have the range of a standard interval. Portfolio convexity means that, with a given list of assets, combining a portfolio with a better-diversified one does not lead to worse diversification than the original portfolio, reflecting a fundamental principle in economics (Mas-Colell et al. 1995). The financial interpretation of a DQ is that it quantifies the improvement of a risk-level parameter (such as the parameter in VaR or ES) caused by pooling assets, and this is discussed in Section 3.4.

A detailed analysis of the properties of DQ based on general risk measures is discussed in Section 4, which reveals that DQs have many appealing features both theoretically and practically. In addition to standard operational properties (Proposition 2), DQs have intuitive behavior for several benchmark portfolio scenarios (Theorem 3). Moreover, the DQ allows for consistency with stochastic dominance (Proposition 3) and a fair comparison across portfolio dimensions (Proposition 4). We proceed to focus on VaR and ES in Section 5. It turns out that DQs based on VaR and ES have convenient alternative formulations (Theorem 4) and a natural range of $[0,n]$ and $[0,1]$, respectively (Proposition 5). Further, they report intuitive comparisons between normal and t-models, and they have the nice feature that they can capture heavy tails and common shocks.

In Section 6, efficient algorithms for DQs based on VaR and ES in portfolio optimization based on empirical observations are obtained (Proposition 6). Our new diversification index is applied to financial data in Section 7, where several empirical observations highlight the advantages of DQs. We conclude the paper in Section 8 by discussing a number of implications and promising future directions for DQs. Some additional results, proofs, and some omitted numerical results are relegated to the e-companion.

Notation. Throughout this paper, $(\Omega,\mathcal{F},\mathbb{P})$ is an atomless probability space on which almost surely equal random variables are treated as identical. A risk measure ϕ is a mapping from $\mathcal{X}$ to $\mathbb{R}$, where $\mathcal{X}$ is a convex cone of random variables on $(\Omega,\mathcal{F},\mathbb{P})$ representing losses faced by a financial institution or an investor (i.e., a sign flip from Artzner et al. 1999), and $\mathcal{X}$ is assumed to include all constants (i.e., degenerate random variables). For $p\in(0,\infty)$, denote by $L^p = L^p(\Omega,\mathcal{F},\mathbb{P})$ the set of all random variables X with $\mathbb{E}[|X|^p] < \infty$, where $\mathbb{E}$ is the expectation under $\mathbb{P}$. Furthermore, $L^\infty = L^\infty(\Omega,\mathcal{F},\mathbb{P})$ is the space of all (essentially) bounded random variables, and $L^0 = L^0(\Omega,\mathcal{F},\mathbb{P})$ is the space of all random variables. Write $X \sim F$ if the random variable X has the distribution function F under $\mathbb{P}$, and $X \overset{d}{=} Y$ if two random variables, X and Y, have the same distribution. Further, denote $\mathbb{R}_+ = [0, \infty)$ and $\overline{\mathbb{R}} = [-\infty, \infty]$. Terms such as increasing or decreasing functions are in the nonstrict sense. For $X \in L^0$, ess-sup(X) and ess-inf(X) are the essential supremum and the essential infimum of X, respectively. Let n be a fixed positive integer representing the number of assets in a portfolio, and write $[n] = \{1, \ldots, n\}$. It does not hurt to think about $n \geqslant 2$, although our results hold also (trivially) for $n = 1$. The vector $\mathbf{0}$ represents the n-vector of zeros, and we always write $\mathbf{X} = (X_1, \ldots, X_n)$ and $\mathbf{Y} = (Y_1, \ldots, Y_n)$.

2. Preliminaries and Motivation

The main object of the paper, a *diversification index D*, is a mapping from $\mathcal{X}^n$ to $\overline{\mathbb{R}}$, which is used to quantify the magnitude of diversification of a risk vector $\mathbf{X} \in \mathcal{X}^n$ representing portfolio losses. Our convention is that a smaller value of $D(\mathbf{X})$ represents a stronger diversification in a sense specified by the design of D.

As the evaluation of diversification is closely related to that of risk, diversification indices in the literature are often defined through risk measures. An example of a diversification index is the diversification ratio mentioned in Section 1, based on measures of variability such as the standard deviation and variance:

$$\mathrm{DR}^{\mathrm{SD}}(\mathbf{X}) = \frac{\mathrm{SD}(\sum_{i=1}^n X_i)}{\sum_{i=1}^n \mathrm{SD}(X_i)} \quad \text{and}$$

$$\mathrm{DR}^{\mathrm{var}}(\mathbf{X}) = \frac{\mathrm{var}(\sum_{i=1}^n X_i)}{\sum_{i=1}^n \mathrm{var}(X_i)},$$

with the convention $0/0 = 0$. We refer to Rockafellar et al. (2006), Furman et al. (2017), and Bellini et al. (2022) for general measures of variability. DRs based on SD and var satisfy the three simple properties below, which can be easily checked.

[+] (Nonnegativity). $D(\mathbf{X}) \geqslant 0$ *for all* $\mathbf{X} \in \mathcal{X}^n$.

[LI] (Location Invariance). $D(\mathbf{X} + \mathbf{c}) = D(\mathbf{X})$ *for all* $\mathbf{c} = (c_1, \ldots, c_n) \in \mathbb{R}^n$ *and all* $\mathbf{X} \in \mathcal{X}^n$.

[SI] (Scale Invariance). $D(\lambda \mathbf{X}) = D(\mathbf{X})$ *for all* $\lambda > 0$ *and all* $\mathbf{X} \in \mathcal{X}^n$.

The first property, [+], simply means that diversification is measured in nonnegative values, where zero typically represents a fully diversified or hedged portfolio (in some sense). The property [LI] means that injecting constant losses or gains to components of a portfolio, or changing the initial price of assets in the portfolio,[3] does not affect its diversification index. The property [SI] means that rescaling a portfolio does not affect its diversification index. The latter two properties are arguably natural, although they are not satisfied by some diversification indices used in the literature (see (2) below).

A diversification index satisfying both [LI] and [SI] is called location-scale invariant.

Next, we define the two popular risk measures in banking and insurance practice. The VaR at level $\alpha \in [0, 1)$ is defined as

$$\mathrm{VaR}_\alpha(X) = \inf\{x \in \mathbb{R} : \mathbb{P}(X \leqslant x) \geqslant 1 - \alpha\}, \quad X \in L^0,$$

and the ES (also called CVaR, TVaR, or AVaR) at level $\alpha \in (0, 1)$ is defined as

$$\mathrm{ES}_\alpha(X) = \frac{1}{\alpha} \int_0^\alpha \mathrm{VaR}_\beta(X) \mathrm{d}\beta, \quad X \in L^1,$$

and $\mathrm{ES}_0(X) = \text{ess-sup}(X) = \mathrm{VaR}_0(X)$, which may be ∞. The probability level α above is typically very small, for example, 0.01 or 0.025 in Basel Committee on Banking Supervision (2019); note that we use the "small α" convention. Artzner et al. (1999) introduced *coherent* risk measures $\phi : \mathcal{X} \to \mathbb{R}$ as those satisfying the following four properties.

[M] (Monotonicity). $\phi(X) \leqslant \phi(Y)$ *for all* $X, Y \in \mathcal{X}$ *with* $X \leqslant Y$.[4]

[CA] (Constant Additivity). $\phi(X + c) = \phi(X) + c$ *for all* $c \in \mathbb{R}$ *and* $X \in \mathcal{X}$.

[PH] (Positive Homogeneity). $\phi(\lambda X) = \lambda \phi(X)$ *for all* $\lambda \in (0, \infty)$ *and* $X \in \mathcal{X}$.

[SA] (Subadditivity). $\phi(X + Y) \leqslant \phi(X) + \phi(Y)$ *for all* $X, Y \in \mathcal{X}$.

ES satisfies all four properties above, whereas VaR does not satisfy [SA]. We say that a risk measure is *monetary* if it satisfies [CA] and [M], and it is *MCP* if it satisfies [M], [CA], and [PH]. For discussions and interpretations of these properties, we refer to Föllmer and Schied (2016).

Some diversification indices are defined via risk measures such as DR (e.g., Bürgi et al. 2008, Mainik and Rüschendorf 2010, Embrechts et al. 2015) and the diversification benefit (DB) (e.g., Embrechts et al. 2009, McNeil et al. 2015). For a risk measure ϕ, the DR and DB based on ϕ are defined as[5]

$$\mathrm{DR}^\phi(\mathbf{X}) = \frac{\phi(\sum_{i=1}^n X_i)}{\sum_{i=1}^n \phi(X_i)} \quad \text{and}$$

$$\mathrm{DB}^\phi(\mathbf{X}) = \sum_{i=1}^n \phi(X_i) - \phi\left(\sum_{i=1}^n X_i\right). \tag{2}$$

In contrast to the DR, a larger value of DB represents a stronger diversification, but this convention can be easily modified by flipping the sign to consider $-\mathrm{DB}^\phi$. By definition, DR is the ratio of the pooled risk to the sum of the individual risks and thus a measurement of how substantially pooling reduces risk; similarly, DB measures the difference instead of the ratio.

The DR has a number of deficiencies. First, the value of DR^{ϕ} is not necessarily nonnegative, violating [+]. Because the risk measure ϕ may take negative values,[6] it would be difficult to interpret the case where either the numerator or denominator in the DR is negative, and this makes optimization of the DR troublesome. An example is a portfolio of credit default losses, where the VaR of individual losses is often zero or negative, but the VaR of the portfolio loss is positive; see McNeil et al. (2015, example 2.25). Second, for common risk measures, the DR violates [LI], meaning that adding a risk-free asset changes the value of the DR. Third, the DR is not necessarily quasiconvex in portfolio weights; this point is subtler and will be explained later. In addition to the above drawbacks, we also find that the DR has wrong incentives for some simple models; for instance, it suggests that an independent and identically distributed (iid) portfolio of t-distributed risks is less diversified than a portfolio with a common shock and the same marginals; see Section 5.2 for details. Similarly to the DR, the index DB satisfies [LI] for ϕ satisfying [CA], but it does not satisfy [SI] for common risk measures, and it may take both positive and negative values.

In financial applications, the risk measures VaR and ES are specified in regulatory documents such as EIOPA (2011) and Basel Committee on Banking Supervision (2019), and therefore, it is beneficial to stick to VaR or ES as the risk measure when assessing diversification. Both $\mathrm{DR}^{\mathrm{VaR}_\alpha}$ and $\mathrm{DR}^{\mathrm{ES}_\alpha}$ satisfy [SI], but they do not satisfy [+] or [LI].[7] It remains unclear how one can define a diversification index based on VaR or ES satisfying these properties. In the remainder of the paper, we will introduce and study a new index of diversification to bridge this gap.

3. Diversification Indices: An Axiomatic Theory

In this section, we fix $\mathcal{X} = L^\infty$ as the standard choice in the literature of axiomatic theory of risk measures. In addition to [+], [LI], and [SI] introduced in Section 2, we propose four new axioms. The first six axioms together characterize a new class of diversification indices, that is, diversification quotients based on MCP risk measures. With the seventh axiom of portfolio convexity, we further pin down the class of DQ based on coherent risk measures.

3.1. Axioms of Rationality, Normalization, and Continuity

We first present three axioms that depend on a risk measure ϕ. These three axioms are standard and weak in the sense that they do not impose a specific functional structure on D other than some forms of monotonicity, normalization, and continuity.

For a risk measure ϕ, we say that two vectors $\mathbf{X}, \mathbf{Y} \in \mathcal{X}^n$ are *ϕ-marginally equivalent* if $\phi(X_i) = \phi(Y_i)$ for each $i \in [n]$, and we denote this by $\mathbf{X} \overset{\mathrm{m}}{\simeq} \mathbf{Y}$. In other words, if an agent evaluates risks using the risk measure ϕ, then the agent would view the individual components of $\mathbf{X}$ and those of $\mathbf{Y}$ as equally risky. Similarly, denote by $\mathbf{X} \overset{\mathrm{m}}{\succeq} \mathbf{Y}$ if $\phi(X_i) \leqslant \phi(Y_i)$ for each $i \in [n]$, and by $\mathbf{X} \overset{\mathrm{m}}{\succ} \mathbf{Y}$ if $\phi(X_i) < \phi(Y_i)$ for each $i \in [n]$. The other three desirable axioms are presented below, and they are built on a given risk measure ϕ, such as VaR or ES, typically specified exogenously by financial regulation.

$[\mathrm{R}]_\phi$ (Rationality). *$D(\mathbf{X}) \leqslant D(\mathbf{Y})$ for $\mathbf{X}, \mathbf{Y} \in \mathcal{X}^n$ satisfying $\mathbf{X} \overset{\mathrm{m}}{\simeq} \mathbf{Y}$ and $\sum_{i=1}^n X_i \leqslant \sum_{i=1}^n Y_i$.*

To interpret the axiom $[\mathrm{R}]_\phi$, consider two portfolios, $\mathbf{X}$ and $\mathbf{Y}$, satisfying $\mathbf{X} \overset{\mathrm{m}}{\simeq} \mathbf{Y}$. If, further, $\sum_{i=1}^n X_i \leqslant \sum_{i=1}^n Y_i$ holds, then the total loss from portfolio $\mathbf{X}$ is always less or equal to that from portfolio $\mathbf{Y}$, making the portfolio $\mathbf{X}$ safer than $\mathbf{Y}$. Because the individual components in $\mathbf{X}$ and those in $\mathbf{Y}$ are equally risky, the fact that $\mathbf{X}$ is safer in aggregation is a result of the different diversification effects in $\mathbf{X}$ and $\mathbf{Y}$, leading to the inequality $D(\mathbf{X}) \leqslant D(\mathbf{Y})$. This axiom is called rationality because a rational agent always prefers to have smaller losses.

Next, we formulate our idea about normalizing representative values of the diversification index. First, we assign the zero portfolio $\mathbf{0}$ the value $D(\mathbf{0}) = 0$, as it carries no risk in every sense.[8] A natural benchmark of a nondiversified portfolio is one in which all components are the same. Such a portfolio $\mathbf{X}^{\mathrm{du}} = (X, \ldots, X)$ will be called a *duplicate* portfolio, and we may, ideally, wish to assign the value $D(\mathbf{X}^{\mathrm{du}}) = 1$. However, because the zero portfolio $\mathbf{0}$ is also duplicate but $D(\mathbf{0}) = 0$, we will require the weaker assumption $D(\mathbf{X}^{\mathrm{du}}) \leqslant 1$ for duplicate portfolios.[9] Lastly, we should understand for what portfolios $D(\mathbf{X}) \geqslant 1$ needs to occur. We say that a portfolio $\mathbf{X}^{\mathrm{wd}} = (X_1, \ldots, X_n)$ is *worse than duplicate* if there exists a duplicate portfolio $\mathbf{X}^{\mathrm{du}} = (X, \ldots, X)$ such that $\mathbf{X}^{\mathrm{wd}} \overset{\mathrm{m}}{\succ} \mathbf{X}^{\mathrm{du}}$ and $\sum_{i=1}^n X_i \geqslant nX$. Intuitively, this means that each component of $\mathbf{X}^{\mathrm{wd}}$ is strictly less risky than X, but putting them together always incurs a larger loss than nX; in this case, diversification creates nothing but a penalty to the risk manager, and we assign $D(\mathbf{X}^{\mathrm{wd}}) \geqslant 1$.[10] Existence of worse-than-duplicate portfolios is characterized in Section EC.3.1 in the e-companion. Using all of the considerations above, we propose the following normalization axiom.

$[\mathrm{N}]_\phi$ (Normalization). *$D(\mathbf{0}) = 0$, $D(\mathbf{X}) \leqslant 1$ if $\mathbf{X}$ is duplicate, and $D(\mathbf{X}) \geqslant 1$ if $\mathbf{X}$ is worse than duplicate.*

Finally, we propose a continuity axiom, which is mainly for technical convenience.

$[\mathrm{C}]_\phi$ (Continuity). *For $\{\mathbf{Y}^k\}_{k \in \mathbb{N}} \subseteq \mathcal{X}^n$ and $\mathbf{X} \in \mathcal{X}^n$ satisfying $\mathbf{Y}^k \overset{\mathrm{m}}{\simeq} \mathbf{X}$ for each k, if $\left(\sum_{i=1}^n X_i - \sum_{i=1}^n Y_i^k\right)_+ \overset{L^\infty}{\longrightarrow} 0$ as $k \to \infty$, then $(D(\mathbf{X}) - D(\mathbf{Y}^k))_+ \to 0$.*

The axiom $[\mathrm{C}]_\phi$ is a special form of semicontinuity. To interpret it, consider portfolios $\mathbf{X}$ and $\mathbf{Y}$ that are marginally equivalent. If the sum of components of $\mathbf{X}$ is not much worse than that of $\mathbf{Y}$ in L^∞, then the axiom says that the diversification of $\mathbf{X}$ is not much worse than the diversification of $\mathbf{Y}$. This property allows for a special form of stability or robustness[11] with respect to statistical errors when estimating the distributions of portfolio losses.

One can check that the axioms [R]$_\phi$, [N]$_\phi$, and [C]$_\phi$ are satisfied by $\mathrm{DR}^{\mathrm{VaR}_\alpha}$ and $\mathrm{DR}^{\mathrm{ES}_\alpha}$ if we only consider positive portfolio vectors. The axioms are not satisfied by $\mathrm{DR}^{\mathrm{SD}}$ because SD is not monotone, and hence, the inequalities $\sum_{i=1}^n X_i \leqslant \sum_{i=1}^n Y_i$ and $\sum_{i=1}^n X_i \geqslant nX$ used in [R]$_\phi$ and [N]$_\phi$ are not relevant for SD.

3.2. Portfolio Convexity

The next axiom, different from the three above, imposes a natural form of convexity on the diversification index. Portfolio diversification is intrinsically connected to convexity of ordering relations. Quoting Mas-Colell et al. (1995, p. 44), "Convexity can also be viewed as the formal expression of a basic inclination of economic agents for diversification." For this purpose, we propose an axiom of portfolio convexity in this section.

Let a random vector $\mathbf{X} \in \mathcal{X}^n$ represent losses from n assets and a vector $\mathbf{w} = (w_1, \dots, w_n) \in \Delta_n$ of portfolio weights, where Δ_n is the standard n-simplex, given by

$$\Delta_n = \{\mathbf{x} \in [0,1]^n : x_1 + \cdots + x_n = 1\}.$$

The total loss of the portfolio is $\mathbf{w}^\top \mathbf{X}$. We write $\mathbf{w} \odot \mathbf{X} = (w_1 X_1, \dots, w_n X_n)$, which is the portfolio loss vector with the weight $\mathbf{w}$. The *portfolio convexity* axiom is formulated below.

[PC] (Portfolio Convexity). *The set* $\{\mathbf{w} \in \Delta_n : D(\mathbf{w} \odot \mathbf{X}) \leqslant d\}$ *is convex for each* $\mathbf{X} \in \mathcal{X}^n$ *and* $d \in \overline{\mathbb{R}}$.

Intuitively, portfolio convexity means that for a given vector $\mathbf{X}$ of assets, combining a portfolio strategy with a better-diversified one on the same set of assets does not result in a portfolio that is less diversified than the original portfolio. As convexity is the decision-theoretic counterpart of diversification, [PC] is desirable for diversification indices.

Remark 1. Axiom [PC] is equivalent to *quasiconvexity* of $\mathbf{w} \mapsto D(\mathbf{w} \odot \mathbf{X})$ for each $\mathbf{X} \in \mathcal{X}^n$; that is, $D((\lambda \mathbf{w} + (1-\lambda)\mathbf{w}') \odot \mathbf{X}) \leqslant D(\mathbf{w} \odot \mathbf{X}) \vee D(\mathbf{w}' \odot \mathbf{X})$ for all $\lambda \in [0,1]$, $\mathbf{w}, \mathbf{w}' \in \Delta_n$ and $\mathbf{X} \in \mathcal{X}^n$.

Remark 2. Convexity or quasiconvexity of $\mathbf{X} \mapsto D(\mathbf{X})$ is not natural or desirable. For instance, combining two diversified portfolios (X, Y) and (Y, X) may result in a duplicate portfolio; see Example EC.1 in Section EC.3.2 of the e-companion. Convexity of $\mathbf{w} \mapsto D(\mathbf{w} \odot \mathbf{X})$, which is stronger than [PC], is not desirable either; see Example EC.2 in Section EC.3.2.

The four axioms introduced above, together with the three in Section 2, lead to a class of diversification indices, which we define next.

3.3. Characterization Results

We first formally introduce the diversification index DQ relying on a parametric class of risk measures, which will be characterized in two results below.

Definition 1. Let $\rho = (\rho_\alpha)_{\alpha \in I}$ be a class of risk measures indexed by $\alpha \in I = (0, \overline{\alpha})$ with $\overline{\alpha} \in (0, \infty]$ such that ρ_α is decreasing in α. For $\mathbf{X} \in \mathcal{X}^n$, the *diversification quotient* based on the class ρ at level $\alpha \in I$ is defined by

$$\mathrm{DQ}_\alpha^\rho(\mathbf{X}) = \frac{\alpha^*}{\alpha}, \quad \text{where } \alpha^* = \inf\left\{\beta \in I : \rho_\beta\left(\sum_{i=1}^n X_i\right) \leqslant \sum_{i=1}^n \rho_\alpha(X_i)\right\}, \tag{3}$$

with the convention $\inf(\varnothing) = \overline{\alpha}$.

We first characterize DQ based on MCP risk measures by six axioms without [PC].

Theorem 1. *A diversification index* $D : \mathcal{X}^n \to \overline{\mathbb{R}}$ *satisfies* [+], [LI], [SI], [R]$_\phi$, [N]$_\phi$, *and* [C]$_\phi$ *for some MCP risk measure* ϕ *if and only if* D *is* DQ_α^ρ *for some* α *and decreasing class* ρ *of MCP risk measures. Moreover, in both directions of the above equivalence, it can be required that* $\rho_\alpha = \phi$.

Theorem 1 gives the first axiomatic characterization of diversification indices to the best of our knowledge. The proof techniques to show the important "only if" statement of Theorem 1 is based on a sophisticated analysis of an auxiliary mapping:

$$R : \mathcal{X} \to [0, \infty], \; R(X) = \inf\left\{D(\mathbf{X}) : X \leqslant \sum_{i=1}^n X_i, \; \mathbf{X} \overset{\mathrm{m}}{\simeq} \mathbf{0}\right\},$$

and this is explained in Section EC.1 in the e-companion.

Next, we incorporate portfolio convexity into our axiomatic framework. For this purpose, it is natural to build the diversification indices based on risk measures with convexity. When formulated on monetary risk measures, convexity represents the idea that diversification reduces the risk; see Föllmer and Schied (2016). For risk measures that are not constant additive, Cerreia-Vioglio et al. (2011) argued that quasiconvexity is more suitable than convexity to reflect the consideration of diversification; moreover, convexity and quasiconvexity are equivalent if [CA] holds. A risk measure is *linear* if it satisfies $\phi(aX + bY) = a\phi(X) + b\phi(Y)$ for all $X, Y \in \mathcal{X}$ and $a, b \in \mathbb{R}$. Because linear risk measures correspond to expectations (under monotonicity) that do not reflect diversification, we will focus on nonlinear ones. The next theorem characterizes DQ based on coherent risk measures.

Theorem 2. *Suppose* $n \geqslant 4$, *and* ϕ *is a nonlinear coherent risk measure. A diversification index* $D : \mathcal{X}^n \to \overline{\mathbb{R}}$ *satisfies* [+], [LI], [SI], [R]$_\phi$, [N]$_\phi$, [C]$_\phi$, *and* [PC] *if and only if* D

is $\mathrm{DQ}^{\rho}_{\alpha}$ *for some* α *and decreasing class* ρ *of coherent risk measures with* $\rho_\alpha = \phi$.

The conditions $n \geqslant 4$ and nonlinearity of ϕ are essential to the proof of Theorem 2. They are harmless for financial applications because typical portfolios have more than a few components, and common risk measures are not linear.

Although portfolio convexity is crucial for diversification indices, making Theorem 2 a central result, we present Theorem 1 separately for the following reasons. First, Theorem 1 reveals the fundamental properties needed to pin down the form of DQ, and this helps to clarify the role of [PC]. Second, the proof of Theorem 2 is technically built on Theorem 1. Third, the class of DQ characterized by Theorem 1 allows for DQ based on VaR, which is popular in financial regulation.

In the next proposition, we show that for sublinear risk measures, DQ satisfies [PC] (thus, the "if" direction of Theorem 2 does not need [M] and [CA] for ρ), and its range is $[0,1]$ under mild conditions, avoiding nondegeneracy. A risk measure is sublinear if it satisfies subadditivity and positive homogeneity (equivalently, convexity and positive homogeneity).

Proposition 1. *Let* $\rho = (\rho_\beta)_{\beta \in I}$ *be a decreasing class of sublinear risk measures and* $\alpha \in I$. *Then,* $\mathrm{DQ}^{\rho}_{\alpha}$ *satisfies* [PC]. *If* $n \geqslant 3$, ρ_α *is nonlinear, and there exists* $X \in \mathcal{X}$ *such that* $\beta \longmapsto \rho_\beta(X)$ *is strictly decreasing, then* $\{\mathrm{DQ}^{\rho}_{\alpha}(\mathbf{X}) : \mathbf{X} \in \mathcal{X}^n\} = [0,1]$.

Given a sublinear risk measure ρ_α, the conditions in Proposition 1 for $\{\mathrm{DQ}^{\rho}_{\alpha}(\mathbf{X}) : \mathbf{X} \in \mathcal{X}^n\} = [0,1]$ are mild and satisfied by, for example, DQs based on the family of ES. In contrast to DQs, DRs based on sublinear risk measures may not satisfy [PC] because the denominator in (2) may be negative. For a clear comparison, we summarize in Table 1 the axioms satisfied by the diversification indices that appear in the paper.

3.4. Interpretation of DQs

DQs based on MCP or coherent risk measures have been characterized axiomatically, but we have not interpreted the meaning of DQ in (3). For an interpretation, consider a decreasing class of risk measures $(\rho_\beta)_{\beta \in I}$. The values of risk measures typically represent the capital requirement of a risky asset, and hence, β is interpreted as a parameter of risk level (as in VaR_β or ES_β); that is, a smaller β means a larger capital requirement for the same risk. Notice from (3) that under mild conditions, α^* is uniquely determined by

$$\rho_{\alpha^*}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \rho_\alpha(X_i).$$

Therefore, α^* is the parameter of risk level achieved by pooling, assuming that the portfolio maintains the same total capital requirement assessed by ρ_α when there is no pooling; that is, $\sum_{i=1}^{n} \rho_\alpha(X_i)$. As $\mathrm{DQ}^{\rho}_{\alpha}(\mathbf{X}) = \alpha^*/\alpha$, the DQ is the ratio of the risk-level parameters before and after pooling. To summarize, *the index DQ quantifies the improvement of the risk-level parameter caused by pooling assets.*

In the most relevant case $\rho_\alpha(\sum_{i=1}^{n} X_i) < \sum_{i=1}^{n} \rho_\alpha(X_i)$, we present in Figure 1 the conceptual symmetry between DQ, which measures the improvement by pooling in the horizontal direction, and DR, which measures an improvement in the vertical direction. In particular, in the case of VaR, DQ measures the probability improvement, whereas DR measures the quantile improvement; see Theorem 4 and (7) below.

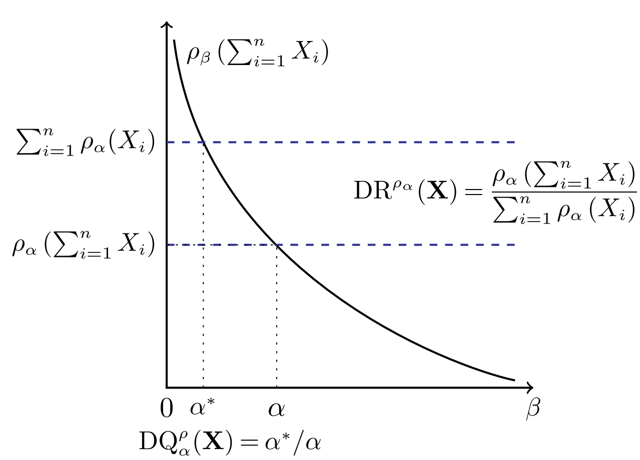
Remark 3. The idea of improvement of risk level is closely related to acceptability indices proposed by Cherny and Madan (2009). More precisely, an acceptability index for a loss $X \in \mathcal{X}$ is defined by $\mathrm{AI}^{\rho}(X) = \sup\{\gamma \in \mathbb{R}_+ : \rho_{1/\gamma}(X) \leqslant 0\}$ for a decreasing class of coherent risk measures $(\rho_\gamma)_{\gamma \in \mathbb{R}_+}$, which has visible similarity to α^* in (3); see Kováčová et al. (2022) for optimization of acceptability indices. If ρ is a class of risk measures

Table 1. Summary of Axioms Satisfied by Diversification Indices DR^{ϕ}, DB^{ϕ}, and $\mathrm{DQ}^{\rho}_{\alpha}$ (with $\phi = \rho_\alpha$)

Index	Domain	[+]	[LI]	[SI]	$[\mathrm{R}]_\phi$	$[\mathrm{N}]_\phi$	$[\mathrm{C}]_\phi$	[PC]
$\mathrm{DR}^{\mathrm{VaR}_\alpha}$ and $\mathrm{DR}^{\mathrm{ES}_\alpha}$	$\mathcal{X}^n$	×	×	√	×	×	×	×
$\mathrm{DR}^{\mathrm{VaR}_\alpha}$	$\mathcal{X}^n_+$	√	×	√	√	√	√	×
$\mathrm{DR}^{\mathrm{ES}_\alpha}$	$\mathcal{X}^n_+$	√	×	√	√	√	√	√
$\mathrm{DR}^{\mathrm{SD}}$	$\mathcal{X}^n$	√	√	√	×	×	×	√
$\mathrm{DR}^{\mathrm{var}}$	$\mathcal{X}^n$	√	√	√	×	×	×	×
$-\mathrm{DB}^{\mathrm{VaR}_\alpha}$	$\mathcal{X}^n$	×	√	×	√	×	√	×
$-\mathrm{DB}^{\mathrm{ES}_\alpha}$	$\mathcal{X}^n$	×	√	×	√	×	√	√
$\mathrm{DQ}^{\mathrm{VaR}}_{\alpha}$	$\mathcal{X}^n$	√	√	√	√	√	√	×
$\mathrm{DQ}^{\mathrm{ES}}_{\alpha}$	$\mathcal{X}^n$	√	√	√	√	√	√	√

Note. $\mathcal{X}_+$ is the set of nonnegative elements in $\mathcal{X}$ and $\alpha \in (0,1)$.

Figure 1. (Color online) Conceptual Symmetry Between DQs and DRs



satisfying [CA], then

$$\mathrm{DQ}^{\rho}_{\alpha}(\mathbf{X}) = \frac{1}{\alpha}\left(\mathrm{AI}^{\rho}\left(\sum_{i=1}^{n}(X_i - \rho_{\alpha}(X_i))\right)\right)^{-1}.$$

Dhaene et al. (2012) studied several methods for capital allocation, among which the quantile allocation principle computes a capital allocation $(C_1, \dots, C_n)$ such that $\sum_{i=1}^{n} C_i = \mathrm{VaR}_{\alpha}(\sum_{i=1}^{n} X_i)$ and $C_i = \mathrm{VaR}_{c\alpha}(X_i)$ for some $c > 0$. The constant c appearing as a nuisance parameter in the above rule has a visible mathematical similarity to $\mathrm{DQ}^{\mathrm{VaR}}_{\alpha}$. Mafusalov and Uryasev (2018) studied the so-called buffered probability of exceedance, which is the inverse of the ES curve $\beta \longmapsto \mathrm{ES}_{\beta}(X)$ at a specific point $x \in \mathbb{R}$; note that α^* in (3) is obtained by inverting the ES curve $\beta \longmapsto \mathrm{ES}_{\beta}(\sum_{i=1}^{n} X_i)$ at $\sum_{i=1}^{n} \mathrm{ES}_{\alpha}(X_i)$.

We close the section with discussions on the construction of DQs. First, DQs can be constructed from any monotonic parametric family of risk measures. All commonly used risk measures belong to a monotonic family, as this includes VaR, ES, expectiles (e.g., Bellini et al. 2014), mean variance (e.g., Markowitz 1952, Maccheroni et al. 2009), and entropic risk measures (e.g., Föllmer and Schied 2016); some choices do not guarantee all axioms to hold. Our results imply that using ES or expectiles guarantees all axioms and nondegeneracy for DQ. In addition, there are ways to construct DQ from a single risk measure; see Section EC.3.3 in the e-companion. DQs can also be axiomatized using preferences instead of risk measures; see Section EC.3.4.

DQs can be used as a normative tool for measuring diversification. In this context, the choice of the parametric family of risk measures is up to the user, and DQs serve as a versatile tool that accommodates various risk attitudes. The choice of risk measures (e.g., VaR, ES) and the determination of the confidence level (α) should be aligned with the risk tolerance, objectives, and regulatory requirements of the decision maker. For instance, conservative investors, prioritizing capital preservation, may gravitate toward the family of ES at a high-level α, which reflects an assessment of downside risk, whereas those with aggressive risk preferences may opt for VaR or ES at a lower-level α. Most generally, we would recommend the use of DQ based on ES, which has a natural and strong connection to financial regulation and tail risk management, and the parameter α allows for flexibility in the assessment of tail risk.

4. Properties of DQs

In this section, we study the properties of DQs defined in Definition 1. For the greatest generality, we do not impose any properties of risk measures in the decreasing family $\rho = (\rho_{\alpha})_{\alpha \in I}$; that is, the family ρ is not limited to MCP or coherent risk measures, so our results can be applied to more flexible contexts in which some of the seven axioms are relaxed. In this section, $\mathcal{X}$ is not restricted to L^{∞}.

4.1. Basic Properties

We first make a few immediate observations by the definition of DQs. From (3), we can see that computing $\mathrm{DQ}^{\rho}_{\alpha}$ is to invert the decreasing function $\beta \longmapsto \rho_{\beta}(\sum_{i=1}^{n} X_i)$ at $\sum_{i=1}^{n} \rho_{\alpha}(X_i)$. For the cases of VaR and ES, $I = (0,1)$, $\alpha^* \in [0,1]$, and DQs have simple formulas; see Theorem 4 in Section 5. For a fixed value of $\sum_{i=1}^{n} \rho_{\alpha}(X_i)$, the DQ is larger if the curve $\beta \longmapsto \rho_{\beta}(\sum_{i=1}^{n} X_i)$ is larger, and the DQ is smaller if the curve $\beta \longmapsto \rho_{\beta}(\sum_{i=1}^{n} X_i)$ is smaller. This is consistent with our intuition that a diversification index is large if there is little or no diversification and thus a large value of the portfolio risk, and a diversification index is small if there is strong diversification.

In Theorem 1, we have seen that the DQ satisfies [SI] and [LI] if ρ is a class of MCP risk measures. These properties of DQs can be obtained based on a more general version of properties [CA] and [PH] of risk measures, allowing us to include SD and the variance. The results are summarized in Proposition 2, which are straightforward to check by definition.

[CA]$_m$ (Constant Additivity with $m \in \mathbb{R}$). $\phi(X + c) = \phi(X) + mc$ *for all* $c \in \mathbb{R}$ *and* $X \in \mathcal{X}$.

[PH]$_\gamma$ (Positive Homogeneity with $\gamma \in \mathbb{R}$). $\phi(\lambda X) = \lambda^{\gamma}\phi(X)$ *for all* $\lambda \in (0,\infty)$ *and* $X \in \mathcal{X}$.

Proposition 2. *Let* $\rho = (\rho_{\alpha})_{\alpha \in I}$ *be a class of risk measures decreasing in* α. *For each* $\alpha \in I$,

(i) *if* ρ_{β} *satisfies* [PH]$_\gamma$ *with the same* γ *across* $\beta \in I$, *then* $\mathrm{DQ}^{\rho}_{\alpha}$ *satisfies* [SI].

(ii) *if* ρ_{β} *satisfies* [CA]$_m$ *with the same* m *across* $\beta \in I$, *then* $\mathrm{DQ}^{\rho}_{\alpha}$ *satisfies* [LI].

(iii) *if* ρ_{α} *satisfies* [SA], *then* $\mathrm{DQ}^{\rho}_{\alpha}$ *takes value in* $[0,1]$.

It is clear that [CA] is [CA]$_m$ with $m = 1$, and [PH] is [PH]$_\gamma$ with $\gamma = 1$. More properties of DQs on the important families of VaR and ES will be discussed in Section 5. In particular, we will see that the ranges of $\mathrm{DQ}^{\mathrm{VaR}}_{\alpha}$ and $\mathrm{DQ}^{\mathrm{ES}}_{\alpha}$ are $[0,n]$ and $[0,1]$, respectively.

Example 1 (Liquidity and Temporal Consistency). In risk management practice, liquidity and time horizons of potential losses need to be taken into account; see Basel Committee on Banking Supervision (2019, p. 89). If liquidity risk is of concern, one may use a risk measure with [PH]$_\gamma$ with $\gamma > 1$ to penalize large exposures of losses. For such risk measures, $\mathrm{DQ}^{\rho}_{\alpha}$ remains scale invariant, as shown by Proposition 2. On the other hand, if the risk associated with the loss $\mathbf{X}(t)$ at different time spots $t > 0$ is scalable by a function $f > 0$ (usually of the order $f(t) = \sqrt{t}$ in standard models such as the Black-Scholes), then DQ is consistent across different horizons in the sense that $\mathrm{DQ}^{\rho}_{\alpha}(\mathbf{X}(t)) = \mathrm{DQ}^{\rho}_{\alpha}(\mathbf{X}(s))$ for two time spots $s, t > 0$ given that $\rho_\beta(X_i(t)) = f(t)\rho_\beta(X_i(1))$ for $i \in [n]$, $t > 0$, and $\beta \in I$.

Next, we explain that the values taken by DQ are consistent with our usual perceptions of portfolio diversification. For a given risk measure ϕ and a portfolio risk vector $\mathbf{X}$, we consider the following three situations that yield intuitive values of DQs.

(i) There is no insolvency risk with pooled individual capital; that is, $\sum_{i=1}^n X_i \leqslant \sum_{i=1}^n \phi(X_i)$ almost surely;

(ii) diversification benefit exists; that is, $\phi(\sum_{i=1}^n X_i) < \sum_{i=1}^n \phi(X_i)$;

(iii) the portfolio relies on a single asset; that is, $\mathbf{X} = (\lambda_1 X, \ldots, \lambda_n X)$ for some $X \in \mathcal{X}$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{R}_+$. A duplicate portfolio relies on a single asset.

The above three situations receive special attention because they intuitively correspond to very strong diversification, some diversification, and no diversification, respectively. Naturally, we would expect the DQ to be very small for (i), smaller than one for (ii), and one for (iii). It turns out that the above intuitions all check out under very weak conditions that are satisfied by commonly used classes of risk measures.

Before presenting this result, we fix some technical terms. For a class ρ of risk measures ρ_α decreasing in α, we say that ρ is *nonflat from the left* at (α, X) if $\rho_\beta(X) > \rho_\alpha(X)$ for all $\beta \in (0, \alpha)$, and ρ is *left continuous* at (α, X) if $\alpha \longmapsto \rho_\alpha(X)$ is left continuous. A random vector $(X_1, \ldots, X_n)$ is *comonotonic* if there exists a random variable Z and increasing functions $f_1, \ldots, f_n$ on $\mathbb{R}$ such that $X_i = f_i(Z)$ a.s. for every $i \in [n]$. A risk measure is *comonotonic additive* if $\phi(X + Y) = \phi(X) + \phi(Y)$ for comonotonic (X, Y). Each of ES and VaR satisfies comonotonic additivity, as well as any distortion risk measures (Yaari 1987, Kusuoka 2001) and signed Choquet integrals (Wang et al. 2020). We denote $\rho_0 = \lim_{\alpha\downarrow 0}\rho_\alpha$. Note that $\rho_0 =$ ess-sup for common classes ρ such as VaR, ES, expectiles, and entropic risk measures.

Theorem 3. *For given $\mathbf{X} \in \mathcal{X}^n$ and $\alpha \in I$, if ρ is left continuous and nonflat from the left at $(\alpha, \sum_{i=1}^n X_i)$, the following hold.*

(i) *Suppose $\rho_0 \leqslant$ ess-sup. If for ρ_α, there is no insolvency risk with pooled individual capital, then $\mathrm{DQ}_\alpha^\rho(\mathbf{X}) = 0$. The converse holds true if $\rho_0 =$ ess-sup.*

(ii) *Diversification benefit exists if and only if $\mathrm{DQ}_\alpha^\rho(\mathbf{X}) < 1$.*

(iii) *If ρ_α satisfies [PH] and $\mathbf{X}$ relies on a single asset, then $\mathrm{DQ}_\alpha^\rho(\mathbf{X}) = 1$.*

(iv) *If ρ_α is comonotonic additive and $\mathbf{X}$ is comonotonic, then $\mathrm{DQ}_\alpha^\rho(\mathbf{X}) = 1$.*

In (i), we see that if there is no insolvency risk with pooled individual capital, then $\mathrm{DQ}_\alpha^\rho(\mathbf{X}) = 0$. In typical models, such as some elliptical models in Section 5.2, $\sum_{i=1}^n X_i$ is unbounded from above unless it is a constant. Hence, for such models and ρ satisfying $\rho_0 =$ ess-sup, $\mathrm{DQ}_\alpha^\rho(\mathbf{X}) = 0$ if and only if $\sum_{i=1}^n X_i$ is a constant; thus, full hedging is achieved. This is also consistent with our intuition of full hedging as the strongest form of diversification. The existence of diversification benefit is the main idea behind the coherent risk measures of Artzner et al. (1999). By (ii), the DQ and DR agree on whether diversification benefit exists under mild conditions, and this intuition is consistent with Artzner et al. (1999).

Remark 4. We require ρ to be left continuous and nonflat from the left to make the inequality in (ii) hold strictly. This requirement excludes, in particular, trivial cases such as $\mathbf{X} = \mathbf{c} \in \mathbb{R}^n$, which gives $\mathrm{DQ}_\alpha^{\mathrm{VaR}}(\mathbf{X}) = 0$ by definition. In case the conditions fail to hold, $\mathrm{DQ}_\alpha^\rho(\mathbf{X}) < 1$ may not guarantee $\rho_\alpha(\sum_{i=1}^n X_i) < \sum_{i=1}^n \rho_\alpha(X_i)$, but it implies the nonstrict inequality $\rho_\alpha(\sum_{i=1}^n X_i) \leqslant \sum_{i=1}^n \rho_\alpha(X_i)$, and thus, the portfolio risk is not worse than the sum of the individual risks.

4.2. Stochastic Dominance and Dependence

In this section, we discuss the consistency of DQ with respect to stochastic dominance, as well as the best and worst cases for DQ among all dependence structures with given marginal distributions of the risk vector.

For a diversification index, monotonicity with respect to stochastic dominance yields consistency with common decision-making criteria such as the expected utility model and the rank-dependent utility model. A random variable X (representing random loss) is dominated by a random variable Y in second-order stochastic dominance (SSD) if $\mathbb{E}[f(X)] \leqslant \mathbb{E}[f(Y)]$ for all decreasing concave functions $f: \mathbb{R} \to \mathbb{R}$ provided that the expectations exist, and we denote this by $X \leqslant_{\mathrm{SSD}} Y$.[12] A risk measure ϕ is *SSD consistent* if $\phi(X) \geqslant \phi(Y)$ for all $X, Y \in \mathcal{X}$ whenever $X \leqslant_{\mathrm{SSD}} Y$. SSD consistency is known as strong risk aversion in the classic decision theory literature (Rothschild and Stiglitz 1970). SSD-consistent monetary risk measures, which include all law-invariant convex risk measures such as ES, admit an ES-based characterization (Mao and Wang 2020).

Proposition 3. *Assume that $\rho = (\rho_\alpha)_{\alpha\in I}$ is a decreasing class of SSD-consistent risk measures. For $\mathbf{X}, \mathbf{Y} \in \mathcal{X}^n$ and $\alpha \in I$, if $\sum_{i=1}^n \rho_\alpha(X_i) \leqslant \sum_{i=1}^n \rho_\alpha(Y_i)$ and $\sum_{i=1}^n X_i \leqslant_{\mathrm{SSD}} \sum_{i=1}^n Y_i$, then $\mathrm{DQ}_\alpha^\rho(\mathbf{X}) \geqslant \mathrm{DQ}_\alpha^\rho(\mathbf{Y})$.*

Proposition 3 follows from the simple observation that

$$\left\{\beta \in I : \rho_\beta\left(\sum_{i=1}^n X_i\right) \leqslant \sum_{i=1}^n \rho_\alpha(X_i)\right\} \subseteq \left\{\beta \in I : \rho_\beta\left(\sum_{i=1}^n Y_i\right) \leqslant \sum_{i=1}^n \rho_\alpha(Y_i)\right\},$$

and we omit the proof.

Assume ρ is a class of SSD-consistent risk measures (e.g., law-invariant convex risk measures). Proposition 3 implies that if the sum of marginal risks is the same for $\mathbf{X}$ and $\mathbf{Y}$ (this holds in particular if $\mathbf{X}$ and $\mathbf{Y}$ have the same marginal distributions), then DQ is decreasing in SSD of the total risk. The dependence structures that maximize or minimize DQ for $\mathbf{X}$ with specified marginal distributions are discussed in Section EC.4.1 in the e-companion. For instance, a comonotonic portfolio has the largest DQ (thus, the smallest diversification) among all portfolios with the same marginal distributions; this observation is related to Proposition 2(iii) and Theorem 3(iv).

4.3. Consistency Across Dimensions

All properties in the previous sections are discussed under the assumption that the dimension $n \in \mathbb{N}$ is fixed. Letting n vary allows for a comparison of diversification between portfolios with different dimensions. In this section, we slightly generalize our framework by considering a diversification index D as a mapping on $\cup_{n\in\mathbb{N}}\mathcal{X}^n$; note that the input vector $\mathbf{X}$ of the DQ and DR can naturally have any dimension n. We present two more useful properties of DQs in this setting. For $\mathbf{X} \in \mathcal{X}^n$ and $c \in \mathbb{R}$, $(\mathbf{X}, c)$ is the $(n+1)$-dimensional random vector obtained by pasting $\mathbf{X}$ and c, and $(\mathbf{X}, \mathbf{X})$ is the $(2n)$-dimensional random vector obtained by pasting $\mathbf{X}$ and $\mathbf{X}$.

[RI] (Riskless Invariance). *$D(\mathbf{X}, c) = D(\mathbf{X})$ for all $n \in \mathbb{N}$, $\mathbf{X} \in \mathcal{X}^n$ and $c \in \mathbb{R}$.*

[RC] (Replication Consistency). *$D(\mathbf{X}, \mathbf{X}) = D(\mathbf{X})$ for all $n \in \mathbb{N}$ and $\mathbf{X} \in \mathcal{X}^n$.*

Riskless invariance means that adding a risk-free asset to the portfolio $\mathbf{X}$ does not affect its diversification. For instance, the Sharpe ratio (SR) of the portfolio does not change under such an operation. Replication consistency means that replicating the same portfolio composition does not affect D. Both properties are arguably desirable in most applications because of their natural interpretations.

Proposition 4. *Let $\rho = (\rho_\alpha)_{\alpha\in I}$ be a class of risk measures decreasing in α. For $\alpha \in I$,*

(i) *if ρ_β satisfies [CA]$_m$ with $m \in \mathbb{R}$ for $\beta \in I$ and $\rho_\alpha(0) = 0$, then DQ^ρ_α satisfies [RI].*

(ii) *If ρ_β satisfies [PH] for $\beta \in I$, then DQ^ρ_α satisfies [RC].*

We further show in Proposition EC.7 in the e-companion that if [RI] is assumed, then the only option for the DR is to use a nonnegative ϕ (which is a subclass of DQs). Thus, if [RI] is considered as desirable, then the DQ becomes useful compared with the DR, as it offers more choices, and in particular, it works for any classes ρ of monetary risk measures with $\rho_\alpha(0) = 0$, including VaR and ES. Both the DQ and DR satisfy [RC] and [RI] for MCP risk measures.

Example 2. Let ϕ be a risk measure satisfying [CA], such as ES_α or VaR_α. Suppose that $\phi(\sum_{i=1}^n X_i) = 1$ and $\sum_{i=1}^n \phi(X_i) = 2$, and thus, $\mathrm{DR}^\phi(\mathbf{X}) = 1/2$. If a nonrandom payoff of $c > 0$ is added to the portfolio, then $\mathrm{DR}^\phi(\mathbf{X}, -c) = (1-c)/(2-c)$, which turns to 0 as $c \uparrow 1$, and it becomes negative as soon as $c > 1$. Hence, DR^ϕ is improved or made negative by including constant payoffs (either as a new asset or added to an existing asset). This creates problematic incentives in optimization. On the other hand, the DQ does not suffer from this problem because of [LI] and [RI].

5. DQs Based on the Classes of VaR and ES

Because VaR and ES are the two most common classes of risk measures in practice, we focus on the theoretical properties of $\mathrm{DQ}^{\mathrm{VaR}}_\alpha$ and $\mathrm{DQ}^{\mathrm{ES}}_\alpha$ in this section. We fix the parameter range $I = (0,1)$, and we choose $\mathcal{X}^n$ to be $(L^0)^n$ when we discuss $\mathrm{DQ}^{\mathrm{VaR}}_\alpha$ and $(L^1)^n$ when we discuss $\mathrm{DQ}^{\mathrm{ES}}_\alpha$, but all results hold true if we fix $\mathcal{X} = L^1$.

5.1. General Properties

We first provide alternative formulations of $\mathrm{DQ}^{\mathrm{VaR}}_\alpha$ and $\mathrm{DQ}^{\mathrm{ES}}_\alpha$. The formulations offer clear interpretations and simple ways to compute the values of DQs. Equation (6) below can be derived from the optimization formulation for the buffered probability of exceedance in proposition 2.2 of Mafusalov and Uryasev (2018).

Theorem 4. *For a given $\alpha \in (0,1)$, $\mathrm{DQ}^{\mathrm{VaR}}_\alpha$ and $\mathrm{DQ}^{\mathrm{ES}}_\alpha$ have the alternative formulas*

$$\mathrm{DQ}^{\mathrm{VaR}}_\alpha(\mathbf{X}) = \frac{1}{\alpha}\mathbb{P}\left(\sum_{i=1}^n X_i > \sum_{i=1}^n \mathrm{VaR}_\alpha(X_i)\right), \quad \mathbf{X} \in \mathcal{X}^n, \tag{4}$$

and

$$\mathrm{DQ}^{\mathrm{ES}}_\alpha(\mathbf{X}) = \frac{1}{\alpha}\mathbb{P}\left(Y > \sum_{i=1}^n \mathrm{ES}_\alpha(X_i)\right), \quad \mathbf{X} \in \mathcal{X}^n, \tag{5}$$

where $Y = \mathrm{ES}_U(\sum_{i=1}^n X_i)$, and $U \sim \mathrm{U}[0,1]$. Furthermore, if $\mathbb{P}(\sum_{i=1}^n X_i > \sum_{i=1}^n \mathrm{ES}_\alpha(X_i)) > 0$, then

$$\mathrm{DQ}^{\mathrm{ES}}_\alpha(\mathbf{X}) = \frac{1}{\alpha}\min_{r\in(0,\infty)} \mathbb{E}\left[\left(r\sum_{i=1}^n (X_i - \mathrm{ES}_\alpha(X_i)) + 1\right)_+\right], \tag{6}$$

and otherwise, $\mathrm{DQ}^{\mathrm{ES}}_\alpha(\mathbf{X}) = 0$.

As a first observation from Theorem 4, it is straightforward to compute $\mathrm{DQ}^{\mathrm{VaR}}_\alpha$ and $\mathrm{DQ}^{\mathrm{ES}}_\alpha$ on real or simulated data by applying (4) and (5) to the empirical distribution of the data.

Theorem 4 also gives $\mathrm{DQ}^{\mathrm{VaR}}_\alpha$ a clear economic interpretation as the improvement of insolvency probability when risks are pooled, making the discussion in Section 3.4

more concrete. Suppose that $X_1, \ldots, X_n$ are continuously distributed and they represent losses from n assets. The total pooled capital is $s_\alpha = \sum_{i=1}^n \mathrm{VaR}_\alpha(X_i)$, which is determined by the marginals of $\mathbf{X}$ but not the dependence structure. An agent investing only in asset X_i with capital computed by VaR_α has an insolvency probability $\alpha = \mathbb{P}(X_i > \mathrm{VaR}_\alpha(X_i))$. On the other hand, by Theorem 4, α^* is the probability that the pooled loss $\sum_{i=1}^n X_i$ exceeds the pooled capital s_α. The improvement from α to α^*, computed by α^*/α, is precisely $\mathrm{DQ}_\alpha^{\mathrm{VaR}}(\mathbf{X})$. From here, it is also clear that $\mathrm{DQ}_\alpha^{\mathrm{VaR}}(\mathbf{X}) < 1$ is equivalent to $\mathbb{P}(\sum_{i=1}^n X_i > s_\alpha) < \alpha$.

To compare $\mathrm{DQ}_\alpha^{\mathrm{VaR}}$ with $\mathrm{DR}^{\mathrm{VaR}_\alpha}$, recall that the two diversification indices can be rewritten as

$$\mathrm{DQ}_\alpha^{\mathrm{VaR}}(\mathbf{X}) = \frac{\mathbb{P}(\sum_{i=1}^n X_i > s_\alpha)}{\alpha} \quad \text{and}$$

$$\mathrm{DR}^{\mathrm{VaR}_\alpha}(\mathbf{X}) = \frac{\mathrm{VaR}_\alpha(\sum_{i=1}^n X_i)}{s_\alpha}. \tag{7}$$

From (7), we can see a clear symmetry between DQ, which measures the probability improvement, and DR, which measures the quantile improvement. DQs and DRs based on ES have a similar comparison.

The range of DQs based on VaR is different from that based on ES, which is $[0,1]$ by Proposition 1. We summarize them below.

Proposition 5. *For $\alpha \in (0,1)$ and $n \geqslant 2$, $\{\mathrm{DQ}_\alpha^{\mathrm{VaR}}(\mathbf{X}) : \mathbf{X} \in \mathcal{X}^n\} = [0, \min\{n, 1/\alpha\}]$ and $\{\mathrm{DQ}_\alpha^{\mathrm{ES}}(\mathbf{X}) : \mathbf{X} \in \mathcal{X}^n\} = [0,1]$.*

Both $\mathrm{DQ}_\alpha^{\mathrm{VaR}}$ and $\mathrm{DQ}_\alpha^{\mathrm{ES}}$ take values on a bounded interval. In contrast, the diversification ratio $\mathrm{DR}^{\mathrm{VaR}_\alpha}$ is unbounded, and $\mathrm{DR}^{\mathrm{ES}_\alpha}$ is bounded above by one only when the ES of the total risk is nonnegative.

Remark 5. It is a coincidence that $\mathrm{DQ}_\alpha^{\mathrm{VaR}}$ for $\alpha < 1/n$ and $\mathrm{DR}^{\mathrm{var}}$ both have a maximum value n. The latter maximum value is attained by a risk vector $(X/n, \ldots, X/n)$ for any $X \in L^2$.

5.2. Capturing Heavy Tails and Common Shocks

In this section, we analyze three simple normal and t-models to illustrate some features of DQ regarding heavy tails and common shocks in the portfolio models. Here, we only present some key observations. A detailed study of DQs based on VaR and ES for elliptical distributions and multivariate regularly varying models, including explicit formulas to compute DQ for these models, can be found in Han et al. (2023).

Let $\mathbf{Z} = (Z_1, \ldots, Z_n)$ be an n-dimensional standard normal random vector, and let ξ^2 have an inverse gamma distribution independent of $\mathbf{Z}$. Denote by $\mathrm{it}_n(\nu)$ the joint distribution with n independent t-marginals $\mathrm{t}(\nu, 0, 1)$, where the parameter ν represents the degrees of freedom; see McNeil et al. (2015) for t-models. The model $\mathbf{Y} = (Y_1, \ldots, Y_n) \sim \mathrm{it}_n(\nu)$ can be stochastically represented by

$$Y_i = \xi_i Z_i, \quad \text{for } i \in [n], \tag{8}$$

where $\xi_1, \ldots, \xi_n$ are iid following the same distribution as ξ, and independent of $\mathbf{Z}$. In contrast, a joint t-distributed random vector $\mathbf{Y}' = (Y_1', \ldots, Y_n') \sim \mathrm{t}(\nu, \mathbf{0}, I_n)$ has a stochastic representation $\mathbf{Y}' = \xi \mathbf{Z}$; that is,

$$Y_i' = \xi Z_i, \quad \text{for } i \in [n]. \tag{9}$$

In other words, $\mathbf{Y}'$ is a standard normal random vector multiplied by a heavy-tailed common shock ξ. All three models $\mathbf{Z}, \mathbf{Y}, \mathbf{Y}'$ have the same correlation matrix, the identity matrix I_n.

Because of the common shock ξ in (9), large losses from components of $\mathbf{Y}'$ are more likely to occur simultaneously compared with $\mathbf{Y}$ in (8), which does not have a common shock. Indeed, $\mathbf{Y}'$ is tail dependent (example 7.39 of McNeil et al. 2015), whereas $\mathbf{Y}$ is tail independent. As such, at least intuitively (if not rigorously), diversification for portfolio $\mathbf{Y}'$ should be considered as weaker than $\mathbf{Y}$, although both models are uncorrelated and have the same marginals.[13] By the central limit theorem, for $\nu > 2$, the component-wise average of $\mathbf{Y}$ (scaled by its variance) is asymptotically normal as n increases, whereas the component-wise average of $\mathbf{Y}'$ is always t-distributed. Hence, one may intuitively expect the order $D(\mathbf{Z}) < D(\mathbf{Y}) < D(\mathbf{Y}')$ to hold.

In Tables 2 and 3, we present DQ and DR for a few different models based on $\mathrm{N}(\mathbf{0}, I_n)$, $\mathrm{t}(\nu, \mathbf{0}, I_n)$, and $\mathrm{it}_n(\nu)$. We choose $n = 10$ and $\nu = 3$ or 4,[14] and thus, we have five models in total. As we see from Tables 2 and 3, DQs based on both VaR and ES report a lower value for $\mathrm{it}_n(\nu)$ and a larger value for $\mathrm{t}(\nu, \mathbf{0}, I_n)$, meaning that diversification is weaker for the common shock t-model (9) than the iid t-model (8). For the iid normal model, the diversification is the strongest according to DQ. In contrast, DR sometimes reports that the iid t-model has a larger diversification than the common shock t-model, which is counterintuitive. In the setting of both Tables 2 and 3, a risk manager governed by $\mathrm{DQ}_\alpha^{\mathrm{VaR}}$ would prefer the iid portfolio over the common shock portfolio, but the preference is flipped if the risk manager uses $\mathrm{DR}^{\mathrm{VaR}_\alpha}$. A more detailed analysis on this phenomenon for varying $\alpha \in (0, 0.1]$ is presented in Figure EC.1 in Section EC.5 in the e-companion, and consistent results are observed.

6. Portfolio Selection with DQ

Next, we focus on the optimal diversification problem

$$\min_{\mathbf{w} \in \Delta_n} \mathrm{DQ}_\alpha^{\mathrm{VaR}}(\mathbf{w} \odot \mathbf{X}) \quad \text{and} \quad \min_{\mathbf{w} \in \Delta_n} \mathrm{DQ}_\alpha^{\mathrm{ES}}(\mathbf{w} \odot \mathbf{X}); \tag{10}$$

recall that a smaller value of DQ means better diversification.[15] Recall from Table 1 that the first optimization is not quasiconvex, and the second one is quasiconvex (Proposition 1). We do not say that optimizing a diversification index has a decision-theoretic benefit; here, we

Table 2. DQs/DRs Based on VaR, ES, SD, and var

D	$\mathrm{DQ}_\alpha^{\mathrm{VaR}}$	$\mathrm{DQ}_\alpha^{\mathrm{ES}}$	$\mathrm{DR}^{\mathrm{VaR}_\alpha}$	$\mathrm{DR}^{\mathrm{ES}_\alpha}$	$\mathrm{DR}^{\mathrm{SD}}$	$\mathrm{DR}^{\mathrm{var}}$
$\mathbf{Z} \sim \mathrm{N}(\mathbf{0}, I_n)$	$\mathbf{2.0 \times 10^{-6}}$	$\mathbf{1.9 \times 10^{-9}}$	**0.3162**	0.3162	**0.3162**	**1**
$\mathbf{Y} \sim \mathrm{it}_n(3)$	0.0235	0.0124	0.3569	**0.2903**	**0.3162**	**1**
$\mathbf{Y}' \sim \mathrm{t}(3, \mathbf{0}, I_n)$	0.0502	0.0340	**0.3162**	0.3162	**0.3162**	**1**
$D(\mathbf{Z}) < D(\mathbf{Y})$	Yes	Yes	Yes	No	No	No
$D(\mathbf{Y}) < D(\mathbf{Y}')$	Yes	Yes	No	Yes	No	No

Notes. $\alpha = 0.05$, $n = 10$, and $\nu = 3$. Numbers in bold indicate the most diversified among $\mathbf{Z}, \mathbf{Y}, \mathbf{Y}'$ according to the index D.

simply illustrate the advantage of DQ in computation and optimization. Whether optimizing diversification is desirable for individual or institutional investors is an open-ended question that goes beyond the current paper; we refer to Van Nieuwerburgh and Veldkamp (2010), Boyle et al. (2012), and Choi et al. (2017) for relevant discussions.

For the portfolio weight $\mathbf{w}$, the DQ based on VaR at level $\alpha \in (0,1)$ is given by

$$\mathrm{DQ}_\alpha^{\mathrm{VaR}}(\mathbf{w} \odot \mathbf{X}) = \frac{1}{\alpha} \inf \left\{ \beta \in (0,1) : \mathrm{VaR}_\beta \left(\sum_{i=1}^n w_i X_i \right) \leqslant \sum_{i=1}^n w_i \mathrm{VaR}_\alpha(X_i) \right\},$$

and DQ based on ES is similar. In what follows, we fix $\alpha \in (0,1)$ and $\mathbf{X} = (X_1, \ldots, X_n) \in \mathcal{X}^n$, where $\mathcal{X}$ is L^0 for VaR and L^1 for ES, as in Section 5. Write $\mathbf{0} = (0, \ldots, 0) \in \mathbb{R}^n$ and $\mathbf{x}_\alpha^\rho = (\rho_\alpha(X_1), \ldots, \rho_\alpha(X_n))$ for a given risk measure ρ.

Proposition 6. *Fix $\alpha \in (0,1)$ and $\mathbf{X} \in \mathcal{X}^n$. The optimization of $\mathrm{DQ}_\alpha^{\mathrm{VaR}}(\mathbf{w} \odot \mathbf{X})$ in (10) can be solved by*

$$\min_{\mathbf{w} \in \Delta_n} \mathbb{P}(\mathbf{w}^\top (\mathbf{X} - \mathbf{x}_\alpha^{\mathrm{VaR}}) > 0). \tag{11}$$

Assuming $\mathbb{P}(X_i > \mathrm{ES}_\alpha(X_i)) > 0$ for each $i \in [n]$, the optimization of $\mathrm{DQ}_\alpha^{\mathrm{ES}}(\mathbf{w} \odot \mathbf{X})$ in (10) can be solved by the convex program

$$\min_{\mathbf{v} \in \mathbb{R}_+^n \setminus \{\mathbf{0}\}} \mathbb{E}[(\mathbf{v}^\top (\mathbf{X} - \mathbf{x}_\alpha^{\mathrm{ES}}) + 1)_+], \tag{12}$$

and the optimal $\mathbf{w}^*$ is given by $\mathbf{v}/\|\mathbf{v}\|_1$.

Proposition 6 offers efficient algorithms to optimize $\mathrm{DQ}_\alpha^{\mathrm{VaR}}$ and $\mathrm{DQ}_\alpha^{\mathrm{ES}}$ in real-data applications. The values of $\mathbf{x}_\alpha^{\mathrm{VaR}}$ and $\mathbf{x}_\alpha^{\mathrm{ES}}$ can be computed by many existing estimators of the individual losses (see, e.g., McNeil et al. 2015). In particular, a simple way to estimate these risk measures is to use an empirical estimator. More specifically, if we have data $\mathbf{X}^{(1)}, \ldots, \mathbf{X}^{(N)}$ sampled from $\mathbf{X}$ satisfying some ergodicity condition (being iid would be sufficient), then the empirical version of the Problem (11) is

$$\text{minimize} \quad \sum_{j=1}^N \mathbb{1}_{\{\mathbf{w}^\top (\mathbf{X}^{(j)} - \hat{\mathbf{x}}_\alpha^{\mathrm{VaR}}) > 0\}} \quad \text{over } \mathbf{w} \in \Delta_n, \tag{13}$$

where $\hat{\mathbf{x}}_\alpha^{\mathrm{VaR}}$ is the empirical estimator of $\mathbf{x}_\alpha^{\mathrm{VaR}}$ based on sample $\mathbf{X}^{(1)}, \ldots, \mathbf{X}^{(N)}$; see McNeil et al. (2015). Write $\mathbf{y}^{(j)} = \mathbf{X}^{(j)} - \hat{\mathbf{x}}_\alpha^{\mathrm{VaR}}$ and $z_j = \mathbb{1}_{\{\mathbf{w}^\top \mathbf{y}^{(j)} > 0\}}$ for $j \in [n]$. Problem (13) involves a chance constraint (see, e.g., Luedtke 2014, Liu et al. 2016). By using the big-M method (see, e.g., Shen et al. 2010) via choosing a sufficient large M (e.g., it is sufficient if M is larger than the components of $\mathbf{y}^{(j)}$ for all j), (13) can be converted into the following linear integer program:

$$\begin{aligned} &\text{minimize} \quad \sum_{j=1}^N z_j \\ &\text{subject to} \quad \mathbf{w}^\top \mathbf{y}^{(j)} - M z_j \leqslant 0, \quad \sum_{i=1}^n w_i = 1, \\ &\qquad\qquad z_j \in \{0,1\}, \quad w_i \geqslant 0 \quad \text{for all } j \in [N] \text{ and } i \in [n]. \end{aligned} \tag{14}$$

Similarly, the Optimization Problem (12) for $\mathrm{DQ}_\alpha^{\mathrm{ES}}$ can be solved by the empirical version of the Problem (12), which is a convex program:

$$\text{minimize} \quad \sum_{j=1}^N \max\{\mathbf{v}^\top (\mathbf{X}^{(j)} - \hat{\mathbf{x}}_\alpha^{\mathrm{ES}}) + 1, 0\} \quad \text{over } \mathbf{v} \in \mathbb{R}_+, \tag{15}$$

where $\hat{\mathbf{x}}_\alpha^{\mathrm{ES}}$ is the empirical estimator of $\mathbf{x}_\alpha^{\mathrm{ES}}$ based on

Table 3. DQs/DRs Based on VaR, ES, SD, and var

D	$\mathrm{DQ}_\alpha^{\mathrm{VaR}}$	$\mathrm{DQ}_\alpha^{\mathrm{ES}}$	$\mathrm{DR}^{\mathrm{VaR}_\alpha}$	$\mathrm{DR}^{\mathrm{ES}_\alpha}$	$\mathrm{DR}^{\mathrm{SD}}$	$\mathrm{DR}^{\mathrm{var}}$
$\mathbf{Z} \sim \mathrm{N}(\mathbf{0}, I_n)$	$\mathbf{2.0 \times 10^{-6}}$	$\mathbf{1.9 \times 10^{-9}}$	**0.3162**	0.3162	**0.3162**	**1**
$\mathbf{Y} \sim \mathrm{it}_n(4)$	0.0050	0.0017	0.3415	**0.2828**	**0.3162**	**1**
$\mathbf{Y}' \sim \mathrm{t}(4, \mathbf{0}, I_n)$	0.0252	0.0138	**0.3162**	0.3162	**0.3162**	**1**
$D(\mathbf{Z}) < D(\mathbf{Y})$	Yes	Yes	Yes	No	No	No
$D(\mathbf{Y}) < D(\mathbf{Y}')$	Yes	Yes	No	Yes	No	No

Notes. $\alpha = 0.05$, $n = 10$, and $\nu = 4$. Numbers in bold indicate the most diversified among $\mathbf{Z}, \mathbf{Y}, \mathbf{Y}'$ according to the index D.

sample $\mathbf{X}^{(1)}, \ldots, \mathbf{X}^{(N)}$. Both Problems (14) and (15) can be efficiently solved by modern optimization programs, such as CVX programming (see, e.g., Matmoura and Penev 2013).

Additional linear constraints, such as those on budget or expected return, can be easily included in (11)–(15), and the corresponding optimization problems can be solved similarly.

Tie breaking needs to be addressed when working with (13) because its objective function takes integer values. In dynamic portfolio selection, it is desirable to avoid adjusting positions too drastically or frequently. Therefore, in the real-data analysis in Section 7.3, among tied optimizers, we pick the closest one (in L^1-norm $\|\cdot\|_1$ on $\mathbb{R}^n$) to a given benchmark $\mathbf{w}_0$, the portfolio weight of the previous trading period. With this tie-breaking rule, we solve

$$\begin{aligned} &\text{minimize} \quad \|\mathbf{w}-\mathbf{w}_0\|_1 \quad \text{over } \mathbf{w} \in \Delta_n \\ &\text{subject to} \quad \sum_{j=1}^{N} \mathbb{1}_{\{\mathbf{w}^\top \mathbf{y}^{(j)} > 0\}} \leqslant m^*, \end{aligned} \tag{16}$$

where m^* is the optimum of (13). A tie breaking for (15) may need to be addressed similarly because (15) is not strictly convex.

7. Numerical Illustrations

To illustrate the performance of DQ, we collect historical asset prices from Yahoo Finance and conduct three sets of numerical experiments based on the data. We use the period from January 3, 2012, to December 31, 2021, with a total of 2,518 observations of daily losses and 500 trading days for the initial training. In Section 7.1, we first compare DQs and DRs based on VaR and ES. In Section 7.2, we calculate the values of $\mathrm{DQ}_\alpha^{\mathrm{VaR}}$ and $\mathrm{DQ}_\alpha^{\mathrm{ES}}$ under different selections of stocks. Finally, we construct portfolios by minimizing $\mathrm{DQ}_\alpha^{\mathrm{VaR}}$, $\mathrm{DQ}_\alpha^{\mathrm{ES}}$, and $\mathrm{DR}^{\mathrm{SD}}$ and by the mean-variance criterion in Section 7.3.

7.1. Comparing the DQ and DR

We first identify the largest stock in each of the S&P 500 sectors ranked by market cap in 2012. Among these stocks, we select the five largest stocks[16] to build our portfolio. We compute $\mathrm{DQ}_\alpha^{\mathrm{VaR}}$, $\mathrm{DQ}_\alpha^{\mathrm{ES}}$, $\mathrm{DR}^{\mathrm{VaR}_\alpha}$, and $\mathrm{DR}^{\mathrm{ES}_\alpha}$ on each day using the empirical distribution in a rolling window of 500 days, where we set $\alpha = 0.05$.

Figure 2 shows that the values of DQ and DR are between zero and one. This corresponds to the observation in Theorem 3 that $\mathrm{DQ}_\alpha^\rho < 1$ is equivalent to $\mathrm{DR}^{\rho_\alpha} < 1$. The DQ has a similar temporal pattern to the DR in the above period of time, with a large jump when COVID-19 exploded, which is more visible for the DQ than for the DR. We remind the reader that the DQ and DR are not meant to be compared on the same scale, and hence, the fact that the DQ has a larger range than the DR should be taken lightly. We also note that the values of $\mathrm{DQ}_\alpha^{\mathrm{VaR}}$ are in discrete grids. This is because the empirical distribution function takes value in multiples of $1/N$, where N is the sample size (500 in this experiment), and hence, $\mathrm{DQ}_\alpha^{\mathrm{VaR}}$ takes the values $k/(N\alpha)$ for an integer k; see (4). If a smooth curve is preferred, then one can employ a smoothed VaR through linear interpolation. This is a standard technique for handling VaR; see McNeil et al. (2015, section 9.2.6) and Li and Wang (2022, remark 8 and appendix B).

7.2. DQs for Different Portfolios

In this section, we fix $\alpha = 0.05$ and calculate the values of $\mathrm{DQ}_\alpha^{\mathrm{VaR}}$ and $\mathrm{DQ}_\alpha^{\mathrm{ES}}$ under different portfolio compositions of stocks. We consider portfolios with the following stock compositions:

(i) the two largest stocks from each of the 10 different sectors of the S&P 500;

(ii) the largest stock from each of five different sectors of the S&P 500 (as in Section 7.1);

(iii) the five largest stocks, AAPL, MSFT, IBM, GOOGL, and ORCL, from the information technology (IT) sector;

Figure 2. (Color online) DQs and DRs Based on VaR and ES with $\alpha = 0.05$

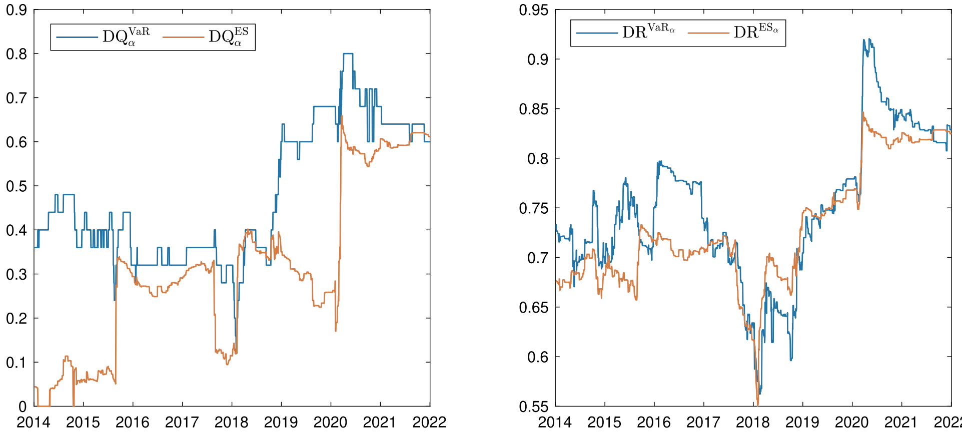
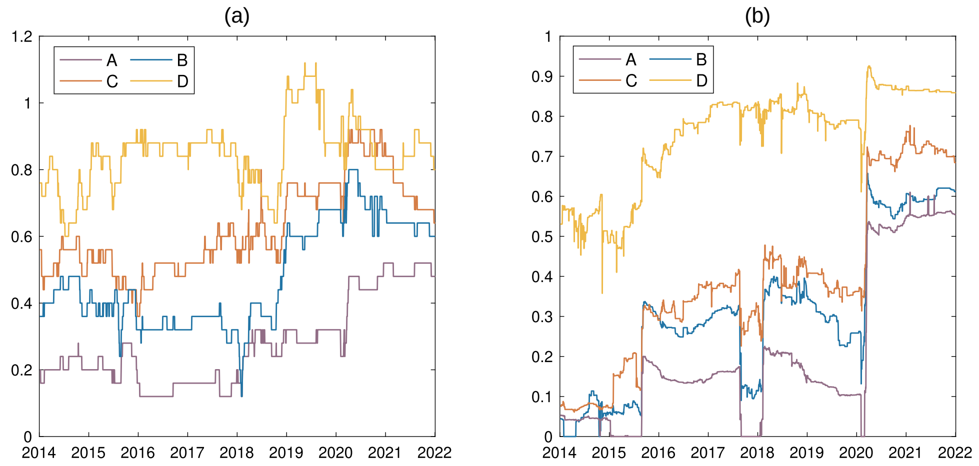


Figure 3. (Color online) DQs Based on VaR and ES with $\alpha = 0.05$



Notes. (a) VaR. (b) ES.

(iv) the five largest stocks, BRK/B, WFC, JPM, C, and BAC, from the financials (FINL) sector.

We make a few observations from Figure 3. Both $\mathrm{DQ}_\alpha^{\mathrm{VaR}}$ and $\mathrm{DQ}_\alpha^{\mathrm{ES}}$ provide similar comparative results. The order (i) $\leqslant$ (ii) $\leqslant$ (iii) $\leqslant$ (iv) is consistent with our intuition.[17] First, portfolio (i) of 20 stocks has the strongest diversification effect among the four compositions. Second, portfolio (ii) across five sectors has stronger diversification than (iii) and (iv) within one sector. Third, portfolio (iii) of five stocks within the IT sector has a stronger diversification than portfolio (iv) of five stocks within the FINL sector, consistent with the fact that the stocks in the IT sector are less correlated. Moreover, $\mathrm{DQ}_\alpha^{\mathrm{VaR}}$ for the FINL sector is larger than one during some period of time, which means that there is no diversification benefit if risk is evaluated by VaR. All DQ curves based on ES show a large upward jump around the COVID-19 outbreak; such a jump also exists for curves based on VaR, but it is less pronounced.

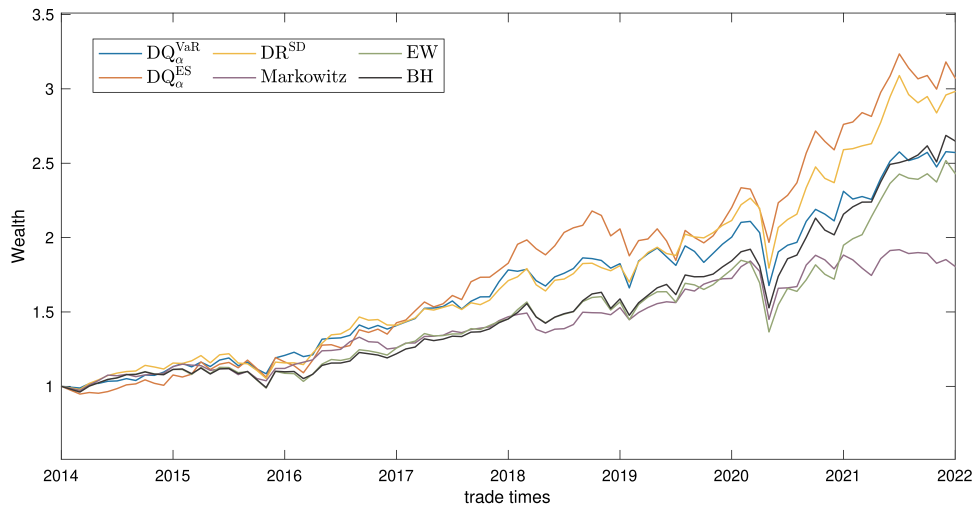
7.3. Optimal Diversified Portfolios

In this section, we fix $\alpha = 0.1$ and build portfolios via $\mathrm{DQ}_\alpha^{\mathrm{VaR}}$, $\mathrm{DQ}_\alpha^{\mathrm{ES}}$, $\mathrm{DR}^{\mathrm{SD}}$, and the mean-variance criterion in the Markowitz (1952) model.[18] The optimal portfolio problems using $\mathrm{DR}^{\mathrm{SD}}$ and the Markowitz model are well studied in literature; see, for example, Choueifaty and Coignard (2008). We compare these portfolio wealth with the equal-weighted (EW) portfolio and the simple buy-and-hold (BH) portfolio. For an analysis on the EW strategy, see DeMiguel et al. (2009).

We apply the algorithms in Proposition 6 to optimize $\mathrm{DQ}_\alpha^{\mathrm{VaR}}$ and $\mathrm{DQ}_\alpha^{\mathrm{ES}}$, which are extremely fast. A tie breaking is addressed for each objective as in (16). Minimization of $\mathrm{DR}^{\mathrm{SD}}$ and the Markowitz model can be solved by existing algorithms. The initial wealth is set to one, and the risk-free rate is $r = 2.84\%$, which is the 10-year yield of the U.S. treasury bill in January 2014. Note that the risk-free asset is not used to construct portfolios but is only used to calculate the Sharpe ratios. The target annual expected return for the Markowitz portfolio is set to 10%. We optimize the portfolio weights in each month with a rolling window of 500 days. That is, in each month, roughly 21 trading days, starting from January 2, 2014, we use the preceding 500 trading days to compute the optimal portfolio weights using the method described above. The portfolio is rebalanced every month. We choose the four largest stocks from each of the 10 different sectors of S&P 500 ranked by market cap in 2012 as the portfolio compositions (40 stocks in total). The portfolio performance is reported in Figure 4, and the cumulative distribution of the sorted portfolio weights, averaged over each month, is shown in Figure 5. Summary statistics, including the annualized return (AR), the annualized volatility (AV), the Sharpe ratio, and the average trading proportion (ATP), are reported in Table 4.[19]

From these results, we can see that the portfolio optimization strategies based on minimizing DQ perform quite well, similar to those based on $\mathrm{DR}^{\mathrm{SD}}$ and better than the Markowitz strategy. Moreover, ATP and portfolio weight distribution are similar across the strategies based on the three diversification indices and the Markowitz strategy. In contrast, the EW and BH strategies have more uniform portfolio weight distributions and smaller ATP, as anticipated. We remark that it is not our intention to analyze which diversification strategy generates the highest return, which is a challenging question that needs a separate study; also, we do not suggest diversification should or should not be optimized in practice. The empirical results here are presented to illustrate how our proposed diversification indices work in the context of portfolio selection. More

Figure 4. (Color online) Wealth Processes for Portfolios, 40 Stocks, January 2014 to December 2021



empirical results with some other data sets and portfolio strategies are given in Section EC.7 in the e-companion, and the results show similar patterns.

8. Concluding Remarks

In this paper, we put forward six axioms to jointly characterize a new class of indices of diversification and a seventh axiom to specialize this class. The new diversification index DQ has favorable features both theoretically and practically, and it is contrasted with its competitors, in particular, the DR. At a high level, because of the conceptual symmetry in Figure 1 (see also (7)), we expect both the DQ and DR to have advantages and disadvantages in different applications, and none should fully dominate the other. Nevertheless, we find many attractive features of DQ through the results in this paper, which suggest that DQ may be a better choice in many situations.

We summarize these features below. Some of these features are shared by the DR, but many are not. (i) DQs defined on a class of MCP risk measures can be uniquely characterized by six intuitive axioms (Theorem 1). DQs defined on a class of coherent risk measures can be uniquely characterized by further adding the axiom of portfolio convexity (Theorem 2). These two results lay an axiomatic foundation for using the DQ as a diversification index. (ii) The DQ further satisfies many properties for common risk measures (Propositions 1–4). These properties are not shared by the corresponding DR. (iii) The DQ is intuitive and interpretable with respect to dependence and common perceptions of diversification (Theorem 3). (iv) The DQ can be applied to a wide range of risk measures, such as the regulatory risk measures

Figure 5. (Color online) Cumulative Portfolio Weights, 40 Stocks, January 2014 to December 2021

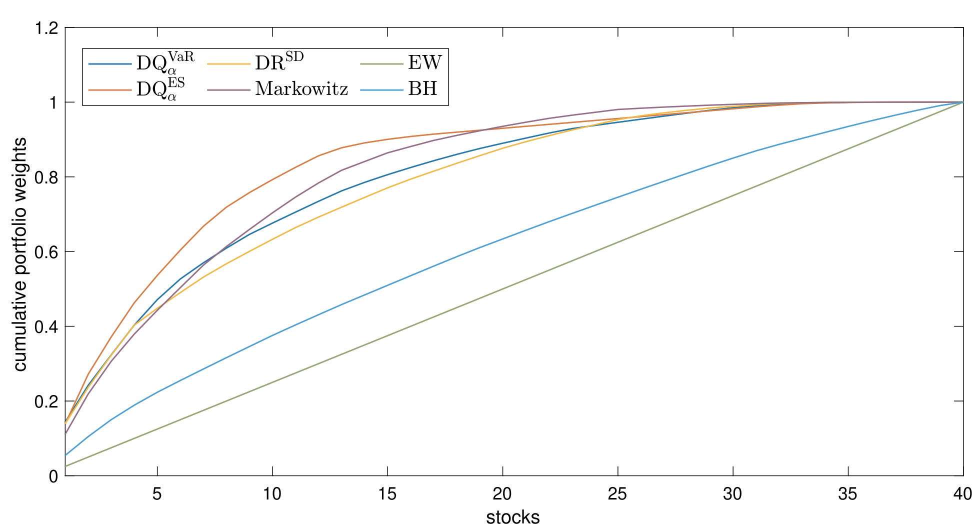


Table 4. Annualized Return, Annualized Volatility, Sharpe Ratio, and Average Trading Proportion for Different Portfolio Strategies from January 2014 to December 2021

%	$\mathrm{DQ}^{\mathrm{VaR}}_{\alpha}$	$\mathrm{DQ}^{\mathrm{ES}}_{\alpha}$	$\mathrm{DR}^{\mathrm{SD}}$	Markowitz	EW	BH
AR	12.56	14.59	14.36	7.93	11.91	12.88
AV	14.64	15.74	14.99	12.98	15.92	14.34
SR	66.40	74.66	76.85	39.22	56.95	70.02
ATP	19.29	14.75	15.61	18.79	4.43	0

VaR and ES, as well as expectiles. In cases of VaR and ES, the DQ has simple formulas and convenient properties (Theorem 4 and Proposition 5). (v) Portfolio optimization of DQs based on VaR and ES can be computed very efficiently (Proposition 6). (vi) The DQ can be easily applied to real data, and it produces results that are consistent with our usual perception of diversification (Section 7).

Among the class of DQs, for most applications, we generally recommend the use of DQs based on ES for the following reasons: (i) they satisfy all seven axioms of intuitive appeal, (ii) they have a simple optimization formula that is very convenient in portfolio optimization, (iii) they are closely connected to financial regulation as ES is the standard risk measure of Basel IV, (iv) they have a flexible parameter α that allows for reflecting the sensitivity to the tail risk of the decision maker, and (vi) they are conceptually easy to interpret as the (usually unique) level β of the ES family such that $\mathrm{ES}_{\beta}(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} \mathrm{ES}_{\alpha}(X_i)$.

We also mention a few interesting questions on DQs that call for thorough future study. (i) DQ is defined through a class of risk measures. It would be interesting to formulate DQ using expected utility or behavioral decision models to analyze the decision-theoretic implications of DQs. For instance, DQs based on entropic risk measures can be equivalently formulated using exponential utility functions. Alternatively, one may also build DQs directly from acceptability indices (see Remark 3). (ii) To compute DQs, one needs to invert the decreasing function $\beta \longmapsto \rho_{\beta}(\sum_{i=1}^{n} X_i)$. In the case of VaR and ES, the formula for this inversion is simple (Theorem 4). For more complicated classes of risk measures, this computation may be complicated and requires detailed analysis. (iii) For general distributions and risk measures other than VaR and ES, finding analytical formulas or efficient algorithms for optimal diversification using either DQs or DRs is a challenging task. (iv) Further analysis of DQs without scale invariance, such as those built on star-shaped risk measures (Castagnoli et al. 2022), may further generalize the domain of application.

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Endnotes

[1] We focus on the one-period losses to establish the theory. This is consistent with the vast majority of literature on risk measures and decision models.

[2] A different list of desirable axioms for diversification indices is studied by Koumou and Dionne (2022). Their framework is mathematically different from ours, as their diversification indices are mappings of portfolio weights instead of mappings of portfolio random vectors. They did not provide axiomatic characterization results.

[3] Recall that X_i represents the loss from asset i. Suppose that two agents purchased the same portfolio of assets but at different prices of each asset. Denote by $\mathbf{X}$ the portfolio loss vector of agent 1. The portfolio loss vector of agent 2 is $\mathbf{X} + \mathbf{c}$, where $\mathbf{c}$ is the vector of differences between their purchase prices. The two agents should have the same level of diversification regardless of their purchase prices, as they hold the same portfolio.

[4] The inequality $X \leqslant Y$ between two random variables X and Y is pointwise.

[5] If the denominator in the definition of $\mathrm{DR}^{\phi}(\mathbf{X})$ is zero, then we use the convention $0/0 = 0$ and $1/0 = \infty$.

[6] A negative value of a risk measure has a concrete meaning as the amount of capital to be withdrawn from a portfolio position while keeping it acceptable; see Artzner et al. (1999).

[7] An impossibility result (Proposition EC.1) is presented in Section EC.2 in the e-companion, which suggests that it is not possible to construct nontrivial diversification indices such as DR and DB satisfying [+], [LI], and [SI].

[8] Indeed, the value of $D(\mathbf{0})$ may be rather arbitrary; this is the case for a DR where $0/0$ occurs.

[9] Theorem 3 gives some mild conditions that yield $D(\mathbf{X}^{\mathrm{du}}) = 1$ for the class D characterized in this section.

[10] Such situations may be regarded as diversification disasters; see Ibragimov et al. (2011).

[11] In the literature on statistical robustness, often a different metric than the L^{∞} metric is used; see Huber and Ronchetti (2009) for a general treatment. Our choice of formulating continuity via the L^{∞} metric is standard in the axiomatic theory of risk mappings on L^{∞}.

[12] If X and Y represent gains instead of losses, then SSD is typically defined via increasing concave functions.

[13] On a related note, as discussed by Embrechts et al. (2002), correlation is not a good measure of diversification in the presence of heavy-tailed and skewed distributions.

[14] Most financial asset log-loss data have a tail index between [3,5], which corresponds to $\nu \in [3,5]$; see, for example, Jansen and De Vries (1991).

[15] A possible alternative formulation to (10) is to use DQ as a constraint instead of an objective in the optimization. This is mathematically similar to a risk measure constraint (e.g., Basak and Shapiro 2001, Rockafellar and Uryasev 2002, Mafusalov and Uryasev 2018), but with a different interpretation, as DQ is not designed to measure or control risk.

[16] XOM from ENR, AAPL from IT, BRK/B from FINL, WMT from CONS, and GE from INDU.

[17] The observations here are consistent with those from applying DR^{SD} (which is also a DQ) in the same setting; see Section EC.7 in the e-companion.

[18] One may try other portfolio criteria other than mean variance. For instance, Levy and Levy (2004) found that portfolio strategies based on prospect theory perform similarly to the mean-variance strategies.

[19] ATP is an approximation of trading costs, and it is computed as the average of $\sum_{t=1}^{T} |w_i^t - w_i^{t-}|$ over $i = 1, \ldots, n$, where $T = 96$ is the total number of months, w_i^t is the portfolio weight of asset i at the beginning of month t, and w_i^{t-} is the portfolio weight of asset i at the end of month $t-1$, with w_i^{1-} set to w_i^1. Note that for BH, ATP is zero because there is no trading, whereas for EW, ATP is positive, as rebalancing occurs at the end of each month.

References

Artzner P, Delbaen F, Eber J-M, Heath D (1999) Coherent measures of risk. *Math. Finance* 9(3):203–228.

Basak S, Shapiro A (2001) Value-at-risk based risk management: Optimal policies and asset prices. *Rev. Financial Stud.* 14(2):371–405.

Basel Committee on Banking Supervision (2019) Minimum capital requirements for market risk. February 2019. Document d457, Bank for International Settlements, Basel, Switzerland.

Bellini F, Fadina T, Wang R, Wei Y (2022) Parametric measures of variability induced by risk measures. *Insurance Math. Econom.* 106:270–284.

Bellini F, Klar B, Müller A, Rosazza Gianin E (2014) Generalized quantiles as risk measures. *Insurance Math. Econom.* 54(1):41–48.

Berger L, Eeckhoudt L (2021) Risk, ambiguity, and the value of diversification. *Management Sci.* 67(3):1639–1647.

Boyle P, Garlappi L, Uppal R, Wang T (2012) Keynes meets Markowitz: The trade-off between familiarity and diversification. *Management Sci.* 58(2):253–272.

Bürgi R, Dacorogna MM, Iles R (2008) Risk aggregation, dependence structure and diversification benefit. Rösch D, Scheule H, eds. *Stress Testing for Financial Institutions* (Risk Books, London), 265–306.

Castagnoli E, Cattelan G, Maccheroni F, Tebaldi C, Wang R (2022) Star-shaped risk measures. *Oper. Res.* 70(5):2637–2654.

Cerreia-Vioglio S, Maccheroni F, Marinacci M, Montrucchio L (2011) Risk measures: Rationality and diversification. *Math. Finance* 21(4):743–774.

Chambers CP (2009) An axiomatization of quantiles on the domain of distribution functions. *Math. Finance* 19(2):335–342.

Cherny A, Madan D (2009) New measures for performance evaluation. *Rev. Financial Stud.* 22(7):2571–2606.

Choi N, Fedenia M, Skiba H, Sokolyk T (2017) Portfolio concentration and performance of institutional investors worldwide. *J. Financial Econom.* 123(1):189–208.

Choueifaty Y, Coignard Y (2008) Toward maximum diversification. *J. Portfolio Management* 35(1):40–51.

D'Acunto F, Prabhala N, Rossi AG (2019) The promises and pitfalls of robo-advising. *Rev. Financial Stud.* 32(5):1983–2020.

DeMiguel V, Garlappi L, Uppal R (2009) Optimal vs. naive diversification: How inefficient is the 1/N portfolio strategy? *Rev. Financial Stud.* 22(5):1915–1953.

Denis DJ, Denis DK, Yost K (2002) Global diversification, industrial diversification, and firm value. *J. Finance* 57(5):1951–1979.

Dhaene J, Tsanakas A, Valdez EA, Vanduffel S (2012) Optimal capital allocation principles. *J. Risk Insurance* 79(1):1–28.

EIOPA (2011) Equivalence assessment of the Swiss supervisory system in relation to articles 172, 227 and 260 of the Solvency II Directive. EIOPA-BoS-11-028, European Insurance and Occupational Pensions Authority, Frankfurt, Germany.

Embrechts P, Furrer H, Kaufmann R (2009) Different kinds of risk. Mikosch T, Kreiß JP, Davis R, Andersen T, eds. *Handbook of Financial Time Series* (Springer, Berlin, Heidelberg), 729–751.

Embrechts P, Liu H, Wang R (2018) Quantile-based risk sharing. *Oper. Res.* 66(4):936–949.

Embrechts P, McNeil A, Straumann D (2002) Correlation and dependence in risk management: Properties and pitfalls. Dempster MAH, ed. *Risk Management: Value at Risk and Beyond* (Cambridge University Press, Cambridge, UK), 176–223.

Embrechts P, Wang B, Wang R (2015) Aggregation-robustness and model uncertainty of regulatory risk measures. *Finance Stochastics* 19(4):763–790.

Embrechts P, Puccetti G, Rüschendorf L, Wang R, Beleraj A (2014) An academic response to Basel 3.5. *Risks* 2(1):25–48.

Emmer S, Kratz M, Tasche D (2015) What is the best risk measure in practice? A comparison of standard measures. *J. Risk* 18(2):31–60.

Fama EF, Miller MH (1972) *The Theory of Finance* (Holt, Rinehart, and Winston, New York).

Föllmer H, Schied A (2016) *Stochastic Finance. An Introduction in Discrete Time*, 4th ed. (Walter de Gruyter, Berlin).

Frittelli M, Rosazza Gianin E (2002) Putting order in risk measures. *J. Banking Finance* 26:1473–1486.

Furman E, Wang R, Zitikis R (2017) Gini-type measures of risk and variability: Gini shortfall, capital allocation and heavy-tailed risks. *J. Banking Finance* 83:70–84.

Gilboa I (2009) *Theory of Decision Under Uncertainty* (Cambridge University Press, Cambridge, UK).

Gilboa I, Postlewaite A, Samuelson L (2019) What are axiomatizations good for? *Theory and Decision* 86(3–4):339–359.

Green RC, Hollifield B (1992) When will mean-variance efficient portfolios be well diversified? *J. Finance* 47(5):1785–1809.

Han X, Lin L, Wang R (2023) Diversification quotients based on VaR and ES. *Insurance Math. Econom.* 113:185–197.

Huber PJ, Ronchetti EM, eds. (2009) *Robust Statistics*, 2nd ed. Wiley Series in Probability and Statistics (Wiley, Hoboken, NJ).

Ibragimov R, Jaffee D, Walden J (2011) Diversification disasters. *J. Financial Econom.* 99(2):333–348.

Jansen DW, De Vries CG (1991) On the frequency of large stock returns: Putting booms and busts into perspective. *Rev. Econom. Statist.* 73(1):18–24.

Koumou GB (2020) Diversification and portfolio theory: A review. *Financial Markets Portfolio Management* 34(3):267–312.

Koumou GB, Dionne G (2022) Coherent diversification measures in portfolio theory: An axiomatic foundation. *Risks* 10(11):205.

Kováčová G, Rudloff B, Cialenco I (2022) Acceptability maximization. *Frontiers Math. Finance* 1(2):219–248.

Kusuoka S (2001) On law invariant coherent risk measures. *Adv. Math. Econom.* 3:83–95.

Levy H, Levy M (2004) Prospect theory and mean-variance analysis. *Rev. Financial Stud.* 17(4):1015–1041.

Levy H, Sarnat M (1970) International diversification of investment portfolios. *Amer. Econom. Rev.* 60(4):668–675.

Li H, Wang R (2022) PELVE: Probability equivalent level of VaR and ES. *J. Econometrics* 234(1):353–370.

Liu X, Küçükyavuz S, Luedtke J (2016) Decomposition algorithms for two-stage chance-constrained programs. *Math. Programming* 157(1):219–243.

Luedtke J (2014) A branch-and-cut decomposition algorithm for solving chance-constrained mathematical programs with finite support. *Math. Programming* 146(1):219–244.

Maccheroni F, Marinacci M, Rustichini A, Taboga M (2009) Portfolio selection with monotone mean-variance preferences. *Math. Finance* 19(3):487–521.

Mafusalov A, Uryasev S (2018) Buffered probability of exceedance: Mathematical properties and optimization. *SIAM J. Optim.* 28(2):1077–1103.

Mainik G, Rüschendorf L (2010) On optimal portfolio diversification with respect to extreme risks. *Finance Stochastics* 14(4):593–623.

Mao T, Wang R (2020) Risk aversion in regulatory capital calculation. *SIAM J. Financial Math.* 11(1):169–200.

Markowitz H (1952) Portfolio selection. *J. Finance* 7(1):77–91.

Mas-Colell A, Whinston MD, Green JR (1995) *Microeconomic Theory* (Oxford University Press, Oxford, UK).

Matmoura Y, Penev S (2013) Multistage optimization of option portfolio using higher order coherent risk measures. *Eur. J. Oper. Res.* 227(1):190–198.

McNeil AJ, Frey R, Embrechts P (2015) *Quantitative Risk Management: Concepts, Techniques and Tools*, revised edition (Princeton University Press, Princeton, NJ).

Rockafellar RT, Uryasev S (2002) Conditional value-at-risk for general loss distributions. *J. Banking Finance* 26(7):1443–1471.

Rockafellar RT, Uryasev S, Zabarankin M (2006) Generalized deviations in risk analysis. *Finance Stochastics* 10(1):51–74.

Rothschild M, Stiglitz JE (1970) Increasing risk: I. A definition. *J. Econom. Theory* 2(3):225–243.

Samuelson PA (1967) General proof that diversification pays. *J. Financial Quant. Anal.* 2(1):1–13.

Sharpe WF (1964) Capital asset prices: A theory of market equilibrium under conditions of risk. *J. Finance* 19(3):425–442.

Shen S, Smith JC, Ahmed S (2010) Expectation and chance-constrained models and algorithms for insuring critical paths. *Management Sci.* 56(10):1794–1814.

Tasche D (2007) Capital allocation to business units and sub-portfolios: The Euler principle. Preprint, submitted August 19, https://arxiv.org/abs/0708.2542.

Tu J, Zhou G (2011) Markowitz meets Talmud: A combination of sophisticated and naive diversification strategies. *J. Financial Econom.* 99(1):204–215.

Van Nieuwerburgh S, Veldkamp L (2010) Information acquisition and under-diversification. *Rev. Econom. Stud.* 77(2):779–805.

Wakker PP (2010) *Prospect Theory: For Risk and Ambiguity* (Cambridge University Press, Cambridge, UK).

Wang R, Zitikis R (2021) An axiomatic foundation for the expected shortfall. *Management Sci.* 67(3):1413–1429.

Wang R, Wei Y, Willmot GE (2020) Characterization, robustness and aggregation of signed Choquet integrals. *Math. Oper. Res.* 45(3):993–1015.

Yaari ME (1987) The dual theory of choice under risk. *Econometrica* 55(1):95–115.