Finite element and integral equation methods to conical diffraction by imperfectly conducting gratings

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Abstract

In this paper we study the variational method and integral equation methods for a conical diffraction problem for imperfectly conducting gratings modeled by the impedance boundary value problem of the Helmholtz equation in periodic structures. We justify the strong ellipticity of the sesquilinear form corresponding to the variational formulation and prove the uniqueness of solutions at any frequency. Convergence of the finite element method using the transparent boundary condition (Dirichlet-to-Neumann mapping) is verified. The boundary integral equation method is also discussed.

Keywords: Diffraction gratings, conical diffraction, variational methods, integral equation methods, finite element analysis, well-posedness.

1 Introduction

Grating diffraction problems have been extensively studied in the literature via variational and integral equation methods. We refer to [3, 10] and references therein for mathematical analysis and numerical treatment. In the polarization case, one must assume that the diffraction grating is periodic in one direction $(x_1$ -direction), invariant in another direction $(x_3$ -direction) and that the incident direction of a time-harmonic electromagnetic plane wave is orthogonal to the ox_1x_3 plane. In the TE (resp. TM) polarization, the electric (resp. magnetic) field is parallel to the x_3 -direction. In this paper we suppose that a time-harmonic plane wave incident obliquely on an imperfectly conducting grating, which leads to the so-called conical diffraction problems. The impedance boundary condition will be used to model the boundary behavior of the wave fields between a highly conducting material and an isotropic, homogeneous and lossless background medium.

In periodic structures, Elschner, Hinder, Penzel and Schmidit [5] proved the well-posedness of the conical diffraction problem with transmission conditions via the variational method. Elschner and Schmidt [6] studied stability of the conical diffraction problem with respect to variation of the grating profile and obtained explicit formulas for the derivatives of reflection and transmission

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coefficients with respect to perturbations of interfaces. If the scattering object is an infinitely long cylinder, the conical diffraction is also referred to as oblique scattering. In [9], Wang and Nakamura applied the integral equation method to prove the well-posedness of the oblique scattering problem in a homogeneous medium. In an inhomogeneous medium, the uniqueness and existence of the oblique problem were also studied through the Lax-Phillips method; see Nakamura [12]. In this paper, we consider the conical diffraction problem in periodic structures under the impedance boundary condition and investigate both finite element and boundary integral equation methods.

The outline of the paper is organized as follows. In Section 2, we formulate the conical diffraction problem by deriving a coupled Helmholtz system with the impedance boundary condition from Maxwell's system. In Section 3, we state the variational formulation in one periodic cell with the DtN operator imposed on the artificial boundary. An energy formula is verified to prove the uniqueness of the truncated boundary value problem. The strong ellipticity of the variational formulation is shown and the well-posedness of the diffraction problem follows from the Fredholm theory. In Section 4, we show the convergence of the finite element method based on the variational formulation. Finally, the integral equation method will be briefly discussed in Section 5.

2 Conical diffraction problem

Assume an incoming time-harmonic plane wave of the form

$$(\mathcal{E}^{in}, \mathcal{H}^{in}) = (\mathbf{p}e^{i\alpha x_1 - i\beta x_2 + i\gamma x_3}, \mathbf{q}e^{i\alpha x_1 - i\beta x_2 + i\gamma x_3})e^{-i\omega t} =: (\mathbf{E}^{in}, \mathbf{H}^{in})e^{-i\omega t},$$
(2.1)

is incident on an imperfectly conducting grating with a high conductivity embedded in an isotropic homogeneous medium in \mathbb{R}^3 . Denote by $\tilde{\Gamma}$ the grating profile and $\tilde{\Omega}$ the unbounded domain above $\tilde{\Gamma}$. The diffraction problem can be modeled by the reduced Maxwell's system

$$\nabla \times \mathbf{E} = i\omega\mu \mathbf{H}, \quad \nabla \times \mathbf{H} = -i\omega\epsilon \mathbf{E} \quad \text{in} \quad \tilde{\Omega}, \tag{2.2}$$

where the total fields (\mathbf{E}, \mathbf{H}) are the sum of the incident waves $(\mathbf{E}^{in}, \mathbf{H}^{in})$ and the outgoing scattered waves $(\mathbf{E}^{sc}, \mathbf{H}^{sc})$ in $\tilde{\Omega}$. In (2.2), ω denotes the angular frequency. The dielectric coefficient ϵ and the magnetic permeability μ of the homogeneous medium in $\tilde{\Omega}$ are both assumed to be positive constants. Set $k = \omega \sqrt{\epsilon \mu}$ as the wavenumber of the background medium. We enforce the impedance boundary condition on $\tilde{\Gamma}$

$$\nu \times \mathbf{E} \times \nu = \lambda \ (\nu \times \mathbf{H}) \quad \text{on} \quad \Gamma, \tag{2.3}$$

where $\nu = (\nu_1, \nu_2, \nu_3) \in \mathbb{S}^2 := \{x \in \mathbb{R}^3 : |x| = 1\}$ is normal to $\tilde{\Gamma}$ directed into the exterior of $\tilde{\Omega}$ and $\lambda < 0$ is the impedance coefficient which is assumed to be a constant. The problem (2.1)-(2.3) is called a conical diffraction problem if the incident direction $\mathbf{k} =: (\alpha, -\beta, \gamma)$ is not orthogonal to the x_2 -direction, i.e., $\gamma \neq 0$. For conical diffraction problems, the wave vectors of the reflected or transmitted propagating modes lie on the surface of a cone whose axis is parallel to the x_3 -direction [11]. We refer to Figure 1 for an illustration of the grating conical diffraction problem.

In order for $(\mathbf{E}^{in}, \mathbf{H}^{in})$ given in (2.1) to satisfy (2.2), the constant amplitude vector \mathbf{p} must be perpendicular to the wave vector $\mathbf{k} = (\alpha, -\beta, \gamma)$, that is $\mathbf{p} \cdot \mathbf{k} = 0$. Furthermore $\mathbf{k} \cdot \mathbf{k} = k^2 = \omega^2 \epsilon \mu$



Figure 1: Geometry of the three-dimensional conical diffraction problem in one periodic cell. ϕ is the angle between incident direction **k** and (x_1, x_2) -plane. θ is the angle between $(\alpha, -\beta)$ and the x_2 -axis.

and $\mathbf{q} = (\omega \mu)^{-1} \mathbf{k} \times \mathbf{p}$. We can express the wave vector \mathbf{k} as

$$\mathbf{k} = (\alpha, -\beta, \gamma) := k(\sin\theta\cos\phi, -\cos\theta\cos\phi, \sin\phi),$$

in terms of the angles of incidence $\theta, \phi \in (-\pi/2, \pi/2)$.

Assume that Γ remains invariant in x_3 and is 2π -periodic in x_1 . If the incoming wave is of the form (2.1), we make an ansatz on the total field

$$(\mathbf{E},\mathbf{H})(x_1,x_2,x_3) = (E(x_1,x_2),H(x_1,x_3)) e^{i\gamma x_3},$$

with $E = (E_1, E_2, E_3), H = (H_1, H_2, H_3) : \mathbb{R}^2 \to \mathbb{C}^3$. The Maxwell equations (2.2) can be reduced to two Helmholtz equations for the total fields $u = E_3$ and $v = H_3$ (see [5]):

$$\Delta u + \kappa^2 u = 0, \quad \Delta v + \kappa^2 v = 0 \quad \text{in} \quad \Omega, \quad \kappa^2 = k^2 - \gamma^2.$$

Here Ω denotes the restriction of the cross-section of $\tilde{\Omega}$ by the (x_1, x_2) -plane to one periodic cell $(0, 2\pi)$. Analogously, denote by Γ the counter part of $\tilde{\Gamma}$ in the periodic cell $(0, 2\pi)$. The reduced geometry of the conical diffraction problem is shown in Figure 2. Next, we turn to the reduction of the boundary condition (2.3) in \mathbb{R}^2 . Obviously, we have $\nu_3 = 0$ and

$$(\nu \times E) \times \nu = (-\nu_2(\nu_1 E_2 - \nu_2 E_1), -\nu_1(\nu_1 E_2 - \nu_2 E_1), E_3),$$

$$\nu \times H = (\nu_2 H_3, -\nu_1 H_3, \nu_1 H_2 - \nu_2 H_1).$$

$$(2.4)$$

Moreover, there holds (see [5])

$$\nu_1 E_2 - \nu_2 E_1 = \frac{i\gamma}{\kappa^2} \frac{\partial E_3}{\partial \tau} - \frac{i\omega\mu}{\kappa^2} \frac{\partial H_3}{\partial n}, \quad \nu_1 H_2 - \nu_2 H_1 = \frac{i\gamma}{\kappa^2} \frac{\partial H_3}{\partial \tau} + \frac{i\omega\epsilon}{\kappa^2} \frac{\partial E_3}{\partial n}, \tag{2.5}$$

with

$$n = (\nu_1, \nu_2) \in \mathbb{S}, \ \tau = (-\nu_2, \nu_1) \in \mathbb{S}, \quad \partial_n = \nu_1 \partial_1 + \nu_2 \partial_2, \ \partial_\tau = -\nu_2 \partial_1 + \nu_1 \partial_2, \ \partial_j = \frac{\partial}{\partial x_j}.$$



Figure 2: Geometry of the conical diffraction problem.

Meanwhile, for the reduced Helmholtz equation, the incoming time-harmonic plane wave (2.1) takes the form

$$u^i = p_3 e^{i\alpha x_1 + i\beta x_2}, \quad v^i = p_3 e^{i\alpha x_1 + i\beta x_2}$$

Combining (2.4)-(2.5) and the impedance boundary condition (2.3), we get

$$-\frac{i\gamma}{\kappa^2}\frac{\partial E_3}{\partial \tau} + \frac{i\omega\mu}{\kappa^2}\frac{\partial H_3}{\partial n} = \lambda H_3, \quad E_3 = \lambda \left(\frac{i\gamma}{\kappa^2}\frac{\partial H_3}{\partial \tau} + \frac{i\omega\epsilon}{\kappa^2}\frac{\partial E_3}{\partial n}\right),$$

which, for $u = E_3$ and $v = H_3$, is equivalent to the boundary condition

$$\lambda \frac{\partial u}{\partial n} + \frac{i\kappa^2}{\omega\epsilon} u + \frac{\lambda\gamma}{\omega\epsilon} \frac{\partial v}{\partial\tau} = 0, \quad \frac{\partial v}{\partial n} + \frac{i\lambda\kappa^2}{\omega\mu} v - \frac{\gamma}{\omega\mu} \frac{\partial u}{\partial\tau} = 0 \quad \text{on} \quad \Gamma.$$
(2.6)

Using $\gamma = \omega \sqrt{\epsilon \mu} \sin \phi$ and $\kappa^2 = k^2 \cos^2 \phi = \omega^2 \mu \epsilon \cos^2 \phi$, the previous boundary condition can be written as

$$\begin{cases} \lambda \frac{\partial u}{\partial n} + i\omega\mu\cos^2\phi \ u + \lambda\sin\phi\sqrt{\frac{\mu}{\epsilon}}\frac{\partial v}{\partial \tau} &= 0,\\ \frac{\partial v}{\partial n} + i\lambda\omega\epsilon\cos^2\phi \ v - \sin\phi\sqrt{\frac{\epsilon}{\mu}}\frac{\partial u}{\partial \tau} &= 0, \end{cases} \quad \text{on} \quad \Gamma.$$
(2.7)

Remark 2.1. If $\lambda = 0$, then the boundary condition (2.6) (or (2.7)) reduces to $\frac{\partial v}{\partial n} = u = 0$, which corresponds to the TE or TM polarization of the electromagnetic scattering by perfectly conducting gratings. If $\phi = 0$, then both u and v satisfy the standard impedance boundary condition for the Helmholtz equation.

3 Radiation condition and variational formulation

For $b > \Gamma_{\max} := \max_{x \in \Gamma} \{x_2\}$, define

$$\Gamma_b := \{ (x_1, b) : 0 < x_1 < 2\pi \}, \quad \Omega_b := \{ x \in \Omega : x_2 < b \}.$$

A function $u(x_1, x_2)$ is called α -quasiperiodic if $e^{-i\alpha x_1}u(x_1, x_2)$ is 2π -periodic in x_1 , or equivalently,

$$u(x_1 + 2\pi, x_2) = e^{2i\alpha\pi} u(x_1, x_2).$$

Since the incident field is α -quasiperiodic, the scattered field u^s , v^s are also assumed to be α -quasiperiodic. Then the function $u^s(x_1, x_2)e^{-i\alpha x_1}$, $v^s(x_1, x_2)e^{-i\alpha x_1}$ can be expended as a Fourier series. Inserting these series into the Helmholtz equation, we can express u^s and v^s as a sum of plane waves. Physically, the scattered field (u^s, v^s) remain bounded as $x_2 \to \infty$, leading to the well-known Rayleigh expansion condition:

$$u^{s}(x) = \sum_{n \in \mathbb{Z}} u_{n} e^{i\alpha_{n}x_{1} + i\beta_{n}x_{2}}, \qquad v^{s}(x) = \sum_{n \in \mathbb{Z}} v_{n} e^{i\alpha_{n}x_{1} + i\beta_{n}x_{2}}, \quad x_{2} > \Gamma_{\max},$$
(3.1)

with the Rayleigh coefficients $u_n, v_n \in \mathbb{C}$, where

$$\alpha_n := n + \alpha, \qquad \beta_n := \begin{cases} \sqrt{\kappa^2 - |\alpha_n|^2}, & |\alpha_n| \le \kappa, \\ i\sqrt{|\alpha_n|^2 - \kappa^2}, & |\alpha_n| > \kappa, \end{cases}$$

with $i = \sqrt{-1}$. It is clear that (u^s, v^s) in (3.1) can be split into the finite sum $\sum_{|\alpha_n| \leq k}$ of outgoing plane waves and the infinite sum $\sum_{|\alpha_n| > k}$ of exponentially decaying waves, which are called surface or evanescent waves. We summarize our conical diffraction problem as follows:

$$\begin{cases} \Delta u + \kappa^2 u = 0, \ \Delta v + \kappa^2 v = 0, & \text{in } \Omega, \\ \lambda \frac{\partial u}{\partial n} + \frac{i\kappa^2}{\omega\epsilon} u + \frac{\lambda\gamma}{\omega\epsilon} \frac{\partial v}{\partial \tau} = 0, \ \frac{\partial v}{\partial n} + \frac{i\lambda\kappa^2}{\omega\mu} v - \frac{\gamma}{\omega\mu} \frac{\partial u}{\partial \tau} = 0, & \text{on } \Gamma, \\ u^s \text{ and } v^s \text{ fulfill the Rayleigh expansion (3.1).} \end{cases}$$
(3.2)

Then we introduce the variational space

$$X = \{(u, v) \in H^1(\Omega_b)^2 : u, v \text{ are } \alpha \text{-quasiperiodic}\}.$$

In order to derive the variational formula, we will need Green's formula for functions in $H^1_{\alpha}(\Omega_b)$, for which it is well known.

Lemma 3.1. Assume that $f \in H^2_{\alpha}(\Omega_b)$ and $g \in H^1_{\alpha}(\Omega_b)$, Then

$$\int_{\Omega_b} \nabla f \cdot \nabla \overline{g} + \Delta f \overline{g} \, dx = \int_{\partial \Omega_b} \partial_n f \overline{g} \, ds, \quad \int_{\Omega_b} \nabla f \cdot \nabla^\perp \overline{g} \, dx = -\int_{\partial \Omega_b} \partial_\tau f \overline{g} \, ds,$$

where $\nabla = (\partial_1, \partial_2)$ and $\nabla^{\perp} = (-\partial_2, \partial_1)$.

Let $u, v \in H^1_{\alpha}(\Omega_b)$ solve the conical diffraction problem (3.2). Applying Green's formula to the Helmholtz equations yields

$$0 = \int_{\Omega_b} (\Delta u + \kappa^2 u) \overline{\varphi} \, dx = \int_{\Omega_b} -\nabla u \cdot \nabla \overline{\varphi} + \kappa^2 u \overline{\varphi} \, dx + \int_{\partial \Omega_b} \partial_n u \, \overline{\varphi} \, ds, \tag{3.3}$$

$$\int_{\Omega_b} \nabla v \cdot \nabla^{\perp} \overline{\varphi} \, dx = -\int_{\partial \Omega_b} \partial_\tau v \, \overline{\varphi} \, ds \quad \text{for all } \varphi \in H^1_{\alpha}(\Omega_b). \tag{3.4}$$

Multiplying the equations (3.3) and (3.4) by the constant factors $\frac{\omega\epsilon}{\kappa^2}$ and $\frac{\gamma}{\kappa^2}$, respectively, and taking the difference of the resulting formulas, we get

$$\int_{\partial\Omega_b} \frac{\omega\epsilon}{\kappa^2} \partial_n u \,\overline{\varphi} + \frac{\gamma}{\kappa^2} \partial_\tau v \,\overline{\varphi} \, ds = \int_{\Omega_b} \left[\frac{\omega\epsilon}{\kappa^2} \nabla u \cdot \nabla \overline{\varphi} - \frac{\gamma}{\kappa^2} \nabla v \cdot \nabla^\perp \overline{\varphi} - \omega\epsilon \, u \,\overline{\varphi} \right] \, dx. \tag{3.5}$$

Similarly, we get

$$0 = \int_{\Omega_b} (\Delta v + \kappa^2 v) \overline{\psi} \, dx = \int_{\Omega_b} -\nabla v \cdot \nabla \overline{\psi} + \kappa^2 v \overline{\psi} \, dx + \int_{\partial \Omega_b} \partial_n v \, \overline{\psi} \, ds, \tag{3.6}$$

$$\int_{\Omega_b} \nabla u \cdot \nabla^{\perp} \overline{\psi} \, dx = -\int_{\partial \Omega_b} \partial_{\tau} u \, \overline{\psi} \, ds, \quad \text{for all } \psi \in H^1_{\alpha}(\Omega_b). \tag{3.7}$$

Multiplying the equations (3.6) and (3.7) by the constant factors $\frac{\omega\mu}{\kappa^2}$ and $\frac{\gamma}{\kappa^2}$, respectively, then taking the sum of the two formulas, we get

$$\int_{\partial\Omega_b} \frac{\omega\mu}{\kappa^2} \partial_n v \,\overline{\psi} - \frac{\gamma}{\kappa^2} \partial_\tau u \,\overline{\psi} \, ds = \int_{\Omega_b} \left[\frac{\omega\mu}{\kappa^2} \nabla v \cdot \nabla \overline{\psi} + \frac{\gamma}{\kappa^2} \nabla u \cdot \nabla^\perp \overline{\psi} - \omega\mu \, v \,\overline{\psi} \right] \, dx. \tag{3.8}$$

Recalling the boundary conditions (2.6) on Γ , the left-hand terms of (3.5) and (3.8) over the integral Γ can be reformulated as

$$\int_{\Gamma} \frac{\omega\epsilon}{\kappa^2} \partial_n u \,\overline{\varphi} + \frac{\gamma}{\kappa^2} \partial_\tau v \,\overline{\varphi} \, ds = \int_{\Gamma} \frac{\omega\epsilon}{\lambda\kappa^2} \left(-\frac{i\kappa^2}{\omega\epsilon} u \right) \overline{\varphi} \, ds = -\frac{i}{\lambda} \int_{\Gamma} u \,\overline{\varphi} \, ds,$$
$$\int_{\Gamma} \frac{\omega\mu}{\kappa^2} \partial_n v \,\overline{\psi} - \frac{\gamma}{\kappa^2} \partial_\tau u \,\overline{\psi} \, ds = \int_{\Gamma} \frac{\omega\mu}{\kappa^2} \left(-\frac{i\lambda\kappa^2}{\omega\mu} v \right) \overline{\psi} \, ds = -i\lambda \int_{\Gamma} v \,\overline{\psi} \, ds.$$

Therefore, we need to find $(u, v) \in X$ such that for all $(\varphi, \psi) \in X$,

$$0 = \frac{i}{\lambda} \int_{\Gamma} u \,\overline{\varphi} \, ds + \int_{\Omega_b} \left[\frac{\omega \epsilon}{\kappa^2} \nabla u \cdot \nabla \overline{\varphi} - \frac{\gamma}{\kappa^2} \nabla v \cdot \nabla^{\perp} \overline{\varphi} - \omega \epsilon \, u \,\overline{\varphi} \right] \, dx - \int_{\Gamma_b} \left[\frac{\omega \epsilon}{\kappa^2} \partial_n u \,\overline{\varphi} + \frac{\gamma}{\kappa^2} \partial_\tau v \,\overline{\varphi} \right] \, ds,$$
(3.9)
$$0 = i\lambda \int_{\Gamma} v \,\overline{\psi} \, ds + \int_{\Omega_b} \left[\frac{\omega \mu}{\kappa^2} \nabla v \cdot \nabla \overline{\psi} + \frac{\gamma}{\kappa^2} \nabla u \cdot \nabla^{\perp} \overline{\psi} - \omega \mu \, v \,\overline{\psi} \right] \, dx - \int_{\Gamma_b} \left[\frac{\omega \mu}{\kappa^2} \partial_n v \,\overline{\psi} - \frac{\gamma}{\kappa^2} \partial_\tau u \,\overline{\psi} \right] \, ds.$$
(3.10)

Combining (3.9) and (3.10), we get

$$\int_{\Gamma} \frac{i}{\lambda} u \,\overline{\varphi} + i\lambda v \,\overline{\psi} \, ds + \int_{\Omega_b} \left[\frac{\omega \epsilon}{\kappa^2} \nabla u \cdot \nabla \overline{\varphi} - \frac{\gamma}{\kappa^2} \nabla v \cdot \nabla^{\perp} \overline{\varphi} - \omega \epsilon \, u \,\overline{\varphi} + \frac{\omega \mu}{\kappa^2} \nabla v \cdot \nabla \overline{\psi} + \frac{\gamma}{\kappa^2} \nabla u \cdot \nabla^{\perp} \overline{\psi} - \omega \mu \, v \,\overline{\psi} \right] \, dx - \int_{\Gamma_b} \frac{1}{\kappa^2} \left(\begin{array}{c} \omega \epsilon \partial_n u + \gamma \partial_\tau v \\ \omega \mu \partial_n v - \gamma \partial_\tau u \end{array} \right) \cdot \left(\begin{array}{c} \overline{\varphi} \\ \overline{\psi} \end{array} \right) \, ds = 0. \quad (3.11)$$

Definition 3.2 (DtN map). The Dirichlet-to-Neumann (DtN) map T is defined by

$$T: (g_1, g_2)^\top \to -\left(\frac{\omega\epsilon}{\kappa^2}\partial_n w_1 + \frac{\gamma}{\kappa^2}\partial_\tau w_2, \frac{\omega\mu}{\kappa^2}\partial_n w_2 - \frac{\gamma}{\kappa^2}\partial_\tau w_1\right)^\top \quad on \ \Gamma_b,$$

where $w_j (j = 1, 2)$ is the unique radiation solution to the Helmholtz equation $\Delta w_j + \kappa^2 w_j = 0$ in $x_2 > b$ with the Dirichlet boundary condition $w_j = g_j$ on Γ_b . Now we want to derive an analytical expression of the DTN map T. For the α quasiperiodic vector function $g = (g_1, g_2)^\top \in H^{1/2}_{\alpha}(\Gamma_b)^2$, we can get its Fourier expansion $g(x_1) = \sum_{n \in \mathbb{Z}} \hat{g}_n e^{i\alpha_n x_1}$, where $\hat{g}_n = (\hat{g}_{n,1}, \hat{g}_{n,2})^\top$. It is easy to deduce that

$$w_j(x) = \sum_{n \in \mathbb{Z}} \hat{g}_{n,j} e^{i\alpha_n x_1 + i\beta_n (x_2 - b)}, \quad x_2 > b, \ j = 1, 2,$$

where w_i is the function specified in the Definition 3.2. Direct calculations show

$$-\left(\frac{\omega\epsilon}{\kappa^{2}}\partial_{n}w_{1}+\frac{\gamma}{\kappa^{2}}\partial_{\tau}w_{2},\frac{\omega\mu}{\kappa^{2}}\partial_{n}w_{2}-\frac{\gamma}{\kappa^{2}}\partial_{\tau}w_{1}\right)\Big|_{\Gamma_{b}}$$

$$= -\frac{1}{\kappa^{2}}\left(\omega\epsilon\sum_{n\in\mathbb{Z}}i\beta_{n}\hat{g}_{n,1}e^{i\alpha_{n}x_{1}}+\gamma\sum_{n\in\mathbb{Z}}(-i\alpha_{n})\hat{g}_{n,2}e^{i\alpha_{n}x_{1}},\frac{\omega\mu\sum_{n\in\mathbb{Z}}i\beta_{n}\hat{g}_{n,2}e^{i\alpha_{n}x_{1}}-\gamma\sum_{n\in\mathbb{Z}}(-i\alpha_{n})\hat{g}_{n,1}e^{i\alpha_{n}x_{1}}\right)$$

$$= -\frac{1}{\kappa^{2}}\sum_{n\in\mathbb{Z}}\left(\begin{array}{c}i\omega\epsilon\beta_{n}&-i\gamma\alpha_{n}\\i\gamma\alpha_{n}&i\omega\mu\beta_{n}\end{array}\right)\left(\begin{array}{c}\hat{g}_{n,1}\\\hat{g}_{n,2}\end{array}\right)e^{i\alpha_{n}x_{1}}$$

$$= \sum_{n\in\mathbb{Z}}M_{n}\hat{g}_{n}e^{i\alpha_{n}x_{1}},$$

where

$$M_n = \frac{1}{\kappa^2} \begin{pmatrix} -i\omega\epsilon\beta_n & i\gamma\alpha_n \\ -i\gamma\alpha_n & -i\omega\mu\beta_n \end{pmatrix}.$$
 (3.12)

Hence, the operator T acting on the α -quasiperiodic vector function $w \in H^{1/2}_{\alpha}(\Gamma_b)^2$ can be expressed as

$$(Tw)(x) = \sum_{n \in \mathbb{Z}} M_n \hat{w}_n e^{i\alpha_n x}, \quad \hat{w}_n = \frac{1}{2\pi} \int_0^{2\pi} w(x) e^{-i\alpha_n x} \, dx \in \mathbb{C}^2.$$

Lemma 3.3. (see [2]) The DtN operator $T: H^{1/2}_{\alpha}(\Gamma_b)^2 \to H^{-1/2}_{\alpha}(\Gamma_b)^2$ is continuous, i.e., there exists a positive constant C such that

$$||Tw||_{H^{-1/2}_{\alpha}(\Gamma_b)^2} \le C ||w||_{H^{1/2}_{\alpha}(\Gamma_b)^2} \quad \text{for all } w \in H^{1/2}_{\alpha}(\Gamma_b)^2.$$

Then we come back to the last term of the left-hand side of (3.11). Direct calculations show

$$T\left(\begin{array}{c}u^{s}|_{\Gamma_{b}}\\v^{s}|_{\Gamma_{b}}\end{array}\right) = \sum_{n\in\mathbb{Z}}M_{n}\left(\begin{array}{c}u_{n}\\v_{n}\end{array}\right)e^{i\alpha_{n}x_{1}+i\beta_{n}b} = \sum_{n\in\mathbb{Z}}\frac{1}{\kappa^{2}}\left(\begin{array}{c}-i\omega\epsilon\beta_{n}u_{n}+i\gamma\alpha_{n}v_{n}\\-i\gamma\alpha_{n}u_{n}-i\omega\mu\beta_{n}v_{n}\end{array}\right)e^{i\alpha_{n}x_{1}+i\beta_{n}b},$$
$$T\left(\begin{array}{c}u^{i}|_{\Gamma_{b}}\\v^{i}|_{\Gamma_{b}}\end{array}\right) = M_{0}\left(\begin{array}{c}p_{3}\\q_{3}\end{array}\right)e^{i\alpha x_{1}-i\beta b} = -\frac{1}{\kappa^{2}}\left(\begin{array}{c}i\omega\epsilon\beta p_{3}-i\gamma\alpha q_{3}\\i\gamma\alpha p_{3}+i\omega\mu\beta q_{3}\end{array}\right)e^{i\alpha x_{1}-i\beta b}.$$

Therefore,

$$\frac{1}{\kappa^{2}} \begin{pmatrix} \omega\epsilon\partial_{\nu}u + \gamma\partial_{\tau}v \\ \omega\mu\partial_{\nu}v - \gamma\partial_{\tau}u \end{pmatrix}\Big|_{\Gamma_{b}} \\
= \sum_{n\in\mathbb{Z}} \frac{1}{\kappa^{2}} \begin{pmatrix} i\omega\epsilon\beta_{n} & -i\gamma\alpha_{n} \\ i\gamma\alpha_{n} & i\omega\mu\beta_{n} \end{pmatrix} \begin{pmatrix} u_{n} \\ v_{n} \end{pmatrix} e^{i\alpha_{n}x_{1}+i\beta_{n}b} - \frac{1}{\kappa^{2}} \begin{pmatrix} i\omega\epsilon\beta p_{3}+i\gamma\alpha q_{3} \\ i\omega\mu\beta q_{3}-i\gamma\alpha p_{3} \end{pmatrix} e^{i\alpha x_{1}-i\beta b} \\
= -T \begin{pmatrix} u^{s}|_{\Gamma_{b}} \\ v^{s}|_{\Gamma_{b}} \end{pmatrix} - \frac{1}{\kappa^{2}} \begin{pmatrix} i\omega\epsilon\beta p_{3}+i\gamma\alpha q_{3} \\ i\omega\mu\beta q_{3}-i\gamma\alpha p_{3} \end{pmatrix} e^{i\alpha x_{1}-i\beta b} \\
= -T \begin{pmatrix} u|_{\Gamma_{b}} \\ v|_{\Gamma_{b}} \end{pmatrix} + T \begin{pmatrix} u^{i}|_{\Gamma_{b}} \\ v^{i}|_{\Gamma_{b}} \end{pmatrix} - \frac{1}{\kappa^{2}} \begin{pmatrix} i\omega\epsilon\beta p_{3}-i\gamma\alpha q_{3} \\ i\omega\mu\beta q_{3}-i\gamma\alpha p_{3} \end{pmatrix} e^{i\alpha x_{1}-i\beta b} \\
= -T \begin{pmatrix} u|_{\Gamma_{b}} \\ v|_{\Gamma_{b}} \end{pmatrix} - \frac{1}{\kappa^{2}} \begin{pmatrix} i\omega\epsilon\beta p_{3}-i\gamma\alpha q_{3} \\ i\gamma\alpha p_{3}+i\omega\mu\beta q_{3} \end{pmatrix} e^{i\alpha x_{1}-i\beta b} - \frac{1}{\kappa^{2}} \begin{pmatrix} i\omega\epsilon\beta p_{3}+i\gamma\alpha q_{3} \\ i\omega\mu\beta q_{3}-i\gamma\alpha p_{3} \end{pmatrix} e^{i\alpha x_{1}-i\beta b} \\
= -T \begin{pmatrix} u|_{\Gamma_{b}} \\ v|_{\Gamma_{b}} \end{pmatrix} - \frac{2}{\kappa^{2}} \begin{pmatrix} i\omega\epsilon\beta p_{3} \\ i\omega\mu\beta q_{3} \end{pmatrix} e^{i\alpha x_{1}-i\beta b}.$$
(3.13)

Note that in deriving (3.13), we have used the expression of (u, v) given by

$$\begin{split} u &= p_3 e^{i\alpha x_1 - i\beta x_2} + \sum_{n \in \mathbb{Z}} u_n e^{i\alpha_n x_1 + i\beta_n x_2}, \\ v &= q_3 e^{i\alpha x_1 - i\beta x_2} + \sum_{n \in \mathbb{Z}} v_n e^{i\alpha_n x_1 + i\beta_n x_2}, \qquad x_2 > \Gamma_{\max}. \end{split}$$

Inserting (3.13) into (3.11), we get the variational formulation

$$B(u, v; \varphi, \psi) = F(\varphi, \psi) \quad \text{for all } (\varphi, \psi) \in X,$$
(3.14)

where

$$B(u,v;\varphi,\psi) := \int_{\Gamma} \frac{i}{\lambda} u \,\overline{\varphi} + i\lambda v \,\overline{\psi} \, ds + \int_{\Omega_b} \left[\frac{\omega\epsilon}{\kappa^2} \nabla u \cdot \nabla \overline{\varphi} - \frac{\gamma}{\kappa^2} \nabla v \cdot \nabla^{\perp} \overline{\varphi} + \frac{\omega\mu}{\kappa^2} \nabla v \cdot \nabla \overline{\psi} + \frac{\gamma}{\kappa^2} \nabla u \cdot \nabla^{\perp} \overline{\psi} - \omega\epsilon \, u \,\overline{\varphi} - \omega\mu \, v \,\overline{\psi} \right] \, dx + \int_{\Gamma_b} T \left(\begin{array}{c} u \\ v \end{array} \right) \cdot \left(\begin{array}{c} \overline{\varphi} \\ \overline{\psi} \end{array} \right) \, ds, \quad (3.15)$$

$$2i\omega\epsilon\beta e^{-i\beta b} \quad f \quad = -i$$

$$F(\varphi,\psi) := -\frac{2i\omega\epsilon\beta e^{-i\beta b}}{\kappa^2} \int_{\Gamma_b} (\epsilon p_3\overline{\varphi} + \mu q_3\overline{\psi}) e^{i\alpha x_1} \, ds.$$
(3.16)

Below we prove an energy formula under the impedance boundary condition.

Lemma 3.4. Let $u, v \in H^1_{\alpha}(\Omega_b)$ be the total fields to our conical diffraction problem. We have the energy formula

$$\frac{2\pi\omega}{\kappa^2} \sum_{|\alpha_n| \le \kappa} \beta_n(\epsilon |u_n|^2 + \mu |v_n|^2) = \int_{\Gamma} \frac{1}{\lambda} |u|^2 + \lambda |v|^2 \, ds + \frac{2\pi\omega\beta}{\kappa^2} (\epsilon |p_3|^2 + \mu |q_3|^2). \tag{3.17}$$

Proof. By (3.11) and taking $\varphi = u, \psi = v$, we have

$$0 = \int_{\Gamma} \frac{i}{\lambda} |u|^2 + i\lambda |v|^2 \, ds - \frac{1}{\kappa^2} \int_{\Gamma_b} \left(\frac{\omega \epsilon \partial_n u + \gamma \partial_\tau v}{\omega \mu \partial_n v - \gamma \partial_\tau u} \right) \cdot \left(\frac{\overline{u}}{\overline{v}} \right) \, ds \\ + \int_{\Omega_b} \left[\frac{\omega \epsilon}{\kappa^2} |\nabla u|^2 - \frac{\gamma}{\kappa^2} \nabla v \cdot \nabla^\perp \overline{u} - \omega \epsilon |u|^2 + \frac{\omega \mu}{\kappa^2} |\nabla v|^2 + \frac{\gamma}{\kappa^2} \nabla u \cdot \nabla^\perp \overline{v} - \omega \mu |v|^2 \right] \, dx.$$
(3.18)

We want to calculate the imaginary part of (3.18). First, we have

$$\operatorname{Im} \int_{\Gamma_{b}} \begin{pmatrix} \omega \epsilon \partial_{n} u + \gamma \partial_{\tau} v \\ \omega \mu \partial_{n} v - \gamma \partial_{\tau} u \end{pmatrix} \cdot \begin{pmatrix} \overline{u} \\ \overline{v} \end{pmatrix} ds$$
$$= \operatorname{Im} \int_{0}^{2\pi} \begin{pmatrix} \omega \epsilon \partial_{2} u - \gamma \partial_{1} v \\ \omega \mu \partial_{2} v + \gamma \partial_{1} u \end{pmatrix} \Big|_{\Gamma_{b}} \cdot \begin{pmatrix} \overline{u} \\ \overline{v} \end{pmatrix} \Big|_{\Gamma_{b}} dx_{1}$$

For the total field u and v, we can easily get

$$\begin{split} \operatorname{Im} & \int_{0}^{2\pi} \left[\omega \epsilon (-i\beta p_{3}e^{i\alpha x_{1}-i\beta b}+\sum_{n\in\mathbb{Z}}i\beta_{n}u_{n}e^{i\alpha_{n}x_{1}+i\beta_{n}b}) -\gamma(i\alpha q_{3}e^{i\alpha x_{1}-i\beta b}\right.\\ &+\sum_{n\in\mathbb{Z}}i\alpha_{n}v_{n}e^{i\alpha_{n}x_{1}+i\beta_{n}b}) \right] \left(\overline{p_{3}e^{i\alpha x_{1}-i\beta b}+\sum_{m\in\mathbb{Z}}u_{m}e^{i\alpha_{m}x_{1}+i\beta_{m}b}} \right)\\ &+\left[\omega\mu(-i\beta q_{3}e^{i\alpha x_{1}-i\beta b}+\sum_{n\in\mathbb{Z}}i\beta_{n}v_{n}e^{i\alpha_{n}x_{1}+i\beta_{n}b}) +\gamma(i\alpha p_{3}e^{i\alpha x_{1}-i\beta b}\right.\\ &+\sum_{n\in\mathbb{Z}}i\alpha_{n}u_{n}e^{i\alpha_{n}x_{1}+i\beta_{n}b}) \right] \left(\overline{q_{3}e^{i\alpha x_{1}-i\beta b}+\sum_{m\in\mathbb{Z}}v_{m}e^{i\alpha_{m}x_{1}+i\beta_{m}b}} \right) dx_{1}\\ &=\operatorname{Im} 2\pi \left[\omega\epsilon \left(-i\beta |p_{3}|^{2}+\sum_{n\in\mathbb{Z}}i\beta_{n}|u_{n}|^{2} \right) -\gamma \left(i\alpha q_{3}\bar{p}_{3}+\sum_{n\in\mathbb{Z}}i\alpha_{n}v_{n}\bar{u}_{n} \right) \right.\\ &+\omega\mu \left(-i\beta |q_{3}|^{2}+\sum_{n\in\mathbb{Z}}i\beta_{n}|v_{n}|^{2} \right) +\gamma \left(i\alpha p_{3}\bar{q}_{3}+\sum_{n\in\mathbb{Z}}i\alpha_{n}u_{n}\bar{v}_{n} \right) \right]. \end{split}$$

Therefore,

$$\operatorname{Im} \int_{\Gamma_{b}} \begin{pmatrix} \omega \epsilon \partial_{n} u + \gamma \partial_{\tau} v \\ \omega \mu \partial_{n} v - \gamma \partial_{\tau} u \end{pmatrix} \cdot \begin{pmatrix} \overline{u} \\ \overline{v} \end{pmatrix} ds$$
$$= -2\pi \omega \beta \left(\epsilon |p_{3}|^{2} + \mu |q_{3}|^{2} \right) + 2\pi \sum_{|\alpha_{n}| \leq \kappa} \omega \beta_{n} \left(\epsilon |u_{n}|^{2} + \mu |v_{n}|^{2} \right).$$
(3.19)

In addition,

$$\operatorname{Im} \int_{\Gamma_{b}} \nabla v \cdot \nabla^{\perp} \overline{u} - \nabla u \cdot \nabla^{\perp} \overline{v} \, dx$$

=
$$\operatorname{Im} \int_{\Gamma_{b}} -\partial_{1} v \partial_{2} \overline{u} + \partial_{2} v \partial_{1} \overline{u} - (-\partial_{1} u \partial_{2} \overline{v} + \partial_{2} u \partial_{1} \overline{v}) \, dx$$

=
$$\operatorname{Im} \int_{\Gamma_{b}} -(\partial_{1} v \partial_{2} \overline{u} + \partial_{2} u \partial_{1} \overline{v}) + (\partial_{2} v \partial_{1} \overline{u} + \partial_{1} u \partial_{2} \overline{v}) \, dx$$

=
$$0. \qquad (3.20)$$

Taking the imaginary part of (3.18) and using (3.19) and (3.20), we obtain

$$0 = \int_{\Gamma} \frac{1}{\lambda} |u|^2 + \lambda |v|^2 \, ds - \operatorname{Im} \frac{1}{\kappa^2} \int_0^{2\pi} \left(\begin{array}{c} \omega \epsilon \partial_2 u - \gamma \partial_1 v \\ \omega \mu \partial_2 v + \gamma \partial_1 u \end{array} \right) \Big|_{\Gamma_b} \cdot \left(\begin{array}{c} \overline{u} \\ \overline{v} \end{array} \right) \Big|_{\Gamma_b} \, dx_1$$
$$= \int_{\Gamma} \frac{1}{\lambda} |u|^2 + \lambda |v|^2 \, ds + \frac{2\pi \omega \beta}{\kappa^2} \left(\epsilon |p_3|^2 + \mu |q_3|^2 \right) - \frac{2\pi \omega}{\kappa^2} \sum_{|\alpha_n| \le \kappa} \beta_n \left(\epsilon |u_n|^2 + \mu |v_n|^2 \right),$$

which completes the proof.

Theorem 3.5. Suppose that Γ is a Lipschitz curve, $k^2 \neq \gamma^2$ and the impedance coefficient $\lambda < 0$. Then, the variational problem (3.14) has at most one solution $(u, v) \in X$.

Proof. To prove uniqueness, we assume $u^i = v^i = 0$, i.e. $p_3 = q_3 = 0$. Choosing $\varphi = u, \psi = v$ in (3.14) and taking the imaginary part, we have

$$\int_{\Gamma} \frac{1}{\lambda} |u|^2 + \lambda |v|^2 \, ds + \operatorname{Im} \, \int_{\Gamma_b} T\left(\begin{array}{c} u\\ v \end{array}\right) \cdot \left(\begin{array}{c} \overline{u}\\ \overline{v} \end{array}\right) \, ds = 0. \tag{3.21}$$

Next, we calculate the second term of (3.21). By the definition of T (see Definition 3.2), we have for $w = (u, v)^{\top}$ that

$$\operatorname{Im} \int_{\Gamma_b} Tw \cdot \overline{w} \, ds = \operatorname{Im} \int_{\Gamma_b} \sum_{n \in \mathbb{Z}} M_n \hat{w}_n e^{i\alpha_n x} \cdot \overline{\sum_{m \in \mathbb{Z}} \hat{w}_m e^{i\alpha_m x}} \, ds$$
$$= \operatorname{Im} 2\pi \sum_{n \in \mathbb{Z}} M_n \hat{w}_n \cdot \overline{\hat{w}_n}$$
$$= 2\pi \sum_{n \in \mathbb{Z}} (\operatorname{Im} M_n) \hat{w}_n \cdot \overline{\hat{w}_n},$$

where $\hat{w}_n = (\hat{w}_{n,1}, \hat{w}_{n,2}) = (u_n, v_n)$. Recalling the expression of M_n , we have

$$\operatorname{Im} M_{n} = \frac{1}{2i} (M_{n} - M_{n}^{*})$$

$$= \frac{1}{2i} \frac{1}{\kappa^{2}} \left[\begin{pmatrix} -i\omega\epsilon\beta_{n} & i\gamma\alpha_{n} \\ -i\gamma\alpha_{n} & -i\omega\mu\beta_{n} \end{pmatrix} - \begin{pmatrix} i\omega\epsilon\overline{\beta_{n}} & i\gamma\alpha_{n} \\ -i\gamma\alpha_{n} & i\omega\mu\overline{\beta_{n}} \end{pmatrix} \right]$$

$$= \frac{1}{\kappa^{2}} \begin{pmatrix} \operatorname{Im} (-i\omega\epsilon\beta_{n}) & 0 \\ 0 & \operatorname{Im} (-i\omega\mu\beta_{n}) \end{pmatrix}$$

$$= \begin{cases} \frac{1}{\kappa^{2}} \begin{pmatrix} -\omega\epsilon\beta_{n} & 0 \\ 0 & -\omega\mu\beta_{n} \end{pmatrix}, & |\alpha_{n}| \leq \kappa, \\ 0, & |\alpha_{n}| > \kappa. \end{cases}$$

Therefore,

$$\operatorname{Im} \int_{\Gamma_b} Tw \cdot \overline{w} \, ds = 2\pi \sum_{|\alpha_n| \le \kappa} \frac{1}{\kappa^2} \begin{pmatrix} -\omega\epsilon\beta_n & 0\\ 0 & -\omega\mu\beta_n \end{pmatrix} \begin{pmatrix} \hat{w}_{n1} \\ \hat{w}_{n2} \end{pmatrix} \cdot \begin{pmatrix} \overline{\hat{w}_{n1}} \\ \overline{\hat{w}_{n2}} \end{pmatrix}$$
$$= -\frac{2\pi\omega}{\kappa^2} \sum_{|\alpha_n| \le \kappa} \beta_n(\epsilon |\hat{w}_{n1}|^2 + \mu |\hat{w}_{n2}|^2) \le 0.$$

Inserting these results into (3.21), we have

$$\int_{\Gamma} \frac{1}{\lambda} |u|^2 + \lambda |v|^2 \, ds - \frac{2\pi\omega}{\kappa^2} \sum_{|\alpha_n| \le \kappa} \beta_n(\epsilon |u_n|^2 + \mu |v_n|^2) = 0.$$

Noting that $\lambda < 0$, we have u = v = 0 on Γ . By the impedance radiation condition (2.6), we have $\partial_n u = \partial_n v = 0$ on Γ . By Holmgren theorem, u = v = 0 in Ω .

Remark 3.6. We can also prove the uniqueness result by taking the imaginary part of the energy formula (3.17) with $p_3 = q_3 = 0$.

The proof of Theorem 3.5 provides an alternative approach to the proof of the energy formula via matrix operations.

Definition 3.7 (Strong ellipticity). We call a bounded sesquilinear form $B(\cdot, \cdot)$ given on some Hilbert space X strongly elliptic if there exists a complex number θ , $|\theta| = 1$ and a compact form $q(\cdot, \cdot)$ such that

$$\operatorname{Re}\left(\theta B(u, u)\right) \ge c \|u\|_X^2 - q(u, u) \qquad \text{for all } u \in X,$$

for some constant c > 0.

The following theorem establishes the strong ellipticity of the form (3.15) and leads, together with Theorem 3.5 or Remark 3.6, to the solvability results for the conical diffraction problem.

Theorem 3.8. The sesquilinear form B defined in (3.14) is strongly elliptic over X.

We divide the proof of Theorem 3.8 into several lemmas. It is convenient to reformulate the variational form (3.15) as follows (see [5])

$$B(u,v;\varphi,\psi) = A(u,v;\varphi,\psi) + B_1(u,v;\varphi,\psi) + C(u,v;\varphi,\psi) + D(u,v;\varphi,\psi),$$

where

$$\begin{array}{lll} A(u,v;\varphi,\psi) &:=& \displaystyle \int_{\Gamma} \frac{i}{\lambda} u \,\overline{\varphi} + i\lambda v \,\overline{\psi} \, ds, \\ C(u,v;\varphi,\psi) &:=& \displaystyle \int_{\Omega_b} \omega \epsilon \, u \,\overline{\varphi} + \omega \mu \, v \, \overline{\psi} \, dx, \\ D(u,v;\varphi,\psi) &:=& \displaystyle \int_{\Gamma_b} T \left(\begin{array}{c} u \\ v \end{array} \right) \cdot \left(\begin{array}{c} \overline{\varphi} \\ \overline{\psi} \end{array} \right) \, ds, \end{array}$$

and

$$B_{1}(u,v;\varphi,\psi) := \int_{\Omega_{b}} \left[\frac{\omega\epsilon}{\kappa^{2}} \nabla u \cdot \nabla \overline{\varphi} - \frac{\gamma}{\kappa^{2}} \nabla v \cdot \nabla^{\perp} \overline{\varphi} + \frac{\omega\mu}{\kappa^{2}} \nabla v \cdot \nabla \overline{\psi} + \frac{\gamma}{\kappa^{2}} \nabla u \cdot \nabla^{\perp} \overline{\psi} \right] dx$$
$$= \int_{\Omega_{b}} \mathcal{D}(\partial_{1}u, \partial_{1}v, \partial_{2}u, \partial_{2}v)^{\mathsf{T}} \cdot \overline{(\partial_{1}u, \partial_{1}v, \partial_{2}u, \partial_{2}v)^{\mathsf{T}}} dx,$$

with the matrix \mathcal{D} given by (see [5])

$$\mathcal{D} = \frac{1}{\kappa^2} \begin{pmatrix} \omega \epsilon & 0 & 0 & -\gamma \\ 0 & \omega \mu & \gamma & 0 \\ 0 & \gamma & \omega \epsilon & 0 \\ -\gamma & 0 & 0 & \omega \mu \end{pmatrix}.$$

We can further write B_1 into the matrix form

$$B_1(u,v;\varphi,\psi) = \int_{\Omega_b} N^+ \partial^+ \begin{pmatrix} u \\ v \end{pmatrix} \cdot \overline{\partial^+ \begin{pmatrix} \varphi \\ \psi \end{pmatrix}} + N^- \partial^- \begin{pmatrix} u \\ v \end{pmatrix} \cdot \overline{\partial^- \begin{pmatrix} \varphi \\ \psi \end{pmatrix}} \, ds \tag{3.22}$$

where

$$N^{\pm} = \frac{1}{\kappa^2} \begin{pmatrix} \omega \epsilon & \pm i\gamma \\ \mp i\gamma & \omega \mu \end{pmatrix}, \quad \partial^+ := \frac{1}{\sqrt{2}} (-i\partial_1 + \partial_2), \quad \partial^- := \frac{1}{\sqrt{2}} (\partial_1 - i\partial_2).$$

To study the form B, we need the following lemma.

Lemma 3.9. Choose $\theta = \frac{i+\delta}{|i+\delta|}$ with $\delta > 0$ sufficiently small. (i) For any $\xi \in \mathbb{C}^2$, we have $\operatorname{Re}(\theta N^{\pm}\xi \cdot \overline{\xi}) \geq C_N |\xi|^2$, where

$$C_N = \frac{1}{2\omega\epsilon\mu\cos^2\phi} \operatorname{Re}\theta \left[(\epsilon+\mu) - \sqrt{(\epsilon-\mu)^2 + 4\epsilon\mu\sin^2\phi} \right] \ge 0.$$
(3.23)

(ii) Let $M_n \in \mathbb{C}^{2 \times 2}$ be defined by (3.12). It holds that $\operatorname{Re}(\theta M_n) \geq 0$ for all $n \in \mathbb{Z} \setminus \mathcal{A}$, where the index set \mathcal{A} is defined by

$$\mathcal{A} = \{ n \in \mathbb{Z} : -k(1 + \sin\theta\cos\phi) < n \le -k\cos\phi(1 + \sin\theta) \\ or \ k\cos\phi(1 - \sin\theta) \le n < k(1 - \sin\theta\cos\phi) \}.$$
(3.24)

Proof. (i) By the definition of N^{\pm} , we have

$$\operatorname{Re}\left(\theta N^{\pm}\right) = \frac{\theta N^{\pm} + (\theta N^{\pm})^{*}}{2} = \frac{1}{\kappa^{2}} \begin{pmatrix} \omega \epsilon \operatorname{Re}\theta & \pm i\gamma \operatorname{Re}\theta \\ \mp i\gamma \operatorname{Re}\theta & \omega \mu \operatorname{Re}\theta \end{pmatrix},$$

which is a Hermitian matrix. Recalling that $\gamma = k \sin \phi = \omega \sqrt{\epsilon \mu} \sin \phi$ and $\kappa^2 = k^2 \cos^2 \phi$, we compute the eigenvalues of $\operatorname{Re}(\theta N^{\pm})$ as following

$$\lambda_{1} = \frac{1}{2\omega\epsilon\mu\cos^{2}\phi}\operatorname{Re}\theta\left[(\epsilon+\mu) + \sqrt{(\epsilon-\mu)^{2} + 4\epsilon\mu\sin^{2}\phi}\right] > 0,$$

$$\lambda_{2} = \frac{1}{2\omega\epsilon\mu\cos^{2}\phi}\operatorname{Re}\theta\left[[(\epsilon+\mu) - \sqrt{(\epsilon-\mu)^{2} + 4\epsilon\mu\sin^{2}\phi}\right] \ge 0.$$

Defining $C_N = \lambda_2 < \lambda_1$, and by [7, Theorem 4.2.2], we complete the first part of the proof. (ii) Recalling the definition of M_n , we have

$$\theta M_n = \frac{1}{\kappa^2} \begin{pmatrix} -i\omega\epsilon\beta_n\theta & i\gamma\alpha_n\theta \\ -i\gamma\alpha_n\theta & -i\omega\mu\beta_n\theta \end{pmatrix} = \frac{1}{\kappa^2} \begin{pmatrix} -i(\omega\mu)^{-1}k^2\beta_n\theta & i\gamma\alpha_n\theta \\ -i\gamma\alpha_n\theta & -i\omega\mu\beta_n\theta \end{pmatrix}.$$

Case 1. $|\alpha_n|<\kappa,$ i.e., $\beta_n\in\mathbb{R}$ is real number. We have

$$(\theta M_n)^* = \frac{1}{\kappa^2} \left(\begin{array}{cc} i(\omega\mu)^{-1}k^2\beta_n\bar{\theta} & i\gamma\alpha_n\bar{\theta} \\ -i\gamma\alpha_n\bar{\theta} & i\omega\mu\beta_n\bar{\theta} \end{array} \right).$$

Therefore,

$$\operatorname{Re}\left(\theta M_{n}\right) = \frac{\theta M_{n} + (\theta M_{n})^{*}}{2} = \frac{1}{\kappa^{2}} \left(\begin{array}{c} (\omega \mu)^{-1} k^{2} \beta_{n} \operatorname{Im} \theta & i \gamma \alpha_{n} \operatorname{Re} \theta \\ -i \gamma \alpha_{n} \operatorname{Re} \theta & \omega \mu \beta_{n} \operatorname{Im} \theta \end{array} \right).$$

In this case, $\operatorname{Re}(\theta M_n) \geq 0$ if and only if the following two conditions are satisfied:

$$\operatorname{Im} \theta \geq 0,$$

$$det(\operatorname{Re}(\theta M_n)) = \frac{1}{\kappa^4} \left[(\omega \mu)^{-1} k^2 \beta_n \operatorname{Im} \theta \omega \mu \beta_n \operatorname{Im} \theta - \gamma^2 \alpha_n^2 (\operatorname{Re} \theta)^2 \right]$$

$$= \frac{1}{\kappa^4} \left[k^2 \beta_n^2 - \gamma^2 \alpha_n^2 \delta^2 \right] (\operatorname{Im} \theta)^2 \geq 0.$$
(3.25)

The conditions in (3.25) obviously hold due to the definition of θ with a small $\delta > 0$.

Case 2. $|\alpha_n| \ge \kappa$, i.e., β_n is a pure imaginary number. We have

$$(\theta M_n)^* = \frac{1}{\kappa^2} \begin{pmatrix} (\omega \mu)^{-1} k^2 |\beta_n| \bar{\theta} & i\gamma \alpha_n \bar{\theta} \\ -i\gamma \alpha_n \bar{\theta} & \omega \mu |\beta_n| \bar{\theta} \end{pmatrix}.$$

Therefore,

$$\operatorname{Re}\left(\theta M_{n}\right) = \frac{\theta M_{n} + (\theta M_{n})^{*}}{2} = \frac{1}{\kappa^{2}} \left(\begin{array}{cc} (\omega \mu)^{-1} k^{2} |\beta_{n}| \operatorname{Re} \theta & i \gamma \alpha_{n} \operatorname{Re} \theta \\ -i \gamma \alpha_{n} \operatorname{Re} \theta & \omega \mu |\beta_{n}| \operatorname{Re} \theta \end{array}\right).$$

In this case, $\operatorname{Re}(\theta M_n) \geq 0$ if and only if the following two conditions are satisfied:

$$\operatorname{Re} \theta \geq 0,$$

$$det(\operatorname{Re} (\theta M_n)) = \frac{1}{\kappa^4} \left[(\omega \mu)^{-1} k^2 |\beta_n| (\operatorname{Re} \theta) \omega \mu |\beta_n| \operatorname{Re} \theta - \gamma^2 \alpha_n^2 (\operatorname{Re} \theta)^2 \right]$$

$$= \frac{1}{\kappa^4} \left[k^2 |\beta_n|^2 - \gamma^2 \alpha_n^2 \right] (\operatorname{Re} \theta)^2 \geq 0.$$

The first condition is obvious. The second condition can be fulfilled if $k^2 |\beta_n|^2 - \gamma^2 \alpha_n^2 \ge 0$. Recalling that $|\beta_n|^2 = \alpha_n^2 - (k^2 - \gamma^2)$, we have $\alpha_n^2 \ge k^2$. Combining the above two cases, we get that when

$$n \in \mathcal{B} := \{ n \in \mathbb{Z} : k^2 - \gamma^2 \le \alpha_n^2 < k^2 \},\$$

Re (θM_n) is not positive definite. We should point out that $n \in \mathcal{B}$ if $\beta_n = 0$ (i.e. $|\alpha_n| = \kappa$). Next, we continue to simplify the set \mathcal{B} . Recalling $\alpha = k \sin \theta \cos \phi$ and $\alpha_n = n + \alpha$, we have

$$\begin{aligned} \mathcal{B} &= \{ n \in \mathbb{Z} : k^2 - \gamma^2 \le \alpha_n^2 < k^2 \} \\ &= \{ n \in \mathbb{Z} : k^2 - k^2 \sin^2 \phi \le (k \sin \theta \cos \phi + n)^2 < k^2 \} \\ &= \{ n \in \mathbb{Z} : k \cos \phi \le |k \sin \theta \cos \phi + n| < k \} \\ &= \{ n \in \mathbb{Z} : -k(1 + \sin \theta \cos \phi) < n < k(1 - \sin \theta \cos \phi) \} \cap \\ &\{ n \in \mathbb{Z} : n \ge k \cos \phi (1 - \sin \theta) \text{ or } n \le -k \cos \phi (1 + \sin \theta) \} \\ &= \{ n \in \mathbb{Z} : -k(1 + \sin \theta \cos \phi) < n \le -k \cos \phi (1 + \sin \theta) \\ &\text{ or } k \cos \phi (1 - \sin \theta) \le n < k(1 - \sin \theta \cos \phi) \}. \end{aligned}$$

This coincides with the set \mathcal{A} given by (3.24).

$$u|_{\Gamma_{b}} = \sum_{n \in \mathbb{Z}} \tilde{u}_{n} e^{i\alpha_{n}x_{1}}, v|_{\Gamma_{b}} = \sum_{n \in \mathbb{Z}} \tilde{v}_{n} e^{i\alpha_{n}x_{1}}, \text{ we define}$$

$$q(u, v; u, v) = 2\pi \operatorname{Re} \sum_{n \in \mathcal{A}} \theta M_{n} \begin{pmatrix} \tilde{u}_{n} \\ \tilde{v}_{n} \end{pmatrix} \cdot \begin{pmatrix} \overline{\tilde{u}_{n}} \\ \overline{\tilde{v}_{n}} \end{pmatrix}, \qquad (3.26)$$

where the set \mathcal{A} is defined by (3.24).

For

Proof of Theorem 3.8. Choose $\theta = \frac{i+\delta}{|i+\delta|}$. By the definition of \mathcal{A} ,

$$\operatorname{Re}\left(\theta A(u,v;u,v)\right) = \operatorname{Re}\int_{\Gamma} \frac{i+\delta}{|i+\delta|} \left(\frac{i}{\lambda}|u|^2 + i\lambda|v|^2\right) ds$$
$$= -\int_{\Gamma} \frac{1}{|i+\delta|} \left(\frac{1}{\lambda}|u|^2 + \lambda|v|^2\right) ds \ge 0$$

Before calculating $\operatorname{Re}(\theta B_1(u,v;u,v))$, we compute the following relation:

$$\begin{aligned} \partial^+ \begin{pmatrix} u \\ v \end{pmatrix} \cdot \overline{\partial^+ \begin{pmatrix} u \\ v \end{pmatrix}} &= \frac{1}{2} \begin{pmatrix} -i\partial_1 u + \partial_2 u \\ -i\partial_1 v + \partial_2 v \end{pmatrix} \cdot \begin{pmatrix} i\partial_1 \bar{u} + \partial_2 \bar{u} \\ i\partial_1 \bar{v} + \partial_2 \bar{v} \end{pmatrix} \\ &= \frac{1}{2} \left[|\partial_1 u|^2 + |\partial_2 u|^2 + i\partial_1 \bar{u}\partial_2 u - i\partial_1 u\partial_2 \bar{u} + |\partial_1 v|^2 + |\partial_2 v|^2 + i\partial_1 \bar{v}\partial_2 v - i\partial_1 v\partial_2 \bar{v} \right] \\ &= \frac{1}{2} \left[|\nabla u|^2 + |\nabla v|^2 + i(\partial_1 \bar{u}\partial_2 u + \partial_1 \bar{v}\partial_2 v) - i(\partial_1 u\partial_2 \bar{u} + \partial_1 v\partial_2 \bar{v}) \right]. \end{aligned}$$

Similarly,

$$\partial^{-} \begin{pmatrix} u \\ v \end{pmatrix} \cdot \overline{\partial^{-} \begin{pmatrix} u \\ v \end{pmatrix}} = \frac{1}{2} \left[|\nabla u|^{2} + |\nabla v|^{2} + i(\partial_{1}u\partial_{2}\bar{u} + \partial_{1}v\partial_{2}\bar{v}) - i(\partial_{1}\bar{u}\partial_{2}u + \partial_{1}\bar{v}\partial_{2}v) \right].$$

Therefore,

$$\partial^+ \left(\begin{array}{c} u \\ v \end{array}\right) \cdot \overline{\partial^+ \left(\begin{array}{c} u \\ v \end{array}\right)} + \partial^- \left(\begin{array}{c} u \\ v \end{array}\right) \cdot \overline{\partial^- \left(\begin{array}{c} u \\ v \end{array}\right)} = |\nabla u|^2 + |\nabla v|^2.$$

Then, by (3.22) and Lemma 3.9, we have

$$\operatorname{Re}\left(\theta B_{1}(u,v;u,v)\right)$$

$$=\operatorname{Re}\left(\theta \int_{\Omega_{b}} N^{+}\partial^{+}\begin{pmatrix} u \\ v \end{pmatrix} \cdot \overline{\partial^{+}\begin{pmatrix} u \\ v \end{pmatrix}} + N^{-}\partial^{-}\begin{pmatrix} u \\ v \end{pmatrix} \cdot \overline{\partial^{-}\begin{pmatrix} u \\ v \end{pmatrix}} dx\right)$$

$$\geq C_{N} \int_{\Omega_{b}} \partial^{+}\begin{pmatrix} u \\ v \end{pmatrix} \cdot \overline{\partial^{+}\begin{pmatrix} u \\ v \end{pmatrix}} + \partial^{-}\begin{pmatrix} u \\ v \end{pmatrix} \cdot \overline{\partial^{-}\begin{pmatrix} u \\ v \end{pmatrix}} dx$$

$$= C_{N} \int_{\Omega_{b}} |\nabla u|^{2} + |\nabla v|^{2} dx$$

$$= C_{N} \left(\|u\|_{H^{1}(\Omega_{b})}^{2} + \|v\|_{H^{1}(\Omega_{b})}^{2} \right) - C_{N} \int_{\Omega_{b}} |u|^{2} + |v|^{2} dx,$$

where $C_N \ge 0$ is defined by (3.23). It is obvious that

$$\operatorname{Re}\left(\theta C(u,v;u,v)\right) = \operatorname{Re}\int_{\Omega_b} \frac{i+\delta}{|i+\delta|} \left(\omega\epsilon |u|^2 + \omega\mu |v|^2\right) dx$$
$$= \frac{\delta}{|i+\delta|} \int_{\Omega_b} \omega\epsilon |u|^2 + \omega\mu |v|^2 dx \ge 0.$$

Then we need to consider the term $B(u,v;\varphi,\psi).$ Suppose that

$$u|_{\Gamma_b} = \sum_{n \in \mathbb{Z}} \tilde{u}_n e^{i\alpha_n x_1}, \quad v|_{\Gamma_b} = \sum_{n \in \mathbb{Z}} \tilde{v}_n e^{i\alpha_n x_1}.$$

We get

$$\begin{aligned} \operatorname{Re}\left(\theta D(u,v;u,v)\right) &= \operatorname{Re}\left(\theta \int_{\Gamma_{b}} T\left(\begin{array}{c} u\\ v\end{array}\right) \cdot \left(\begin{array}{c} \overline{u}\\ \overline{v}\end{array}\right) \, ds\right) \\ &= 2\pi \operatorname{Re}\left(\sum_{n \in \mathbb{Z}} \theta M_{n}\left(\begin{array}{c} \tilde{u}_{n}\\ \tilde{v}_{n}\end{array}\right) \cdot \left(\begin{array}{c} \overline{\tilde{u}_{n}}\\ \overline{\tilde{v}_{n}}\end{array}\right)\right) \\ &= 2\pi \operatorname{Re}\left(\sum_{n \in \mathbb{Z}/\mathcal{A}} \theta M_{n}\left(\begin{array}{c} \tilde{u}_{n}\\ \tilde{v}_{n}\end{array}\right) \cdot \left(\begin{array}{c} \overline{\tilde{u}_{n}}\\ \overline{\tilde{v}_{n}}\end{array}\right)\right) + q(u,v;u,v) \\ &\geq q(u,v;u,v), \end{aligned}$$

where we have used Lemma 3.9 (ii). Note that q(u, v; u, v) is a compact form because \mathcal{A} is a finite set. Therefore, by Lemma 3.9, we have

$$\operatorname{Re}\left(\theta B(u, v; u, v)\right)$$

= $\operatorname{Re}\left(\theta A(u, v; u, v)\right) + \operatorname{Re}\left(\theta B_{1}(u, v; u, v)\right) + \operatorname{Re}\left(\theta C(u, v; u, v)\right) + \operatorname{Re}\left(\theta D(u, v; u, v)\right)$
$$\geq C_{N}\left(\left\|u\right\|_{H^{1}(\Omega_{b})}^{2} + \left\|v\right\|_{H^{1}(\Omega_{b})}^{2}\right) - Q(u, v; u, v),$$

where $C_N \ge 0$ is defined by (3.23) and

$$Q(u,v;u,v) := C_N \int_{\Omega_b} |u|^2 + |v|^2 \, dx - \frac{\delta}{|i+\delta|} \int_{\Omega_b} \omega \epsilon |u|^2 + \omega \mu |v|^2 \, dx - q(u,v;u,v)$$

is a compact form over $X \times X$. By Definition 3.7, we finish the proof.

Theorem 3.10. Suppose that Γ is a Lipschitz curve, $k^2 \neq \gamma^2$ and that the impedance coefficient $\lambda < 0$. Then, the variational problem (3.9)-(3.10) admits a unique solution $(u, v) \in X$.

Proof. Under the assumption of Theorem 3.8, the operator defined in (3.15) is a Fredholm operator with index zero. Using Theorem 3.5, we obtain the existence and uniqueness result as a consequence of the Fredholm alternative.

4 Finite element analysis

We study the finite element approximation of the variational problem (3.14). Let $\{X_h^2 : h \in (0,1)\}$ be a family of finite dimensional subspaces of $H^1_{\alpha}(\Omega_b)^2$, where h stands for the maximum mesh size after partitioning Ω_b into simple domains, for example, a regular triangulation of Ω_b . We make a general assumption [4] on the subspace X_h^2 for $(\varphi, \psi) \in H^{\rho}_{\alpha}(\Omega_b)^2$, $\rho \geq 2$,

$$\inf_{\substack{(\xi,\eta)\in X_{h}^{2}}} \left(\|(\varphi,\psi)-(\xi,\eta)\|_{L^{2}(\Omega_{b})^{2}} + h\|(\nabla\varphi,\nabla\psi)-(\nabla\xi,\nabla\eta)\|_{L^{2}(\Omega_{b})^{2}} \\
+ h^{1/2}\|(\varphi,\psi)-(\xi,\eta)\|_{L^{2}(\Gamma_{b})^{2}} + h\|(\varphi,\psi)-(\xi,\eta)\|_{H^{1/2}(\Gamma_{b})^{2}} \\
+ h^{1/2}\|(\varphi,\psi)-(\xi,\eta)\|_{L^{2}(\Gamma_{b})^{2}} + h\|(\varphi,\psi)-(\xi,\eta)\|_{H^{1/2}(\Gamma)^{2}} \right) \\
\leq Ch^{l}\|(\varphi,\psi)\|_{H^{l}(\Omega)^{2}}, \quad l \in [2,\rho]$$
(4.1)

where the positive constant C is independent of h and (φ, ψ) . The finite element approximation to the variational (3.14) is to find $(u_h, v_h) \in X_h^2$ such that

$$B(u_h, v_h; \varphi_h, \psi_h) = F(\varphi_h, \psi_h), \quad \text{for all } (\varphi_h, \psi_h) \in X_h^2, \tag{4.2}$$

where B is defined by (3.15) and F is defined by (3.16). The finite element method consists of the following steps to solve (4.2):

- (1) Choose a finite set of basis functions $\{\phi_1, \phi_2, \cdots, \phi_m\}$ of X_h ;
- (2) Let $u_h = c_1\phi_1 + c_2\phi_2 + \dots + c_m\phi_m$, $v_h = d_1\phi_1 + d_2\phi_2 + \dots + d_m\phi_m$. Substitute the expression into (4.2) and choose $(\varphi_h, \psi_h) = (\phi_i, 0)$, $(0, \phi_i)$, $i = 1, 2, \dots, m$ to get a system of linear equations;
- (3) Solve the linear system for the coefficients $c_1, c_2, \dots, c_m, d_1, d_2, \dots, d_m$ and get the approximation of (u, v) in X_h^2 .

More precisely, we have

$$\begin{split} B(u_h, v_h; \phi_i, 0) \\ = \int_{\Omega_b} \left[\frac{\omega \epsilon}{\kappa^2} \left(\sum_{j=1}^m c_j \nabla \phi_j \right) \cdot \nabla \overline{\phi}_i - \frac{\gamma}{\kappa^2} \left(\sum_{j=1}^m d_j \nabla \phi_j \right) \cdot \nabla^{\perp} \overline{\phi}_i - \omega \epsilon \left(\sum_{j=1}^m c_j \phi_j \right) \overline{\phi}_i \right] dx \\ + \int_{\Gamma} \frac{i}{\lambda} \left(\sum_{j=1}^m c_j \phi_j \right) \overline{\phi}_i ds + \int_{\Gamma_b} T \left(\begin{array}{c} \sum_{j=1}^m c_j \phi_j \\ \sum_{j=1}^m d_j \phi_j \end{array} \right) \cdot \left(\begin{array}{c} \overline{\phi}_i \\ 0 \end{array} \right) ds, \end{split}$$

$$B(u_h, v_h; 0, \phi_i) = \int_{\Omega_b} \left[\frac{\omega \mu}{\kappa^2} \left(\sum_{j=1}^m d_j \nabla \phi_j \right) \cdot \nabla \overline{\phi}_i + \frac{\gamma}{\kappa^2} \left(\sum_{j=1}^m c_j \nabla \phi_j \right) \cdot \nabla^{\perp} \overline{\phi}_i - \omega \mu \left(\sum_{j=1}^m d_j \phi_j \right) \overline{\phi}_i \right] dx + \int_{\Gamma} i\lambda \left(\sum_{j=1}^m d_j \phi_j \right) \overline{\phi}_i ds + \int_{\Gamma_b} T \left(\sum_{j=1}^m c_j \phi_j \\ \sum_{j=1}^m d_j \phi_j \right) \cdot \left(\frac{0}{\overline{\phi}_i} \right) ds.$$

In order to deduce the stiffness matrix, we need to define the following inner product.

$$\langle f,g \rangle_{\Omega_b} = \int_{\Omega_b} f \bar{g} \, dx, \quad \langle f,g \rangle_{\Gamma} = \int_{\Gamma} f \bar{g} \, ds, \quad \langle f,g \rangle_{\Gamma_b} = \int_{\Gamma_b} f \bar{g} \, ds.$$

Let $\mathcal{B} \in \mathbb{C}^{2m \times 2m}$ be the stiffness matrix with the entries

$$B_{ij} = \begin{cases} \frac{\omega\epsilon}{\kappa^2} \langle \nabla\phi_j, \nabla\phi_i \rangle_{\Omega_b} - \omega\epsilon \langle \phi_j, \phi_i \rangle_{\Omega_b} + \frac{i}{\lambda} \langle \phi_j, \phi_i \rangle_{\Gamma} \\ + \left\langle T \begin{pmatrix} \phi_j \\ 0 \end{pmatrix}, \begin{pmatrix} \phi_i \\ 0 \end{pmatrix} \right\rangle_{\Gamma_b}, & 1 \le i, j \le m, \\ \\ \frac{\gamma}{\kappa^2} \langle \nabla\phi_{j-m}, \nabla^{\perp}\phi_i \rangle_{\Omega_b} + \left\langle T \begin{pmatrix} 0 \\ \phi_{j-m} \end{pmatrix}, \begin{pmatrix} \phi_i \\ 0 \end{pmatrix} \right\rangle_{\Gamma_b}, & 1 \le i \le m, m+1 \le j \le 2m, \\ \\ \frac{\gamma}{\kappa^2} \langle \nabla\phi_j, \nabla^{\perp}\phi_{i-m} \rangle_{\Omega_b} + \left\langle T \begin{pmatrix} \phi_j \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \phi_{i-m} \end{pmatrix} \right\rangle_{\Gamma_b}, & m+1 \le i \le 2m, 1 \le j \le m, \\ \\ \frac{\omega\mu}{\kappa^2} \langle \nabla\phi_{j-m}, \nabla\phi_{i-m} \rangle_{\Omega_b} - \omega\mu \langle \phi_{j-m}, \phi_{i-m} \rangle_{\Omega_b} \\ + \frac{i}{\lambda} \langle \phi_{j-m}, \phi_{i-m} \rangle_{\Gamma} + \left\langle T \begin{pmatrix} 0 \\ \phi_{j-m} \end{pmatrix}, \begin{pmatrix} 0 \\ \phi_{i-m} \end{pmatrix} \right\rangle_{\Gamma_b}, & m+1 \le i, j \le 2m, \end{cases}$$

and let $F \in \mathbb{C}^{2m}$ be a vector whose components are given by

$$F_{i} = \begin{cases} -\frac{2i\omega\epsilon\beta e^{-i\beta b}}{\kappa^{2}} \int_{\Gamma_{b}} \epsilon p_{3}\overline{\phi}_{i}e^{i\alpha x_{1}} ds, & 1 \leq i \leq m \\ -\frac{2i\omega\epsilon\beta e^{-i\beta b}}{\kappa^{2}} \int_{\Gamma_{b}} \mu q_{3}\overline{\phi}_{i-m}e^{i\alpha x_{1}} ds, & m+1 \leq i \leq 2m. \end{cases}$$

Then we get the system of linear equations

$$\sum_{j=1}^{2m} B_{ij} a_j = F_i, \quad 1 \le i \le 2m.$$
(4.3)

Having obtained $\{a_j\}_{j=1}^{2m}$ from (4.3), we can get u_h and v_h by setting $c_j = a_j$, $d_j = a_{j+m}$ for $1 \le j \le m$.

Below we prove the well-posedness of the finite element approximation problem (4.2) and an error estimate of the finite element solution. Denote $e_h = (u - u_h, v - v_h)$. It is obvious that e_h is α -quasiperiodic. Define the projection operator $P: L^2(\Gamma_b)^2 \to L^2(\Gamma_b)^2$ by

$$(Pf)(x_1) = \sum_{n \in \mathcal{A}} f_n e^{i\alpha_n x_1}, \quad f = \sum_{n \in \mathbb{Z}} f_n e^{i\alpha_n x_1} \in L^2(\Gamma_b)^2,$$

where the set \mathcal{A} is defined by (3.24).

Lemma 4.1. There exists a constant $h_1 \in (0,1)$ such that for $h \in (0,h_1)$ the following estimate holds:

$$\|e_h\|_{H^1(\Omega_b)^2}^2 \le C\left(h^{2\rho-2}\|(u,v)\|_{H^\rho(\Omega_b)^2}^2 + \|e_h\|_{L^2(\Omega_b)^2}^2 + \|Pe_h\|_{L^2(\Gamma_b)^2}^2\right),$$

where the constant C depends on ρ but is independent of h and (u, v).

Proof. It follows from the sesquilinear form (4.2) that

$$B(e_h; e_h) := \int_{\Omega_b} \left[\frac{\omega \epsilon}{\kappa^2} |\nabla(u - u_h)|^2 - \frac{\gamma}{\kappa^2} \nabla(v - v_h) \cdot \nabla^{\perp} \overline{(u - u_h)} - \omega \epsilon |u - u_h|^2 + \frac{\omega \mu}{\kappa^2} |\nabla(v - v_h)|^2 + \frac{\gamma}{\kappa^2} \nabla(u - u_h) \cdot \nabla^{\perp} \overline{(v - v_h)} - \omega \mu |v - v_h|^2 \right] dx + \int_{\Gamma} \frac{i}{\lambda} |u - u_h|^2 + i\lambda |v - v_h|^2 ds + \int_{\Gamma_b} Te_h \cdot \overline{e_h} ds.$$

$$(4.4)$$

Multiplying both sides of (4.4) by $\theta = \frac{i+\delta}{|i+\delta|}$ and taking the real part, we get

$$\operatorname{Re}\left[\theta B(e_{h};e_{h})\right] = \operatorname{Re}\left\{\theta \int_{\Omega_{b}} \left[\frac{\omega\epsilon}{\kappa^{2}}|\nabla(u-u_{h})|^{2} - \frac{\gamma}{\kappa^{2}}\nabla(v-v_{h})\cdot\nabla^{\perp}\overline{(u-u_{h})} - \omega\epsilon |u-u_{h}|^{2} + \frac{\omega\mu}{\kappa^{2}}|\nabla(v-v_{h})|^{2} + \frac{\gamma}{\kappa^{2}}\nabla(u-u_{h})\cdot\nabla^{\perp}\overline{(v-v_{h})} - \omega\mu |v-v_{h}|^{2}\right] dx\right\} + \operatorname{Re}\left\{\theta \int_{\Gamma}\frac{i}{\lambda}|u-u_{h}|^{2} + i\lambda|v-v_{h}|^{2} ds\right\} + \left\{\operatorname{Re}\theta \int_{\Gamma_{b}}Te_{h}\cdot\overline{e_{h}} ds\right\}.$$

$$(4.5)$$

From the strongly elliptic analysis (see the proof of Theorem 3.8), we can easily get

Re
$$\left\{ \theta \int_{\Gamma} \frac{i}{\lambda} |u - u_h|^2 + i\lambda |v - v_h|^2 \, ds \right\} \ge 0,$$

and

$$\operatorname{Re}\left\{\theta\int_{\Gamma_b}Te_h\cdot\overline{e_h}\,ds+q(e_h;e_h)\right\}\geq 0,$$

where the compact form q is defined by (3.26), that is, for $e_h = \sum_{n \in \mathbb{Z}} A_n e^{i\alpha_n x_1}$, we have

$$q(e_h; e_h) = 2\pi \operatorname{Re} \sum_{n \in \mathcal{A}} \theta M_n A_n \cdot \overline{A_n} \le C \|Pe_h\|_{L^2(\Gamma_b)^2}^2.$$

Therefore, by (4.5),

$$\begin{aligned} &\operatorname{Re}\left\{\theta\int_{\Omega_{b}}\left[\frac{\omega\epsilon}{\kappa^{2}}|\nabla(u-u_{h})|^{2}-\frac{\gamma}{\kappa^{2}}\nabla(v-v_{h})\cdot\nabla^{\perp}\overline{(u-u_{h})}\right.\\ &+\frac{\omega\mu}{\kappa^{2}}|\nabla(v-v_{h})|^{2}+\frac{\gamma}{\kappa^{2}}\nabla(u-u_{h})\cdot\nabla^{\perp}\overline{(v-v_{h})}\right]\,dx\right\}\\ &=\operatorname{Re}\left(\theta B(e_{h};e_{h})\right)-\operatorname{Re}\left\{\theta\int_{\Gamma}\frac{i}{\lambda}|u-u_{h}|^{2}+i\lambda|v-v_{h}|^{2}\,ds\right\}-\operatorname{Re}\left\{\theta\int_{\Gamma_{b}}Te_{h}\cdot\overline{e_{h}}\,ds\right\}\\ &+\operatorname{Re}\left\{\theta\int_{\Omega_{b}}\omega\epsilon\,|u-u_{h}|^{2}+\omega\mu\,|v-v_{h}|^{2}\,dx\right\}\\ &\leq\operatorname{Re}\left(\theta B(e_{h};e_{h})\right)+\operatorname{Re}\left\{\theta\int_{\Omega_{b}}\omega\epsilon\,|u-u_{h}|^{2}+\omega\mu\,|v-v_{h}|^{2}\,dx\right\}+q(e_{h};e_{h}),\end{aligned}$$

Using Lemma 3.9 (i) we get

$$C_1 \|e_h\|_{H^1(\Omega_b)^2}^2 \le |B(e_h; e_h)| + C_2 \|e_h\|_{L^2(\Omega_b)^2}^2 + C \|Pe_h\|_{L^2(\Gamma_b)^2}^2.$$
(4.6)

Observing for any $(\xi, \eta) \in X_h^2$ that

$$B(u, v; \xi - u_h, \eta - v_h) = F(\xi - u_h, \eta - v_h), \quad B(u_h, v_h; \xi - u_h, \eta - v_h) = F(\xi - u_h, \eta - v_h),$$

we obtain

we obtain

$$B(u - u_h, v - v_h; \xi - u_h, \eta - v_h) = 0.$$
(4.7)

Therefore for any $(\xi,\eta)\in X_h^2$, we have

$$B(u - u_h, v - v_h; u - u_h, v - v_h) = B(u - u_h, v - v_h; u - \xi, v - \eta).$$
(4.8)

Since X_h^2 is of finite dimensions, it is complete and therefore closed. Hence, the infimum in (4.1) is actually attained for $(\varphi, \psi) = (u, v)$ in (4.1). For any small positive constants ϵ_i (i = 1, 2, 3, 4), it follows from (4.8) and Young's inequality that

$$\begin{split} |B(e_{h};e_{h})| &= |B(u-u_{h},v-v_{h};u-\xi,v-\eta)| \\ &= \left| \int_{\Gamma} \frac{i}{\lambda} (u-u_{h}) \overline{(u-\xi)} + i\lambda(v-v_{h}) \overline{(v-\eta)} \, ds + \int_{\Omega_{b}} \frac{\omega\epsilon}{\kappa^{2}} \nabla(u-u_{h}) \cdot \nabla\overline{(u-\xi)} \right| \\ &- \frac{\gamma}{\kappa^{2}} \nabla(v-v_{h}) \cdot \nabla^{\perp} \overline{(v-\eta)} - \omega\epsilon (u-u_{h}) \overline{(u-\xi)} + \frac{\omega\mu}{\kappa^{2}} \nabla(v-v_{h}) \cdot \overline{\nabla(v-\eta)} \right| \\ &+ \frac{\gamma}{\kappa^{2}} \nabla(u-u_{h}) \cdot \nabla^{\perp} \overline{(v-\eta)} - \omega\mu (v-v_{h}) \overline{(v-\eta)} \, dx \\ &+ \int_{\Gamma_{b}} T \left(\frac{u-u_{h}}{v-v_{h}} \right) \cdot \left(\frac{\overline{u-\xi}}{\overline{v-\eta}} \right) \, ds \right| \\ &\leq \frac{1}{|\lambda|} \left(h ||u-u_{h}||^{2}_{L^{2}(\Gamma)} + \frac{1}{h} ||u-\xi||^{2}_{L^{2}(\Gamma)} \right) + |\lambda| \left(h ||v-v_{h}||^{2}_{L^{2}(\Gamma)} + \frac{1}{h} ||v-\eta||^{2}_{L^{2}(\Gamma)} \right) \\ &+ \frac{\omega\epsilon}{\kappa^{2}} \left(\epsilon_{1} ||\nabla u - \nabla u_{h}||^{2}_{L^{2}(\Omega_{b})} + \frac{1}{4\epsilon_{1}} ||\nabla u - \nabla \xi||^{2}_{L^{2}(\Omega_{b})} \right) \\ &+ \frac{|\gamma|}{\kappa^{2}} \left(\epsilon_{2} ||\nabla v - \nabla v_{h}||^{2}_{L^{2}(\Omega_{b})} + \frac{1}{4\epsilon_{2}} ||\nabla u - \nabla \xi||^{2}_{L^{2}(\Omega_{b})} \right) \\ &+ \frac{\omega\mu}{\kappa^{2}} \left(\epsilon_{3} ||\nabla v - \nabla v_{h}||^{2}_{L^{2}(\Omega_{b})} + \frac{1}{4\epsilon_{3}} ||\nabla v - \nabla \eta||^{2}_{L^{2}(\Omega_{b})} \right) \\ &+ \frac{|\gamma|}{\kappa^{2}} \left(\epsilon_{4} ||\nabla u - \nabla u_{h}||^{2}_{L^{2}(\Omega_{b})} + \frac{1}{4\epsilon_{4}} ||\nabla v - \nabla \eta||^{2}_{L^{2}(\Omega_{b})} \right) \\ &+ \omega\mu \left(h^{2} ||v-v_{h}||^{2}_{L^{2}(\Omega_{b})} + h^{-2} ||v-\eta||^{2}_{L^{2}(\Omega_{b})} \right) \\ &+ \left| \int_{\Gamma_{b}} T \left(\begin{array}{c} u-u_{h} \\ v-v_{h} \end{array} \right) \cdot \left(\begin{array}{c} \overline{u-\xi} \\ \overline{v-\eta} \end{array} \right) \, ds \right|. \end{split}$$

Using the continuity of the DtN map T (see Lemma 3.3), trace theorem and Young's inequality, we have

$$\left| \int_{\Gamma_{b}} T\left(\begin{array}{c} u - u_{h} \\ v - v_{h} \end{array} \right) \cdot \left(\begin{array}{c} \overline{u - \xi} \\ \overline{v - \eta} \end{array} \right) ds \right| \\
\leq \left\| T\left(\begin{array}{c} u - u_{h} \\ v - v_{h} \end{array} \right) \right\|_{H^{-1/2}(\Gamma_{b})^{2}} \left\| \left(\begin{array}{c} u - \xi \\ v - \eta \end{array} \right) \right\|_{H^{1/2}(\Gamma_{b})^{2}} \\
\leq C \left\| e_{h} \right\|_{H^{1/2}(\Gamma_{b})^{2}} \left\| \left(\begin{array}{c} u - \xi \\ v - \eta \end{array} \right) \right\|_{H^{1/2}(\Gamma_{b})^{2}} \\
\leq C \left(\epsilon_{5} \| e_{h} \|_{H^{1}(\Omega_{b})^{2}}^{2} + \frac{1}{4\epsilon_{5}} \left\| \left(\begin{array}{c} u - \xi \\ v - \eta \end{array} \right) \right\|_{H^{1/2}(\Gamma_{b})^{2}} \right). \quad (4.10)$$

One deduces from (4.1) and (4.9) - (4.10) that

$$|B(e_h; e_h)| \le Ch \|e_h\|_{L^2(\Gamma)^2}^2 + \sigma \|e_h\|_{H^1(\Omega_b)^2}^2 + C_1 h^2 \|e_h\|_{L^2(\Omega_b)^2}^2 + C(\sigma) h^{2\rho-2} \|(u, v)\|_{H^{\rho}(\Omega_b)^2}^2,$$
(4.11)

where $\sigma = \sigma(\epsilon_1, \dots, \epsilon_5) > 0$. Combining (4.6) and (4.11) leads to

$$C_{1} \|e_{h}\|_{H^{1}(\Omega_{b})^{2}}^{2} \leq |B(e_{h};e_{h})| + C_{2} \|e_{h}\|_{L^{2}(\Omega_{b})^{2}}^{2} + |q(e_{h};e_{h})|$$

$$\leq C_{3}h \|e_{h}\|_{L^{2}(\Gamma)^{2}}^{2} + \sigma \|e_{h}\|_{H^{1}(\Omega_{b})^{2}}^{2} + C_{4}h^{2} \|e_{h}\|_{L^{2}(\Omega_{b})^{2}}^{2}$$

$$+ C(\sigma)h^{2\rho-2} \|(u,v)\|_{H^{\rho}(\Omega_{b})^{2}}^{2} + C_{2} \|e_{h}\|_{L^{2}(\Omega_{b})^{2}}^{2} + C_{5} \|Pe_{h}\|_{L^{2}(\Gamma_{b})^{2}}^{2}.$$

Using the estimate

$$||e_h||_{L^2(\Gamma)^2} \le ||e_h||_{H^{1/2}(\Gamma)^2} \le C ||e_h||_{H^1(\Omega_b)^2}, \quad C > 0.$$

we get from (4.12) that

$$C_1 \|e_h\|_{H^1(\Omega_b)^2}^2 \le (CC_3h + \sigma + C_4h^2) \|e_h\|_{H^1(\Omega_b)^2}^2 + C(\sigma)h^{2\rho-2} \|(u,v)\|_{H^{\rho}(\Omega_b)^2}^2 + C_2 \|e_h\|_{L^2(\Omega_b)^2}^2 + C_5 \|Pe_h\|_{L^2(\Gamma_b)^2}^2.$$

Now choose σ sufficiently small and let h_1 be a constant such that $\sigma + CC_3h_1 + C_4h_1^2 < C_1$. Then for $h \in (0, h_1)$, we obtain the desired estimate of this lemma.

We next estimate the L^2 -norm of e_h in Ω_b .

Lemma 4.2. There exists a constant $h_2 \in (0, 1)$ such that

$$||e_h||_{L^2(\Omega_b)^2} \le C(h + C_1 h^{3/2}) ||e_h||_{H^1(\Omega_b)^2} \quad \text{for all } h \in (0, h_2),$$

where the constants C, C_1 depend on ρ but are independent of h and (u, v).

Proof. We use the duality argument. By the definition

$$\|e_h\|_{L^2(\Omega_b)^2} = \sup_{(\phi,\zeta)\in C_0^\infty(\Omega_b)^2} \frac{(e_h, (\phi,\zeta))_{L^2(\Omega_b)^2}}{\|(\phi,\zeta)\|_{L^2(\Omega_b)^2}},$$
(4.12)

where

$$(e_h, (\phi, \zeta))_{L^2(\Omega_b)^2} := \int_{\Omega_b} \frac{\omega\epsilon}{\kappa^2} (u - u_h) \bar{\phi} + \frac{\omega\mu}{\kappa^2} (v - v_h) \bar{\zeta} \, dx.$$

$$(4.13)$$

Consider a quasi-periodic solution (w, z) of the following boundary value problem:

$$\begin{cases}
\Delta w + k^2 w = -\bar{\phi} & \text{in } \Omega_b, \\
\Delta z + k^2 z = -\bar{\zeta} & \text{in } \Omega_b, \\
\lambda \partial_n w - \frac{i\kappa^2}{\omega_{\mu}} w + \frac{\lambda\gamma}{\omega_{\epsilon}} \partial_\tau z = 0 & \text{on } \Gamma, \\
\partial_n z - \frac{i\lambda\kappa^2}{\omega_{\mu}} z - \frac{\lambda}{\omega_{\mu}} \partial_\tau w = 0 & \text{on } \Gamma, \\
T^* \begin{pmatrix} w \\ z \end{pmatrix} = \sum_{n \in \mathbb{Z}} M_n^* \begin{pmatrix} \hat{w}_n \\ \hat{z}_n \end{pmatrix} e^{i\alpha_n x_1} & \text{on } \Gamma_b,
\end{cases}$$
(4.14)

where T^* is the adjoint operator of T. We can easily get the variational formulation of (4.14) that for all $(\varphi, \psi) \in X$,

$$\int_{\Omega_b} \left[\frac{\omega \epsilon}{\kappa^2} \nabla w \cdot \nabla \varphi - \frac{\gamma}{\kappa^2} \nabla z \cdot \nabla^{\perp} \varphi - \omega \epsilon \, w \, \varphi + \frac{\omega \mu}{\kappa^2} \nabla z \cdot \nabla \psi + \frac{\gamma}{\kappa^2} \nabla w \cdot \nabla^{\perp} \psi - \omega \mu \, z \, \psi \right] \, dx$$

$$+\int_{\Gamma}\frac{i}{\lambda}w\,\varphi + i\lambda z\,\psi\,ds - \int_{\Gamma_b}T^*\left(\begin{array}{c}w\\z\end{array}\right)\cdot\left(\begin{array}{c}\varphi\\\psi\end{array}\right)\,ds = \int_{\Omega_b}\frac{\omega\epsilon}{\kappa^2}\bar{\phi}\varphi + \frac{\omega\mu}{\kappa^2}\bar{\zeta}\psi.$$
(4.15)

Taking $\varphi = u - u_h$, $\psi = v - v_h$ in (4.15), using the definition of $(e_h, (\phi, \zeta))_{L^2(\Omega_b)^2}$ in (4.13) and recalling the form $B(u, v; \varphi, \psi)$ in (3.15), we get

$$B(u - u_h, v - v_h; \bar{w}, \bar{z}) = (e_h, (\phi, \zeta))_{L^2(\Omega_b)^2}.$$
(4.16)

The well-posedness of the problem (4.14) can be established by the same argument as the proof for the variational problem (3.14). Moreover, we have

$$\|(w,z)\|_{H^2(\Omega_b)^2} \le C \|(\phi,\zeta)\|_{L^2(\Omega_b)^2}.$$
(4.17)

Using the orthogonal formula (4.7), we have

$$B(u - u_h, v - v_h; \bar{w} - \xi, \bar{z} - \eta) = B(u - u_h, v - v_h; \bar{w}, \bar{z}) - B(u - u_h, v - v_h; \xi, \eta)$$

= $B(u - u_h, v - v_h; \bar{w}, \bar{z}).$ (4.18)

Combining (4.16) and (4.18) gives

$$|(e_h, (\phi, \zeta))_{L^2(\Omega_b)^2}| = |B(e_h, (\bar{w}, \bar{z}))| = |B(e_h, (\bar{w}, \bar{z}) - (\xi, \eta))| \quad \text{for all } (\xi, \eta) \in X_h^2.$$
(4.19)

In particular, (ξ, η) can be chosen in such a way that the infimum is attained for $(\varphi, \psi) = (w, z)$ in (4.1). By arguing analogously to the proof of Lemma 4.1, we deduce from (4.9) - (4.10) that

$$\begin{split} &|B(e_h,(\bar{w},\bar{z})-(\xi,\eta))|\\ &= \left| \int_{\Gamma} \frac{i}{\lambda} (u-u_h) \overline{(\bar{w}-\xi)} + i\lambda(v-v_h) \overline{(\bar{z}-\eta)} \, ds + \int_{\Omega_b} \frac{\omega\epsilon}{\kappa^2} \nabla(u-u_h) \cdot \nabla \overline{(\bar{w}-\xi)} \right. \\ &\left. - \frac{\gamma}{\kappa^2} \nabla(v-v_h) \cdot \nabla^{\perp} \overline{(\bar{w}-\xi)} - \omega\epsilon \left(u-u_h\right) \overline{(\bar{w}-\xi)} + \frac{\omega\mu}{\kappa^2} \nabla(v-v_h) \cdot \nabla \overline{(\bar{z}-\eta)} \right. \\ &\left. + \frac{\gamma}{\kappa^2} \nabla(u-u_h) \cdot \nabla^{\perp} \overline{(\bar{z}-\eta)} - \omega\mu \left(v-v_h\right) \overline{(\bar{z}-\eta)} \, dx \right. \\ &\left. + \int_{\Gamma_b} T \left(\begin{array}{c} u-u_h \\ v-v_h \end{array} \right) \cdot \left(\begin{array}{c} \overline{\bar{w}-\xi} \\ \overline{\bar{z}-\eta} \end{array} \right) \, ds \right|. \end{split}$$

Using the Cauchy-Schwarz inequality, we continue to estimate the above equation by

$$\begin{aligned} &|B(e_{h},(\bar{w},\bar{z})-(\xi,\eta))| \\ \leq & \frac{1}{|\lambda|} \left(h^{-1/2} \|u-u_{h}\|_{L^{2}(\Gamma)} h^{1/2} \|\bar{w}-\xi\|_{L^{2}(\Gamma)}\right) + |\lambda| \left(h^{-1/2} \|v-v_{h}\|_{L^{2}(\Gamma)} h^{1/2} \|\bar{z}-\eta\|_{L^{2}(\Gamma)}\right) \\ &+ \frac{\omega\epsilon}{\kappa^{2}} \left(\frac{1}{h} \|\nabla u-\nabla u_{h}\|_{L^{2}(\Omega_{b})} h |\bar{w}-\xi|_{H^{1}(\Omega_{b})}\right) + \frac{|\gamma|}{\kappa^{2}} \left(\frac{1}{h} \|\nabla v-\nabla v_{h}\|_{L^{2}(\Omega_{b})} h |\bar{w}-\xi|_{H^{1}(\Omega_{b})}\right) \\ &+ \omega\epsilon \left(\|u-u_{h}\|_{L^{2}(\Omega_{b})} \|\bar{w}-\xi\|_{L^{2}(\Omega_{b})}\right) + \frac{\omega\mu}{\kappa^{2}} \left(\frac{1}{h} \|\nabla v-\nabla v_{h}\|_{L^{2}(\Omega_{b})} h |\bar{z}-\eta|_{H^{1}(\Omega_{b})}\right) \\ &+ \frac{|\gamma|}{\kappa^{2}} \left(\frac{1}{h} \|\nabla u-\nabla u_{h}\|_{L^{2}(\Omega_{b})} h |\bar{z}-\eta|_{H^{1}(\Omega_{b})}\right) + \omega\mu \left(\|v-v_{h}\|_{L^{2}(\Omega_{b})} \|\bar{z}-\eta\|_{L^{2}(\Omega_{b})}\right) \\ &+ \frac{1}{h} \|e_{h}\|_{H^{1}(\Omega_{b})^{2}} h \left\| \left(\frac{u-\xi}{v-\eta}\right) \right\|_{H^{1/2}(\Gamma_{b})^{2}} \\ \leq & C_{1}h\|(w,z)\|_{H^{2}(\Omega_{b})^{2}} \left[h^{1/2}\|e_{h}\|_{H^{1}(\Omega_{b})^{2}} + C_{2}\|\nabla e_{h}\|_{L^{2}(\Omega_{b})^{2}} \\ &+ C_{3}h\|e_{h}\|_{L^{2}(\Omega_{b})^{2}} + C_{4}\|e_{h}\|_{H^{1}(\Omega_{b})^{2}}\right],
\end{aligned}$$

$$(4.20)$$

where the constants C_j (j = 1, 2, 3, 4) depend on k but independent of h. Combining (4.12) and (4.17), (4.19) and (4.20), we can find a positive constant $h_2 \leq 1$ such that for all $h \in (0, h_2)$,

$$\begin{split} \|e_{h}\|_{L^{2}(\Omega_{b})^{2}} &= \sup_{(\phi,\zeta)\in C_{0}^{\infty}(\Omega_{b})^{2}} \frac{(e_{h},(\phi,\zeta))_{L^{2}(\Omega_{b})^{2}}}{\|(\phi,\zeta)\|_{L^{2}(\Omega_{b})^{2}}} \\ &\leq \sup_{(\phi,\zeta)\in C_{0}^{\infty}(\Omega_{b})^{2}} \frac{C_{1}h\|(w,s)\|_{H^{2}(\Omega_{b})^{2}} \left[h^{1/2}\|e_{h}\|_{H^{1}(\Omega_{b})^{2}} + C_{2}\|e_{h}\|_{H^{1}(\Omega_{b})^{2}} + C_{3}h\|e_{h}\|_{L^{2}(\Omega_{b})^{2}}\right]}{\|(\phi,\zeta)\|_{L^{2}(\Omega_{b})^{2}}} \\ &\leq \sup_{(\phi,\zeta)\in C_{0}^{\infty}(\Omega_{b})^{2}} \frac{C_{1}h\|(\phi,\zeta)\|_{L^{2}(\Omega_{b})^{2}} \left[h^{1/2}\|e_{h}\|_{H^{1}(\Omega_{b})^{2}} + C_{2}\|e_{h}\|_{H^{1}(\Omega_{b})^{2}} + C_{3}h\|e_{h}\|_{L^{2}(\Omega_{b})^{2}}\right]}{\|(\phi,\zeta)\|_{L^{2}(\Omega_{b})^{2}}} \\ &= Ch\|e_{h}\|_{H^{1}(\Omega_{b})^{2}} + C_{1}h^{2}\|e_{h}\|_{L^{2}(\Omega_{b})^{2}} + C_{2}h^{3/2}\|e_{h}\|_{H^{1}(\Omega_{b})^{2}}. \end{split}$$

Now let h_2 be a constant that satisfies $1 - C_1 h_2^2 > 0$. We then obtain the estimate of this lemma for all $h \in (0, h_2)$.

We proceed with the estimate of the L^2 -norm of Pe_h on Γ_b .

Lemma 4.3. There exists a constant C such that

$$||Pe_h||_{L^2(\Gamma_b)^2} \le C ||e_h||_{L^2(\Omega_b)^2},$$

where the positive constant C is independent of h and (u, v).

Proof. Define

$$D = \{ (x_1, x_2) \in \mathbb{R}^2 : 0 < x_1 < 2\pi, b - \epsilon < x_2 < b \},\$$

where the constant $\epsilon > 0$ is chosen to satisfy $b - \epsilon > \max_{x \in \Gamma} \{x_2\}$. Suppose

$$e_h(x) = \sum_{n \in \mathbb{Z}} A_n e^{i(\alpha_n x_1 + \beta_n x_2)}, \quad x \in D.$$

Then $P(e_h|_{\Gamma_b}) = \sum_{n \in \mathcal{A}} A_n e^{i(\alpha_n x_1 + \beta_n b)}$. Direct calculations show that

$$\begin{split} \|e_{h}\|_{L^{2}(\Omega_{b})^{2}}^{2} &\geq \|e_{h}\|_{L^{2}(D)^{2}}^{2} = \int_{0}^{2\pi} \int_{b-\epsilon}^{b} |e_{h}|^{2} dx_{2} dx_{1} \\ &= \int_{0}^{2\pi} \int_{b-\epsilon}^{b} \sum_{n \in \mathbb{Z}} A_{n} e^{i(\alpha_{n}x_{1}+\beta_{n}x_{2})} \cdot \sum_{m \in \mathbb{Z}} \overline{A_{m}} e^{-i\alpha_{m}x_{1}-i\overline{\beta_{m}}x_{2}} dx_{2} dx_{1} \\ &= \sum_{n,m \in \mathbb{Z}} A_{n} \cdot \overline{A_{m}} \int_{0}^{2\pi} e^{i(\alpha_{n}-\alpha_{m})x_{1}} dx_{1} \int_{b-\epsilon}^{b} e^{i(\beta_{n}-\overline{\beta_{m}})x_{2}} dx_{2} \\ &= 2\pi \sum_{n \in \mathbb{Z}} |A_{n}|^{2} \int_{b-\epsilon}^{b} e^{i(\beta_{n}-\overline{\beta_{n}})x_{2}} dx_{2} \\ &= 2\pi \left(\sum_{|\alpha_{n}| \leq \kappa} \epsilon |A_{n}|^{2} + \sum_{|\alpha_{n}| > \kappa} |A_{n}|^{2} \int_{b-\epsilon}^{b} e^{-2|\beta_{n}|x_{2}} dx_{2} \right) \\ &= 2\pi \left(\sum_{|\alpha_{n}| \leq \kappa} \epsilon |A_{n}|^{2} + \sum_{|\alpha_{n}| > \kappa} |A_{n}|^{2} C_{n} \right), \end{split}$$

where $C_n = -\frac{\epsilon}{2|\beta_n|} \left(e^{-2|\beta_n|b} - e^{-2|\beta_n|(b-1)} \right) > 0$. By (3.24) and the proof of Theorem 3.9, we can easily get that \mathcal{A} coincides with the set $\{n \in \mathbb{Z} : \kappa \leq |\alpha_n| < k\}$. Then we have

$$\mathcal{A} \subset \mathcal{C} := \{ n \in \mathbb{Z} : |\alpha_n| \ge \kappa \}.$$

Therefore,

$$2\pi \left(\sum_{|\alpha_n| \le \kappa} \epsilon |A_n|^2 + \sum_{|\alpha_n| > \kappa} |A_n|^2 C_n \right) \ge 2\pi C \left(\sum_{n \in \mathcal{A}} |A_n|^2 \right) = C \|Pe_h\|_{L^2(\Gamma_b)^2}^2$$

where $C = \min\{\epsilon, \min_{n \in \mathcal{A}} C_n\}.$

The main result of this section is stated below.

Theorem 4.4. Suppose that $(u, v) \in H^{\rho}(\Omega_b)^2$, $\rho \geq 2$, satisfies the variational problem (3.14). Suppose also that the family of finite element spaces $\{X_h^2\}$ satisfies the assumption (4.1). Then there exists $h_0 \in (0, 1)$ such that for $h \in (0, h_0)$, the problem (4.2) admits a unique solution (u_h, v_h) with the estimates

$$\|(u,v) - (u_h,v_h)\|_{L^2(\Omega_b)^2} \le C \left(h^{\rho} + C_1 h^{\rho+1/2}\right) \|(u,v)\|_{H^{\rho}(\Omega_b)^2},$$

$$\|(u,v) - (u_h,v_h)\|_{H^1(\Omega_b)^2} \le C h^{\rho-1} \|(u,v)\|_{H^{\rho}(\Omega_b)^2},$$

where the positive constant C depends on ρ but is independent of h and (u, v).

Proof. Let h_1 and h_2 be specified as in Lemmas 4.1 and 4.2 and set $h_0 = \min\{h_1, h_2\}$. For $h \in (0, h_0)$, we deduce from Lemmas 4.1-4.3 that

$$\begin{aligned} \|e_h\|_{H^1(\Omega_b)^2}^2 &\leq C\left(h^{2\rho-2}\|(u,v)\|_{H^{\rho}(\Omega_b)^2}^2 + C\|e_h\|_{L^2(\Omega_b)^2}^2\right) \\ &\leq C\left(h^{2\rho-2}\|(u,v)\|_{H^{\rho}(\Omega_b)^2}^2 + C_1(h+C_2h^{3/2})^2\|e_h\|_{H^1(\Omega_b)^2}^2\right). \end{aligned}$$

Now letting h_0 be a constant such that $1 - C_1(h_0 + C_2 h_0^{3/2})^2 > 0$, we obtain

$$||e_h||_{H^1(\Omega_b)^2} \le Ch^{\rho-1} ||(u,v)||_{H^{\rho}(\Omega_b)^2}$$
 for all $h \in (0,h_0)$.

Therefore, using Lemma 4.2,

$$\|e_h\|_{L^2(\Omega_b)^2} \le C(h+C_1h^{3/2})\|e_h\|_{H^1(\Omega_b)^2} \le C\left(h^{\rho}+C_1h^{\rho+1/2}\right)\|(u,v)\|_{H^{\rho}(\Omega_b)^2},$$

which completes the proof.

5 Integral equation methods

The aim of this section is to develop an integral equation method for the conical diffraction problem (3.2). We make the following assumption.

Assumption (A): The grating profile Γ is the graph of some 2π -periodic function $x_2 = f(x_1)$, $x_1 \in \mathbb{R}$, where f is either C^2 -smooth or piecewise linear with a finite number of corner points in one periodic cell.

Introduce the α -quasiperiodic fundamental solution to the Helmholtz equation $(\Delta + \kappa^2)u = 0$ by

$$G(x,y) = \frac{i}{4} \sum_{n \in \mathbb{Z}} \exp(-i\alpha 2\pi n) H_0^{(1)} \left(k \sqrt{(x_1 + 2n\pi - y_1)^2 + (x_2 - y_2)^2} \right)$$
$$= \frac{i}{4\pi} \sum_{n \in \mathbb{Z}} \frac{1}{\beta_n} \exp\left(i\alpha_n (x_1 - y_1) + i\beta_n |x_2 - y_2|\right)$$

for $x - y \neq n(2\pi, 0)$, with $H_0^{(1)}(t)$ being the first kind Hankel function of order zero. Define the single-layer potential by

$$(\mathcal{S}g)(x) = 2 \int_{\Gamma} G(x, y)g(y)ds(y), \quad x \in \Omega,$$

with the density g. We make the ansatz for the solution (u^s, v^s) in the form

$$u^s = \mathcal{S}g_1, \quad v^s = \mathcal{S}g_2.$$

Further, define the single- and double-layer operators S and K by

$$(S\rho)(x) := 2 \int_{\Gamma} G(x,y)\rho(y)ds(y), \quad x \in \Gamma,$$

$$(K\rho)(x) := 2 \int_{\Gamma} \frac{\partial G(x,y)}{\partial \nu(y)}\rho(y)ds(y), \quad x \in \Gamma,$$

and the normal and tangential derivative operators K' and H' by

$$\begin{aligned} & (K'\rho)(x) & := 2 \int_{\Gamma} \frac{\partial G(x,y)}{\partial \nu(x)} \rho(y) ds(y), \quad x \in \Gamma, \\ & (H'\rho)(x) & := 2 \int_{\Gamma} \frac{\partial G(x,y)}{\partial \tau(x)} \rho(y) ds(y), \quad x \in \Gamma, \end{aligned}$$

where ν denotes the unit normal vector to the boundary Γ directed into the exterior of Ω and τ denotes the unit tangential vector to Γ .

Lemma 5.1. Let g_1, g_2 be the density functions of u^s, v^s , respectively, then the following jump relations hold

$$\begin{split} u^{s}(x) &= 2 \int_{\Gamma} G(x,y)g_{1}(y)ds(y) = Sg_{1}, \quad x \in \Gamma, \\ v^{s}(x) &= 2 \int_{\Gamma} G(x,y)g_{2}(y)ds(y) = Sg_{2}, \quad x \in \Gamma, \\ \frac{\partial u^{s}_{\pm}}{\partial \nu}(x) &= 2 \int_{\Gamma} \frac{\partial G(x,y)}{\partial \nu(x)}g_{1}(y)ds(y) \pm g_{1}(x) = K'g_{1}(x) \pm g_{1}(x), \quad x \in \Gamma, \\ \frac{\partial v^{s}_{\pm}}{\partial \nu}(x) &= 2 \int_{\Gamma} \frac{\partial G(x,y)}{\partial \nu(x)}g_{2}(y)ds(y) \pm g_{2}(x) = K'g_{2}(x) \pm g_{2}(x), \quad x \in \Gamma, \\ \frac{\partial u^{s}}{\partial \tau}(x) &= 2 \int_{\Gamma} \frac{\partial G(x,y)}{\partial \tau(x)}g_{1}(y)ds(y) = H'g_{1}(x), \quad x \in \Gamma, \\ \frac{\partial v^{s}}{\partial \tau}(x) &= 2 \int_{\Gamma} \frac{\partial G(x,y)}{\partial \tau(x)}g_{2}(y)ds(y) = H'g_{2}(x), \quad x \in \Gamma, \end{split}$$

where

$$\begin{aligned} \frac{\partial u_{\pm}^s}{\partial \nu}(x) &:= \lim_{h \to +0} \nu(x) \cdot \nabla u^s(x \pm h\nu(x)), \quad \frac{\partial v_{\pm}^s}{\partial \nu}(x) := \lim_{h \to +0} \nu(x) \cdot \nabla v^s(x \pm h\nu(x)), \\ \frac{\partial u^s}{\partial \tau}(x) &:= \tau(x) \cdot \nabla u^s(x), \qquad \qquad \frac{\partial v^s}{\partial \tau}(x) := \tau(x) \cdot \nabla v^s(x). \end{aligned}$$

Proof. We refer to [1] for the mapping properties of the single- and double-layer operators. By reference [8], we only need to prove the continuity of $\frac{\partial u}{\partial \tau}(x)$ on Γ .

$$\lim_{x \to \Gamma^{\pm}} \nabla u^s(x) = 2 \int_{\Gamma} \nabla_x G(x, y) g_1(y) ds(y) \pm g_1(y) \nu(y).$$

Therefore,

$$\lim_{x \to \Gamma^{\pm}} \frac{\partial u^{s}}{\partial \tau}(x) = \lim_{x \to \Gamma^{\pm}} \tau(x) \cdot \nabla u^{s}(x)$$
$$= \tau(x) \cdot 2 \int_{\Gamma} \nabla_{x} G(x, y) g_{1}(y) ds(y) \pm \tau(x) \cdot g_{1}(y) \nu(y)$$
$$= 2 \int_{\Gamma} \frac{\partial G(x, y)}{\partial \tau(x)} g_{1}(y) ds(y).$$

Using the jump relations for (2.7), we have on Γ that

$$\lambda \frac{\partial u^s}{\partial \nu} + i\omega\mu\cos^2\phi \ u^s + \lambda\sin\phi\sqrt{\frac{\mu}{\epsilon}}\frac{\partial v^s}{\partial \tau}$$
$$= \lambda(K'g_1 + g_1) + i\omega\mu\cos^2\phi Sg_1 + \lambda\sin\phi\sqrt{\frac{\mu}{\epsilon}}H'g_2 = h_1, \tag{5.1}$$

$$\frac{\partial v^s}{\partial n} + i\lambda\omega\epsilon\cos^2\phi \,v^s - \sin\phi\sqrt{\frac{\epsilon}{\mu}}\frac{\partial u^s}{\partial\tau}$$
$$= (K'g_2 + g_2) + i\lambda\omega\epsilon\cos^2\phi Sg_2 - \sin\phi\sqrt{\frac{\epsilon}{\mu}}H'g_1 = h_2,$$
(5.2)

where

$$h_1 := -\left(\lambda \frac{\partial u^i}{\partial \nu} + i\omega\mu \cos^2\phi \ u^i + \lambda \sin\phi \sqrt{\frac{\mu}{\epsilon}} \frac{\partial v^i}{\partial \tau}\right),$$
$$h_2 := -\left(\frac{\partial v^i}{\partial \nu} + i\lambda\omega\epsilon \cos^2\phi \ v^i - \sin\phi \sqrt{\frac{\epsilon}{\mu}} \frac{\partial u^i}{\partial \tau}\right).$$

Combining (5.1) with (5.2), we obtain the integral equations

$$\begin{pmatrix} \lambda(K'+I) & \lambda \sin \phi \sqrt{\mu/\epsilon} H' \\ -\sin \phi \sqrt{\epsilon/\mu} H' & K'+I \end{pmatrix} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} + \begin{pmatrix} i\omega\mu\cos^2\phi S & 0 \\ 0 & i\lambda\omega\epsilon\cos^2\phi S \end{pmatrix} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}.$$

Therefore the equivalent system is

$$Ag + Bg := \begin{pmatrix} \lambda(K'+I) & dH' \\ -cH' & K'+I \end{pmatrix} g + \begin{pmatrix} iaS & 0 \\ 0 & ibS \end{pmatrix} g = h,$$
(5.3)

with $g = (g_1, g_2)^{\top}, h = (h_1, h_2)^{\top} \in H^{-1/2}(\Gamma)^2$, and

$$d = \lambda \sin \phi \sqrt{\mu/\epsilon}, \ c = \sin \phi \sqrt{\epsilon/\mu}, \ a = \omega \mu \cos^2 \phi, \ b = \lambda \omega \epsilon \cos^2 \phi$$

Under the Assumption (A), the single-layer operator S is invertible form $H^{-1/2}(\Gamma) \rightarrow H^{1/2}(\Gamma)$.

Theorem 5.2. Suppose that Assumption (A) holds. Then the operator A + B defined in (5.3) is a Fredholm operator with an index zero. Moreover, the system (5.3) admits a unique solution if $k^2 \neq \gamma^2$ and $\lambda < 0$.

Proof. It suffices to prove the Fredholm property of A+B, since the second assertion of Theorem 5.2 follows directly from the Fredholm alternative combined with Theorem 3.5. To do this, we introduce the adjoint operator H of H', given by

$$(Hg)(x) := 2 \int_{\Gamma} \frac{\partial G(x,y)}{\partial \tau(y)} g(y) ds(y), \quad x \in \Gamma.$$

It is known that the adjoint operator of K is just K'. Since the operator S is compact from $H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma)$, we only need to justify the Fredholm property of the adjoint operator A^* of A, given by

$$A^* = \begin{pmatrix} \lambda(I+K) & -cH \\ dH & I+K \end{pmatrix} : \quad H^{1/2}(\Gamma)^2 \to H^{1/2}(\Gamma)^2.$$

It is easy to see that the operator $H_1 = H + j$ with the rank 1 operator

$$ju = (u, e)_{L^2(\Gamma)} e, \quad e = 1 \in \mathbb{C},$$

is invertible in $H^{1/2}(\Gamma)$. We will show that the operator

$$A_1 := \begin{pmatrix} \lambda(I+K) & -cH_1 \\ dH_1 & I+K \end{pmatrix} : \quad H^{1/2}(\Gamma)^2 \to H^{1/2}(\Gamma)^2$$

is a Fredholm operator with an index zero. Simple calculations show that the operator

$$B_1 := \begin{pmatrix} -(dH_1)^{-1}(I+K) & I \\ I & 0 \end{pmatrix} = \begin{pmatrix} 0 & I \\ I & (dH_1)^{-1}(I+K) \end{pmatrix}^{-1}$$

is invertible, and that

$$A_1 B_1 = \begin{pmatrix} -\lambda (I+K)(dH_1)^{-1}(I+K) - cH_1 & \lambda (I+K) \\ 0 & dH_1 \end{pmatrix}.$$
 (5.4)

Using the relations HK = -KH and (I + K)e = 0 (see [8]), we get

$$(I+K)H_1 = H_1(I-K) - j(I-K),$$

and thus

$$(dH_1)^{-1}(I+K) = d^{-1}(I-K)H_1^{-1} - (dH_1)^{-1}[j(I-K)]H_1^{-1}.$$
(5.5)

Inserting (5.5) into (5.4) gives

$$A_1B_1 = \begin{pmatrix} -d^{-1}[\lambda(I-K^2) + cdH_1^2]H_1^{-1} + j_1 & \lambda(I+K) \\ 0 & dH_1 \end{pmatrix}$$

with $j_1 := \lambda (I + K) (dH_1)^{-1} [j(I - K)] H_1^{-1}$ being a rank one operator. Hence, A_1 is Fredholm with index zero if this is true for the operator $\lambda (I - K^2) + cdH_1^2$. Making use of $K^2 - H^2 = I$ and the definitions of c and d, we find

$$\lambda(I - K^2) + cdH_1^2 = (cd - \lambda)H_1^2 + j_2 = -\lambda\cos^2\phi H_1^2 + j_2,$$
(5.6)

where j_2 is some operator with rank one. Since $|\phi| < \pi/2$, we finally conclude that the operator (5.6) is Fredholm with index zero. Theorem 5.2 is thus proven.

Acknowledgments

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