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Stability of grating diffraction problems for plane wave incidence: Explicit dependence on wavenumbers and incident angles

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ABSTRACT

Suppose a time-harmonic plane wave is incident onto an impenetrable grating profile of Dirichlet or Impedance kind in two dimensions. The grating profile is assumed to be given by a Lipschitz function. We derive a stability estimate of this grating diffraction problem using the variational method with a transparent boundary condition. An explicit dependence of solutions on the incident wavenumber and the incident angle is obtained, which carries over to transmission problems for penetrable gratings. Our approach relies heavily on using Rellich's identities in periodic structures.

1. Introduction

Diffraction gratings have a long history and are widely used in a wide variety of scientific and technological fields [3,22]. We refer to the books [4,22,24] for their physical and mathematical background and to [1,6,8,10] and the references therein for mathematical analysis and numerical methods on electromagnetic scattering by diffraction gratings. There are two fundamental polarizations in electromagnetic scattering problems. The first is the transverse electric (TE) polarization where the electric field is parallel to the grooves of the grating. As for the second transverse magnetic (TM) polarization, the magnetic field is parallel to the grooves of the grating. In both TE or TM polarization cases, the well-posedness of the grating diffraction problem with the Rayleigh expansion radiation condition has been sufficiently studied using variational and integral equation methods. It is well known that the diffraction problem is uniquely solvable for all frequencies except for a discrete set with the only accumulation point at infinity [1,2,6,13,14,19,23]. However, uniqueness does not hold in general due to the existence of guided wave modes to the homogeneous problem. Uniqueness can be proved (i.e., guided waves are excluded) if additional conditions are imposed on the incident wavenumbers,

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scattering interfaces and material parameters; see e.g., [2,6,11,13,19,23]. We also refer to recent studies [16,20] where uniqueness and existence are justified for all frequencies using sharp Rayleigh expansion conditions derived from different limiting absorption principles.

In this paper, we assume that the grating is 2π-periodic in the x_1-direction and invariant in the x_3-direction. The grating interface is supposed to be given by a Lipschitz function. We investigate the stability estimate of the scattering of time-harmonic (with the time variation $e^{-i\omega t}$, $w > 0$) plane waves by a perfectly reflective grating or by a penetrable grating in an isotropic lossless medium. One feature of our stability estimates is the explicit dependence of the solution on the incident wavenumber $k > 0$, the incident angle θ as well as the Lipschitz constant of the profile function. Wavenumber-dependent estimates were derived in [7] for rough surface scattering due to a compactly supported source term and also in [5] for the acoustic scattering of plane waves by a rectangular cavity. However, to the best of our knowledge, both incident wavenumber- and angle-dependent estimates are not available in the literature. Our approach relies heavily on the use of Rellich's type identities, which were also used in [6,19] for proving the uniqueness of solutions to grating diffractions for all frequencies; see also [7,14,18]. We also refer to [9,12,15,17] for the application of Rellich's identities to Maxwell equations and the Lame system.

This article is organized as follows. In Section 2 we mathematically formulate grating diffraction problems by introducing some basic notations. Section 3 is devoted to the variational method for the Dirichlet boundary value problem with a Lipschitz grating profile. We derive a stability result with explicit dependence on the incident angle and also on the wavenumber. In Section 4 we discuss the impedance boundary value problem for impenetrable gratings and in Section 5 the transmission conditions for penetrable gratings, with the same kind of stability estimate. The proof of some preliminary lemmas in Section 3 will be postponed to Section 6.

2. Grating diffraction problems

Let the diffraction grating profile be described by the curve

$$\tilde{\Gamma} := \{x \in \mathbb{R}^2 : x_2 = f(x_1),\ x_1 \in \mathbb{R}\},$$

with $f \in C_p^{0,1}$, i.e., f is a 2π-periodic Lipschitz function. We assume $f(x_1) > f_-$ for all $x_1 \in (0, 2\pi)$. Denote by $L > 0$ the Lipschitz constant of f. Suppose that the space above $\tilde{\Gamma}$ is filled with a homogeneous and isotropic medium. This implies that the unbounded region

$$\tilde{\Omega} := \{x \in \mathbb{R}^2 : x_2 > f(x_1),\ x_1 \in \mathbb{R}\}$$

can be characterized by a constant index of refraction (or wave number) $k > 0$. We use the following notations in one period of the grating profile

$$\Gamma = \Gamma_f := \{x \in \mathbb{R}^2 : x_2 = f(x_1),\ 0 < x_1 < 2\pi\},$$
$$\Omega = \Omega_f := \{x \in \mathbb{R}^2 : x_2 > f(x_1),\ 0 < x_1 < 2\pi\}.$$

Introduce the artificial boundary

$$\Gamma_R := \{(x_1, R) : 0 \le x_1 \le 2\pi\}, \qquad R > \Gamma_{\max} := \max_{0 \le t \le 2\pi} |f(t)|,$$

and the bounded domain

$$\Omega_R = \Omega_{R,f} := \{x \in \mathbb{R}^2 : f(x_1) < x_2 < R,\ 0 < x_1 < 2\pi\}.$$

Assume that a plane wave given by

$$u^i(x) = \gamma e^{i\alpha x_1 - i\beta x_2}, \qquad \gamma \in \mathbb{C}, \qquad i = \sqrt{-1} \tag{2.1}$$

is incident onto Γ from above, where $\alpha = k\sin\theta$, $\beta = k\cos\theta$ and $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is the incident angle with the positive x_2-axis. Here we have assumed that $|u^i| = |\gamma|$, in order to see the uniqueness of the diffraction problem from our stability estimates. Then the scattered field $u^s = u - u^i$ satisfies the Helmholtz equation

$$\Delta u^s + k^2 u^s = 0 \qquad\qquad \text{in} \quad \Omega. \tag{2.2}$$

Moreover, the scattered field u^s is assumed to be α-quasiperiodic in the sense that $u^s(x_1, x_2)e^{-i\alpha x_1}$ is 2π-periodic in x_1. By this definition, we have

$$u^s(x_1 + 2\pi, x_2) = e^{2i\alpha\pi} u^s(x_1, x_2) \quad \text{ for all } x \in \Omega. \tag{2.3}$$

It is obvious that the incident field u^i is α-quasiperiodic. Under the assumption (2.3), the function $u^s(x_1, x_2)e^{-i\alpha x_1}$ can be expanded into the Fourier series:

$$u^s(x_1, x_2)e^{-i\alpha x_1} = \sum_{n\in\mathbb{Z}} u_n(x_2)e^{inx_1}, \qquad x_2 > \Gamma_{\max}.$$

Substituting u^s into the Helmholtz equation (2.2) in Ω and applying the method of separation of variables, we get the expression of u^s as a sum of plane waves:

$$u^s(x_1, x_2) = \sum_{n\in\mathbb{Z}} A_n e^{i\alpha_n x_1 + i\beta_n x_2} + B_n e^{i\alpha_n x_1 - i\beta_n x_2}, \qquad A_n, B_n \in \mathbb{C},$$

where

$$\alpha_n = n + \alpha, \qquad \beta_n = \begin{cases} \sqrt{k^2 - |\alpha_n|^2}, & |\alpha_n| \le k, \\ i\sqrt{|\alpha_n|^2 - k^2}, & |\alpha_n| > k. \end{cases} \tag{2.4}$$

Physically, the scattered field remains bounded as $x_2 \to \infty$. Hence, u^s is only composed of bounded outgoing waves in Ω, leading to the well-known Rayleigh expansion condition:

$$u^s(x) = \sum_{n\in\mathbb{Z}} u_n e^{i\alpha_n x_1 + i\beta_n x_2}, \qquad x_2 > \Gamma_{\max}, \tag{2.5}$$

with the Rayleigh coefficients $u_n \in \mathbb{C}$. Obviously, u^s in (2.5) can be split into the finite sum $\sum_{|\alpha_n|\le k}$ of outgoing plane waves and the infinite sum $\sum_{|\alpha_n|>k}$ of exponentially decaying waves, which are called surface or evanescent waves.

We now introduce periodic and quasiperiodic Sobolev spaces to be used in this paper. For $s \in \mathbb{R}$, $s \ge 0$, the Sobolev space $H^s(0, 2\pi)$ of periodic functions is defined as the completion of $\left\{u\big|_{[0,2\pi]} : u \text{ is trigonometric polynomial}\right\}$ with respect to the inner product

$$\langle u, v\rangle := \sum_{n\in\mathbb{Z}} (k^2 + n^2)^s u_n \bar{v}_n,$$

where u_n and v_n are Fourier coefficients of u and v, respectively. The periodic Sobolev space $H^s_p(\Gamma)$ and the α-quasiperiodic Sobolev space $H^s_\alpha(\Gamma)$ can be defined, respectively, by

$$H_p^s(\Gamma) := \{u : \Gamma \to \mathbb{C},\ u(x_1, f(x_1)) \in H^s(0, 2\pi)\},$$
$$H_\alpha^s(\Gamma) := \left\{u : \Gamma \to \mathbb{C},\ u(x_1, f(x_1))e^{-i\alpha x_1} \in H_p^s(0, 2\pi)\right\}.$$

We also define

$$H_\alpha^1(\Omega_R) := \left\{u \in H^1(\Omega_R) : u(x_1, x_2)e^{-i\alpha x_1} \text{ is } 2\pi\text{-periodic with respect to } x_1\right\}.$$
$$H_{\mathrm{loc},\alpha}^1(\Omega) := \left\{u : u \in H_\alpha^1(\Omega_b) \text{ for any } b > \Gamma_{\max}\right\}.$$

3. Dirichlet boundary value problem

In this section, we consider the TE polarization of electromagnetic scattering from a perfectly conducting grating. The problem we wish to analyze is to find $u \in H^1_{loc,\alpha}(\Omega)$ such that

$$\text{(DBVP)}: \qquad \begin{cases} \Delta u + k^2 u = 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \Gamma, \\ u - u^i = \sum_{n\in\mathbb{Z}} u_n e^{i\alpha_n x_1 + i\beta_n x_2} & x_2 > \Gamma_{\max}. \end{cases} \tag{3.1}$$

The purpose of this section is to derive a stability estimate of the Dirichlet problem (3.1) via the variational approach. Define the space

$$X_R = X(\Omega_R) := \{u \in H_\alpha^1(\Omega_R) : u = 0 \text{ on } \Gamma\},$$

equipped with the norm

$$\|u\|_{X_R}^2 = k^2\|u\|_{L^2(\Omega_R)}^2 + \|\nabla u\|_{L^2(\Omega_R)}^2.$$

We define the Dirichlet-to-Neumann map $T : H_\alpha^{1/2}(\Gamma_R) \to H_\alpha^{-1/2}(\Gamma_R)$ on the artificial boundary Γ_R by

$$(Tg)(x_1) := \sum_{n\in\mathbb{Z}} i\beta_n g_n e^{i\alpha_n x_1}, \qquad g(x_1) := \sum_{n\in\mathbb{Z}} g_n e^{i\alpha_n x_1} \in H_\alpha^{1/2}(\Gamma_R),$$

which corresponds to the Rayleigh expansion (2.5). The operator T is well defined and bounded because $\beta_n = i|\alpha_n| + O\left(\frac{1}{|n|}\right)$ as $|n| \to +\infty$. Clearly,

$$T\left(u^s\big|_{x_2=R}\right) = \frac{\partial u^s}{\partial x_2}\Big|_{\Gamma_R}, \qquad T\left(u^i\big|_{x_2=R}\right) = i\beta e^{-i\beta R}\gamma e^{i\alpha x_1}.$$

This implies that

$$T\left(u^i\big|_{\Gamma_R}\right) - \frac{\partial u^i}{\partial \nu}\Big|_{\Gamma_R} = 2i\beta e^{-i\beta R}\gamma e^{i\alpha x_1}, \tag{3.2}$$

where $\nu = (0, 1)^\top$ is the normal direction on Γ_R pointing into Ω.

Theorem 3.1. *Let u^i be a plane wave given by (2.1) and suppose that $f_- + 1 < \Gamma_{\min} := \min_{0\le t\le 2\pi} |f(t)|$. Then there exists a unique solution $u \in X_R$ to the Dirichlet problem (3.1) satisfying*

$$\|u\|_{X_R} \le 2\sqrt{2\pi}\cos\theta|\gamma|C, \tag{3.3}$$

where

$$C := \sqrt{kM^2 + 4k^4(R - f_-)^3},$$
$$M := 4k^3(R - f_-)^3 + 2k^2(R - f_-)^2 + 4k^3(R - f_-)^3 \cos\theta + 1.$$

Remark 3.2. (i) The stability estimate (3.3) obviously yields uniqueness of (DBVP), that is, $u \equiv 0$ if $\gamma = 0$. If the incident angle θ tends to $\frac{\pi}{2}$ or $-\frac{\pi}{2}$, the energy of the solution will become smaller and smaller; note that $|\theta| \neq \pi/2$. At the normal incidence (i.e. $\theta = 0$), it implies that the energy is larger than for other incident angles.

(ii) At high wavenumbers, Theorem 3.1 shows that $\|u\|_{X_R} \sim k^{7/2}$ for a plane wave in the Dirichlet case.

The proof of Theorem 3.1 relies on the variational framework [7] where a wave-number-dependent stability estimate was derived for rough surface scattering problems caused by an L^2 source term. We adapt the arguments of [7] to grating diffraction problems with a plane wave incidence and deduce the dependence on both wavenumbers and incident angles. It is easy to get the following variational formulation in one periodic cell: find $u \in X_R$ such that

$$a(u, v) = F(v) \quad \text{for all } v \in X_R, \tag{3.4}$$

where

$$a(u, v) := \int_{X_R} \nabla u \cdot \nabla \bar{v} - k^2 u\bar{v}\, dx - \int_{\Gamma_R} (Tu)\bar{v}\, ds, \tag{3.5}$$

$$F(v) := -2i\beta e^{-i\beta R}\gamma \int_0^{2\pi} e^{i\alpha x_1}\bar{v}(x_1, R)\, dx_1.$$

Noting that on Γ_R, we have used (3.2) and the following identity:

$$\partial_\nu u = \partial_\nu u^s + \partial_\nu u^i = Tu^s + \partial_\nu u^i = Tu - Tu^i + \partial_\nu u^i.$$

We also need the following estimate (see also [7]).

Lemma 3.3. *If $u \in X_R$ is a solution to (3.1), then we have*

$$\left\|\frac{\partial u}{\partial x_2}\right\|^2_{L^2(\Omega_R)} \le (R - f_-)\int_{\Gamma_R} \left|\frac{\partial u}{\partial x_2}\right|^2 - \left|\frac{\partial u}{\partial x_1}\right|^2 + k^2|u|^2\, ds + \int_{\Omega_R} |\nabla u|^2 - k^2|u|^2\, dx. \tag{3.6}$$

Proof. For grating profile functions of Lipschitz kind (i.e., $f \in C_p^{0,1}$), there exists a sequence of infinitely smooth functions f_j, $j \in \mathbb{N}$ such that $\|f_j - f\|_{C_p^{0,1}} \to 0$ as $j \to \infty$. Define

$$\Gamma_j := \{x \in \mathbb{R} : x_2 = f_j(x_1), 0 < x_1 < 2\pi\},$$
$$\Omega_{R,j} := \{x \in \mathbb{R}^2 : f_j(x_1) < x_2 < R,\ 0 < x_1 < 2\pi\},$$

and let $u_j \in X(\Omega_{R,j}) \subset X(\Omega_R)$ be the unique solution to the Dirichlet problem (3.1). By Remark 3.2 in [14], we have $u_j \to u$, $j \to \infty$ in X_R. Using the Rellich identity (see Corollary 6.2 with $c = f_-$ in the Appendix) and the fact that $\nu_2 \ge 0$ on Γ, we obtain

$$\begin{aligned}\left\|\frac{\partial u_j}{\partial x_2}\right\|^2_{L^2(\Omega_{R,j})} &\leq \int\limits_{\Gamma_j}(x_2-f_-)\nu_2\left|\frac{\partial u_j}{\partial \nu}\right|^2 ds + 2\int\limits_{\Omega_{R,j}}\left|\frac{\partial u_j}{\partial x_2}\right|^2 dx \\ &= (R-f_-)\int\limits_{\Gamma_R}\left|\frac{\partial u_j}{\partial x_2}\right|^2 - \left|\frac{\partial u_j}{\partial x_1}\right|^2 + k^2|u_j|^2\,ds + \int\limits_{\Omega_{R,j}}|\nabla u_j|^2 - k^2|u_j|^2\,dx. \end{aligned} \tag{3.7}$$

Hence passing the limit $j\to\infty$ in the inequality (3.7), we obtain (3.6). This completes the proof. □

The proofs of Lemmas 3.4 and 3.5 below will be given in the Appendix.

Lemma 3.4. *If $v\in X_R$, then*

$$\sqrt{2\pi}\|v\|_{H^{1/2}_\alpha(\Gamma_R)} \leq \|v\|_{X_R}.$$

Lemma 3.5. *If $v\in H^1_\alpha(\Omega_R)$, then*

$$\|v\|^2_{L^2(\Omega_R)} \leq (R-f_-)^2\left\|\frac{\partial v}{\partial x_2}\right\|^2_{L^2(\Omega_R)} + 2(R-f_-)\|v\|^2_{L^2(\Gamma)}.$$

From the Rayleigh expansion (2.5), we may rewrite the restriction of u to Γ_R as

$$u|_{\Gamma_R} = \sum_{n\in\mathbb{Z}} u_n e^{i\alpha_n x_1 + i\beta_n R} + \gamma e^{i(\alpha x_1 - \beta R)} = \sum_{n\in\mathbb{Z}}\tilde{u}_n e^{i\alpha_n x_1}, \tag{3.8}$$

where

$$\tilde{u}_n := \begin{cases} u_n e^{i\beta_n R}, & n\neq 0,\\ u_0 e^{i\beta R} + \gamma e^{-i\beta R}, & n=0.\end{cases}$$

Using these notations, we have the following lemma.

Lemma 3.6. *Let $u=u^i+u^s\in X_R$ be a solution to (3.1). Then we have*

$$\int\limits_{\Gamma_R}\left|\frac{\partial u}{\partial x_2}\right|^2 - \left|\frac{\partial u}{\partial x_1}\right|^2 + k^2|u|^2\,ds \leq 2k\mathrm{Im}\int\limits_{\Gamma_R}(Tu)\bar{u}\,ds - 8\pi\mathrm{Re}\,(u_0\bar{\gamma}e^{i2\beta R})|\beta|^2.$$

Here u_0 denotes the zeroth Rayleigh coefficient of u^s.

Proof. Recall that

$$u(x_1,x_2) = \sum_{n\in\mathbb{Z}} u_n e^{i\alpha_n x_1 + i\beta_n x_2} + \gamma e^{i(\alpha x_1 - \beta x_2)}, \quad x_2 > \Gamma_{\max}.$$

Using (3.8), we have

$$\begin{aligned}&\int\limits_{\Gamma_R}\left|\frac{\partial u}{\partial x_2}\right|^2 - \left|\frac{\partial u}{\partial x_1}\right|^2 + k^2|u|^2\,ds \\ &= \int\limits_0^{2\pi}\left|\sum_{n\in\mathbb{Z}} u_n i\beta_n e^{i(\alpha_n x_1 + \beta_n R)} - \gamma i\beta e^{i(\alpha x_1 - \beta R)}\right|^2 - \left|\sum_{n\in\mathbb{Z}} i\alpha_n\tilde{u}_n e^{i\alpha_n x_1}\right|^2 + k^2\left|\sum_{n\in\mathbb{Z}}\tilde{u}_n e^{i\alpha_n x_1}\right|^2 dx_1\end{aligned}$$

$$
\begin{aligned}
&= \int_0^{2\pi} \left| \sum_{n\neq 0} i\beta_n \tilde{u}_n e^{i\alpha_n x_1} + i\beta e^{i\alpha x_1}(u_0 e^{i\beta R} - \gamma e^{-i\beta R}) \right|^2 dx_1 - 2\pi \sum_{n\in\mathbb{Z}} |\alpha_n|^2 |\tilde{u}_n|^2 + 2\pi k^2 \sum_{n\in\mathbb{Z}} |\tilde{u}_n|^2 \\
&\le 2\pi \left(\sum_{n\neq 0} |\beta_n|^2 |\tilde{u}_n|^2 + |\beta|^2 |u_0 e^{i\beta R} - \gamma e^{-i\beta R}|^2 \right) - 2\pi \sum_{n\in\mathbb{Z}} |\alpha_n|^2 |\tilde{u}_n|^2 + 2\pi k^2 \sum_{n\in\mathbb{Z}} |\tilde{u}_n|^2 \\
&= 2\pi \sum_{n\in\mathbb{Z}} (|\beta_n|^2 - |\alpha_n|^2 + k^2)|\tilde{u}_n|^2 + 2\pi|\beta|^2(|u_0 e^{i\beta R} - \gamma e^{-i\beta R}|^2 - |\tilde{u}_0|^2). \qquad (3.9)
\end{aligned}
$$

We next calculate the first and second terms on the right-hand side of (3.9). Recalling the definition of β_n in (2.4),

$$
\sum_{n\in\mathbb{Z}} (|\beta_n|^2 - |\alpha_n|^2 + k^2)|\tilde{u}_n|^2 = 2 \sum_{|\alpha_n|\le k} |\beta_n|^2 |\tilde{u}_n|^2.
$$

By the definition of $\tilde{u}_0$, we have

$$
\begin{aligned}
|u_0 e^{i\beta R} - \gamma e^{-i\beta R}|^2 - |\tilde{u}_0|^2 &= |u_0 e^{i\beta R} - \gamma e^{-i\beta R}|^2 - |u_0 e^{i\beta R} + \gamma e^{-i\beta R}|^2 \\
&= (|u_0 e^{i\beta R}|^2 + |\gamma e^{-i\beta R}|^2 - 2\mathrm{Re}\,(u_0 e^{i\beta R} \bar{\gamma} e^{i\beta R})) \\
&\quad - (|u_0 e^{i\beta R}|^2 + |\gamma e^{-i\beta R}|^2 + 2\mathrm{Re}\,(u_0 e^{i\beta R} \bar{\gamma} e^{i\beta R})) \\
&= -4\mathrm{Re}\,(u_0 \bar{\gamma} e^{2i\beta R}).
\end{aligned}
$$

Inserting the previous two equations into (3.9) gives

$$
\begin{aligned}
\int_{\Gamma_R} \left|\frac{\partial u}{\partial x_2}\right|^2 - \left|\frac{\partial u}{\partial x_1}\right|^2 + k^2|u|^2\, ds &\le 4\pi \sum_{|\alpha_n|\le k} |\beta_n|^2|\tilde{u}_n|^2 + 2\pi|\beta|^2[-4\mathrm{Re}\,(u_0\bar{\gamma}e^{2i\beta R})] \\
&= 4\pi \sum_{|\alpha_n|\le k} |\beta_n|^2|\tilde{u}_n|^2 - 8\pi|\beta|^2 \mathrm{Re}\,(u_0\bar{\gamma}e^{2i\beta R}). \qquad (3.10)
\end{aligned}
$$

Furthermore, by definition of the Dirichlet-to-Neumann map T and the definition of β_n, we have

$$
\int_{\Gamma_R} (Tu)\bar{u}\, ds = \int_0^{2\pi} \left(\sum_{n\in\mathbb{Z}} i\beta_n \tilde{u}_n e^{i\alpha_n x_1} \right) \left(\sum_{m\in\mathbb{Z}} \bar{\tilde{u}}_m e^{-i\alpha_m x_1} \right) dx_1 = 2\pi \sum_{n\in\mathbb{Z}} i\beta_n |\tilde{u}_n|^2, \qquad (3.11)
$$

and therefore

$$
\mathrm{Im} \int_{\Gamma_R} (Tu)\bar{u}\, ds = 2\pi \sum_{|\alpha_n|\le k} \beta_n |\tilde{u}_n|^2. \qquad (3.12)
$$

Consequently, combining (3.10) with (3.12) yields

$$
\begin{aligned}
& 2k\mathrm{Im} \int_{\Gamma_R} (Tu)\bar{u}\, ds - 8\pi \mathrm{Re}\,(u_0\bar{\gamma}e^{i2\beta R})|\beta|^2 - \int_{\Gamma_R} \left|\frac{\partial u}{\partial x_2}\right|^2 - \left|\frac{\partial u}{\partial x_1}\right|^2 + k^2|u|^2\, ds \\
&\ge 4k\pi \sum_{|\alpha_n|\le k} \beta_n |\tilde{u}_n|^2 - 4\pi \sum_{|\alpha_n|\le k} |\beta_n|^2 |\tilde{u}_n|^2 \\
&= 4\pi \sum_{|\alpha_n|\le k} \beta_n (k - \beta_n)|\tilde{u}_n|^2 \ge 0,
\end{aligned}
$$

which completes the proof. □

Proof of Theorem 3.1. Choosing $v = u$ in the variational formula (3.4) and then taking the real and imaginary parts respectively, we get

$$\int_{\Omega_R} |\nabla u|^2 - k^2|u|^2\,dx - \mathrm{Re}\int_{\Gamma_R} (Tu)\bar{u}\,ds = \mathrm{Re}\,F(u), \tag{3.13}$$

$$-\mathrm{Im}\int_{\Gamma_R} (Tu)\bar{u}\,ds = \mathrm{Im}\,F(u). \tag{3.14}$$

By (3.11), we can easily get

$$\mathrm{Re}\int_{\Gamma_R} (Tu)\bar{u}\,ds = -2\pi \sum_{|\alpha_n|>k} |\beta_n||\tilde{u}_n|^2 \le 0. \tag{3.15}$$

Using equation (3.14) and Lemma 3.6, we can bound the left-hand side of the inequality in Lemma 3.6 by

$$\int_{\Gamma_R} \left|\frac{\partial u}{\partial x_2}\right|^2 - \left|\frac{\partial u}{\partial x_1}\right|^2 + k^2|u|^2\,ds \le -2k\mathrm{Im}\,F(u) - 8\pi\mathrm{Re}\,(u_0\bar{\gamma}e^{i2\beta R})|\beta|^2. \tag{3.16}$$

Inserting equations (3.13) and (3.16) into (3.6) and using (3.15), we get

$$\begin{aligned}
\left\|\frac{\partial u}{\partial x_2}\right\|^2_{L^2(\Omega_R)} &\le (R-f_-)\int_{\Gamma_R} \left|\frac{\partial u}{\partial x_2}\right|^2 - \left|\frac{\partial u}{\partial x_1}\right|^2 + k^2|u|^2\,ds + \int_{\Omega_R} |\nabla u|^2 - k^2|u|^2\,dx \\
&\le (R-f_-)\left[-2k\mathrm{Im}\,F(u) - 8\pi\mathrm{Re}\,(u_0\bar{\gamma}e^{2i\beta R})|\beta|^2\right] \\
&\quad +\mathrm{Re}\int_{\Gamma_R}(Tu)\bar{u}\,ds + \mathrm{Re}\,F(u) \\
&\le (R-f_-)2k|F(u)| + |F(u)| + 8\pi(R-f_-)|\beta|^2|\mathrm{Re}\,(\bar{\gamma}u_0e^{2i\beta R})| \\
&\le [2k(R-f_-)+1]\,|F(u)| + 8\pi(R-f_-)|\beta|^2(|\gamma||\tilde{u}_0| + |\gamma|^2).
\end{aligned} \tag{3.17}$$

In the last term of the above inequality, we have used the following identity,

$$\bar{\gamma}\tilde{u}_0e^{i\beta R} = \bar{\gamma}\left(u_0e^{2i\beta R} + \gamma\right) = \bar{\gamma}u_0e^{2i\beta R} + |\gamma|^2,$$

which implies,

$$\left|\mathrm{Re}\,\left(\bar{\gamma}u_0e^{2i\beta R}\right)\right| \le \left|\mathrm{Re}\,\left(\bar{\gamma}\tilde{u}_0e^{i\beta R}\right) - |\gamma|^2\right| \le |\gamma||\tilde{u}_0| + |\gamma|^2.$$

On the other hand, since

$$\bar{u}\big|_{\Gamma_R} = \sum_{n\in\mathbb{Z}} \bar{\tilde{u}}_n e^{-i\alpha_n x_1}, \quad \alpha_n = n+\alpha,$$

we get

$$
\begin{aligned}
|F(u)| &= \left| -2i\beta e^{-i\beta R}\gamma \int_0^{2\pi} e^{i\alpha x_1}\bar{u}(x_1,R)\,dx_1 \right| \\
&= \left| 2\beta\gamma \int_0^{2\pi} \sum_{n\in\mathbb{Z}} \bar{\tilde{u}}_n e^{i(\alpha-\alpha_n)x_1}\,dx_1 \right| \\
&= |4\pi\beta\gamma\tilde{u}_0|. \qquad (3.18)
\end{aligned}
$$

Hence, (3.17) can be estimated by

$$
\begin{aligned}
\left\| \frac{\partial u}{\partial x_2} \right\|^2_{L^2(\Omega_R)} &\le [2k(R-f_-)+1]\, 4\pi|\beta||\gamma||\tilde{u}_0| + 8\pi(R-f_-)|\beta|^2|\gamma||\tilde{u}_0| \\
&\quad + 8\pi(R-f_-)|\beta|^2|\gamma|^2 \\
&= C_1|\tilde{u}_0||\gamma| + 8\pi(R-f_-)|\beta|^2|\gamma|^2, \qquad (3.19)
\end{aligned}
$$

where

$$
C_1 = 4\pi|\beta|\left[2k(R-f_-)+2|\beta|(R-f_-)+1\right].
$$

With the help of (3.13) and (3.15), we get

$$
\begin{aligned}
\|u\|^2_{X_R} &= \|\nabla u\|^2_{L^2(\Omega_R)} + k^2\|u\|^2_{L^2(\Omega_R)} \\
&= 2k^2\|u\|^2_{L^2(\Omega_R)} + \mathrm{Re}\int_{\Gamma_R} (Tu)\bar{u}\,ds + \mathrm{Re}\,F(u) \\
&\le 2k^2\|u\|^2_{L^2(\Omega_R)} + |F(u)|. \qquad (3.20)
\end{aligned}
$$

Applying Lemma 3.5 (note that $u=0$ on Γ), (3.18) and (3.19), we continue to estimate $\|u\|^2_{X_R}$ in (3.20) by

$$
\begin{aligned}
\|u\|^2_{X_R} &\le 2k^2(R-f_-)^2 \left\| \frac{\partial u}{\partial x_2} \right\|^2_{L^2(\Omega_R)} + 4\pi|\beta||\gamma||\tilde{u}_0| \\
&\le 2k^2(R-f_-)^2\left[C_1|\gamma||\tilde{u}_0| + 8\pi(R-f_-)|\beta|^2|\gamma|^2\right] + 4\pi|\beta|\gamma||\tilde{u}_0| \\
&= C_2|\gamma||\tilde{u}_0| + C_3|\gamma|^2, \qquad (3.21)
\end{aligned}
$$

where

$$
\begin{aligned}
C_3 &= 16\pi k^2(R-f_-)^3|\beta|^2, \\
C_2 &= 2k^2(R-f_-)^2C_1 + 4\pi|\beta| = 4\pi|\beta|M, \\
M &:= 4k^3(R-f_-)^3 + 2k^2(R-f_-)^2 + 4k^2(R-f_-)^3|\beta| + 1.
\end{aligned}
$$

Recall Young's inequality

$$
C_2|\gamma||\tilde{u}_0| \le C_2\left(\epsilon|\tilde{u}_0|^2 + \frac{|\gamma|^2}{4\epsilon}\right), \qquad (3.22)
$$

for any $\epsilon > 0$. Using the definition of $\|u\|_{H^{1/2}_\alpha(\Gamma_R)}$ and Lemma 3.4, we have

$$\|u\|_{X_R}^2 \geq 2\pi\|u\|_{H_\alpha^{1/2}(\Gamma_R)}^2 = 2\pi \sum_{n\in\mathbb{Z}} (k^2+\alpha_n^2)^{1/2}|\tilde{u}_n|^2 \geq 2\pi k|\tilde{u}_0|^2. \tag{3.23}$$

Combining (3.22) and (3.23) gives

$$C_2|\gamma||\tilde{u}_0| \leq \frac{C_2\epsilon}{2\pi k}\|u\|_{X_R}^2 + \frac{C_2}{4\epsilon}|\gamma|^2.$$

Thus, (3.21) becomes

$$\|u\|_{X_R}^2 \leq \frac{C_2\epsilon}{2\pi k}\|u\|_{X_R}^2 + \frac{C_2}{4\epsilon}|\gamma|^2 + C_3|\gamma|^2,$$

which implies

$$\|u\|_{X_R}^2 \leq \frac{2\pi k\left(\frac{C_2}{4\epsilon}+C_3\right)}{2\pi k - C_2\epsilon}|\gamma|^2.$$

Taking $\epsilon = \frac{\pi k}{C_2}$, we get

$$\|u\|_{X_R}^2 \leq \left(\frac{C_2}{2\epsilon}+2C_3\right)|\gamma|^2 = \left(\frac{C_2^2}{2\pi k}+2C_3\right)|\gamma|^2.$$

Furthermore, by $\beta = k\cos\theta$ and the definitions of C_2 and C_3, we have

$$\frac{C_2^2}{2\pi k} + 2C_3 = 8\pi k\cos^2\theta M^2 + 32\pi k^4(R-f_-)^3\cos^2\theta = 8\pi\cos^2\theta C^2,$$

where $C > 0$ is given as in Theorem 3.1. □

4. Impedance boundary value problem

In this section, we suppose that the grating surface is coated by a thin layer of material with the positive surface impedance $\lambda > 0$. Consider the impedance boundary value problem

$$(\text{IBVP}): \quad \begin{cases} \Delta u + k^2 u = 0 & \text{in} \quad \Omega, \\ \partial_\nu u + i\lambda u = 0 & \text{on} \quad \Gamma, \\ u - u^i = \sum_{n\in\mathbb{Z}} u_n e^{i\alpha_n x_1 + i\beta_n x_2} & x_2 > \Gamma_{\max}. \end{cases} \tag{4.1}$$

Here ν is the normal direction at Γ pointing to Ω. We keep the notations used in Section 3, except for the new solution space $X_R := H_\alpha^1(\Omega_R)$. From [19] we know that the problem (4.1) is uniquely solvable in $H_\alpha^1(\Omega_R)$. The main stability estimate is stated below.

Theorem 4.1. *Let u^i be a plane wave given by (2.1) and suppose that $R - 1 > \Gamma_{\max}$ and $f_- + 1 < \Gamma_{\min}$. Then impedance boundary value problem (4.1) has a unique solution $u \in H_\alpha^1(\Omega_R)$ with the stability estimate*

$$\|u\|_{H_\alpha^1(\Omega_R)} \leq 2\sqrt{2\pi}\cos\theta|\gamma|C^*,$$

where

$$C^* = \sqrt{k\left(1+\frac{4k^2\tilde{C}^2}{\lambda}\right)^2 + 8k^4\tilde{C}^2},$$

$$\tilde{C}^2 = \frac{4(R-f_-)^2 + (2k+1)(R-f_-)^3[2k(R-f_-)+1]}{2\min\left\{\frac{R-f_--1}{(2k+1)(R-f_-)^3}, \frac{1}{\sqrt{1+L^2}}\right\}}.$$

Here L is the Lipschitz constant of f.

The variational formulation of the impedance problem is the same as (3.4), except that the right-hand side of (3.5) is subtracted by $\int_\Gamma i\lambda u\bar{v}\,ds$, i.e.,

$$a(u,v) = F(v) \quad \text{for all } v \in H^1_\alpha(\Omega_R)$$

where

$$a(u,v) = \int_{\Omega_R} \nabla u\cdot\nabla\bar{v} - k^2u\bar{v}\,dx - \int_\Gamma i\lambda u\bar{v}\,ds - \int_{\Gamma_R}(Tu)\bar{v}\,ds,$$

$$F(v) = -2i\beta e^{-i\beta R}\gamma\int_0^{2\pi} e^{i\alpha x_1}\bar{v}(x_1,R)\,dx_1.$$

Again we take the real and imaginary parts of the variational formula with $v = u$ to get

$$\int_{\Omega_R} |\nabla u|^2 - k^2|u|^2\,dx - \operatorname{Re}\int_{\Gamma_R}(Tu)\bar{u}\,ds = \operatorname{Re}F(u), \tag{4.2}$$

$$-\int_\Gamma \lambda|u|^2\,ds - \operatorname{Im}\int_{\Gamma_R}(Tu)\bar{u}\,ds = \operatorname{Im}F(u). \tag{4.3}$$

Analogously to the Dirichlet problem, it holds that

$$\operatorname{Re}\int_{\Gamma_R}(Tu)\bar{u}\,ds = -2\pi\sum_{|\alpha_n|>k}|\beta_n||\tilde{u}_n|^2 \le 0, \tag{4.4}$$

$$\operatorname{Im}\int_{\Gamma_R}(Tu)\bar{u}\,ds = 2\pi\sum_{|\alpha_n|\le k}\beta_n|\tilde{u}_n|^2 \ge 0. \tag{4.5}$$

Using (4.2) and (4.4),

$$\begin{aligned}\int_{\Omega_R}|\nabla u|^2\,dx &= k^2\int_{\Omega_R}|u|^2\,dx + \operatorname{Re}\int_{\Gamma_R}(Tu)\bar{u}\,ds + \operatorname{Re}F(u),\\ &\le k^2\int_{\Omega_R}|u|^2\,dx + |F(u)|.\end{aligned} \tag{4.6}$$

Using (4.3) and (4.5), we get

$$\lambda\int_\Gamma |u|^2\,ds = -\operatorname{Im}\int_{\Gamma_R}(Tu)\bar{u}\,ds - \operatorname{Im}F(u) \le |F(u)|,$$

that is

$$\int_{\Gamma} |u|^2\, ds \leq \frac{1}{\lambda}|F(u)|. \tag{4.7}$$

To estimate $\|u\|_{L^2(\Omega_R)}$, we need to consider the following auxiliary problem: find $w \in H^1_{-\alpha}(\Omega_R)$ such that

$$(\mathrm{AP}): \qquad \begin{cases} \Delta w + k^2 w = \bar{u} & \text{in} \quad \Omega, \\ w = 0 & \text{on} \quad \Gamma, \\ \hat{T}w = \partial_2 w & \text{on} \quad \Gamma_R, \end{cases} \tag{4.8}$$

where the operator $\hat{T}: H^{1/2}_{-\alpha}(\Gamma_R) \to H^{-1/2}_{-\alpha}(\Gamma_R)$ is defined as

$$(\hat{T}g)(x_1) := \sum_{n\in\mathbb{Z}} i\hat{\beta}_n \hat{g}_n e^{i\hat{\alpha}_n x_1}, \qquad g(x_1) = \sum_{n\in\mathbb{Z}} \hat{g}_n e^{i\hat{\alpha}_n x_1} \in H^{1/2}_{-\alpha}(\Gamma_R)$$

with

$$\hat{\alpha}_n = -\alpha + n, \quad \hat{\beta}_n = \begin{cases} \sqrt{k^2 - |\hat{\alpha}_n|^2}, & |\hat{\alpha}_n| \leq k, \\ i\sqrt{|\hat{\alpha}_n|^2 - k^2}, & |\hat{\alpha}_n| > k. \end{cases}$$

Lemma 4.2. *Let $w \in H^1_{-\alpha}(\Omega_R)$ be a solution to (4.8). Assume $f_- + 1 < \Gamma_{\min}$, then we have*

$$\|w\|_{L^2(\Omega_R)} + \|\partial_\nu w\|_{L^2(\Gamma)} \leq \tilde{C}\|u\|_{L^2(\Omega_R)},$$

where

$$\tilde{C}^2 = \frac{4(R-f_-)^2 + (2k+1)(R-f_-)^3[2k(R-f_-)+1]}{2\min\left\{\frac{R-f_- -1}{(2k+1)(R-f_-)^3}, \frac{1}{\sqrt{1+L^2}}\right\}}.$$

Proof. First, we assume that Γ is an infinitely smooth curve such that $\max_{x_1\in(0,2\pi)} |f'(x_1)| < L$. Then it holds that $w \in H^2_{-\alpha}(\Omega_R)$. By the Rellich identity (see Corollary 6.2 with $c = f_-$ in the Appendix), we get

$$2\int_{\Omega_R} |\partial_2 w|^2\, dx + \int_{\Gamma} (x_2 - f_-)\nu_2 |\partial_\nu w|^2\, ds = I_1 + I_2 + I_3, \tag{4.9}$$

where

$$I_1 = -2\mathrm{Re} \int_{\Omega_R} (x_2 - f_-)\partial_2 \bar{w}\bar{u}\, dx,$$

$$I_2 = (R - f_-)\int_{\Gamma_R} |\partial_2 w|^2 - |\partial_1 w|^2 + k^2|w|^2\, ds,$$

$$I_3 = \int_{\Omega_R} |\nabla w|^2 - k^2|w|^2\, dx.$$

Below we shall estimate I_1, I_2 and I_3 in (4.9) separately. First, I_1 can be bounded by

$$|I_1| \leq 2(R - f_-)\|\partial_2 w\|_{L^2(\Omega_R)}\|u\|_{L^2(\Omega_R)}. \tag{4.10}$$

Expanding w into the Rayleigh series

$$w = \sum_{n\in\mathbb{Z}} w_n e^{i(\hat{\alpha}_n x_1 + \hat{\beta}_n x_2)}, \qquad x_2 > \Gamma_{\max},$$

we find

$$\mathrm{Re}\int_{\Gamma_R} (\hat{T}w)\bar{w}\,ds = -2\pi \sum_{|\hat{\alpha}_n|>k} |\hat{\beta}_n||w_n|^2,\ \mathrm{Im}\int_{\Gamma_R} (\hat{T}w)\bar{w}\,ds = 2\pi \sum_{|\hat{\alpha}_n|\leq k} \hat{\beta}_n|w_n|^2, \tag{4.11}$$

$$\int_{\Gamma_R} |\partial_2 w|^2 - |\partial_1 w|^2 + k^2|w|^2\,ds \leq 2k\mathrm{Im}\int_{\Gamma_R} (\hat{T}w)\bar{w}\,ds. \tag{4.12}$$

Taking the real and imaginary parts of the variational formulation of (4.8), we get

$$\int_{\Omega_R} |\nabla w|^2 - k^2|w|^2\,dx - \mathrm{Re}\int_{\Gamma_R} (\hat{T}w)\bar{w}\,ds = -\mathrm{Re}\int_{\Omega_R} \bar{u}\bar{w}\,dx, \tag{4.13}$$

$$\mathrm{Im}\int_{\Gamma_R} (\hat{T}w)\bar{w}\,ds = \mathrm{Im}\int_{\Omega_R} \bar{u}\bar{w}\,dx. \tag{4.14}$$

Combining (4.12) and (4.14)

$$\int_{\Gamma_R} |\partial_2 w|^2 - |\partial_1 w|^2 + k^2|w|^2\,ds \leq 2k\mathrm{Im}\int_{\Omega_R} \bar{u}\bar{w}\,dx \leq 2k\|u\|_{L^2(\Omega_R)}\|w\|_{L^2(\Omega_R)}. \tag{4.15}$$

Inserting (4.15) into I_2 gives the estimate of I_2:

$$|I_2| \leq 2k(R - f_-)\|u\|_{L^2(\Omega_R)}\|w\|_{L^2(\Omega_R)}. \tag{4.16}$$

Using (4.13) and (4.11), we have

$$I_3 = \mathrm{Re}\int_{\Gamma_R} (\hat{T}w)\bar{w}\,ds - \mathrm{Re}\int_{\Omega_R} \bar{u}\bar{w}\,dx \leq \|u\|_{L^2(\Omega_R)}\|w\|_{L^2(\Omega_R)}. \tag{4.17}$$

Next, we derive a lower bound of the left-hand side of (4.9). The unit normal vector to Γ is given by $\nu = (\nu_1, \nu_2)^\top = \frac{(-f'(x_1),1)^\top}{\sqrt{1+|f'(x_1)|^2}}$, implying $\nu_2 \geq C_L$, where $C_L = \frac{1}{\sqrt{1+L^2}} \leq 1$. Owing to $f(x_1) - f_- \geq 1$, we have

$$2\int_{\Omega_R} |\partial_2 w|^2\,dx + \int_{\Gamma} (x_2 - f_-)\nu_2|\partial_\nu w|^2\,ds \geq 2\|\partial_2 w\|^2_{L^2(\Omega_R)} + C_L\|\partial_\nu w\|^2_{L^2(\Gamma)}. \tag{4.18}$$

Substituting (4.18), (4.10), (4.16), (4.17) into (4.9), we get

$$2\|\partial_2 w\|^2_{L^2(\Omega_R)} + C_L\|\partial_\nu w\|^2_{L^2(\Gamma)}$$
$$\leq 2(R - f_-)\left(\epsilon_1\|\partial_2 w\|^2_{L^2(\Omega_R)} + \frac{1}{4\epsilon_1}\|u\|^2_{L^2(\Omega_R)}\right) + 2k(R - f_-)\left(\epsilon_2\|w\|^2_{L^2(\Omega_R)}\right.$$

$$
\begin{aligned}
&\quad + \frac{1}{4\epsilon_2}\|u\|^2_{L^2(\Omega_R)}\Big) + \left(\epsilon_2\|w\|^2_{L^2(\Omega_R)} + \frac{1}{4\epsilon_2}\|u\|^2_{L^2(\Omega_R)}\right) \\
&\le 2\epsilon_1(R-f_-)\|\partial_2 w\|^2_{L^2(\Omega_R)} + \epsilon_2[2k(R-f_-)+1]\|w\|^2_{L^2(\Omega_R)} \\
&\quad + \left(\frac{R-f_-}{2\epsilon_1} + \frac{2k(R-f_-)+1}{4\epsilon_2}\right)\|u\|^2_{L^2(\Omega_R)} \\
&= C_4(\epsilon_1)\|\partial_2 w\|^2_{L^2(\Omega_R)} + C_5(\epsilon_2)\|w\|^2_{L^2(\Omega_R)} + C_6(\epsilon_1,\epsilon_2)\|u\|^2_{L^2(\Omega_R)}, \qquad (4.19)
\end{aligned}
$$

where

$$
\begin{aligned}
C_4(\epsilon_1) &= 2\epsilon_1(R-f_-), \quad C_5(\epsilon_2) = \epsilon_2[2k(R-f_-)+1], \\
C_6(\epsilon_1,\epsilon_2) &= \frac{R-f_-}{2\epsilon_1} + \frac{2k(R-f_-)+1}{4\epsilon_2},
\end{aligned}
$$

and $\epsilon_1, \epsilon_2 > 0$ are arbitrary. Rewriting (4.19) into the form

$$
(2 - C_4(\epsilon_1))\|\partial_2 w\|^2_{L^2(\Omega_R)} - C_5(\epsilon_2)\|w\|^2_{L^2(\Omega_R)} + C_L\|\partial_\nu w\|^2_{L^2(\Gamma)} \le C_6(\epsilon_1,\epsilon_2)\|u\|^2_{L^2(\Omega_R)}. \qquad (4.20)
$$

Using Lemma 3.5 and noting again $w = 0$ on Γ yield

$$
\|\partial_2 w\|^2_{L^2(\Omega_R)} \ge \frac{1}{(R-f_-)^2}\|w\|^2_{L^2(\Omega_R)}. \qquad (4.21)
$$

Substituting (4.21) into (4.20), we get

$$
C_7(\epsilon_1,\epsilon_2)\|w\|^2_{L^2(\Omega_R)} + C_L\|\partial_\nu w\|^2_{L^2(\Gamma)} \le C_6(\epsilon_1,\epsilon_2)\|u\|^2_{L^2(\Omega_R)}, \qquad (4.22)
$$

where

$$
C_7(\epsilon_1,\epsilon_2) = \frac{2 - C_4(\epsilon_1)}{(R-f_-)^2} - C_5(\epsilon_2).
$$

Taking appropriate ϵ_1 and ϵ_2 to satisfy $C_7(\epsilon_1,\epsilon_2) > 0$ and introduce $C_8(\epsilon_1,\epsilon_2) = \min\{C_7(\epsilon_1,\epsilon_2), C_L\}$, one deduces from (4.22) that

$$
\frac{C_8}{2}\left(\|w\|_{L^2(\Omega_R)} + \|\partial_\nu w\|_{L^2(\Gamma)}\right)^2 \le C_8\left(\|w\|^2_{L^2(\Omega_R)} + \|\partial_\nu w\|^2_{L^2(\Gamma)}\right) \le C_6\|u\|^2_{L^2(\Omega_R)}.
$$

That is,

$$
\|w\|_{L^2(\Omega_R)} + \|\partial_\nu w\|_{L^2(\Gamma)} \le \tilde{C}(\epsilon_1,\epsilon_2)\|u\|_{L^2(\Omega_R)}, \quad \tilde{C}(\epsilon_1,\epsilon_2) = \sqrt{\frac{2C_6(\epsilon_1,\epsilon_2)}{C_8(\epsilon_1,\epsilon_2)}}.
$$

Next, we choose sufficiently small ϵ_1 and ϵ_2 to ensure $C_7(\epsilon_1,\epsilon_2) > 0$, that is,

$$
\frac{2 - 2\epsilon_1(R-f_-)}{(R-f_-)^2} - \epsilon_2[2k(R-f_-)+1] > 0.
$$

Choosing $\epsilon_1 = \frac{1}{2(R-f_-)}$, we need to require

$$
\epsilon_2 < \frac{1}{2k(R-f_-)^3 + (R-f_-)^2}.
$$

Recalling that $R - f_- > 1$ and taking $\epsilon_2 = \frac{1}{(2k+1)(R-f_-)^3}$, we obtain

$$C_4 = 1, \quad C_5 = \frac{2k(R-f_-)+1}{(2k+1)(R-f_-)^3},$$
$$C_6 = (R-f_-)^2 + \frac{(2k+1)(R-f_-)^3[2k(R-f_-)+1]}{4},$$
$$C_7 = \frac{R-f_- - 1}{(2k+1)(R-f_-)^3}, \quad C_8 = \min\left\{\frac{R-f_- - 1}{(2k+1)(R-f_-)^3}, \frac{1}{\sqrt{1+L^2}}\right\}.$$

Hence, the constant $\tilde{C}$ is given by

$$\tilde{C}^2 := \frac{2C_6}{C_8} = \frac{4(R-f_-)^2 + (2k+1)(R-f_-)^3[2k(R-f_-)+1]}{2\min\left\{\frac{R-f_- - 1}{(2k+1)(R-f_-)^3}, \frac{1}{\sqrt{1+L^2}}\right\}}. \tag{4.23}$$

Now we consider the class of Lipschitz profile functions $f \in C_p^{0,1}$, following the arguments in the proof of [14, Theorem 3.3]. We choose C^∞-smooth profiles $\Gamma_j = \Gamma_{f_j}$ such that $\Omega_{R,j} \subset \Omega_R$, $\|f_j - f\|_{C_p^{0,1}} \to 0$, $f_- \le f_j(x_1) \le R$, for all j, x_1 and that the Lipschitz constants are not greater than L. We also assume $\min_{x_1 \in (0,2\pi)} f_j(x_1) - f_- \ge 1$. Let $w_j \in H^1_{-\alpha}(\Omega_{R,j}) \subset H^1_{-\alpha}(\Omega_R)$ be the solution of problem (4.8) in $\Omega_{R,j}$. By arguing analogously to [14, Theorem 3.1] in the case of an infinitely smooth profile, we obtain

$$\left(\int_{\Omega_{R,j}} |w_j|^2\right)^{1/2} + \left(\int_{\Gamma_j} |\partial_\nu w_j|^2\right)^{1/2} \le \tilde{C}\left(\int_{\Omega_{R,j}} |u|^2\right)^{1/2}, \tag{4.24}$$

where $\tilde{C}$ is given by (4.23). From (4.24) we then get $\partial_\nu w_j \rightharpoonup v$ in $L^2(0, 2\pi)$ (for some subsequence) and $w_j \to w$ in $H^1_{-\alpha}(\Omega_R)$ by Remark 3.2 in [14]. It remains to check that v coincides with the $H^{-1/2}_{-\alpha}$ trace of $\partial_\nu w$ on Γ. Remark 2.4.5 in [21] implies that $\phi|_{\Gamma_j} \to \phi|_\Gamma$ in $L^2(0, 2\pi)$ for any $\phi \in H^1_{-\alpha}(\Omega)$. From the variational formulations of w_j and for all $\phi \in H^1_{-\alpha}(\Omega)$, we have

$$\int_{\Omega_{R,j}} \nabla w_j \cdot \nabla\bar{\phi} - k^2 w_j\bar{\phi}\, dx - \int_{\Gamma_R} (\hat{T} w_j)\bar{\phi}\, ds + \int_{\Gamma_j} (\partial_\nu w_j)\bar{\phi}\, ds = -\int_{\Omega_{R,j}} \bar{u}\bar{\phi}\, dx. \tag{4.25}$$

Recalling the variational formulation of (4.8), we obtain

$$\int_{\Omega_R} \nabla w \cdot \nabla\bar{\phi} - k^2 w\bar{\phi}\, dx - \int_{\Gamma_R} (\hat{T} w)\bar{\phi}\, ds + \int_{\Gamma} (\partial_\nu w)\bar{\phi}\, ds = -\int_{\Omega_R} \bar{u}\bar{\phi}\, dx. \tag{4.26}$$

Passing to the limit $j \to \infty$ of (4.25) and using $\partial_\nu w_j \rightharpoonup v$, together with (4.26), we obtain $v = \partial_\nu w$ on Γ. Finally, taking $j \to \infty$ in (4.24) gives the desired result. □

The following result will be used in the proof of Theorem 4.1.

Lemma 4.3. *Assume $R - 1 > \Gamma_{\max}$. Given $w \in H^1_{-\alpha}(\Omega_R)$, we have $|w_0| \le \frac{1}{\sqrt{2\pi}}\|w\|_{L^2(\Omega_R)}$, where $w_0 = \frac{1}{2\pi}\int_0^{2\pi} w(x_1, R)e^{-i\alpha x_1}e^{-i\beta R}\, dx$.*

Proof. Define $D := (0, 2\pi) \times (R-1, R)$, then $D \subset \Omega_R$ and

$$\int_D |w|^2\,dx = \int_0^{2\pi}\int_{R-1}^{R} |w|^2\,dx_2dx_1 = 2\pi\sum_{n\in\mathbb{Z}}|w_n|^2\int_{R-1}^{R} e^{i(\hat\beta_n-\bar{\hat\beta}_n)x_2}\,dx_2$$
$$= 2\pi\left[\sum_{|\hat\alpha_n|\le k}|w_n|^2 + \sum_{|\hat\alpha_n|>k}|w_n|^2\frac{1}{2|\hat\beta_n|}(e^{-2|\hat\beta_n|(R-1)} - e^{-2|\hat\beta_n|R})\right]$$
$$\ge 2\pi|w_0|^2.$$

Therefore, $\|w\|_{L^2(\Omega_R)} \ge \|w\|_{L^2(D)} \ge \sqrt{2\pi}|w_0|$, which completes the proof. $\square$

Proof of Theorem 4.1. Let $w \in H^1_{-\alpha}(\Omega_R)$ be the unique solution to (4.8). Using Green's formula, the Helmholtz equation for $u = u^i + u^s$ and the fact that $w = 0$ on Γ, we have

$$\|u\|^2_{L^2(\Omega_R)} = \int_{\Omega_R} u\bar u\,dx = \int_{\Omega_R} u(\Delta w + k^2 w)\,dx$$
$$= \int_{\partial\Omega_R}(\partial_\nu w u - \partial_\nu u w)\,ds + \int_{\Omega_R} w(\Delta u + k^2 u)\,dx$$
$$= \int_{\Gamma_R}(\partial_\nu w u - \partial_\nu u w)\,ds + \int_{\Gamma}(\partial_\nu w u - \partial_\nu u w)\,ds$$
$$= \int_{\Gamma_R}\left[\partial_\nu w(u^i + u^s) - \partial_\nu(u^i+u^s)w\right]\,ds - \int_\Gamma \partial_\nu w u\,ds$$
$$= \int_{\Gamma_R}(\partial_\nu w u^s - \partial_\nu u^s w)\,ds + \int_{\Gamma_R}\left(\partial_\nu w u^i - \partial_\nu u^i w\right)\,ds - \int_\Gamma \partial_\nu w u\,ds. \tag{4.27}$$

The first integral on the right-hand side of (4.27) vanishes, because

$$\int_{\Gamma_R}(\partial_\nu w u^s - \partial_\nu u^s w)\,ds = 2\pi\sum_{n\in\mathbb{Z}} i\hat\beta_n w_n u_{-n}e^{i2\hat\beta_n R} - 2\pi\sum_{n\in\mathbb{Z}} i\beta_n u_n w_{-n}e^{i2\beta_n R}$$
$$= 2\pi\sum_{n\in\mathbb{Z}}\left(i\beta_n u_n w_{-n}e^{i2\beta_n R} - i\beta_n u_n w_{-n}e^{i2\beta_n R}\right) = 0. \tag{4.28}$$

Here, we have used the $-\alpha$-quasiperiodic and α-quasiperiodic Rayleigh expansions for w and u^s, respectively. The second integral can be further calculated as

$$\left|\int_{\Gamma_R}\left(\partial_\nu w u^i - \partial_\nu u^i w\right)\,ds\right| = \left|\int_0^{2\pi}\sum_{n\in\mathbb{Z}} i w_n\gamma e^{i(\hat\alpha_n x_1 + \hat\beta_n R)}e^{i(\alpha x_1 - \beta R)}(\hat\beta_n + \beta)\,dx_1\right|$$
$$= \left|\int_0^{2\pi}\sum_{n\in\mathbb{Z}} i w_n\gamma(\hat\beta_n+\beta)e^{inx_1}\;e^{i(\hat\beta_n-\beta)R}\,dx_1\right|$$
$$= \left|2\pi w_0\gamma(\hat\beta_0+\beta)\right| = |4\pi k\cos\theta w_0\gamma|\,. \tag{4.29}$$

Substituting (4.28), (4.29) and Lemma 4.3 into (4.27) and inserting Lemma 4.2 into the estimate of $\|u\|_{L^2(\Omega_R)}$, we get by applying Young's inequality that

$$\begin{aligned}\|u\|^2_{L^2(\Omega_R)} &\le |4\pi k\cos\theta w_0\gamma| + \|\partial_\nu w\|_{L^2(\Gamma)}\|u\|_{L^2(\Gamma)}\\ &\le C_9\|w\|_{L^2(\Omega_R)}|\gamma| + \|\partial_\nu w\|_{L^2(\Gamma)}\|u\|_{L^2(\Gamma)}\\ &\le C_9\tilde{C}\|u\|_{L^2(\Omega_R)}|\gamma| + \tilde{C}\|u\|_{L^2(\Omega_R)}\|u\|_{L^2(\Gamma)}\\ &\le C_9\left(\epsilon_3|\tilde{C}|^2\|u\|^2_{L^2(\Omega_R)} + \frac{1}{4\epsilon_3}|\gamma|^2\right) + \frac{1}{4}\|u\|^2_{L^2(\Omega_R)} + |\tilde{C}|^2\|u\|^2_{L^2(\Gamma)},\end{aligned}$$

where $\epsilon_3 > 0$ is arbitrary and $C_9 = 2\sqrt{2\pi}k\cos\theta$. Taking $\epsilon_3 = \frac{1}{4C_9|\tilde{C}|^2}$, we obtain

$$\begin{aligned}\|u\|^2_{L^2(\Omega_R)} &\le \frac{1}{4}\|u\|^2_{L^2(\Omega_R)} + C_9^2|\tilde{C}|^2|\gamma|^2 + \frac{1}{4}\|u\|^2_{L^2(\Omega_R)} + |\tilde{C}|^2\|u\|^2_{L^2(\Gamma)}\\ &= \frac{1}{2}\|u\|^2_{L^2(\Omega_R)} + |\tilde{C}|^2\|u\|^2_{L^2(\Gamma)} + C_9^2|\tilde{C}|^2|\gamma|^2.\end{aligned}$$

Using (4.7), we deduce from the previous relation that

$$\frac{1}{2}\|u\|^2_{L^2(\Omega_R)} \le |\tilde{C}|^2\|u\|^2_{L^2(\Gamma)} + C_9^2|\tilde{C}|^2|\gamma|^2 \le \frac{|\tilde{C}|^2}{\lambda}|F(u)| + C_9^2|\tilde{C}|^2|\gamma|^2,$$

that is,

$$\|u\|^2_{L^2(\Omega_R)} \le \frac{2|\tilde{C}|^2}{\lambda}|F(u)| + 2C_9^2|\tilde{C}|^2|\gamma|^2. \tag{4.30}$$

In view of (4.6) and (4.30),

$$\begin{aligned}\|u\|^2_{H^1_\alpha(\Omega_R)} &= k^2\|u\|^2_{L^2(\Omega_R)} + \|\nabla u\|^2_{L^2(\Omega_R)} \le 2k^2\|u\|^2_{L^2(\Omega_R)} + |F(u)|\\ &\le 2k^2\left(\frac{2|\tilde{C}|^2}{\lambda}|F(u)| + 2C_9^2|\tilde{C}|^2|\gamma|^2\right) + |F(u)|\\ &= \left(1 + \frac{4k^2|\tilde{C}|^2}{\lambda}\right)|F(u)| + 4k^2C_9^2|\tilde{C}|^2|\gamma|^2.\end{aligned} \tag{4.31}$$

Similarly to the Dirichlet case, we can also obtain (3.18), that is $|F(u)| = |4\pi\beta\gamma\tilde{u}_0|$. Recalling $\beta = k\cos\theta$, using Cauchy-Schwarz inequality and again using (3.23) (which in this case is $\|u\|_{H^1_\alpha(\Omega_R)} \ge \sqrt{2\pi k}|\tilde{u}_0|$), we deduce from (4.31) that

$$\begin{aligned}\|u\|^2_{H^1_\alpha(\Omega_R)} &\le \left(1 + \frac{4k^2|\tilde{C}|^2}{\lambda}\right)4\pi|k\cos\theta||\gamma||\tilde{u}_0| + 4k^2C_9^2|\tilde{C}|^2|\gamma|^2\\ &= C_{10}|\gamma||\tilde{u}_0| + C_{11}|\gamma|^2\\ &\le \frac{C_{10}\epsilon_4\|u\|^2_{H^1_\alpha(\Omega_R)}}{2\pi k} + \frac{C_{10}}{4\epsilon_4}|\gamma|^2 + C_{11}|\gamma|^2,\end{aligned} \tag{4.32}$$

where $\epsilon_4 > 0$ is arbitrary and

$$C_{10} = \left(1 + \frac{4k^2|\tilde{C}|^2}{\lambda}\right)4\pi|k\cos\theta|, \quad C_{11} = 4k^2C_9^2|\tilde{C}|^2 = 32\pi k^4\cos^2\theta|\tilde{C}|^2.$$

It follows from (4.32) that

$$\left(1-\frac{C_{10}\epsilon_4}{2\pi k}\right)\|u\|^2_{H^1_\alpha(\Omega_R)} \le \left(\frac{C_{10}}{4\epsilon_4}+C_{11}\right)|\gamma|^2.$$

By taking $\epsilon_4 = \frac{\pi k}{C_{10}}$,

$$\|u\|^2_{H^1_\alpha(\Omega_R)} \le \left(\frac{C_{10}}{2\epsilon_4}+2C_{11}\right)|\gamma|^2 = \left(\frac{C_{10}^2}{2\pi k}+2C_{11}\right)|\gamma|^2.$$

Furthermore,

$$\begin{aligned}\frac{C_{10}^2}{2\pi k}+2C_{11} &= \left(1+\frac{4k^2|\tilde{C}|^2}{\lambda}\right)^2 8\pi k\cos^2\theta + 64\pi k^4\cos^2\theta|\tilde{C}|^2\\ &= 8\pi k\cos^2\theta\left[\left(1+\frac{4k^2|\tilde{C}|^2}{\lambda}\right)^2+8k^3|\tilde{C}|^2\right].\end{aligned}$$

Therefore,

$$\|u\|^2_{H^1_\alpha(\Omega_R)} \le 8\pi k\cos^2\theta|\gamma|^2 C^{*2}, \quad C^* = \sqrt{\left(1+\frac{4k^2\tilde{C}^2}{\lambda}\right)^2+8k^3|\tilde{C}|^2}. \quad \square$$

5. Transmission problems

The transmission problem is to find the total field $u = u(x_1, x_2)$ such that

$$(TP): \quad \begin{cases}\Delta u + k^2 u = 0 & \text{in} \quad \Omega_\Gamma^+,\\ \Delta u + k_1^2 u = 0 & \text{in} \quad \Omega_\Gamma^-,\\ u^+ = u^-,\ \partial_\nu u^+ = \lambda\partial_\nu u^- & \text{on} \quad \Gamma,\\ u = u^i + u^s & \text{in} \quad \Omega_\Gamma^+.\end{cases} \tag{5.1}$$

Here $\Omega_\Gamma^\pm$ denote the region above (+) and below (−) the Lipschitz grating profile Γ, respectively. $\lambda > 0$ is a constant and ν is the normal direction at Γ pointing to Ω_Γ^+. We suppose that $k \neq k_1$ are constants. For notational convenience, we set $k_+ = k,\ k_- = k_1,\ k(x) = k_\pm$ in $\Omega_\Gamma^\pm$. The upward and downward Rayleigh expansion radiation conditions as $x_2 \to \pm\infty$ can be formulated as

$$\begin{aligned}u^s &= \sum_{n\in\mathbb{Z}} u_n^+ e^{i\alpha_n x_1 + i\beta_n^+ x_2}, \quad \text{for} \quad x_2 > \Gamma^+_{\max} := \max_{(x_1,x_2)\in\Gamma} x_2,\\ u &= \sum_{n\in\mathbb{Z}} u_n^- e^{i\alpha_n x_1 - i\beta_n^- x_2}, \quad \text{for} \quad x_2 < \Gamma^-_{\min} := \min_{(x_1,x_2)\in\Gamma} x_2,\end{aligned}$$

where $\alpha := n + \alpha$ and

$$\beta_n^+ = \begin{cases}\sqrt{k_+^2-|\alpha_n|^2}, & |\alpha_n| \le k_+,\\ i\sqrt{|\alpha_n|^2-k_+^2}, & |\alpha_n| > k_+,\end{cases} \quad \beta_n^- = \begin{cases}\sqrt{k_-^2-|\alpha_n|^2}, & |\alpha_n| \le k_-,\\ i\sqrt{|\alpha_n|^2-k_-^2}, & |\alpha_n| > k_-.\end{cases}$$

Throughout this section, we suppose that $f_- < \min_{x\in\Gamma} x_2 < \max_{x\in\Gamma} x_2 < f_+$, where $f_- < f_+$. Then we introduce the notations

$$S_R = S_{R,f} := \{x \in \mathbb{R}^2 : |f(x_1)| < |x_2| < R, 0 < x_1 < 2\pi\} \quad \text{for some } R > \max\{|f_+|, |f_-|\},$$

$S_R^\pm := \{x \in \Omega_\Gamma^\pm : -R < x_2 < R\}$ and $\Gamma_R^\pm := \{(x_1, \pm R) : 0 < x_1 < 2\pi\}$. Define the DtN mappings $T^\pm : H_\alpha^{1/2}(\Gamma_R^\pm) \to H_\alpha^{-1/2}(\Gamma_R^\pm)$ as

$$T^\pm g(x_1) := \pm \sum_{n\in\mathbb{Z}} i\beta_n^\pm g_n e^{i\alpha_n x_1}, \qquad g(x_1) = \sum_{n\in\mathbb{Z}} g_n e^{i\alpha_n x_1} \in H_\alpha^{1/2}(\Gamma_R^\pm).$$

We also define a piecewise constant function $a(x)$ such that $a(x) = 1$ in S_R^+, $a(x) = \lambda$ in S_R^-. Therefore,

$$\|\partial_2 u\|^2_{L^2(S_R)} = \int_{S_R} a(x)|\partial_2 u|^2\, dx, \quad \|u\|^2_{L^2(S_R)} = \int_{S_R} a(x)|u|^2\, dx.$$

Then the Sobolev space $H^1_\alpha(S_R)$ is equipped with the norm

$$\|u\|^2_{H^1_\alpha(S_R)} := \int_{S_R} a(x)|\nabla u|^2 + a(x)k^2(x)|u|^2\, dx. \tag{5.2}$$

Theorem 5.1. *Let u^i be a plane wave given by (2.1) and suppose that $u \in H^1_\alpha(S_R)$ is a solution to the transmission problem (5.1). Choose $f_+, f_- \in \mathbb{R}$ such that $\min_{x_1\in(0,2\pi)} f(x_1) - f_- > 1$ and $\max_{x_1\in(0,2\pi)} f(x_1) - f_+ < -1$.*
(i). If $\lambda \geq 1,\ k_+^2 > \lambda k_-^2$, we have

$$\|u\|_{H^1_\alpha(S_R)} \leq 2\sqrt{2\pi k}\cos\theta C_{12}|\gamma|,$$

where

$$C_{12} = \frac{2k_+^2[2k_+(R-f_-)+1]}{C_T} + 1,$$
$$C_T = \min\left\{\frac{2}{(R-f_-)^2}, \frac{(k_+^2-\lambda k_-^2)}{2(R-f_-)\sqrt{1+L^2}}\right\}.$$

(ii). If $\lambda \leq 1,\ k_+^2 < \lambda k_-^2$, we have

$$\|u\|_{H^1_\alpha(S_R)} \leq 2\sqrt{2\pi k}\cos\theta C_{13}|\gamma|,$$

where

$$C_{13} = \frac{2\lambda k_-^2[2k_+(R-f_+)+1]}{C_S} + 1,$$
$$C_S = \min\left\{\frac{2}{(R-f_-)^2}, \frac{(\lambda k_-^2-k_+^2)}{2(R-f_-)\sqrt{1+L^2}}\right\}.$$

(iii). If $k_+^2 = \lambda k_-^2$, $k_+ \neq k_-$, we have

$$\|u\|_{H^1_\alpha(S_R)} \leq 2\sqrt{2\pi k}\cos\theta C_{14}|\gamma|,$$

where

$$C_{14} = \frac{2k_+^2(1+M)[2k_+(R-f_-)+1]}{C_Z} + 1,$$

$$C_Z = \min\left\{\frac{2}{(R-f_-)^2}, \frac{k_+^2}{2(R-f_-)\sqrt{1+L^2}}\right\}, \quad M = 1 + \frac{\sqrt{1+L^2}(f_+-f_-)}{(\lambda-1)}$$

for $k_+ > k_-$, *and*

$$\|u\|_{H^1_\alpha(S_R)} \le 2\sqrt{2\pi k}\cos\theta C_{15}|\gamma|,$$

where

$$C_{15} = \frac{2k_+^2(1+N)[2k_+(R-f_-)+1]}{C_Z} + 1,$$

$$C_Z = \min\left\{\frac{2}{(R-f_-)^2}, \frac{k_+^2}{2(R-f_-)\sqrt{1+L^2}}\right\}, \quad N = 1 + \frac{\sqrt{1+L^2}(f_+-f_-)}{\lambda(1-\lambda)}$$

for $k_+ < k_-$. *Here* $L > 0$ *is the Lipschitz constant of* f.

Remark 5.2. The results of Theorem 5.1 cover both the TE polarization case (i.e., $\lambda = 1$, $k_+ \neq k_-$) and the TM polarization case (i.e., $\lambda = (k_+/k_-)^2$, $k_+ \neq k_-$).

The variational formulation for the transmission problem can be written as: find $u \in H^1_\alpha(S_R)$ such that

$$a(u,v) = F(v) \quad \text{for all } v \in H^1_\alpha(S_R), \tag{5.3}$$

where

$$a(u,v) := \int_{S_R} a(x)\nabla u \cdot \nabla \bar{v} - a(x)k^2(x)u\bar{v}\,dx - \int_{\Gamma_R^+}(T^+u)\bar{v}\,ds + \lambda\int_{\Gamma_R^-}(T^-u)\bar{v}\,ds,$$

$$F(v) := -2i\beta e^{-i\beta R}\gamma\int_0^{2\pi} e^{i\alpha x_1}\bar{v}(x_1,R)\,dx_1.$$

Taking the real and imaginary parts on both sides of (5.3) with $v = u$, we get

$$\int_{S_R}\left[a(x)|\nabla u|^2 - a(x)k^2(x)|u|^2\right]dx - \mathrm{Re}\int_{\Gamma_R^+}(T^+u)\bar{u}\,ds + \lambda\mathrm{Re}\int_{\Gamma_R^-}(T^-u)\bar{u}\,ds = \mathrm{Re}\,F(u), \tag{5.4}$$

$$-\mathrm{Im}\int_{\Gamma_R^+}(T^+u)\bar{u}\,ds + \lambda\mathrm{Im}\int_{\Gamma_R^-}(T^-u)\bar{u}\,ds = \mathrm{Im}\,F(u). \tag{5.5}$$

By definition of the DtN maps $T^\pm$, we can easily get for $u\big|_{\Gamma_R^\pm} = \sum_{n\in\mathbb{Z}}\hat{u}_n^\pm e^{i\alpha_n x_1}$ that

$$\mathrm{Re}\int_{\Gamma_R^\pm} \pm(T^\pm u)\bar{u} = \mathrm{Re}\,2\pi\sum_{n\in\mathbb{Z}} i\beta_n^\pm|\hat{u}_n^\pm|^2 = -2\pi\sum_{|\alpha_n|>k_\pm}|\beta_n^\pm||\hat{u}_n^\pm|^2 \le 0, \tag{5.6}$$

$$\mathrm{Im}\int_{\Gamma_R^\pm} \pm(T^\pm u)\bar{u} = \mathrm{Im}\,2\pi\sum_{n\in\mathbb{Z}} i\beta_n^\pm|\hat{u}_n^\pm|^2 = 2\pi\sum_{|\alpha_n|\le k_\pm}|\beta_n^\pm||\hat{u}_n^\pm|^2 \ge 0. \tag{5.7}$$

Proof of Theorem 5.1. There exists a sequence of infinitely smooth profile functions $f_j, j \in \mathbb{N}$ such that $\|f_j - f\|_{C_p^{0,1}} \to 0$ as $j \to \infty$. The approximation arguments of Sect. 4 can be adapted to the transmission problem, but the technical details are omitted here. When $\Gamma = \Gamma_f$ is an infinitely smooth profile, we have $u \in H^2_\alpha(S_R^+) \cap H^2_\alpha(S_R^-)$. Applying the Rellich identity (see Lemma 6.1 in the Appendix) yields

$$\begin{aligned} 0 = &\left(\int\limits_{\Gamma_R^\pm} \mp \int\limits_{\Gamma}\right) (x_2 - c)\left[-\nu_2|\nabla u^\pm|^2 + \nu_2 k_\pm^2 |u|^2 + 2\mathrm{Re}\left(\partial_2 \bar{u}^\pm \partial_\nu u^\pm\right)\right] ds \\ &+ \int\limits_{S_R^\pm} |\nabla u|^2 - k_\pm^2|u|^2 - 2|\partial_2 u|^2\, dx := I_c^\pm. \end{aligned} \tag{5.8}$$

Therefore,

$$\begin{aligned} 0 &= I_c^+ + \lambda I_c^- \\ &= \left(\int\limits_{\Gamma_R^+} - \int\limits_{\Gamma}\right) (x_2 - c)\left[-\nu_2|\nabla u^+|^2 + \nu_2 k_+^2 |u|^2 + 2\mathrm{Re}\left(\partial_2 \bar{u}^+ \partial_\nu u^+\right)\right] ds \\ &\quad + \lambda \left(\int\limits_{\Gamma_R^-} + \int\limits_{\Gamma}\right) (x_2 - c)\left[-\nu_2|\nabla u^-|^2 + \nu_2 k_-^2 |u|^2 + 2\mathrm{Re}\left(\partial_2 \bar{u}^- \partial_\nu u^-\right)\right] ds \\ &\quad + \int\limits_{S_R^+} |\nabla u|^2 - k_+^2|u|^2 - 2|\partial_2 u|^2\, dx + \lambda \int\limits_{S_R^-} |\nabla u|^2 - k_-^2|u|^2 - 2|\partial_2 u|^2\, dx. \end{aligned} \tag{5.9}$$

(i) Suppose $\lambda \geq 1,\ k_+^2 > \lambda k_-^2$. Straightforward calculations show that for $u\big|_{\Gamma_R^\pm} = \sum_{n\in\mathbb{Z}} \hat{u}_n^\pm e^{i\alpha_n x_1}$, we obtain

$$\begin{aligned} &\int\limits_{\Gamma_R^\pm} (x_2 - f_-)\left[-\nu_2|\nabla u^\pm|^2 + \nu_2 k_\pm^2 |u|^2 + 2\mathrm{Re}\left(\partial_2 \bar{u}^\pm \partial_\nu u^\pm\right)\right] ds \\ &= \int\limits_{\Gamma_R^\pm} (\pm R - f_-)\left[|\partial_2 u^\pm|^2 - |\partial_1 u^\pm|^2 + k_\pm^2|u|^2\right] ds \\ &= (\pm R - f_-) 4\pi \sum_{|\alpha_n| \leq k_\pm} |\beta_n^\pm|^2 |u_n^\pm|^2. \end{aligned} \tag{5.10}$$

Simple calculations show $\mathrm{Re}\left(\partial_2 \bar{u}^\pm \partial_\nu u^\pm\right) = \nu_2 |\partial_\nu u^\pm|^2$ on Γ, due to $u^+ = u^-$ on Γ. By the transmission conditions, we have

$$\begin{aligned} &-|\nabla u^+|^2 + \lambda|\nabla u^-|^2 + 2|\partial_\nu u^+|^2 - 2\lambda|\partial_\nu u^-|^2 \\ &= -\left(|\partial_\nu u^+|^2 + |\partial_\tau u^+|^2\right) + \lambda\left(|\partial_\nu u^-|^2 + |\partial_\tau u^-|^2\right) + 2|\partial_\nu u^+|^2 - 2\lambda|\partial_\nu u^-|^2 \\ &= -\left(\lambda^2|\partial_\nu u^-|^2 + |\partial_\tau u^-|^2\right) + \lambda\left(|\partial_\nu u^-|^2 + |\partial_\tau u^-|^2\right) + 2\lambda^2|\partial_\nu u^-|^2 - 2\lambda|\partial_\nu u^-|^2 \\ &= (\lambda^2 - \lambda)|\partial_\nu u^-|^2 + (\lambda - 1)|\partial_\tau u^-|^2. \end{aligned} \tag{5.11}$$

Taking $c = f_-$ in (5.9) and using (5.10) and (5.11),

$$
\begin{aligned}
0 = (R - f_-)4\pi \sum_{|\alpha_n| \le k_+} |\beta_n^+|^2 |u_n^+|^2 + \lambda(-R - f_-)4\pi \sum_{|\alpha_n| \le k_-} |\beta_n^-|^2 |u_n^-|^2 \\
- \int_\Gamma \left[\lambda(\lambda - 1)|\partial_\nu u^-|^2 + (\lambda - 1)|\partial_\tau u^-|^2 + (k_+^2 - \lambda k_-^2)|u|^2\right] \nu_2(x_2 - f_-)\, ds \\
- 2\int_{S_R} a(x)|\partial_2 u|^2\, dx + \int_{S_R} a(x)|\nabla u|^2 - a(x)k^2(x)|u|^2\, dx.
\end{aligned} \tag{5.12}
$$

Using (5.4), one deduces from (5.12) that

$$
\begin{aligned}
&\int_\Gamma \left[\lambda(\lambda - 1)|\partial_\nu u^-|^2 + (\lambda - 1)|\partial_\tau u^-|^2 + (k_+^2 - \lambda k_-^2)|u|^2\right] \nu_2(x_2 - f_-)\, ds + 2\int_{S_R} a(x)|\partial_2 u|^2\, dx \\
&= 4\pi(R - f_-) \sum_{|\alpha_n| \le k_+} |\beta_n^+|^2 |\hat{u}_n^+|^2 + 4\pi\lambda(-R - f_-) \sum_{|\alpha_n| \le k_-} |\beta_n^-|^2 |\hat{u}_n^-|^2 \\
&\quad + \mathrm{Re} \int_{\Gamma_R^+} (T^+ u)\bar{u}\, ds - \lambda \mathrm{Re} \int_{\Gamma_R^-} (T^- u)\bar{u}\, ds + \mathrm{Re}\, F(u) \\
&\le 4\pi(R - f_-) \sum_{|\alpha_n| \le k_+} |\beta_n^+|^2 |\hat{u}_n^+|^2 + 4\pi\lambda(-R - f_-) \sum_{|\alpha_n| \le k_-} |\beta_n^-|^2 |\hat{u}_n^-|^2 + \mathrm{Re}\, F(u).
\end{aligned} \tag{5.13}
$$

Recalling the conditions $\lambda \ge 1$, $k_+^2 > \lambda k_-^2$, $\nu_2 \ge C_L := \frac{1}{\sqrt{1+L^2}}$ and $\min_{x_1 \in (0,2\pi)} f(x_1) - f_- > 1$, we can get a lower bound of the left-hand side of (5.13),

$$
\text{Left hand side of (5.13)} \ge (k_+^2 - \lambda k_-^2) C_L \|u\|_{L^2(\Gamma)}^2 + 2\|\partial_2 u\|_{L^2(S_R)}^2.
$$

Using (5.5), (5.6) and (5.7), we can bound the two sums on the right hand of (5.13) by

$$
4\pi(R - f_-) \sum_{|\alpha_n| \le k_+} |\beta_n^+|^2 |\hat{u}_n^+|^2 + 4\pi\lambda(-R - f_-) \sum_{|\alpha_n| \le k_-} |\beta_n^-|^2 |\hat{u}_n^-|^2 \le -2k_+(R - f_-)\mathrm{Im}\, F(u).
$$

Therefore, combining the previous three relations yields

$$
(k_+^2 - \lambda k_-^2) C_L \|u\|_{L^2(\Gamma)}^2 + 2\|\partial_2 u\|_{L^2(S_R)}^2 \le [2k_+(R - f_-) + 1]|F(u)|. \tag{5.14}
$$

Using Lemma 3.5, the L^2-norm of u can be bounded by (5.14) as follows:

$$
\begin{aligned}
C_T \|u\|_{L^2(S_R)}^2 &\le C_T \left[(R - f_-)^2 \|\partial_2 u\|_{L^2(S_R)}^2 + 2(R - f_-)\|u\|_{L^2(\Gamma)}^2\right] \\
&\le 2\|\partial_2 u\|_{L^2(S_R)}^2 + (k_+^2 - \lambda k_-^2) C_L \|u\|_{L^2(\Gamma)}^2,
\end{aligned}
$$

where $C_T = \min\left\{\frac{2}{(R-f_-)^2}, \frac{(k_+^2 - \lambda k_-^2)C_L}{2(R-f_-)}\right\}$. Therefore, using (5.14),

$$
C_T \|u\|_{L^2(S_R)}^2 \le [2k_+(R - f_-) + 1]|F(u)|,
$$

from which we obtain

$$
\|u\|_{L^2(S_R)}^2 \le \frac{[2k_+(R - f_-) + 1]}{C_T}|F(u)|.
$$

Then by (5.2) and (5.4), we have

$$\begin{aligned}\|u\|^2_{H^1_\alpha(S_R)} &= \int_{S_R} a(x)|\nabla u|^2 + a(x)k^2(x)|u|^2\,dx \\ &= \int_{S_R} 2a(x)k^2(x)|u|^2\,dx + \mathrm{Re}\int_{\Gamma_R^+}(T^+u)\bar{u}\,ds - \lambda\mathrm{Re}\int_{\Gamma_R^-}(T^-u)\bar{u}\,ds + \mathrm{Re}\,F(u) \\ &\le C_{12}|F(u)| \end{aligned} \tag{5.15}$$

where $C_{12} = \frac{2k_+^2[2k_+(R-f_-)+1]}{C_T} + 1$. As done in the Dirichlet case, we can estimate $|F(u)|$ by (see (3.18) and (3.23))

$$|F(u)| = 4\pi\beta|\gamma\tilde{u}_0| \le 4\pi\beta\left(\epsilon_5|\tilde{u}_0|^2 + \frac{1}{4\epsilon_5}|\gamma|^2\right) \le 2\epsilon_5\cos\theta\|u\|^2_{H^1_\alpha(S_R)} + \frac{\pi\beta}{\epsilon_5}|\gamma|^2.$$

Therefore,

$$\|u\|^2_{H^1_\alpha(S_R)} \le C_{12}\left(2\epsilon_5\cos\theta\|u\|^2_{H^1_\alpha(S_R)} + \frac{\pi\beta}{\epsilon_5}|\gamma|^2\right).$$

Taking $\epsilon_5 = \frac{1}{4\cos\theta C_{12}}$, we have

$$\|u\|^2_{H^1_\alpha(S_R)} \le 8\pi k\cos^2\theta C_{12}^2|\gamma|^2.$$

(ii) Suppose that $\lambda \le 1$ and $k_+^2 < \lambda k_-^2$. In this case, we take $c = f_+$ in (5.9). Hence, similarly to (5.10) and (5.12), we get

$$\int_{\Gamma_R^\pm}(x_2 - f_+)\left[-\nu_2|\nabla u^\pm|^2 + \nu_2 k_\pm^2|u|^2 + 2\mathrm{Re}\,(\partial_2\bar{u}^\pm\partial_\nu u^\pm)\right]\,ds = (\pm R - f_+)4\pi\sum_{|\alpha_n|\le k_\pm}|\beta_n^\pm|^2|\hat{u}_n^\pm|^2,$$

and

$$\begin{aligned} 0 &= 4\pi(R - f_+)\sum_{|\alpha_n|\le k_+}|\beta_n^+|^2|\hat{u}_n^+|^2 + 4\pi\lambda(-R - f_+)\sum_{|\alpha_n|\le k_-}|\beta_n^-|^2|\hat{u}_n^-|^2 \\ &\quad - 2\int_{S_R} a(x)|\partial_2 u|^2\,dx + \int_{S_R} a(x)|\nabla u|^2 - a(x)k^2(x)|u|^2\,dx \\ &\quad - \int_\Gamma\left[\lambda(\lambda-1)|\partial_\nu u^-|^2 + (\lambda-1)|\partial_\tau u^-|^2 + (k_+^2 - \lambda k_-^2)|u|^2\right]\nu_2(x_2 - f_+)\,ds. \end{aligned}$$

Therefore, similarly to (5.13), we get

$$\begin{aligned} &\int_\Gamma\left[\lambda(\lambda-1)|\partial_\nu u^-|^2 + (\lambda-1)|\partial_\tau u^-|^2 + (k_+^2 - \lambda k_-^2)|u|^2\right]\nu_2(x_2 - f_+)\,ds + 2\int_{S_R} a(x)|\partial_2 u|^2\,dx \\ &= 4\pi(R - f_+)\sum_{|\alpha_n|\le k_+}|\beta_n^+|^2|\hat{u}_n^+|^2 + 4\pi\lambda(-R - f_+)\sum_{|\alpha_n|\le k_-}|\beta_n^-|^2|\hat{u}_n^-|^2 \\ &\quad + \mathrm{Re}\int_{\Gamma_R^+}(T^+u)\bar{u}\,ds - \lambda\mathrm{Re}\int_{\Gamma_R^-}(T^-u)\bar{u}\,ds + \mathrm{Re}\,F(u) \end{aligned}$$

$$\le -2k_+(R-f_+)\mathrm{Im}\,F(u)+\mathrm{Re}\,F(u),$$

which together with $\max_{x_1\in(0,2\pi)} f(x_1)-f_+<-1$ yields

$$(\lambda k_-^2-k_+^2)C_L\|u\|^2_{L^2(\Gamma)}+2\|\partial_2 u\|^2_{L^2(S_R)}\le[2k_+(R-f_+)+1]|F(u)|.$$

Again using Lemma 3.5 gives

$$\|u\|^2_{L^2(S_R)}\le\frac{[2k_+(R-f_+)+1]}{C_S}|F(u)|,\quad C_S=\min\left\{\frac{2}{(R-f_-)^2},\frac{(\lambda k_-^2-k_+^2)C_L}{2(R-f_-)}\right\}.$$

Then by (5.4), we have (see, e.g. (5.15))

$$\|u\|^2_{H^1_\alpha(S_R)}\le 2\max\{k_+,\lambda k_-\}\|u\|^2_{L^2(S_R)}+|F(u)|\le C_{13}|F(u)|$$

where $C_{13}=\frac{2\lambda k_-^2[2k_+(R-f_+)+1]}{C_S}+1$. As in the previous step, we then estimate $|F(u)|$ by

$$|F(u)|=4\pi\beta|\gamma\tilde u_0|\le 4\pi\beta\left(\epsilon_6|\tilde u_0|^2+\frac{1}{4\epsilon_6}|\gamma|^2\right)\le 2\epsilon_6\cos\theta\|u\|^2_{H^1_\alpha(S_R)}+\frac{\pi\beta}{\epsilon_6}|\gamma|^2.$$

Finally, taking $\epsilon_6=\frac{1}{4\cos\theta C_{13}}$ gives the desired result

$$\|u\|^2_{H^1_\alpha(S_R)}\le 8\pi k\cos^2\theta C_{13}^2|\gamma|^2.$$

(iii) Suppose that $k_+^2=\lambda k_-^2$ and $k_+>k_-$. Using (5.13) and the relation $\lambda>1$ (i.e., $\lambda(\lambda-1)>\lambda-1$) and $\nu_2\ge C_L$, one can obtain the L^2-estimate of $\partial_2 u$ over S_R,

$$2\|\partial_2 u\|^2_{L^2(S_R)}\le[2k_+(R-f_-)+1]|F(u)| \tag{5.16}$$

and also the gradient of u over Γ:

$$C_L(\lambda-1)\|\nabla u\|^2_{L^2(\Gamma)}\le[2k_+(R-f_-)+1]|F(u)|. \tag{5.17}$$

Recalling the Rellich identity (5.8) with $c=f_-$ and using $C_L\le\nu_2\le 1$ and (5.17), we obtain

$$\begin{aligned}
&C_L k_+^2\|u\|^2_{L^2(\Gamma)}\\
&\le\int_\Gamma(x_2-f_-)[-\nu_2|\nabla u^+|^2+\nu_2k_+^2|u|^2+2\mathrm{Re}(\partial_2\bar u^+\partial_\nu u^+)]\,ds\\
&\quad+\int_{S_R^+}2|\partial_2 u|^2\,dx+(f_+-f_-)\|\nabla u\|^2_{L^2(\Gamma)}\\
&=\int_{\Gamma_R^+}(x_2-f_-)[-\nu_2|\nabla u^+|^2+\nu_2k_+^2|u|^2+2\mathrm{Re}(\partial_2\bar u^+\partial_\nu u^+)]\,ds\\
&\quad+\int_{S_R^+}|\nabla u|^2-k_+^2|u|^2\,dx+(f_+-f_-)\|\nabla u\|^2_{L^2(\Gamma)}\\
&\le(R-f_-)4\pi\sum_{|\alpha_n|\le k_+}|\beta_n^+|^2|\hat u_n^+|^2+\mathrm{Re}\int_{\Gamma_R^+}(T^+u)\bar u\,ds+\mathrm{Re}\,F(u)+(f_+-f_-)\|\nabla u\|^2_{L^2(\Gamma)}
\end{aligned}$$

$$\leq M[2k_+(R-f_-)+1]|F(u)|, \tag{5.18}$$

where $M = 1 + \frac{f_+ - f_-}{C_L(\lambda-1)}$. Combining (5.16), (5.17) and (5.18) yields a similar estimate to (5.14) and thus also the estimate of $\|u\|^2_{H^1_\alpha(S_R)}$:

$$\begin{aligned} C_Z\|u\|^2_{L^2(S_R)} &\leq C_Z\left[(R-f_-)^2\|\partial_2 u\|^2_{L^2(S_R)} + 2(R-f_-)\|u\|^2_{L^2(\Gamma)}\right] \\ &\leq 2\|\partial_2 u\|^2_{L^2(S_R)} + k_+^2 C_L\|u\|^2_{L^2(\Gamma)} \\ &\leq (1+M)[2k_+(R-f_-)+1]|F(u)|, \end{aligned}$$

where $C_Z = \min\left\{\frac{2}{(R-f_-)^2}, \frac{k_+^2 C_L}{2(R-f_-)}\right\}$. Then we have

$$\|u\|^2_{L^2(S_R)} \leq \frac{(1+M)[2k_+(R-f_-)+1]}{C_Z}|F(u)|.$$

Therefore,

$$\begin{aligned} \|u\|^2_{H^1_\alpha(S_R)} &= \int_{S_R} a(x)|\nabla u|^2 + a(x)k^2(x)|u|^2\,dx \\ &= \int_{S_R} 2a(x)k^2(x)|u|^2\,dx + \mathrm{Re}\int_{\Gamma_R^+}(T^+u)\bar{u}\,ds - \lambda\mathrm{Re}\int_{\Gamma_R^-}(T^-u)\bar{u}\,ds + \mathrm{Re}\,F(u) \\ &\leq C_{14}|F(u)|, \end{aligned}$$

where $C_{14} = \frac{2k_+^2(1+M)[2k_+(R-f_-)+1]}{C_Z} + 1$. Then we get

$$\|u\|_{H^1_\alpha(S_R)} \leq 2\sqrt{2\pi k}\cos\theta C_{14}|\gamma|.$$

The case $k_+ < k_-$ (i.e., $\lambda < 1$) can be treated analogously by taking $c = f_+$ in the Rellich identity (5.8). We can easily get

$$2\|\partial_2 u\|^2_{L^2(S_R)} \leq [2k_+(R-f_+)+1]|F(u)| \leq [2k_+(R-f_-)+1]|F(u)|, \tag{5.19}$$

$$C_L\lambda(1-\lambda)\|\nabla u\|^2_{L^2(\Gamma)} \leq [2k_+(R-f_-)+1]|F(u)|, \tag{5.20}$$

$$C_L k_+^2\|u\|^2_{L^2(\Gamma)} \leq N[2k_+(R-f_-)+1]|F(u)|, \tag{5.21}$$

where $N = 1 + \frac{f_+ - f_-}{C_L\lambda(1-\lambda)}$. Combining (5.19), (5.20) and (5.21) yields

$$\|u\|^2_{L^2(S_R)} \leq \frac{(1+N)[2k_+(R-f_-)+1]}{C_Z}|F(u)|.$$

Therefore,

$$\|u\|_{H^1_\alpha(S_R)} \leq 2\sqrt{2\pi k}\cos\theta C_{15}|\gamma|,$$

where $C_{15} = \frac{2k_+^2(1+N)[2k_+(R-f_-)+1]}{C_Z} + 1$. Combining these two cases yields the desired result. □

6. Appendix

Lemma 6.1. *If $v \in H^2_\alpha(\Omega_R)$, then we have the Rellich identity*

$$2\mathrm{Re}\int\limits_{\Omega_R}(x_2-c)\partial_2\bar{v}(\Delta v+k^2v)\,dx-\int\limits_{\Omega_R}|\nabla v|^2-k^2|v|^2-2|\partial_2 v|^2\,dx$$
$$=\left(\int\limits_{\Gamma_R}-\int\limits_{\Gamma}\right)(x_2-c)[-\nu_2|\nabla v|^2+\nu_2k^2|v|^2+2\mathrm{Re}\,(\partial_2\bar{v}\partial_\nu v)]\,ds,$$

where c is a constant and $\nu=(\nu_1,\nu_2)$ is the normal direction at $\Gamma\cup\Gamma_R$ pointing upward.

Proof. Using the Green's formula and the relation $2\mathrm{Re}\,(\partial_2\bar{v}v)=\partial_2|v|^2$ (note that $\nu=(0,1)$ on Γ_{R}), we have

$$2\mathrm{Re}\int\limits_{\Omega_R}(x_2-c)\partial_2\bar{v}(\Delta v+k^2v)\,dx$$
$$=2\mathrm{Re}\int\limits_{\Omega_R}-\nabla((x_2-c)\partial_2\bar{v})\cdot\nabla v+k^2(x_2-c)\partial_2\bar{v}v\,dx+2\mathrm{Re}\int\limits_{\partial\Omega_R}(x_2-c)\partial_2\bar{v}\partial_\nu v\,ds$$
$$=-2\mathrm{Re}\int\limits_{\Omega_R}(0,\partial_2\bar{v})\cdot(\partial_1 v,\partial_2 v)^\top+(x_2-c)\partial_2\nabla\bar{v}\cdot\nabla v\,dx+\int\limits_{\Omega_R}k^2(x_2-c)\partial_2|v|^2\,dx$$
$$+2\mathrm{Re}\int\limits_{\Gamma_R}(x_2-c)\nu_2|\partial_\nu v|^2\,ds-2\mathrm{Re}\int\limits_{\Gamma}(x_2-c)\nu_2|\partial_\nu v|^2\,ds$$
$$=-2\int\limits_{\Omega_R}|\partial_2 v|^2\,dx-\int\limits_{\Omega_R}(x_2-c)\partial_2|\nabla v|^2\,dx+\int\limits_{\Omega_R}k^2(x_2-c)\partial_2|v|^2\,dx$$
$$+2\int\limits_{\Gamma_R}(x_2-c)\nu_2|\partial_\nu v|^2\,ds-2\int\limits_{\Gamma}(x_2-c)\nu_2|\partial_\nu v|^2\,ds. \tag{6.1}$$

Furthermore, integrating by parts, we have

$$\int\limits_{\Omega_R}(x_2-c)\partial_2|\nabla v|^2\,dx=\int\limits_{\partial\Omega_R}(x_2-c)\nu_2|\nabla v|^2\,ds-\int\limits_{\Omega_R}|\nabla v|^2\,dx$$
$$=\int\limits_{\Gamma_R}(x_2-c)\nu_2|\nabla v|^2\,ds-\int\limits_{\Gamma}(x_2-c)\nu_2|\nabla v|^2\,ds-\int\limits_{\Omega_R}|\nabla v|^2\,dx, \tag{6.2}$$
$$\int\limits_{\Omega_R}k^2(x_2-c)\partial_2|v|^2\,dx=\int\limits_{\partial\Omega_R}k^2|v|^2(x_2-c)\nu_2\,ds-\int\limits_{\Omega_R}k^2|v|^2\,dx$$
$$=\int\limits_{\Gamma_R}k^2|v|^2\nu_2(x_2-c)\,ds-\int\limits_{\Gamma}k^2|v|^2\nu_2(x_2-c)\,ds-\int\limits_{\Omega_R}k^2|v|^2\,dx. \tag{6.3}$$

Substituting (6.2), (6.3) into (6.1) and using

$$-|\nabla v|^2+k^2|v|^2+2|\partial_2 v|^2=|\partial_2 v|^2-|\partial_1 v|^2+k^2|v|^2,$$

we have

$$
\begin{aligned}
&2\mathrm{Re}\int_{\Omega_R}(x_2-c)\partial_2\bar{v}(\Delta v+k^2 v)\,dx\\
&=-2\int_{\Omega_R}|\partial_2 v|^2\,dx-\int_{\Gamma_R}(x_2-c)\nu_2|\nabla v|^2\,ds+\int_{\Gamma}(x_2-c)\nu_2|\nabla v|^2\,ds+\int_{\Omega_R}|\nabla v|^2\,dx\\
&\quad+\int_{\Gamma_R}k^2|v|^2\nu_2(x_2-c)\,ds-\int_{\Gamma}k^2|v|^2\nu_2(x_2-c)\,ds-\int_{\Omega_R}k^2|v|^2\,dx\\
&\quad+2\int_{\Gamma_R}(x_2-c)\nu_2|\partial_\nu v|^2\,ds-2\int_{\Gamma}(x_2-c)\nu_2|\partial_\nu v|^2\,ds\\
&=\left(\int_{\Gamma_R}-\int_{\Gamma}\right)(x_2-c)[-\nu_2|\nabla v|^2+\nu_2 k^2|v|^2+2\mathrm{Re}\,(\partial_2\bar{v}\partial_\nu v)]\,ds\\
&\quad+\int_{\Omega_R}|\nabla v|^2-k^2|v|^2-2|\partial_2 v|^2\,dx
\end{aligned}
$$

which completes the proof of Lemma 6.1. □

The following relation follows directly from Lemma 6.1.

Corollary 6.2. *If $v\in H^2_\alpha(\Omega_R)$ and $v=0$ on Γ, then we have the Rellich identity*

$$
\begin{aligned}
&2\mathrm{Re}\int_{\Omega_R}(x_2-c)\partial_2\bar{v}(\Delta v+k^2v)\,dx+\int_{\Gamma}(x_2-c)\nu_2\left|\frac{\partial v}{\partial\nu}\right|^2ds+2\int_{\Omega_R}\left|\frac{\partial v}{\partial x_2}\right|^2dx\\
&=(R-c)\int_{\Gamma_R}\left|\frac{\partial v}{\partial x_2}\right|^2-\left|\frac{\partial v}{\partial x_1}\right|^2+k^2|v|^2\,ds+\int_{\Omega_R}|\nabla v|^2-k^2|v|^2\,dx,
\end{aligned}
$$

where c is a constant and $\nu=(\nu_1,\nu_2)$ is the normal direction pointing to Ω_R.

Proof of Lemma 3.4. Define the half-plane

$$U_{f_-}=\{x\in\mathbb{R}^2: x_2>f_-,\ x_1\in\mathbb{R}\}.$$

For

$$v\in(C_0^\infty(\tilde{\Omega})\cap X_R)\subset(C_0^\infty(U_{f_-})\cap X_R),$$

we have

$$v(x)=\sum_{n\in\mathbb{Z}}v_n(x_2)e^{i\alpha_n x_1},\quad x_2>f_-,$$

where $v_n(x_2)$ is the Fourier coefficient of $v(x)$. Thus,

$$|v_n(R)|^2 = \int_{f_-}^{R} \frac{\partial}{\partial x_2}|v_n(x_2)|^2\, dx_2 = 2\mathrm{Re} \int_{f_-}^{R} \bar{v}_n(x_2)\frac{\partial}{\partial x_2}v_n(x_2)\, dx_2. \tag{6.4}$$

Using the Cauchy-Schwarz inequality and (6.4), we have

$$\begin{aligned}
\|v\|^2_{H^{1/2}_\alpha(\Gamma_R)} &= \sum_{n\in\mathbb{Z}}(k^2+\alpha_n^2)^{1/2}|v_n(R)|^2 \\
&\le 2\sum_{n\in\mathbb{Z}}(k^2+\alpha_n^2)^{1/2}\left(\int_{f_-}^{R}|v_n(x_2)|^2\,dx_2\right)^{1/2}\left(\int_{f_-}^{R}\left|\frac{\partial}{\partial x_2}v_n(x_2)\right|^2\,dx_2\right)^{1/2} \\
&\le 2\left(\sum_{n\in\mathbb{Z}}(k^2+\alpha_n^2)\int_{f_-}^{R}|v_n(x_2)|^2\,dx_2\right)^{1/2}\left(\sum_{n\in\mathbb{Z}}\int_{f_-}^{R}\left|\frac{\partial}{\partial x_2}v_n(x_2)\right|^2\,dx_2\right)^{1/2} \\
&\le \sum_{n\in\mathbb{Z}}(k^2+\alpha_n^2)\int_{f_-}^{R}|v_n(x_2)|^2\,dx_2 + \sum_{n\in\mathbb{Z}}\int_{f_-}^{R}\left|\frac{\partial}{\partial x_2}v_n(x_2)\right|^2\,dx_2.
\end{aligned} \tag{6.5}$$

Note that in the last inequality, we have used the inequality $a+b \ge 2\sqrt{ab}$ for $a, b > 0$. Then it suffices to prove that the right-hand side of (6.5) is exactly $\|v\|^2_X/(2\pi)$. Define $U^R_{f_-} = (0, 2\pi)\times(f_-, R)$. It follows that

$$\begin{aligned}
\|v\|^2_{L^2(\Omega_R)} &= \left\|\sum_{n\in\mathbb{Z}} v_n(x_2)e^{i\alpha_n x_1}\right\|^2_{L^2(\Omega_R)} \\
&= \int_{U^R_{f_-}}\left|\sum_{n\in\mathbb{Z}} v_n(x_2)e^{i\alpha_n x_1}\right|^2\,dx \\
&= \int_{f_-}^{R}\int_0^{2\pi}\left(\sum_{n\in\mathbb{Z}} v_n(x_2)e^{i\alpha_n x_1}\right)\left(\sum_{m\in\mathbb{Z}} \bar{v}_m(x_2)e^{-i\alpha_m x_1}\right)\,dx_1dx_2 \\
&= 2\pi\sum_{n\in\mathbb{Z}}\int_{f_-}^{R}|v_n(x_2)|^2\,dx_2.
\end{aligned} \tag{6.6}$$

Analogously, we have

$$\begin{aligned}
\|\nabla v\|^2_{L^2(\Omega_R)} &= \left\|\sum_{n\in\mathbb{Z}} v_n(x_2)i\alpha_n e^{i\alpha_n x_1}\right\|^2_{L^2(\Omega_R)} + \left\|\sum_{n\in\mathbb{Z}} \frac{\partial}{\partial x_2}v_n(x_2)e^{i\alpha_n x_1}\right\|^2_{L^2(\Omega_R)} \\
&= \int_{U^R_{f_-}}\left|\sum_{n\in\mathbb{Z}} \alpha_n v_n(x_2)e^{i\alpha_n x_1}\right|^2\,dx + \int_{U^R_{f_-}}\left|\sum_{n\in\mathbb{Z}} \frac{\partial}{\partial x_2}v_n(x_2)e^{i\alpha_n x_1}\right|^2\,dx \\
&= 2\pi\sum_{n\in\mathbb{Z}}\alpha_n^2\int_{f_-}^{R}|v_n(x_2)|^2\,dx_2 + 2\pi\sum_{n\in\mathbb{Z}}\int_{f_-}^{R}\left|\frac{\partial}{\partial x_2}v_n(x_2)\right|^2\,dx_2.
\end{aligned} \tag{6.7}$$

Inserting (6.6) and (6.7) into (6.5) and using the definition of $\|v\|_{X_R}$, we have

$$\|v\|_{H_\alpha^{1/2}(\Gamma_R)} \le \|v\|_{X_R}/(2\pi)^{1/2}.$$

Since set $\{v|_{\Omega_R} : v \in C_0^\infty(\tilde{\Omega}) \cap X_R\}$ is dense in X_R, we complete the proof. □

Proof of Lemma 3.5. Since $C^\infty(\Omega_R)$ is dense in $H_\alpha^1(\Omega)$, we only need to consider smooth $v \in C^\infty(\Omega_R) \cap H_\alpha^1(\Omega)$. For $x \in \Omega_R$, we have

$$v(x) = \int_{f(x_1)}^{x_2} \frac{\partial v(x_1,\tau)}{\partial \tau}\, d\tau + v(x_1, f(x_1)).$$

Therefore, by the Cauchy-Schwarz inequality we have

$$\begin{aligned} |v(x_1,x_2)|^2 &\le 2\left|\int_{f(x_1)}^{x_2} \frac{\partial v(x_1,\tau)}{\partial \tau}\, d\tau\right|^2 + 2|v(x_1,f(x_1))|^2 \\ &\le 2(x_2 - f(x_1)) \int_{f(x_1)}^{R} \left|\frac{\partial v(x_1,\tau)}{\partial \tau}\right|^2 d\tau + 2|v(x_1,f(x_1))|^2. \end{aligned}$$

Thus,

$$\begin{aligned} \|v\|^2_{L^2(\Omega_R)} = \int_{\Omega_R} |v|^2\, dx &= \int_0^{2\pi}\int_{f(x_1)}^{R} |v(x_1,x_2)|^2\, dx_2 dx_1 \\ &\le \int_0^{2\pi} \left[\left(\int_{f(x_1)}^{R} 2(x_2 - f(x_1))\, dx_2\right)\left(\int_{f(x_1)}^{R} \left|\frac{\partial v(x_1,\tau)}{\partial \tau}\right|^2 d\tau\right)\right] dx_1 \\ &\quad + \int_0^{2\pi} 2(R - f_-)|v(x_1,f(x_1))|^2\, dx_1. \end{aligned} \tag{6.8}$$

The first term on the right-hand side of the above formula can be estimated by

$$\begin{aligned} &\int_0^{2\pi} \left[\left(\int_{f(x_1)}^{R} 2(x_2 - f(x_1))\, dx_2\right)\left(\int_{f(x_1)}^{R} \left|\frac{\partial v(x_1,\tau)}{\partial \tau}\right|^2 d\tau\right)\right] dx_1 \\ &\le \int_0^{2\pi} (R - f(x_1))^2 \left(\int_{f(x_1)}^{R} \left|\frac{\partial v(x_1,\tau)}{\partial \tau}\right|^2 d\tau\right) dx_1 \\ &\le (R - f_-)^2 \int_0^{2\pi}\int_{f(x_1)}^{R} \left|\frac{\partial v(x_1,\tau)}{\partial \tau}\right|^2 d\tau\, dx_1 \\ &= (R - f_-)^2 \left\|\frac{\partial v}{\partial x_2}\right\|^2_{L^2(\Omega_R)}. \end{aligned} \tag{6.9}$$

On the other hand,

$$\|v\|^2_{L^2(\Gamma)} = \int_0^{2\pi} |v(x_1, f(x_1))|^2 \sqrt{1 + (f'(x_1))^2}\, dx_1 \geq \int_0^{2\pi} |v(x_1, f(x_1))|^2\, dx_1. \tag{6.10}$$

Inserting (6.9) and (6.10) into (6.8), we complete the proof. □

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