# TIME-HARMONIC ELASTIC SCATTERING BY UNBOUNDED DETERMINISTIC AND RANDOM ROUGH SURFACES IN THREE DIMENSIONS 

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#### Abstract

In this paper, we investigate well-posedness of time-harmonic scattering of elastic waves by unbounded rigid rough surfaces in three dimensions. The elastic scattering is caused by an $L^{2}$ function with a compact support in the $x_{3}$-direction, and both deterministic and random surfaces are investigated via the variational approach. The rough surface in a deterministic setting is assumed to be Lipschitz and lie within a finite distance of a flat plane, and the scattering is caused by an inhomogeneous term in the elastic wave equation whose support lies within some finite distance of the boundary. For the deterministic case, a stability estimate of elastic scattering by rough surface is shown at an arbitrary frequency. It is noticed that all constants in a priori bounds are bounded by explicit functions of the frequency and geometry of rough surfaces. Furthermore, based on this explicit dependence on the frequency together with the measurability and $\mathbb{P}$-essentially separability of the randomness, we obtain a similar bound for the solution of the scattering by random surfaces.


## 1. Introduction

This paper is concerned with the mathematical analysis of the time-harmonic elastic scattering from unbounded deterministic and random rough surfaces in three dimensions. The phrase rough means surface is a (usually nonlocal) perturbation of an infinite plane such that the whole surface lies within a finite distance of the original plane. Rough surface scattering problems have important applications in diverse scientific areas such as remote sensing, geophysics, outdoor sound propagation, radar techniques (see e.g., [1, 2] and the references cited therein). In linear elasticity, the existence and uniqueness of solution were studied in via the boundary integral equation method [3, 4, 5. . The variational approach was proposed in [7, 9] to handle well-posedenss of the scattering problems in periodic structures by using the Rayleigh expansion condition (REC) and in [8, 10] for general rigid rough surfaces by using the angular spectrum representation (ASR).

Recently, in [11] a mathematical formulation of the elastic rough surface scattering problems was presented in three dimensions. Based on a Rellich-type identity, the uniqueness of weak solutions to the variational problem was proved if the rigid surface was the graph of a uniformly Lipschitz continuous function. The existence of solutions was also proved for the case of locally perturbed scattering problems. However, the well-posedness problem for the scattering by a general rough surface remains unsolved. Later, the authors in [14] further derived an a priori bound which was explicitly dependent on the frequency. The main goal of this paper is three-fold. First, we present a variational formulation of the elastic scattering in three dimensions by a Lipschitz-type rough surface and prove its well-posedness. Second,

[^0]we derive an a priori bound which is explicitly dependent on the frequency. Third, we utilize the explicit bound to derive the well-posedness for scattering by random rough surfaces as [13, 14]. As discussed in [6], we expect that the variational formulation will be suitable for numerical solution via finite element discretization. Furthermore, the explicit bounds we obtain should be useful in establishing the dependence of the constants in a priori error estimates for finite element schemes on the frequency and the geometry of the domain.

This paper utilizes methods and results contained in [8, 11, 14. As was pointed out in [8], the elastic problem is more complicated than the acoustic case due to the coexistence of compressional and shear waves. As a consequence, the Dirichlet-to-Neumann map for the elastic wave equation is tensor-valued which does not have a definite real part. This brings difficulties in deriving the a priori estimates of solutions via Rellich identities for arbitrary frequencies. We prove that the variational problem is well-posed by the theory of semi-Fredholm used in [8]. To this end, we first consider the case of small frequencies in which the Lax-Milgram theorem can be applied. Then we establish several a priori estimates. During this process, we carefully trace the dependence of the coefficients of these bounds on the frequency. In this way, we arrive at an a priori bound for the solution to the variational problem which is explicitly dependent on the frequency. Afterwords, inspired by the framework for scattering by random medium in [12] and random surfaces in [13, 14], we can obtain the well-posedness for a stochastic variation problem with an explicit a priori bound.

The rest of this paper is outlined as follows. In Section 2 we present the variational formulation for the elastic scattering problem. Section 3.3 is devoted to the well-posedness of the variational problem for small frequencies. In Section 4 we derive a priori bounds and trace the explicit dependence on the frequency and on the geometry of the domain. For random cases, a similar bound is derived in Section 5. Conclusions are presented in Section 6

## 2. Problem formulation

This section is devoted to the mathematical formulation of the three-dimensional elastic wave scattering by unbounded rigid rough surfaces. Let $D \subset \mathbb{R}^{3}$ be an unbounded connected open set such that, for some constants $m<M$,

$$
U_{M} \subset D \subset U_{m}, \quad U_{h}:=\left\{x=\left(x^{\prime}, x_{3}\right): x_{3}>h\right\}, \quad x^{\prime}:=\left(x_{1}, x_{2}\right) .
$$

The space $D$ is supposed to be filled with a homogeneous and isotropic elastic medium with unit mass density. We assume that $\Gamma:=\partial D$ is an unbounded rough surface, which is supposed to be the graph of a uniformly Lipschitz continuous function $f$. More precisely, we assume

$$
\Gamma=\left\{x \in \mathbb{R}^{3}: x_{3}=f\left(x^{\prime}\right), x^{\prime}=\left(x_{1}, x_{2}\right) \in \mathbb{R}^{2}\right\}
$$

and there exists a constant $L>0$ such that

$$
\begin{equation*}
\left|f\left(x^{\prime}\right)-f\left(y^{\prime}\right)\right| \leq L\left|x^{\prime}-y^{\prime}\right| \quad \text { for all } \quad x^{\prime}, y^{\prime} \in \mathbb{R}^{2} \tag{2.1}
\end{equation*}
$$

Throughout the paper we fix some $h>M$. Let $\Gamma_{h}=\left\{x \in \mathbb{R}^{3}: x_{3}=h\right\}$ and $S_{h}=D \backslash \bar{U}_{h}$. Denote the unit normal vector on $\Gamma \cup \Gamma_{h}$ by $\nu:=\left(\nu_{1}, \nu_{2}, \nu_{3}\right)$ pointing into the region of $x_{3}>h$ on $\Gamma_{h}$ and into the exterior of $D$ on $\Gamma$. Assume that $g \in L^{2}(D)^{3}$ is an elastic source term with $\operatorname{supp}(g) \subset S_{h}$. Consider the following Navier equation in three dimensions

$$
\begin{equation*}
\Delta^{*} u+\omega^{2} u=g \quad \text { in } D \tag{2.2}
\end{equation*}
$$

where $\Delta^{*}=\mu \Delta+(\lambda+\mu) \nabla \nabla \cdot, u=\left(u_{1}, u_{2}, u_{3}\right)^{\top}$ is the elastic displacement and $\omega>0$ is the angular frequency. Here $\lambda$ and $\mu$ denote the Lamé constants characterizing the medium above $\Gamma$ satisfying $\mu>0, \lambda+2 \mu / 3>0$. Since $\Gamma$ is physically rigid, there holds the Dirichlet boundary condition

$$
\begin{equation*}
u=0 \quad \text { on } \Gamma . \tag{2.3}
\end{equation*}
$$

As the domain $D$ is unbounded, a proper radiation condition should be imposed on $u$ at infinity. In this paper we utilize the elastic Upward Propagation Radiation Condition (UPRC) at infinity to ensure the well-posedness of the boundary value problem (2.2)-(2.3). Below we briefly introduce this radiation condition and refer to [11, 8] for the details. We begin with the decomposition of the wave fields into a sum of compressional and shear parts

$$
\begin{equation*}
u=\frac{1}{\mathrm{i}}(\nabla \varphi+\nabla \times \psi), \quad \nabla \cdot \psi=0 \quad \text { in } \quad x_{3}>h \tag{2.4}
\end{equation*}
$$

where the scalar function $\varphi$ and the vector function $\psi$ satisfy the homogeneous Helmholtz equations

$$
\begin{equation*}
\Delta \varphi+k_{\mathrm{p}}^{2} \varphi=0, \quad \Delta \psi+k_{\mathrm{s}}^{2} \psi=0 \quad \text { in } \quad x_{3}>h \tag{2.5}
\end{equation*}
$$

Here, $k_{\mathrm{p}}$ and $k_{\mathrm{s}}$ are compressional and shear wave numbers, respectively, defined by

$$
k_{\mathrm{p}}:=\frac{\omega}{\sqrt{\lambda+2 \mu}}, \quad k_{\mathrm{s}}:=\frac{\omega}{\sqrt{\mu}} .
$$

Denote by $\hat{v}$ the Fourier transform of $v$ in $\mathbb{R}^{2}$, i.e.,

$$
\hat{v}(\xi)=\mathcal{F} v(\xi):=\frac{1}{2 \pi} \int_{\mathbb{R}^{2}} v\left(x^{\prime}\right) e^{-\mathrm{i} x^{\prime} \cdot \xi} \mathrm{d} x^{\prime}, \quad \xi=\left(\xi_{1}, \xi_{2}\right) \in \mathbb{R}^{2}
$$

Taking the Fourier transform of (2.5) and assuming that $\varphi, \psi$ fulfill the Upward Angular Spectrum Representation (UASR) of the Helmholtz equation in $U_{h}$ (see [6]), we obtain for $x_{3} \geq h$ that

$$
\begin{align*}
& \varphi\left(x^{\prime}, x_{3}\right)=\frac{1}{2 \pi} \int_{\mathbb{R}^{2}} \hat{\varphi}(\xi, h) e^{\mathrm{i} \beta(\xi)\left(x_{3}-h\right)} e^{\mathrm{i} \xi \cdot x^{\prime}} \mathrm{d} \xi \\
& \psi\left(x^{\prime}, x_{3}\right)=\frac{1}{2 \pi} \int_{\mathbb{R}^{2}} \hat{\psi}(\xi, h) e^{\mathrm{i} \gamma(\xi)\left(x_{3}-h\right)} e^{\mathrm{i} \xi \cdot x^{\prime}} \mathrm{d} \xi \tag{2.6}
\end{align*}
$$

where

$$
\beta(\xi):= \begin{cases}\left(k_{\mathrm{p}}^{2}-|\xi|^{2}\right)^{1 / 2}, & |\xi|<k_{\mathrm{p}} \\ \mathrm{i}\left(|\xi|^{2}-k_{\mathrm{p}}^{2}\right)^{1 / 2}, & |\xi|>k_{\mathrm{p}}\end{cases}
$$

and

$$
\gamma(\xi):= \begin{cases}\left(k_{\mathrm{s}}^{2}-|\xi|^{2}\right)^{1 / 2}, & |\xi|<k_{\mathrm{s}} \\ \mathrm{i}\left(|\xi|^{2}-k_{\mathrm{s}}^{2}\right)^{1 / 2}, & |\xi|>k_{\mathrm{s}} .\end{cases}
$$

Denote the Fourier transform of $\varphi\left(x^{\prime}, h\right)$ and $\psi\left(x^{\prime}, h\right)$ by

$$
A_{\mathrm{p}}(\xi)=\hat{\varphi}(\xi, h), \quad \tilde{\boldsymbol{A}}_{\mathrm{s}}(\xi)=\hat{\psi}(\xi, h)
$$

respectively. Noting that $\operatorname{div} \psi=0$, we have $(\xi, \gamma(\xi)) \cdot \tilde{\boldsymbol{A}}_{\mathbf{s}}(\xi)^{\top}=0$. For notational convenience we omit the dependence of $\beta$ and $\gamma$ on $\xi$ in the subsequent context.

Substituting (2.6) into (2.4), we obtain for $x_{3} \geq h$ that

$$
\begin{equation*}
u(x)=\frac{1}{2 \pi} \int_{\mathbb{R}^{2}}\left[A_{\mathrm{p}}(\xi)(\xi, \beta)^{\top} e^{\mathrm{i} \beta\left(x_{3}-h\right)}+\boldsymbol{A}_{\mathrm{s}}(\xi) e^{\mathrm{i} \gamma\left(x_{3}-h\right)}\right] e^{\mathrm{i} \xi \cdot x^{\prime}} \mathrm{d} \xi \tag{2.7}
\end{equation*}
$$

where $\boldsymbol{A}_{\mathrm{s}}=\left(A_{\mathrm{s}}^{(1)}, A_{\mathrm{s}}^{(2)}, A_{\mathrm{s}}^{(3)}\right)^{\top}(\xi):=(\xi, \gamma)^{\top} \times \tilde{\boldsymbol{A}}_{\mathrm{s}}(\xi)$. It follows from (2.7) and the orthogonality $(\xi, \gamma) \cdot \boldsymbol{A}_{\mathrm{s}}^{\top}=0$ that

$$
\left[\begin{array}{c}
\hat{u}(\xi, h) \\
0
\end{array}\right]=\left[\begin{array}{cccc}
\xi_{1} & 1 & 0 & 0 \\
\xi_{2} & 0 & 1 & 0 \\
\beta & 0 & 0 & 1 \\
0 & \xi_{1} & \xi_{2} & \gamma
\end{array}\right]\left[\begin{array}{c}
A_{\mathrm{p}}(\xi) \\
\boldsymbol{A}_{\mathrm{s}}(\xi)
\end{array}\right]:=\widetilde{\mathbb{D}}(\xi) \boldsymbol{A}(\xi)
$$

which gives

$$
\boldsymbol{A}(\xi)=\left[\begin{array}{c}
A_{\mathrm{p}}  \tag{2.8}\\
\boldsymbol{A}_{\mathrm{s}}
\end{array}\right](\xi)=\widetilde{\mathbb{D}}^{-1}(\xi)\left[\begin{array}{c}
\hat{u}(\xi, h) \\
0
\end{array}\right]=\mathbb{D}(\xi) \hat{u}(\xi, h) .
$$

Here $\mathbb{D}$ is a $4 \times 3$ matrix given by

$$
\mathbb{D}(\xi)=\frac{1}{\beta \gamma+|\xi|^{2}}\left[\begin{array}{ccc}
\xi_{1} & \xi_{2} & \gamma \\
\beta \gamma+\xi_{2}^{2} & -\xi_{1} \xi_{2} & -\xi_{1} \gamma \\
-\xi_{1} \xi_{2} & \beta \gamma+\xi^{2} & -\xi_{2} \gamma \\
-\xi_{1} \beta & -\xi_{2} \beta & |\xi|^{2}
\end{array}\right]
$$

Using (2.7)-(2.8) yields the expression of $u$ in $U_{h}$ :

$$
\begin{equation*}
u(x)=\frac{1}{2 \pi} \int_{\mathbb{R}^{2}}\left\{\frac{1}{\beta \gamma+|\xi|^{2}}\left(M_{\mathrm{p}}(\xi) e^{\mathrm{i}\left(\xi \cdot x^{\prime}+\beta\left(x_{3}-h\right)\right)}+M_{\mathrm{s}}(\xi) e^{\mathrm{i}\left(\xi \cdot x^{\prime}+\gamma\left(x_{3}-h\right)\right)}\right) \hat{u}^{\mathrm{sc}}(\xi, h)\right\} \mathrm{d} \xi \tag{2.9}
\end{equation*}
$$

where

$$
M_{\mathrm{p}}(\xi)=:\left[\begin{array}{ccc}
\xi_{1}^{2} & \xi_{1} \xi_{2} & \xi_{1} \gamma \\
\xi_{1} \xi_{2} & \xi_{2}^{2} & \xi_{2} \gamma \\
\xi_{1} \beta & \xi_{2} \beta & \beta \gamma
\end{array}\right] \text { and } M_{\mathrm{s}}(\xi)=\left[\begin{array}{ccc}
\beta \gamma+\xi_{2}^{2} & -\xi_{1} \xi_{2} & -\gamma \xi_{1} \\
-\xi_{1} \xi_{2} & \beta \gamma+\xi_{1}^{2} & -\gamma \xi_{2} \\
-\xi_{1} \beta & -\xi_{2} \beta & |\xi|^{2}
\end{array}\right] .
$$

The representation (2.9) will be referred to as the upward radiation condition for rough surface scattering problems in linear elasticity.

Define the surface traction operator

$$
\begin{equation*}
T u:=2 \mu \partial_{\nu} u+\lambda(\nabla \cdot u) \nu+\mu \nu \times(\nabla \times u), \tag{2.10}
\end{equation*}
$$

where $\nu=\left(\nu_{1}, \nu_{2}, \nu_{3}\right)$ stands for the normal vector on the surface. Plugging (2.9) into (2.10) yields the Dirichlet-to-Neumann (DtN) operator on $\Gamma_{h}$ (cf [11])

$$
\begin{equation*}
T u=\mathcal{T} u\left(x^{\prime}\right):=\frac{\mathrm{i}}{2 \pi} \int_{\mathbb{R}^{2}} M(\xi) \hat{u}(\xi) e^{\mathrm{i} \xi \cdot x^{\prime}} \mathrm{d} \xi \tag{2.11}
\end{equation*}
$$

where $\mathcal{M}(\xi)$ is given by

$$
\begin{aligned}
\mathcal{M}(\xi)= & \frac{1}{|\xi|^{2}+\beta \gamma} \\
& \times\left[\begin{array}{ccc}
\mu\left[(\gamma-\beta) \xi_{2}^{2}+k_{\mathrm{s}}^{2} \beta\right] & -\mu \xi_{1} \xi_{2}(\gamma-\beta) & \left(2 \mu|\xi|^{2}-\omega^{2}+2 \mu \beta \gamma\right) \xi_{1} \\
-\mu \xi_{1} \xi_{2}(\gamma-\beta) & \mu\left[(\gamma-\beta) \xi_{1}^{2}+k_{\mathrm{s}}^{2} \beta\right] & \left(2 \mu|\xi|^{2}-\omega^{2}+2 \mu \beta \gamma\right) \xi_{2} \\
-\left(2 \mu|\xi|^{2}-\omega^{2}+2 \mu \beta \gamma\right) \xi_{1} & -\left(2 \mu|\xi|^{2}-\omega^{2}+2 \mu \beta \gamma\right) \xi_{2} & \gamma \omega^{2}
\end{array}\right]
\end{aligned}
$$

The boundary operator $\mathcal{T}$ is non-local and is equivalent to the upward radiation condition (2.9). It is also called the transparent boundary condition (TBC) for time-harmonic scattering problems in unbounded domains.

Based on the above DtN operator, the wave scattering problem (2.2)-(2.3) can be reduced to a boundary value problem over $S_{h}$ :

$$
\begin{aligned}
\mu \Delta u+(\lambda+\mu) \nabla \nabla \cdot u+\omega^{2} u=g & \text { in } S_{h} \\
u=0 & \text { on } \Gamma \\
T u=\mathcal{T} u & \text { on } \Gamma_{h} .
\end{aligned}
$$

To introduce the variational formulation, we introduce the energy space $V_{h}$ for $h>M$ as the closure of $C_{0}^{\infty}\left(S_{h} \cup \Gamma_{h}\right)^{3}$ in the $H^{1}$ norm

$$
\|u\|_{V_{h}}=\left(\|\nabla u\|_{L^{2}\left(S_{h}\right)^{3}}^{2}+\|u\|_{L^{2}\left(S_{h}\right)^{3}}\right)^{1 / 2}
$$

Multiplying the Navier equation in (2.2) by the complex conjugate of the test function $v \in V_{h}$ and using Betti's formula yield

$$
\int_{S_{h}} \mathcal{E}(u, \bar{v})-\omega^{2} u \cdot \bar{v} \mathrm{~d} x-\int_{\Gamma_{h}} \bar{v} \cdot T u \mathrm{~d} s=\int_{S_{h}} g \cdot \bar{v} \mathrm{~d} x
$$

where the bilinear form $\mathcal{E}(\cdot, \cdot)$ is defined by

$$
\mathcal{E}(u, v):=2 \mu \sum_{j, k=1}^{3} \partial_{k} u_{j} \partial_{k} v_{j}+\lambda \nabla \cdot u \nabla \cdot v-\mu \nabla \times u \cdot \nabla \times v, \quad \forall u, v \in V_{h}
$$

Define the sesquilinear form $B: V_{h} \times V_{h} \rightarrow \mathbb{C}$ by

$$
\begin{equation*}
B(u, v)=\int_{S_{h}} \mathcal{E}(u, \bar{v})-\omega^{2} u \cdot \bar{v} \mathrm{~d} x-\int_{\Gamma_{h}} \bar{v} \cdot \mathcal{T} u \mathrm{~d} s \tag{2.12}
\end{equation*}
$$

Now we can formulate the variational problem as follows:
Variational Problem I: find $u \in V_{h}$ such that

$$
\begin{equation*}
B(u, v)=-\int_{S_{h}} g \cdot \bar{v} \mathrm{~d} x \quad \text { for all } v \in V_{h} \tag{2.13}
\end{equation*}
$$

The variational problem is equivalent to the boundary value problem: given $g \in L^{2}(D)^{3}$, with $\operatorname{supp}(g) \subset S_{h}$ for some $h>M$, find $u \in H_{l o c}^{1}(D)^{3}$ such that $\left.u\right|_{S_{h}} \in V_{h}$ for every $h>M$ (implying $u=0$ on $\Gamma$ ), the Navier equation $\left(\triangle^{*}+\omega^{2}\right) u=g$ in $D$ holds in a distributional sense, and the radiation condition (2.9) is satisfied with $\left.u\right|_{\Gamma_{h}} \in H^{1 / 2}\left(\Gamma_{h}\right)^{3}$ by the trace theorem.

The main theorem of this paper can now be stated as follows.
Theorem 2.1. For any $\omega>0$, the Variational Problem $I$ (2.13) is uniquely solvable in $V_{h}$. Moreover, there exists a constant $C$ independent of $\omega, h$ and the Lipschitz constant $L$ of $f$ such that the solution satisfies the estimate

$$
\begin{equation*}
\|u\|_{V_{h}} \leq(h-m+2)\left(C_{4}(\omega, h)+C_{5}(\omega, h)^{2}+C_{6}(\omega, h, L)\right)\|g\|_{V_{h}} \tag{2.14}
\end{equation*}
$$

where

$$
C_{4}(\omega, h)=C(h+1-m) \omega, \quad C_{5}=C \sqrt{1+\omega^{-1}} C_{3}(\omega, h)
$$

and

$$
C_{6}=C\left(\omega^{-1}+1\right) C_{1}(\omega, h, L) C_{2}(\omega, h, L)^{2} .
$$

Here

$$
\begin{aligned}
C_{1}(\omega, h, L) & =C \omega^{3}\left(1+L^{2}\right)^{1 / 2}(h-m+1) \\
C_{2}(\omega, h, L) & =C\left(1+L^{2}\right)^{1 / 4} \sqrt{h+1-m}(1+\omega(h+1-m)), \\
C_{3}(\omega, h) & =C(h+1-m)(1+\omega(h+1-m))^{2} / \omega
\end{aligned}
$$

The constants $C_{1}-C_{6}$ are derived from a priori bounds of the variational solution, which exhibit explicit dependence on the frequency $\omega$ and the geometry of the rough surface. They lead to the explicit a priori bound of the solution of the elastic scattering problem in three dimensions.

By the semi-Fredholm theory in [8], the results of Theorem 2.1] follow from the wellposedness of the variational problem at small frequencies (cf Theorem 3.3) and an a priori bound of the solution to the variational problem at an arbitrary frequency (cf Theorem 4.3). Thus, in the subsequent two sections we shall focus on mathematical analysis at small frequencies and a priori estimate at an arbitrary frequency.

## 3. Analysis of the variational problem for small frequency

We first investigate mapping properties the $\operatorname{DtN}$ operator in three dimensions. For a matrix $\mathcal{M}(\xi) \in \mathbb{C}^{3 \times 3}$ depending on $\xi$, let $\operatorname{Re} \mathcal{M}(\xi):=\left(\mathcal{M}(\xi)+\mathcal{M}(\xi)^{*}\right) / 2$. We shall write $\operatorname{Re} \mathcal{M}(\xi)>0$ if $\operatorname{Re} \mathcal{M}(\xi)$ is positive definite. Here $M^{*}(\xi)$ is the adjoint of $M$ with respect to the scalar product $(\cdot, \cdot)_{\mathbb{C}^{3 \times 3}}$ in $\mathbb{C}^{3 \times 3}$.

Lemma 3.1. Let $\mathcal{M}(\xi)$ be defined in (2.11) and let $h>M$.
(1) There exists a constant $K$ independent of $\omega$ such that $\operatorname{Re}(-\mathrm{i} M)(\xi)>0$ for all $|\xi|>$ $K \omega$, where

$$
K=\frac{\lambda+2 \mu}{\mu \sqrt{\lambda+\mu}}>\frac{1}{\sqrt{\mu}}
$$

(2) The DtN map $\mathcal{T}$ is a bounded operator from $H^{1 / 2}\left(\Gamma_{h}\right)^{3}$ to $H^{-1 / 2}\left(\Gamma_{h}\right)^{3}$.
(3) For $|\xi|<K \omega$ there holds

$$
\begin{equation*}
\|\mathcal{M}(\xi)\| \leq C_{K} \omega \tag{3.1}
\end{equation*}
$$

where

$$
C_{K}=2(\lambda+4 \mu) K+\left(\mu(\lambda+2 \mu) K^{2}+2(\lambda+2 \mu) / \mu\right) \sqrt{\frac{\lambda+\mu}{\mu(\lambda+2 \mu)}}
$$

is a constant independent of $\omega, \xi$ and the norm

$$
\|\mathcal{M}(\xi)\|=\max _{1 \leq i, j \leq 3}\left|M_{i j}(\xi)\right|
$$

Here $K$ is the constant specified in item (1) and $M_{i j}(\xi), 1 \leq i, j \leq 3$ denote the entries of $\mathcal{M}(\xi)$.

Remark 3.2. In comparison with properties of the matrix $M$ in two dimensions, we provide explicit constants $K$ and $C_{K}$ in terms of the Lame coefficients.

Proof. Item (2) has been proved in [11, Lemma 3.2]. Thus we only need to prove items (1) and (3).
(1) Since $|\xi|>K \omega>k_{s}$, we have $\beta=\mathrm{i}|\beta|$ and $\gamma=\mathrm{i}|\gamma|$, which implies

$$
i \mathcal{M}(\xi)=\frac{-1}{|\xi|^{2}-|\beta||\gamma|}\left[\begin{array}{ccc}
a_{1}(\xi) & b(\xi) & -i c(\xi) \xi_{1}  \tag{3.2}\\
b(\xi) & a_{2}(\xi) & -i c(\xi) \xi_{2} \\
i c(\xi) \xi_{1} & i c(\xi) \xi_{2} & a_{3}(\xi)
\end{array}\right]:=\frac{-1}{|\xi|^{2}-|\beta||\gamma|} \mathcal{M}^{\prime}(\xi)
$$

with

$$
\begin{aligned}
a_{1}(\xi) & =\mu\left[\xi_{2}^{2}(|\gamma|-|\beta|)+k_{s}^{2}|\beta|\right], \quad a_{2}(\xi)=\mu\left[\xi_{1}^{2}(|\gamma|-|\beta|)+k_{s}^{2}|\beta|\right], \quad a_{3}(\xi)=\omega^{2}|\gamma|, \\
b(\xi) & =-\mu \xi_{1} \xi_{2}(|\gamma|-|\beta|), \quad c(\xi)=2 \mu|\xi|^{2}-\omega^{2}+2 \mu|\beta||\gamma| .
\end{aligned}
$$

It is obvious that $a_{i}(\xi), b(\xi), c(\xi) \in \mathbb{R}$. Then from (3.2) we obtain

$$
\Re(-\mathrm{i} \mathcal{M}(\xi))=\frac{1}{|\rho|} \mathcal{M}^{\prime}(\xi)
$$

with $\rho(\xi)=|\xi|^{2}+\beta \gamma$. Hence it remains to prove $\mathcal{M}^{\prime}(\xi)$ is positive-definite when $|\xi|>K \omega$. To this end, we should verify

$$
\text { i) } \left.a_{1}(\xi)>0, \quad \text { ii) }\left|\begin{array}{cc}
a_{1}(\xi) & b(\xi) \\
b(\xi) & a_{2}(\xi)
\end{array}\right|>0, \quad \text { iii }\right) \operatorname{det} \mathcal{M}^{\prime}(\xi)>0
$$

i) By direct calculation, it is obvious that

$$
\begin{align*}
a_{1}(\xi) & =\mu\left[(|\gamma|-|\beta|) \xi_{2}^{2}+k_{s}^{2}|\beta|\right] \\
& =\mu \frac{\xi_{2}^{2}\left(|\gamma|^{2}-|\beta|^{2}\right)+k_{s}^{2}\left(|\beta|^{2}+|\beta||\gamma|\right)}{|\gamma|+|\beta|} \\
& =\mu \frac{\xi_{1} k_{s}^{2}+\xi_{2} k_{p}^{2}+k_{s}^{2}|\beta||\gamma|-k_{p}^{2} k_{s}^{2}}{|\gamma|+|\beta|} \\
& \geq \mu \frac{\left(|\xi|^{2}-k_{s}^{2}\right) k_{p}^{2}+k_{s}^{2}|\beta||\gamma|}{|\gamma|+|\beta|}>0 . \tag{3.3}
\end{align*}
$$

Here the condition $|\xi|>K \omega>k_{s}$ is used in the last step.
ii) Denote $g(\xi)=(|\gamma|-|\beta|)|\xi|^{2}+k_{s}^{2}|\beta|$. Similar as (3.3) we have $g(\xi)>0$. Then one arrives at

$$
\begin{aligned}
\left|\begin{array}{cc}
a_{1}(\xi) & b(\xi) \\
b(\xi) & a_{2}(\xi)
\end{array}\right| & =a_{1} a_{2}-b^{2} \\
& =\mu^{2}\left[(|\gamma|-|\beta|)\left|\xi_{1}\right|^{2}+k_{s}^{2}|\beta|\right]\left[(|\gamma|-|\beta|)\left|\xi_{2}\right|^{2}+k_{s}^{2}|\beta|\right]-\mu^{2} \xi_{1}^{2} \xi_{2}^{2}(|\gamma|-|\beta|)^{2} \\
& =\mu^{2} k_{s}^{2}|\beta| g(\xi)>0 .
\end{aligned}
$$

iii) Denote $h(\xi)=2 \xi_{1} \xi_{2} b(\xi)-a_{1}(\xi) \xi_{2}^{2}-a_{2}(\xi) \xi_{1}^{2}$, then it can be verified that

$$
\begin{align*}
\operatorname{det}\left(\mathcal{M}^{\prime}(\xi)\right) & =a_{3}(\xi)\left|\begin{array}{cc}
a_{1}(\xi) & b(\xi) \\
b(\xi) & a_{2}(\xi)
\end{array}\right|+\left(-a_{1}(\xi) c(\xi)^{2} \xi_{2}^{2}+2 b(\xi) c(\xi)^{2} \xi_{1} \xi_{2}-a_{2}(\xi) c(\xi)^{2} \xi_{1}^{2}\right) \\
& =\mu^{2} k_{s}^{2}|\beta \| \gamma| g(\xi) \omega^{2}+c(\xi)^{2} h(\xi) \tag{3.4}
\end{align*}
$$

Direct calculation implies

$$
\begin{align*}
h(\xi)= & -2 \mu \xi_{1}^{2} \xi_{2}^{2}(|\gamma|-|\beta|)-\mu \xi_{1}^{2}\left[\xi_{1}^{2}(|\gamma|-|\beta|)+k_{s}^{2}|\beta|\right] \\
& -\mu \xi_{2}^{2}\left[\xi_{2}^{2}(|\gamma|-|\beta|)+k_{s}^{2}|\beta|\right] \\
= & -\mu(|\gamma|-|\beta|)|\xi|^{4}-\mu k_{s}^{2}|\beta||\xi|^{2}=-\mu|\xi|^{2} g(\xi) . \tag{3.5}
\end{align*}
$$

Combining (3.4)-(3.5) gives

$$
\begin{aligned}
\operatorname{det}\left(\mathcal{M}^{\prime}(\xi)\right) & =\mu^{3} g(\xi)\left\{k_{s}^{4}|\beta||\gamma|-|\xi|^{2}\left[2|\gamma|(|\gamma|-|\beta|)+k_{s}^{2}\right]\right\} \\
& =\mu^{3} g(\xi)\left\{k_{s}^{4}|\beta||\gamma|-\left[\frac{2|\gamma|\left(k_{p}^{2}-k_{s}^{2}\right)}{|\beta|+|\gamma|}+k_{s}^{2}\right]^{2}\right\} \\
& =\mu^{3} g(\xi) \frac{d(\xi)}{(|\gamma|+|\beta|)^{2}}
\end{aligned}
$$

with

$$
d(\xi)=k_{s}^{4}\left(|\gamma||\beta|-|\xi|^{2}\right)(|\gamma|+|\beta|)^{2}+4|\gamma|\left(k_{s}^{2}-k_{p}^{2}\right)\left(|\gamma| k_{p}^{2}+|\beta| k_{s}^{2}\right)|\xi|^{2} .
$$

Hence we only need to verify $d(\xi)>0$ for $|\xi|>K \omega$. Taking $|\xi|^{2}=K^{\prime} k_{s}^{2}$ implies

$$
\begin{aligned}
d(\xi) & =k_{s}^{8}\left[\left(\sqrt{\left(K^{\prime}-\alpha\right)\left(K^{\prime}-1\right)}-K^{\prime}\right)\left(\sqrt{K^{\prime}-1}+\sqrt{K^{\prime}-\alpha}\right)^{2}\right. \\
& \left.+4(1-\alpha) K^{\prime} \sqrt{K^{\prime}-1}\left(\sqrt{K^{\prime}-\alpha}+\alpha \sqrt{K^{\prime}-1}\right)\right] \\
& >k_{s}^{8}\left[-\left(\sqrt{K^{\prime}-\alpha}+\sqrt{K^{\prime}-1}\right)^{2}+4(1-\alpha) K^{\prime} \sqrt{K^{\prime}-1} \alpha\left(\sqrt{K^{\prime}-\alpha}+\sqrt{K^{\prime}-1}\right)\right]
\end{aligned}
$$

with $\alpha:=k_{p}^{2} / k_{s}^{2}=\mu /(\lambda+2 \mu)<1$. In order to show $d(\xi)>0$, we will verify

$$
2(1-\alpha) \alpha K^{\prime} \sqrt{K^{\prime}-1}>\sqrt{K^{\prime}-\alpha},
$$

i.e.

$$
\begin{equation*}
K^{\prime} \sqrt{\frac{K^{\prime}-1}{K^{\prime}-\alpha}}>\frac{1}{2(1-\alpha) \alpha} . \tag{3.6}
\end{equation*}
$$

To guarantee (3.6), let

$$
\sqrt{\frac{K^{\prime}-1}{K^{\prime}-\alpha}}>\frac{1}{2}, \quad C>\frac{1}{\alpha(1-\alpha)}
$$

i.e.

$$
K^{\prime}>\max \left\{\frac{4-\alpha}{3}, \frac{1}{\alpha(1-\alpha)}\right\}=\frac{1}{\alpha(1-\alpha)}=\frac{(\lambda+2 \mu)^{2}}{\mu(\lambda+\mu)} .
$$

Hence, supposing that

$$
|\xi|>\sqrt{\frac{K^{\prime}}{\mu}} \omega=\frac{\lambda+2 \mu}{\mu \sqrt{\lambda+\mu}} \omega
$$

guarantees $d(\xi)>0$, which implies $\operatorname{det} \mathcal{M}^{\prime}(\xi)>0$.
(3) For $\rho(\xi)=|\xi|^{2}+\beta \gamma$, direct calculation gives

$$
\begin{cases}k_{p}^{2} & \leq|\rho| \leq k_{p} k_{s}, \quad 0 \leq|\xi| \leq k_{p}  \tag{3.7}\\ k_{p}^{2} & \leq|\rho| \leq k_{s}^{2}, \quad k_{p} \leq|\xi| \leq k_{s} \\ c_{K} \omega^{2} & \leq|\rho| \leq k_{s}^{2}, \quad k_{s} \leq|\xi| \leq K \omega,\end{cases}
$$

with

$$
c_{K}=K^{2}-\sqrt{\left(K^{2}-1 / \mu\right)\left(K^{2}-1 /(\lambda+2 \mu)\right)}>1 /(\lambda+2 \mu) .
$$

Here to derive the inequality for $k_{s} \leq|\xi| \leq K \omega$ we have used the fact that the function

$$
\rho(\xi)=|\xi|^{2}-\sqrt{k_{p}^{2}-|\xi|^{2}} \sqrt{k_{s}^{2}-|\xi|^{2}}
$$

is decreasing with respect to $|\xi|$ for $|\xi| \geq k_{s}$. We also consider $\gamma-\beta$ which is

$$
\gamma-\beta=\sqrt{k_{s}^{2}-|\xi|^{2}}-\sqrt{k_{p}^{2}-|\xi|^{2}}=\left\{\begin{array}{cc}
|\gamma|-|\beta|, & 0<|\xi| \leq k_{p} \\
|\gamma|-i|\beta|, & k_{p}<|\xi| \leq k_{s} \\
i(|\gamma|-|\beta|), & |\xi|>k_{s}
\end{array}\right.
$$

Then we immediately obtain

$$
\begin{cases}|\gamma-\beta| \leq \sqrt{k_{s}^{2}-k_{p}^{2}}, & 0<|\xi| \leq k_{p} \text { or }|\xi|>k_{s}  \tag{3.8}\\ |\gamma-\beta|=\sqrt{|\gamma|^{2}+|\beta|^{2}}=\sqrt{k_{s}^{2}-k_{p}^{2}}, & k_{p}<|\xi| \leq k_{s}\end{cases}
$$

(3) To prove the third result, it suffices to verify the inequality $M_{i j} \leq C \omega$, for $i, j=1,2,3$ and $|\xi| \leq K \omega$. For $M_{33}$, by (3.7) we have

$$
\left|M_{33}\right|=\left|\frac{\gamma \omega^{2}}{\rho}\right| \leq\left\{\begin{array}{cc}
\omega^{2} k_{s} / k_{p}^{2}=\omega(\lambda+2 \mu) / \sqrt{\mu}, & 0 \leq|\xi| \leq k_{p}  \tag{3.9}\\
\omega^{2} \sqrt{k_{s}^{2}-k_{p}^{2}} / k_{p}^{2}=\omega \sqrt{(\lambda+\mu)(\lambda+2 \mu) / \mu}, & k_{p} \leq|\xi| \leq k_{s} \\
\omega \sqrt{K^{2} \omega^{2}-k_{s}^{2}} / c_{K} \omega^{2}=\omega \sqrt{K^{2}-1 / \mu} / c_{K}, & k_{s} \leq|\xi| \leq K \omega .
\end{array}\right.
$$

Similarly, $M_{23}$ and $M_{32}$ can be estimated using (3.7) by

$$
\begin{align*}
\left|M_{23}\right| & =\left|M_{32}\right| \\
& =\left|\frac{2 \mu \rho \xi_{2}-\omega^{2} \xi_{2}}{\rho}\right| \leq\left\{\begin{array}{cc}
2 \mu k_{p}+\omega^{2} / k_{P}=\omega(2 \mu / \sqrt{\lambda+2 \mu}+\sqrt{\lambda+2 \mu}), & 0 \leq|\xi| \leq k_{p} \\
2 \mu k_{s}+\omega k_{s} / k_{p}^{2}=\omega(2 \sqrt{\mu}+(\lambda+2 \mu) / \sqrt{\mu}), & k_{p} \leq|\xi| \leq k_{s} \\
2 \mu K \omega+K \omega^{3} / c_{K} \omega^{2}=\omega\left(2 \mu K+K / c_{K}\right), & k_{s} \leq|\xi| \leq K \omega .
\end{array}\right. \tag{3.10}
\end{align*}
$$

It is obvious that $\left|M_{13}\right|=\left|M_{31}\right|$ can also be estimated by the right-hand side of (3.10). It remains to estimate $M_{11}, M_{22}, M_{12}$ and $M_{21}$. For convenience, denote

$$
\sqrt{k_{s}^{2}-k_{p}^{2}}=\omega \sqrt{\frac{\lambda+\mu}{\mu(\lambda+2 \mu)}}:=C_{\lambda, \mu} \omega .
$$

Combining (3.7)-(3.8) gives

$$
\begin{align*}
\left|M_{11}\right| & \leq \mu\left|\frac{(\gamma-\beta) \xi_{2}^{2}+k_{s}^{2} \beta}{\rho}\right| \\
& \leq\left\{\begin{array}{cc}
\mu\left(\omega C_{\lambda, \mu}+k_{s}^{2} / k_{p}\right)=\omega\left(\mu C_{\lambda, \mu}+\sqrt{\lambda+2 \mu}\right), & 0 \leq|\xi| \leq k_{p} \\
\mu\left(C_{\lambda, \mu} \omega k_{s}^{2} / k_{p}^{2}+\omega C_{\lambda, \mu} k_{s}^{2} / k_{p}^{2}\right)=2 \omega C_{\lambda, \mu}(\lambda+2 \mu) / \mu, & k_{p} \leq|\xi| \leq k_{s} \\
\omega\left(\mu C_{\lambda, \mu} K^{2} / c_{K}+\sqrt{K^{2}-1 /(\lambda+2 \mu)} / c_{K}\right), & k_{s} \leq|\xi| \leq K \omega .
\end{array}\right. \tag{3.11}
\end{align*}
$$

Obviously, $\left|M_{22}\right|$ can also be estimated by the right-hand side of (3.11). For $\left|M_{12}\right|$ and $\left|M_{21}\right|$, we combine (3.7)-(3.8) to obtain

$$
\begin{align*}
\left|M_{12}\right| & =\left|M_{21}\right| \\
& \leq \mu\left|\frac{\xi_{1} \xi_{2}(\gamma-\beta)}{\rho}\right| \leq\left\{\begin{array}{cc}
\omega \mu C_{\lambda, \mu}, & 0 \leq|\xi| \leq k_{p}, \\
\mu k_{s}^{2} \omega C_{\lambda, \mu} / k_{p}^{2}=\omega(\lambda+2 \mu) C_{\lambda, \mu}, & k_{p} \leq|\xi| \leq k_{s}, \\
\mu K^{2} C_{\lambda, \mu} \omega^{3} / c_{K} \omega^{2}=\omega\left(\mu K^{2} C_{\lambda, \mu} / c_{K},\right. & k_{s} \leq|\xi| \leq K \omega .
\end{array}\right. \tag{3.12}
\end{align*}
$$

Combining the above results (3.9)-(3.12), we have

$$
\|M\| \leq \begin{cases}C_{K, 1} \omega, & 0 \leq|\xi| \leq k_{p} \\ C_{K, 2} \omega, & k_{p} \leq|\xi| \leq k_{s}, \\ C_{K, 3} \omega, & k_{s} \leq|\xi| \leq K \omega\end{cases}
$$

with

$$
\begin{aligned}
& C_{K, 1}=\max \left\{\frac{\lambda+2 \mu}{\sqrt{\mu}}, \frac{2 \mu}{\sqrt{\lambda+2 \mu}}+\sqrt{\lambda+2 \mu}, \mu C_{\lambda, \mu}+\sqrt{\lambda+2 \mu}, \mu C_{\lambda, \mu}\right\}, \\
& C_{K, 2}=\max \left\{(\lambda+2 \mu) C_{\lambda, \mu}, 2 \sqrt{\mu}+\frac{\lambda+2 \mu}{\sqrt{\mu}}, \frac{2 C_{\lambda, \mu}(\lambda+2 \mu)}{\mu}\right\}, \\
& C_{K, 3}=\max \left\{\frac{\sqrt{K^{2}-\frac{1}{\mu}}}{c_{K}}, 2 \mu K+\frac{K}{c_{K}}, \frac{\mu K^{2} C_{\lambda, \mu}}{c_{K}}, \frac{\mu K^{2} C_{\lambda, \mu}}{c_{K}}+\frac{\sqrt{K^{2}-\frac{1}{\lambda+2 \mu}}}{c_{K}}\right\} .
\end{aligned}
$$

It can be verified that

$$
C_{K, 1} \leq 2 \frac{\lambda+2 \mu}{\sqrt{\mu}}, \quad C_{K, 2} \leq \frac{\lambda+2 \mu}{\sqrt{\mu}}+2 \frac{\lambda+2 \mu}{\mu} C_{\lambda, \mu}+2 \sqrt{\mu}
$$

and

$$
C_{K, 3} \leq \frac{K}{c_{K}}+\frac{\mu K^{2} C_{\lambda, \mu}}{c_{K}}+2 \mu K
$$

Recalling that $c_{K}>1 /(\lambda+2 \mu)$, we have

$$
\begin{aligned}
\max \left\{C_{K, 1}, C_{K, 2}, C_{K, 3}\right\} & \leq 2(\lambda+4 \mu) K+\left(\mu(\lambda+2 \mu) K^{2}+2(\lambda+2 \mu) / \mu\right) C_{\lambda, \mu} \\
& =2(\lambda+4 \mu) K+\left(\mu(\lambda+2 \mu) K^{2}+2(\lambda+2 \mu) / \mu\right) \sqrt{\frac{\lambda+\mu}{\mu(\lambda+2 \mu)}}
\end{aligned}
$$

The proof is completed.

Recall that there exists a constant $C_{0}=C_{0}(h, L, m, M)>0$ independent of $\omega$ such that

$$
\begin{equation*}
\|\nabla u\|_{L^{2}\left(S_{h}\right)^{3}}^{2} \geq 1 / C_{0}\|u\|_{V_{h}}^{2}, \quad\|u\|_{H^{1 / 2}\left(\Gamma_{h}\right)}^{2} \leq C_{0}\|u\|_{V_{h}}^{2}, \tag{3.13}
\end{equation*}
$$

for all $u \in V_{h}$. The well-posedness result for small frequencies is stated below.
Theorem 3.3. Let $K, C_{K}>0$ be given as in Lemma 3.1. Then there exists a small frequency $\omega_{0}>0$ such that the variational problem admits a unique solution in $V_{h}$ for all $\omega \in\left(0, \omega_{0}\right]$.

Proof. It is clear that $\|\nabla \times u\|_{L^{2}\left(S_{h}\right)^{3}}^{2} \leq\|\nabla u\|_{L^{2}\left(S_{h}\right)^{3}}^{2}$. Now it follows from the definition of $B$ and Lemma 3.1 that

$$
\begin{align*}
\Re B(u, u)= & 2 \mu\|\nabla u\|_{L^{2}\left(S_{h}\right)^{3}}^{2}+\lambda\|\nabla \cdot u\|_{L^{2}\left(S_{h}\right)^{3}}^{2}-\mu\|\nabla \times u\|_{L^{2}\left(S_{h}\right)^{3}}^{2} \\
& \quad-\omega^{2}\|u\|_{L^{2}\left(S_{h}\right)^{3}}^{2}-\Re \int_{\Gamma_{h}} \bar{u} \cdot \mathcal{T} u \mathrm{~d} s \\
= & 2 \mu\|\nabla u\|_{L^{2}\left(S_{h}\right)^{3}}^{2}+\lambda\|\nabla \cdot u\|_{L^{2}\left(S_{h}\right)^{3}}^{2}-\mu\|\nabla \times u\|_{L^{2}\left(S_{h}\right)^{3}}^{2}-\omega^{2}\|u\|_{L^{2}\left(S_{h}\right)^{3}}^{2}  \tag{3.14}\\
& \quad+\int_{|\xi| \leq K \omega} \operatorname{Re}(-\mathrm{i} \mathcal{M}(\xi)) \hat{u} \cdot \overline{\hat{u}} \mathrm{~d} \xi+\int_{|\xi|>K \omega} \operatorname{Re}(-\mathrm{i} \mathcal{M}(\xi)) \hat{u} \cdot \overline{\hat{u}} \mathrm{~d} \xi \\
\geq & \mu\|\nabla u\|_{L^{2}\left(S_{h}\right)^{3}}^{2}-\omega^{2}\|u\|_{L^{2}\left(S_{h}\right)^{3}}^{2}+\int_{|\xi| \leq K \omega} \operatorname{Re}(-\mathrm{i} \mathcal{M}(\xi)) \hat{u} \cdot \overline{\hat{u}} \mathrm{~d} \xi \\
\geq & \mu\|\nabla u\|_{L^{2}\left(S_{h}\right)^{3}}^{2}-\omega^{2}\|u\|_{L^{2}\left(S_{h}\right)^{3}}^{2}-C_{K} C_{0} \omega\|u\|_{V_{h}}^{2}, \tag{3.15}
\end{align*}
$$

where the constant $C_{K}>0$ is given by Lemma 3.1 (3) and the constant $C_{0}$ is specified in (3.13). By Lemma 3.4 in [6] we have the following Poincare's inequality

$$
\begin{equation*}
\|u\|_{L^{2}\left(S_{h}\right)^{3}}^{2} \leq(h-m)\left\|\partial_{3} u\right\|_{L^{2}\left(S_{h}\right)^{3}}^{2} \leq(h-m)\|\nabla u\|_{L^{2}\left(S_{h}\right)^{3}}^{2}, \quad u \in V_{h} . \tag{3.16}
\end{equation*}
$$

Using (3.14)-(3.16), we obtain the estimate

$$
\begin{aligned}
\Re B(u, u) & \geq\left(\mu / C_{0}-\omega C_{0} C_{K}-\omega^{2}(h-m)\right)\|u\|_{V_{h}}^{2} \\
& \geq\left(\mu / C_{0}-\omega_{0} C_{0} C_{K}-\omega_{0}^{2}(h-m)\right)\|u\|_{V_{h}}^{2}
\end{aligned}
$$

for all $u \in V_{h}$ and $\omega \in\left(0, \omega_{0}\right.$ ]. Choose $\omega_{0}$ sufficiently small such that

$$
\mu / C_{0}-\omega_{0} C_{0} C_{K}-\omega_{0}^{2}(h-m)>0 .
$$

The proof is completed by applying the Lax-Milgram theorem.

## 4. An a priori bound for smooth rough surfaces

In this section, we establish an a priori bound for a smooth rough surface at any frequency. The attractive feature is that all constants in the a priori estimates are bounded by explicit functions of $\omega, h, m, M$ and $L$.

Lemma 4.1. Let $u \in V_{h}$ be a variational solution to (2.13) with $g \in V_{h}$. We have

$$
\|\nabla \cdot u\|_{L^{2}(\Gamma)}^{2},\|\nabla \times u\|_{L^{2}(\Gamma)^{3}}^{2} \leq C_{1}\|g\|_{L^{2}\left(S_{h}\right)^{3}}\left\|\partial_{3} u\right\|_{L^{2}\left(S_{h}\right)^{3}}
$$

where $C_{1}=4 \mu^{-1}\left(1+L^{2}\right)^{1 / 2}(\omega / \sqrt{\mu}(h-m)+1)$.
Proof. By [11, Lemma 4.1](see also [9, Lemma 5] for the periodic version) we have the following Rellich identity

$$
\begin{align*}
& 2 \Re \int_{S_{h}}\left(\mu \Delta u+(\lambda+\mu) \nabla \nabla \cdot u+\omega^{2} u\right) \cdot \partial_{3} \bar{u} \mathrm{~d} x \\
& =\left(-\int_{\Gamma}+\int_{\Gamma_{h}}\right)\left\{2 \Re\left(T u \cdot \partial_{3} \bar{u}\right)-\nu_{3} \mathcal{E}(u, \bar{u})+\omega^{2}|u|^{2}\right\} \mathrm{d} s \tag{4.1}
\end{align*}
$$

and

$$
\begin{equation*}
T u \cdot \partial_{3} \bar{u}=\nu_{3} \mathcal{E}(u, \bar{u})=\mu\left|\partial_{\nu} u\right|^{2} \nu_{3}+\nu_{3}(\lambda+\mu)|\nabla \cdot u|^{2} . \tag{4.2}
\end{equation*}
$$

From [11, Lemma 4.2 (ii)] we also have the following two identities

$$
\begin{align*}
& \int_{\Gamma_{h}}\left\{2 \Re\left(T u \cdot \partial_{3} \bar{u}\right)-\mathcal{E}(u, \bar{u})+\omega^{2}|u|^{2}\right\} \mathrm{d} s \\
&= 2 \omega^{2} \int_{|\xi|<k_{\mathrm{p}}} \beta^{2}(\xi)\left|A_{\mathrm{p}}(\xi)\right|^{2} \mathrm{~d} \xi+2 \mu \int_{|\xi|<k_{\mathrm{s}}} \gamma^{2}(\xi)\left|\mathbf{A}_{\mathrm{s}}(\xi)\right|^{2} \mathrm{~d} \xi  \tag{4.3}\\
&= 2 \omega^{2}\left\{\int_{|\xi|<k_{\mathrm{p}}} \beta^{2}(\xi)\left|A_{\mathrm{p}}(\xi)\right|^{2} \mathrm{~d} \xi+\int_{|\xi|<k_{\mathrm{s}}} \gamma^{2}(\xi)\left|\tilde{\mathbf{A}}_{\mathrm{s}}(\xi)\right|^{2} \mathrm{~d} \xi\right\}, \\
& \Im \int_{\Gamma_{h}} T u \cdot \bar{u} \mathrm{~d} s=\int_{|\xi|<k_{\mathrm{p}}} \omega^{2} \beta(\xi)\left|A_{\mathrm{p}}(\xi)\right|^{2} \mathrm{~d} \xi+\int_{|\xi|<k_{\mathrm{s}}} \mu \gamma(\xi)\left|\boldsymbol{A}_{\mathrm{s}}(\xi)\right|^{2} \mathrm{~d} \xi \\
&= \omega^{2}\left\{\int_{|\xi|<k_{\mathrm{p}}} \beta(\xi)\left|A_{\mathrm{p}}(\xi)\right|^{2} \mathrm{~d} \xi+\int_{|\xi|<k_{\mathrm{s}}} \gamma(\xi)\left|\tilde{\boldsymbol{A}}_{\mathrm{s}}(\xi)\right|^{2} \mathrm{~d} \xi\right\} . \tag{4.4}
\end{align*}
$$

Here we have used the relation $\left|\boldsymbol{A}_{\mathrm{s}}(\xi)\right|^{2}=k_{\mathrm{s}}^{2}\left|\tilde{\boldsymbol{A}}_{\mathrm{s}}(\xi)\right|^{2}$. Note that the identity (4.3) corrects a mistake made in [11, Formula (4.1)]. Hence, combing (4.1) and (4.2) gives

$$
\begin{align*}
& -\int_{\Gamma} \mu\left|\partial_{\nu} u\right|^{2} \nu_{3}+\nu_{3}(\lambda+\mu)|\nabla \cdot u|^{2} \mathrm{~d} s \\
= & \int_{\Gamma_{h}} 2 \Re\left(T u \cdot \partial_{3} \bar{u}\right)-\nu_{3} \mathcal{E}(u, \bar{u})+\omega^{2}|u|^{2} \mathrm{~d} s-2 \Re \int_{S_{h}} g \partial_{3} \bar{u} \mathrm{~d} x . \tag{4.5}
\end{align*}
$$

Using (4.3) and (4.4) and taking the imaginary part of (2.13), we get

$$
\begin{align*}
\int_{\Gamma_{h}}\left\{2 \Re\left(T u \cdot \partial_{3} \bar{u}\right)-\mathcal{E}(u, \bar{u})+\omega^{2}|u|^{2}\right\} \mathrm{d} s & \leq 2 k_{\mathrm{s}} \Im \int_{\Gamma_{h}} T u \cdot \bar{u} \mathrm{~d} s \\
& \leq 2 k_{\mathrm{s}} \Im \int_{S_{h}} g \cdot \bar{u} \mathrm{~d} s . \tag{4.6}
\end{align*}
$$

Combing (4.5) and (4.6) then gives the estimates

$$
\begin{align*}
& -\int_{\Gamma} \mu\left|\partial_{\nu} u\right|^{2} \nu_{3}+\nu_{3}(\lambda+\mu)|\nabla \cdot u|^{2} \mathrm{~d} s \\
& \quad \leq 2 k_{\mathrm{s}} \Im \int_{S_{h}} g \cdot \bar{u} \mathrm{~d} x-2 \Re \int_{S_{h}} g \cdot \partial_{3} \bar{u} \mathrm{~d} x \\
& \quad \leq 2\|g\|_{L^{2}\left(S_{h}\right)^{3}}^{2}\left(\frac{\omega}{\sqrt{\mu}}\|u\|_{L^{2}\left(S_{h}\right)^{3}}^{2}+\left\|\partial_{3} u\right\|_{L^{2}\left(S_{h}\right)^{3}}^{2}\right) \\
& \quad \leq 2(\omega / \sqrt{\mu}(h-m)+1)\|g\|_{L^{2}\left(S_{h}\right)^{3}}^{2}\left\|\partial_{3} u\right\|_{L^{2}\left(S_{h}\right)^{3}}^{2} \tag{4.7}
\end{align*}
$$

where the last identity follows from (3.16). Since

$$
\begin{equation*}
\nu_{3}(x)=-\frac{1}{\sqrt{1+\left|\nabla_{x^{\prime}} f\right|^{2}}}<-\left(1+L^{2}\right)^{-1 / 2}<0 \quad \text { on } \Gamma \tag{4.8}
\end{equation*}
$$

from (4.7) we obtain that

$$
\begin{align*}
& \|\nabla \cdot u\|_{L^{2}(\Gamma)}^{2}+\left\|\partial_{\nu} u\right\|_{L^{2}(\Gamma)^{3}}^{2} \\
& \leq 2 \mu^{-1}\left(1+L^{2}\right)^{1 / 2}(\omega / \sqrt{\mu}(h-m)+1)\|g\|_{L^{2}\left(S_{h}\right)^{3}}^{2}\left\|\partial_{3} u\right\|_{L^{2}\left(S_{h}\right)^{3}}^{2} . \tag{4.9}
\end{align*}
$$

Finally, using $u=0$ on $\Gamma$ and the identities in [9, (4.17)] we have that

$$
\nu_{3}|\nabla \times u|^{2}=\nu_{3}\left(|\nabla u|^{2}-|\nabla \cdot u|^{2}\right)=\nu_{3}\left(\left|\partial_{\nu} u\right|^{2}-|\nabla \cdot u|^{2}\right) \quad \text { on } \Gamma .
$$

Thus, $\|\nabla \times u\|_{L^{2}(\Gamma)^{3}}$ can also be bounded by the right-hand side of (4.9) multiplied by two.

We next need to derive estimates for the $L^{2}$ norms of the scalar function $\nabla \cdot u$ and the vector function $\nabla \times u$ on the artificial boundary $\Gamma_{H}$ and the strip $S_{H}$ where $H=h+1$. The derivation is based on the a priori bound for the Helmholtz equation in [6]. By (2.4), we define

$$
\varphi:=-\frac{\mathrm{i}}{k_{p}^{2}} \nabla \cdot u, \quad \psi:=\frac{\mathrm{i}}{k_{s}^{2}} \nabla \times u, \quad \text { in } \quad x_{3}>h .
$$

Since both $\varphi$ and $\psi$ satisfy the Helmholtz equation (2.5) and the UASR (2.6), one has the following Dirichlet-to-Neumann map on the artificial boundary $\Gamma_{H}$ :

$$
\begin{equation*}
\widetilde{\mathcal{T}} w=\mathcal{F}^{-1}(\mathrm{i} \eta \mathcal{F} w), \quad w \in H^{1 / 2}\left(\Gamma_{H}\right), \tag{4.10}
\end{equation*}
$$

where $w=\varphi, \psi$, and $\eta=\beta, \gamma$, respectively. Moreover, $\widetilde{\mathcal{T}}$ is a bounded linear map of $H^{1 / 2}\left(\Gamma_{H}\right)$ to $H^{-1 / 2}\left(\Gamma_{H}\right)$ by [6, Lemma 2.4]. From Lemma 4.1 we can estimate the $L^{2}$ norm of the trace $w$ on $\Gamma$ as

$$
\begin{equation*}
\|w\|_{L^{2}(\Gamma)^{3}}^{2} \leq C_{1}(\omega, h, L)\|g\|_{L^{2}\left(S_{H}\right)^{3}}\left\|\partial_{3} u\right\|_{L^{2}\left(S_{H}\right)^{3}} . \tag{4.11}
\end{equation*}
$$

The following lemma provides estimates for $w$ on $S_{H}$ and the trace of $w$ on $\Gamma_{H}$.
Lemma 4.2. Assume that $w$ satisfies the Helmholtz equation

$$
\begin{equation*}
\Delta w+k^{2} w=g_{0} \quad \text { in } S_{H}, \quad \tilde{\mathcal{T}} w=\mathcal{F}^{-1}\left(\mathrm{i} \sqrt{k^{2}-\xi^{2}} \mathcal{F} w\right) \quad \text { on } \Gamma_{H} \tag{4.12}
\end{equation*}
$$

where $g_{0} \in L^{2}\left(S_{H}\right)$. Then there holds the estimate

$$
\begin{equation*}
\|w\|_{L^{2}\left(\Gamma_{H}\right)^{3}} \leq\|w\|_{L^{2}\left(S_{H}\right)^{3}} \leq \widetilde{C}_{2}(L, k, h)\|w\|_{L^{2}(\Gamma)^{3}}+\widetilde{C}_{3}(k, h)\left\|g_{0}\right\|_{L^{2}\left(S_{H}\right)^{3}} \tag{4.13}
\end{equation*}
$$

with

$$
\widetilde{C}_{2}(L, k, h)=C\left(1+L^{2}\right)^{1 / 4} \sqrt{H-m}(1+k(H-m))
$$

and

$$
\widetilde{C}_{3}(k, h)=C(H-m)(1+k(H-m))^{2} / k .
$$

Proof. Consider the boundary value problem of finding $v \in H^{1}\left(S_{H}\right)$ such that

$$
\begin{equation*}
\left(\triangle+k^{2}\right) v=\bar{w} \quad \text { in } \quad S_{H}, \quad v=0 \quad \text { on } \quad \Gamma, \quad \partial_{3} v=\widetilde{T} v \quad \text { on } \quad \Gamma_{H} . \tag{4.14}
\end{equation*}
$$

By [6, Lemma 4.6] the boundary value problem (4.14) is well-posed with the following estimate

$$
\begin{equation*}
\|\nabla v\|_{L^{2}\left(S_{H}\right)}+k\|v\|_{L^{2}\left(S_{H}\right)} \leq C(1+k(H-m))^{2}(H-m)\|w\|_{L^{2}\left(S_{H}\right)} \tag{4.15}
\end{equation*}
$$

We first prove that $\left\|\partial_{\nu} v\right\|_{L^{2}(\Gamma)^{3}}^{2} \leq C\|w\|_{L^{2}\left(S_{H}\right)^{3}}^{2}$ for some constant $C>0$ depending explicitly on $\omega, H$ and the Lipschitz constant $L$ of $\Gamma$. The Rellich identity for the Helmholtz equation gives:

$$
\begin{align*}
& 2 \Re \int_{S_{H}} \partial_{3} \bar{v}\left(\Delta v+k^{2} v\right) \mathrm{d} x \\
= & \left(\int_{\Gamma}+\int_{\Gamma_{H}}\right)\left\{2 \Re\left(\partial_{\nu} v \partial_{3} \bar{v}\right)-\nu_{3}|\nabla v|^{2}+\nu_{3} k^{2}|v|^{2}\right\} \mathrm{d} s, \tag{4.16}
\end{align*}
$$

which can be proved in the same way as (4.1). From the proof of [6, Lemma 4.6] it holds that

$$
\begin{align*}
& \int_{\Gamma_{H}}\left\{2 \Re\left(\partial_{\nu} v \partial_{3} \bar{v}-\nu_{3}|\nabla v|^{2}+\nu_{3} k^{2}|v|^{2}\right)\right\} \mathrm{d} s \leq 2 k \Im \int_{\Gamma_{H}} \bar{v} \widetilde{T} v \mathrm{~d} s \\
& \leq 2 k \Im \int_{S_{H}} \bar{v} \bar{w} \mathrm{~d} x . \tag{4.17}
\end{align*}
$$

Moreover, using the identities in (4.17) of 9 on $\Gamma$ and the bound for $\nu_{3}$ in (4.8) one has

$$
\begin{align*}
& -\int_{\Gamma_{H}}\left\{2 \Re\left(\partial_{\nu} v \partial_{3} \bar{v}-\nu_{3}|\nabla v|^{2}+\nu_{3} k^{2}|v|^{2}\right)\right\} \mathrm{d} s=-\int_{\Gamma} \nu_{3}\left|\partial_{\nu} v\right|^{2} \mathrm{~d} s \\
& \geq\left(1+L^{2}\right)^{-1 / 2}\left\|\partial_{\nu} v\right\|_{L^{2}(\Gamma)}^{2} . \tag{4.18}
\end{align*}
$$

Plugging (4.17) and (4.18) into (4.16) and using (4.15) yield the estimate

$$
\begin{align*}
\left\|\partial_{\nu} v\right\|_{L^{2}(\Gamma)}^{2} & \leq\left(1+L^{2}\right)^{1 / 2}\left\{-2 \Re \int_{S_{H}} \bar{w} \partial_{3} v \mathrm{~d} x+2 k \Im \int_{S_{H}} \bar{w} \bar{v} \mathrm{~d} x\right\} \\
& \leq 2\left(1+L^{2}\right)^{1 / 2}\|w\|_{L^{2}\left(S_{H}\right)}\left(k\|v\|_{L^{2}\left(S_{H}\right)}+\|\nabla v\|_{L^{2}\left(S_{H}\right)}\right) \\
& \leq C\left(1+L^{2}\right)^{1 / 2}(H-m)(1+k(H-m))^{2}\|w\|_{L^{2}\left(S_{H}\right)}^{2} \tag{4.19}
\end{align*}
$$

where the constants $C$ is independent of $w$.
Now we prove the second inequality in (4.13). Following the approach of [8, Lemma 7], we obtain that

$$
\begin{aligned}
\int_{S_{H}}\{w \Delta v-v \Delta w\} \mathrm{d} x & =\int_{\Gamma_{H}}\left\{w \partial_{\nu} v-v \partial_{\nu} w\right\} \mathrm{d} s+\int_{\Gamma} w \partial_{\nu} \mathrm{d} s \\
& =\int_{\Gamma_{H}}\{w \widetilde{T} v-v \widetilde{T} w\} \mathrm{d} s+\int_{\Gamma} w \partial_{\nu} v \mathrm{~d} s \\
& =\int_{\Gamma} w \partial_{\nu} v \mathrm{~d} s .
\end{aligned}
$$

Note that $v=0$ on $\Gamma$, and the Dirichlet-to-Neumann operator $\widetilde{T}$ defined in (4.10) is symmetric (see Lemma 3.2 in [6]). Thus,

$$
\begin{aligned}
\int_{S_{H}}|w|^{2} \mathrm{~d} x & =\int_{S_{H}} w\left(\Delta v+k^{2} v\right) \mathrm{d} x \\
& =\int_{S_{H}} v\left(\nabla w+k^{2} w\right) \mathrm{d} x+\int_{\Gamma} w \partial_{\nu} v \mathrm{~d} s \\
& =\int_{S_{H}} v g \mathrm{~d} x+\int_{\Gamma} w \partial_{\nu} v \mathrm{~d} s
\end{aligned}
$$

Noting (4.15) and (4.19) one has

$$
\begin{aligned}
\|w\|_{L^{2}\left(S_{H}\right)}^{2} \leq & \|v\|_{L^{2}\left(S_{H}\right)}^{2}\|g\|_{L^{2}\left(S_{H}\right)}^{2}+\|w\|_{L^{2}(\Gamma)}^{2}\left\|\partial_{\nu} v\right\|_{L^{2}(\Gamma)}^{2} \\
\leq & C \sqrt{H-m}\left(1+L^{2}\right)^{1 / 4}(1+k(H-m))\|w\|_{L^{2}\left(S_{H}\right)}\|w\|_{L^{2}(\Gamma)} \\
& +C(H-m) \frac{(1+k(H-m))^{2}}{k}\|w\|_{L^{2}\left(S_{H}\right)}\left\|g_{0}\right\|_{L^{2}\left(S_{H}\right)} .
\end{aligned}
$$

Then the following inequality is proved

$$
\begin{equation*}
\|w\|_{L^{2}\left(S_{H}\right)} \leq \widetilde{C}_{2}(L, k, h)\|w\|_{L^{2}(\Gamma)}+\widetilde{C}_{3}(k, h)\left\|g_{0}\right\|_{L^{2}\left(S_{H}\right)} . \tag{4.20}
\end{equation*}
$$

To estimate the first inequality in (4.13) we use

$$
\int_{\Gamma_{H}}|w|^{2} \mathrm{~d} s \leq \int_{\Gamma_{c}}|w|^{2} \mathrm{~d} s, \quad \text { for all } c \in(h, H],
$$

which follows from the proof of [6, Lemma 2.2]. Then we have

$$
\begin{equation*}
(H-h) \int_{\Gamma_{H}}|w|^{2} \mathrm{~d} x \leq \int_{S_{H} \backslash S_{h}}|w|^{2} \mathrm{~d} s \leq \int_{S_{H}}|w|^{2} \mathrm{~d} s \tag{4.21}
\end{equation*}
$$

The estimate (4.13) is proved by combing (4.20) and (4.21).

Next we prove the estimates of the $L^{2}$ norms of $\nabla \cdot u$ and $\nabla \times u$ on $S_{H}$ and $\Gamma_{H}$. Using Lemma 4.2 for $v=\varphi$ and $\psi$ with $g_{0}=-\left(\mathrm{i} / \omega^{2}\right) \nabla \cdot g$ and $\left(\mathrm{i} / \omega^{2}\right) \nabla \times g$ in (4.12), respectively, and (4.11), we obtain the estimate

$$
\begin{align*}
\|\nabla \cdot u\|_{L^{2}\left(S_{H}\right)}^{2} & +\|\nabla \times u\|_{L^{2}\left(S_{H}\right)^{3}}^{2} \\
& \leq C_{2}(\omega, h, L)^{2} C_{1}(\omega, h, L)\|g\|_{V_{h}}\left\|\partial_{3} u\right\|_{L^{2}\left(S_{H}\right)^{3}}+C_{3}(\omega, h)^{2}\|g\|_{V_{h}}^{2}, \tag{4.22}
\end{align*}
$$

where

$$
C_{2}(\omega, h, L)=C\left(1+L^{2}\right)^{1 / 4} \sqrt{H-m}(1+\omega(H-m))
$$

and

$$
C_{3}(\omega, h)=C(H-m)(1+\omega(H-m))^{2} / \omega .
$$

In a similar way, from the estimates (4.13) and (4.11) we have the bound

$$
\begin{align*}
\|\nabla \cdot u\|_{L^{2}\left(\Gamma_{H}\right)}^{2} & +\|\nabla \times u\|_{L^{2}\left(\Gamma_{H}\right)^{3}}^{2} \\
& \leq C_{2}(\omega, h, L)^{2} C_{1}(\omega, h, L)\|g\|_{V_{h}}\left\|\partial_{3} u\right\|_{L^{2}\left(S_{H}\right)^{3}}+C_{3}(\omega, h)^{2}\|g\|_{V_{h}}^{2} . \tag{4.23}
\end{align*}
$$

The following theorem provides the a priori bound for the solution to Variational Problem $I$ dependent on the frequency and geometry of the rough surface.

Theorem 4.3. Assume that $\Gamma$ is given by the graph of a Lipschitz function $f$ satisfying (2.1), and that $u \in V_{h}$ is a solution to the variational problem (2.13). Then there exists a constant $C$ independent of $\omega, h$ and the Lipschitz constant $L$ of $f$ such that the following a priori bound holds

$$
\|u\|_{V_{h}} \leq(h-m+2)\left(C_{4}(\omega, h)+C_{5}(\omega, h)+C_{6}(\omega, h, L)\right)\|g\|_{V_{h}},
$$

where

$$
\begin{aligned}
& C_{4}(\omega, h)=C(h+1-m) \omega, \quad C_{5}=C \sqrt{1+\omega^{-1}} C_{3}(\omega, h), \\
& C_{6}=C\left(\omega^{-1}+1\right) C_{1}(\omega, h, L) C_{2}(\omega, h, L)^{2} .
\end{aligned}
$$

Proof. We first assume that $f$ is smooth. Multiplying both sides of the Navier equation by $\left(x_{3}-m\right) \partial_{3} \bar{u}$ and using integration by parts yields

$$
\begin{align*}
& 2 \Re \int_{S_{H}}\left(\triangle^{*}+\omega^{2}\right) u \cdot\left(x_{3}-m\right) \partial_{3} \bar{u} \mathrm{~d} x \\
& \quad=\int_{S_{H}}\left\{\mathcal{E}(u, \bar{u})-2 \Re\left\{\sum_{j=1}^{3} \mathcal{E}\left(u,\left(x_{3}-m\right) e_{j}\right) \partial_{3} \bar{u}_{j}\right\}-\omega^{2}|u|^{2}\right\} \mathrm{d} x \\
& \quad+\left(\int_{\Gamma_{H}}+\int_{\Gamma}\right)\left[-\nu_{3} \mathcal{E}(u, \bar{u})+2 \Re\left(T u \cdot \partial_{3} \bar{u}\right)+\nu_{3} \omega^{2}|u|^{2}\right]\left(x_{3}-m\right) \mathrm{d} s . \tag{4.24}
\end{align*}
$$

Letting $v=u$ in the variational formulation (2.13) gives

$$
\begin{align*}
& \int_{S_{H}}\left\{\mathcal{E}(u, \bar{u})-\omega^{2}|u|\right\} \mathrm{d} x-\Re \int_{|\xi|>K \omega} \mathcal{M}(\xi) \hat{u}(\xi, H) \cdot \overline{\hat{u}}(\xi, H) \mathrm{d} \xi \\
= & -\Re \int_{S_{H}} g \cdot \bar{u} \mathrm{~d} x+\Re \int_{|\xi| \leq K \omega} \mathcal{M}(\xi) \hat{u}(\xi, H) \cdot \overline{\hat{u}}(\xi, H) \mathrm{d} \xi . \tag{4.25}
\end{align*}
$$

Taking the real part and using Lemma 3.1 we have

$$
\begin{align*}
& \int_{S_{H}}\left\{\mathcal{E}(u, \bar{u})-\omega^{2}|u|\right\} \mathrm{d} x \\
\leq & -\Re \int_{S_{H}} g \cdot \bar{u} \mathrm{~d} x+\Re \int_{|\xi| \leq K \omega} \mathcal{M}(\xi) \hat{u}(\xi, H) \cdot \overline{\hat{u}}(\xi, H) \mathrm{d} \xi . \tag{4.26}
\end{align*}
$$

From (4.24) and using (4.26) and (4.2), we have

$$
\begin{align*}
& \int_{S_{H}} 2 \Re\left\{\sum_{j=1}^{3} \mathcal{E}\left(u,\left(x_{3}-m\right) e_{j}\right) \partial_{3} \bar{u}_{j}\right\} \mathrm{d} x \\
& \quad-\int_{\Gamma}\left(x_{3}-m\right)\left\{\mu\left|\partial_{\nu} u\right|^{2}+(\lambda+\mu)|\nabla \times u|^{2}\right\} \nu_{3} \mathrm{~d} s \\
& =\int_{S_{H}}\left\{\mathcal{E}(u, \bar{u})-\omega^{2}|u|^{2}\right\} \mathrm{d} x-2 \Re \int_{S_{H}}\left(\triangle^{*}+\omega^{2}\right) u \cdot\left(x_{3}-m\right) \partial_{3} \bar{u} \mathrm{~d} x \\
& \quad+(H-m) \int_{\Gamma_{H}}\left\{2 \Re\left(T u \cdot \partial_{3} \bar{u}\right)-\mathcal{E}(u, \bar{u})+\omega^{2}|u|^{2}\right\} \mathrm{d} s \\
& \leq \int_{S_{H}}\left\{-g \cdot u+2 \Re\left(g \cdot \partial_{3} \bar{u}\right)\left(x_{3}-m\right)\right\} \mathrm{d} x+\Re \int_{|\xi| \leq K \omega} \mathcal{M}(\xi) \hat{u}_{H}(\xi) \cdot \overline{\hat{u}}_{H}(\xi) \mathrm{d} \xi \\
& \quad+(H-m) \int_{\Gamma_{H}}\left\{2 \Re\left(T u \cdot \partial_{3} \bar{u}\right)-\mathcal{E}(u, \bar{u})+\omega^{2}|u|^{2}\right\} \mathrm{d} s . \tag{4.27}
\end{align*}
$$

As $\|\mathcal{M}(\xi)\| \leq C \omega$ for all $|\xi|<K \omega$, one has

$$
\begin{equation*}
\Re \int_{|\xi| \leq K \omega} \mathcal{M}(\xi) \hat{u}_{H}(\xi) \cdot \overline{\hat{u}}_{H}(\xi) \mathrm{d} \xi \leq C \omega \int_{|\xi| \leq K \omega}\left|\hat{u}_{H}(\xi)\right|^{2} \mathrm{~d} \xi \tag{4.28}
\end{equation*}
$$

Using (4.28) and (4.23) gives

$$
\begin{align*}
& \Re \int_{|\xi| \leq K \omega} \mathcal{M}(\xi) \hat{u}_{H}(\xi) \cdot \overline{\hat{u}}_{H}(\xi) \mathrm{d} \xi \leq C \omega k_{s}^{2} \int_{|\xi| \leq K \omega}\left\{\left|A_{\mathrm{p}}(\xi)\right|^{2}+\left|\boldsymbol{A}_{\mathrm{s}}(\xi)\right|^{2}\right\} \mathrm{d} \xi \\
\leq & C \omega k_{s}^{2}\left(\left\|A_{p}\right\|_{L^{2}\left(\mathbb{R}^{2}\right)}^{2}+\left\|\boldsymbol{A}_{s}\right\|_{L^{2}\left(\mathbb{R}^{2}\right)^{3}}^{2}\right) \\
\leq & C \omega^{-1}\left(C_{2}(\omega, h, L)^{2} C_{1}(\omega, h, L)\|g\|_{V_{h}}\left\|\partial_{3} u\right\|_{L^{2}\left(S_{H}\right)^{3}}+C_{3}(\omega, h)^{2}\|g\|_{V_{h}}^{2}\right) . \tag{4.29}
\end{align*}
$$

From the estimates (4.6) and (3.16) we have the following estimate for the last term in (4.27):

$$
\begin{align*}
\int_{\Gamma_{h}}\left\{2 \Re\left(T u \cdot \partial_{3} \bar{u}-\mathcal{E}(u, \bar{u})+\omega^{2}|u|^{2}\right)\right\} \mathrm{d} s & \leq 2 k_{\mathrm{s}} \Im \int_{S_{H}} g \cdot \bar{u} \mathrm{~d} x \\
& \leq 2 k_{\mathrm{s}}\|g\|_{V_{h}}\|u\|_{L^{2}\left(S_{H}\right)^{3}} \\
& \leq C(H-m) k_{s}\|g\|_{V_{h}}\left\|\partial_{3} u\right\|_{L^{2}\left(S_{H}\right)^{3}} . \tag{4.30}
\end{align*}
$$

Combing (4.29) -(4.30) and (4.27) and noting that the second term in (4.27) is nonnegative, we have

$$
\begin{align*}
& \int_{S_{H}} 2 \Re\left\{\sum_{j=1}^{3} \mathcal{E}\left(u,\left(x_{3}-m\right) e_{j}\right) \partial_{3} \bar{u}_{j}\right\} \mathrm{d} x \\
& \leq C(H-m) \omega\|g\|_{L^{2}\left(S_{H}\right)^{3}}\left\|\partial_{3} u\right\|_{L^{2}\left(S_{H}\right)^{3}}+C\left(\omega^{-1}+1\right) \\
& \quad \times\left(C_{2}(\omega, h, L)^{2} C_{1}(\omega, h, L)\|g\|_{V_{h}}\left\|\partial_{3} u\right\|_{L^{2}\left(S_{H}\right)^{3}}+C_{3}(\omega, h)^{2}\|g\|_{V_{h}}^{2}\right) . \tag{4.31}
\end{align*}
$$

Direct calculations yield

$$
\begin{aligned}
& \mathcal{E}\left(u,\left(x_{3}-m\right) e_{1}\right) \partial_{3} \bar{u}_{1}=2 \mu\left|\partial_{3} u_{1}\right|^{2}-\mu\left(\partial_{3} u_{1}-\partial_{1} u_{3}\right) \partial_{3} \bar{u}_{1}, \\
& \mathcal{E}\left(u,\left(x_{3}-m\right) e_{2}\right) \partial_{3} \bar{u}_{2}=2 \mu\left|\partial_{3} u_{2}\right|^{2}+\mu\left(\partial_{2} u_{3}-\partial_{3} u_{2}\right) \partial_{3} \overline{u_{2}}, \\
& \mathcal{E}\left(u,\left(x_{3}-m\right) e_{3}\right) \partial_{3} \bar{u}_{3}=(\lambda+2 \mu)\left|\partial_{3} u_{3}\right|^{2}+\lambda\left(\partial_{1} u_{1}+\partial_{2} u_{2}\right) \partial_{3} \bar{u}_{3} .
\end{aligned}
$$

Hence,

$$
\begin{align*}
& \int_{S_{H}} 2 \Re\left\{\sum_{j=1}^{3} \mathcal{E}\left(u,\left(x_{3}-m\right) e_{j}\right) \partial_{3} \bar{u}_{j}\right\} \mathrm{d} x \\
& =2(\lambda+2 \mu)\left\|\partial_{3} u_{3}\right\|_{L^{2}\left(S_{H}\right)^{3}}^{2}+4 \mu\left(\left\|\partial_{3} u_{1}\right\|_{L^{2}\left(S_{H}\right)^{3}}^{2}+\left\|\partial_{3} u_{2}\right\|_{L^{2}\left(S_{H}\right)^{3}}^{2}\right) \\
& \quad+2 \lambda\left(\Re \int_{S_{H}} \partial_{1} u_{1} \partial_{3} \bar{u}_{3} \mathrm{~d} x+\Re \int_{S_{H}} \partial_{2} u_{2} \partial_{3} \bar{u}_{3} \mathrm{~d} x\right) \\
& \quad-2 \mu \Re\left\{\int_{S_{H}}\left(\partial_{3} u_{1}-\partial_{1} u_{3}\right) \partial_{3} \bar{u}_{1}-\left(\partial_{2} u_{3}-\partial_{3} u_{2}\right) \partial_{3} \bar{u}_{2} \mathrm{~d} x\right\} \tag{4.32}
\end{align*}
$$

Choosing $C>0$ to be sufficiently large, we get

$$
\begin{align*}
& \int_{S_{H}} 2 \Re\left\{\sum_{j=1}^{3} \mathcal{E}\left(u,\left(x_{3}-m\right) e_{j}\right) \partial_{3} \bar{u}_{j}\right\} \mathrm{d} x+C\|\nabla \cdot u\|_{L^{2}\left(S_{H}\right)}^{2}+C\|\nabla \times u\|_{L^{2}\left(S_{H}\right)^{3}}^{2} \\
& =I_{1}+I_{2}+I_{3}+C\left\|\partial_{1} u_{2}-\partial_{2} u_{1}\right\|_{L^{2}\left(S_{H}\right)}^{2}, \tag{4.33}
\end{align*}
$$

where

$$
\begin{aligned}
& I_{1}:=[C+2(\lambda+2 \mu)]\left\|\partial_{3} u_{3}\right\|_{L^{2}\left(S_{H}\right)^{3}}^{2}+C\left\|\partial_{1} u_{1}\right\|_{L^{2}\left(S_{H}\right)^{3}}^{2}+C\left\|\partial_{2} u_{2}\right\|_{L^{2}\left(S_{H}\right)^{3}}^{2} \\
& +(C+2 \lambda)\left(\Re \int_{S_{H}} \partial_{1} u_{1} \partial_{3} \bar{u}_{3} \mathrm{~d} x+\Re \int_{S_{H}} \partial_{2} u_{2} \partial_{3} \bar{u}_{3} \mathrm{~d} x\right)+C \Re \int_{S_{H}} \partial_{1} u_{1} \partial_{2} \bar{u}_{2} \mathrm{~d} x, \\
& =\int_{S_{H}} A\left[\partial_{1} u_{1}, \partial_{2} u_{2}, \partial_{3} u_{3}\right]^{\top} \cdot\left[\partial_{1} \bar{u}_{1}, \partial_{2} \bar{u}_{2}, \partial_{3} \bar{u}_{3}\right]^{\top} \mathrm{d} x \text {, } \\
& A:=\left(\begin{array}{ccc}
C & C / 2 & \lambda+C / 2 \\
C / 2 & C & \lambda+C / 2 \\
\lambda+C / 2 & \lambda+C / 2 & C+2(\lambda+2 \mu)
\end{array}\right), \\
& I_{2}:=4 \mu\left\|\partial_{3} u_{1}\right\|_{L^{2}\left(S_{H}\right)^{3}}^{2}+C\left\|\partial_{3} u_{1}-\partial_{1} u_{3}\right\|_{L^{2}\left(S_{H}\right)}^{2}-2 \mu \Re \int_{S_{H}}\left(\partial_{3} u_{1}-\partial_{1} u_{3}\right) \partial_{3} \bar{u}_{1} \mathrm{~d} x, \\
& \left.I_{3}:=4 \mu\left\|\partial_{3} u_{2}\right\|_{L^{2}\left(S_{H}\right)^{3}}^{2}\right)+C\left\|\partial_{2} u_{3}-\partial_{3} u_{2}\right\|_{L^{2}\left(S_{H}\right)}^{2}+2 \mu \Re \int_{S_{H}}\left(\partial_{2} u_{3}-\partial_{3} u_{2}\right) \partial_{3} \bar{u}_{2} \mathrm{~d} x .
\end{aligned}
$$

Direct calculations show that $\operatorname{Det}(A) \sim C^{2} / 8$ as $C \rightarrow \infty$. Hence the matrix $A \in \mathbb{R}^{3 \times 3}$ must be strictly positive for sufficiently large $C>0$. This gives

$$
\begin{equation*}
I_{1} \geq C_{0}\left(\left\|\partial_{1} u_{1}\right\|_{L^{2}\left(S_{H}\right)}^{2}+\left\|\partial_{2} u_{2}\right\|_{L^{2}\left(S_{H}\right)}^{2}+\left\|\partial_{3} u_{3}\right\|_{L^{2}\left(S_{H}\right)}^{2}\right) \tag{4.34}
\end{equation*}
$$

where the constant $C_{0}>0$ only depends on $\lambda$ and $\mu$. By arguing in the same manner one has for $C>\mu^{2} / 4$ that

$$
\begin{align*}
I_{2} & \geq C_{0}\left(\left\|\partial_{3} u_{1}\right\|_{L^{2}\left(S_{H}\right)}^{2}+\left\|\partial_{3} u_{1}-\partial_{1} u_{3}\right\|_{L^{2}\left(S_{H}\right)}^{2}\right)  \tag{4.35}\\
I_{3} & \geq C_{0}\left(\left\|\partial_{3} u_{2}\right\|_{L^{2}\left(S_{H}\right)}^{2}+\left\|\partial_{3} u_{2}-\partial_{2} u_{3}\right\|_{L^{2}\left(S_{H}\right)}^{2}\right) \tag{4.36}
\end{align*}
$$

Hence, it follows from (4.33)-(4.36) that

$$
\begin{align*}
& \int_{S_{H}} 2 \Re\left\{\sum_{j=1}^{3} \mathcal{E}\left(u,\left(x_{3}-m\right) e_{j}\right) \partial_{3} \bar{u}_{j}\right\} \mathrm{d} x+C\|\nabla \cdot u\|_{L^{2}\left(S_{H}\right)}^{2}+C\|\nabla \times u\|_{L^{2}\left(S_{H}\right)^{3}}^{2} \\
& \geq C_{0}\left(\left\|\partial_{1} u_{1}\right\|_{L^{2}\left(S_{H}\right)}^{2}+\left\|\partial_{2} u_{2}\right\|_{L^{2}\left(S_{H}\right)}^{2}+\left\|\partial_{3} u_{3}\right\|_{L^{2}\left(S_{H}\right)}^{2}+\left\|\partial_{1} u_{2}-\partial_{2} u_{1}\right\|_{L^{2}\left(S_{H}\right)}^{2}\right) \\
& \quad+C_{0}\left(\left\|\partial_{3} u_{1}\right\|_{L^{2}\left(S_{H}\right)}^{2}+\left\|\partial_{1} u_{3}\right\|_{L^{2}\left(S_{H}\right)}^{2}+\left\|\partial_{3} u_{2}\right\|_{L^{2}\left(S_{H}\right)}^{2}+\left\|\partial_{2} u_{3}\right\|_{L^{2}\left(S_{H}\right)}^{2}\right), \tag{4.37}
\end{align*}
$$

provided $C>0$ is sufficiently large. Combining (4.22), (4.31) and (4.37) and using Young's inequality gives

$$
\begin{equation*}
\text { Right hand side of (4.37) } \leq\left(C_{4}(\omega, h)^{2}+C_{5}(\omega, h)^{2}+C_{6}(\omega, h, L)^{2}\right)\|g\|_{V_{h}}^{2} \text {. } \tag{4.38}
\end{equation*}
$$

However, we still need to estimate $\left\|\partial_{1} u_{2}\right\|_{L^{2}\left(S_{H}\right)}^{2}$ and $\left\|\partial_{2} u_{1}\right\|_{L^{2}\left(S_{H}\right)}^{2}$. Since $\left\|\partial_{3} u\right\|_{L^{2}\left(S_{H}\right)^{3}}^{2}$ can also be bounded by the right hand side of (4.38), we have (see [6, Lemma 3.4])

$$
\begin{equation*}
\|u\|_{L^{2}\left(S_{H}\right)}^{2} \leq C_{0}\left\|\partial_{3} u\right\|_{L^{2}\left(S_{H}\right)^{3}}^{2} \leq\left(C_{4}(\omega, h)^{2}+C_{5}(\omega, h)^{2}+C_{6}(\omega, h, L)^{2}\right)\|g\|_{V_{h}}^{2} . \tag{4.39}
\end{equation*}
$$

Now, using (4.26), (4.29) and (4.39) we arrive at

$$
\begin{equation*}
\mathcal{E}(u, \bar{u}) \leq\left(C_{4}(\omega, h)^{2}+C_{5}(\omega, h)^{2}+C_{6}(\omega, h, L)^{2}\right)\|g\|_{V_{h}}^{2} . \tag{4.40}
\end{equation*}
$$

Recalling the expression of $\mathcal{E}$, we find

$$
2 \mu\left(\partial_{1} u_{2}+\partial_{2} u_{1}\right)=\mathcal{E}(u, u)-\lambda \nabla \cdot u-\mu \nabla \times u-2 \mu\left(\partial_{1} u_{3}+\partial_{1} u_{1}+\partial_{2} u_{2}+\partial_{2} u_{3}+\sum_{j=1}^{3} \partial_{3} u_{j}\right) .
$$

It follows from (4.40), (4.22) and (4.38) that each term on the right hand side of the previous identity can be bounded by the right hand side of (4.40), leading to the same upper bound for $\left\|\partial_{1} u_{2}+\partial_{2} u_{1}\right\|_{L^{2}\left(S_{H}\right)}$. Finally, recalling the upper bound for the difference $\left\|\partial_{1} u_{2}-\partial_{2} u_{1}\right\|_{L^{2}\left(S_{H}\right)}$ (see (4.37)) we obtain the estimates for $\left\|\partial_{1} u_{2}\right\|_{L^{2}\left(S_{H}\right)}^{2},\left\|\partial_{2} u_{1}\right\|_{L^{2}\left(S_{H}\right)}^{2}$ and thus also for $\|\nabla u\|$. Using the $L^{2}$-estimate for $u$ (see (4.39)) we obtain

$$
\|u\|_{V_{h}}^{2} \leq\left(C_{4}(\omega, h)^{2}+C_{5}(\omega, h)^{2}+C_{6}(\omega, h, L)^{2}\right)\|g\|_{V_{h}}^{2} .
$$

Now the a priori bound for $f$ being smooth has been proved. It can be extended to the case of a general Lipschitz function by the method of approximation in [8]. This completes the proof.

## 5. Well-posedness for random rough surfaces

In this section, we investigate the well-posedness of elastic scattering by a random rough surface. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a complete probability space. Denote by $S(\eta)$ a random surface

$$
\Gamma(\eta):=\left\{x \in \mathbb{R}^{3}: x_{3}=f\left(\eta ; x_{1}, x^{\prime}\right), \eta \in \Omega, x^{\prime} \in \mathbb{R}^{2}\right\} .
$$

Similarly, $D(\eta)$ and $S_{h}(\eta)$ represent the random counterparts of $D$ and $S_{h}$, respectively. Assume $f\left(\eta ; x^{\prime}\right)$ is a Lipschitz continuous function with Lipschitz constant $L(\eta)$ for all $\eta \in \Omega$ and it also satisfies $m<f\left(\eta ; x^{\prime}\right)<M$. The random source $g(\eta)$ is assumed to satisfy $g(\eta) \in L^{2}(D(\eta))^{3}$ with its support in $S_{h}(\eta)$. Similarly as the deterministic case, we can give the following random boundary value problem.

$$
\begin{array}{rll}
\Delta^{*} u(\eta ; \cdot)+\omega^{2} u(\eta ; \cdot)=g(\eta ; \cdot) & \text { in } & S_{h}(\eta), \\
u(\eta ; \cdot)=0 & \text { on } & \Gamma(\eta), \\
T u(\eta ; \cdot)=\mathcal{T} u(\eta ; \cdot) & \text { on } & \Gamma_{h} .
\end{array}
$$

For simplicity, let $V_{h}(\eta)=V_{h}\left(S_{h}(\eta)\right)$. Define a sesquilinear form $\tilde{B}_{\eta}$ on $V_{h}(\eta) \times V_{h}(\eta)$ by

$$
\begin{equation*}
\tilde{B}_{\eta}(u, v)=\int_{S_{h}(\eta)} \mathcal{E}(u, \bar{v})-\omega^{2} u \cdot \bar{v} \mathrm{~d} x-\int_{\Gamma_{h}} \mathcal{T} u \cdot \bar{v} \mathrm{~d} s \tag{5.1}
\end{equation*}
$$

and an antilinear functional $\tilde{G}_{\eta}$ on $V_{h}(\eta)$ by

$$
\begin{equation*}
\tilde{G}_{\eta}(v):=-\int_{S_{h}(\eta)} g(\eta) \cdot \bar{v} \mathrm{~d} x \tag{5.2}
\end{equation*}
$$

To define the stochastic variation problem directly is not suitable since $V_{h}(\eta)$ is dependent on $\eta$. We take a variable transform to give a new sesquilinear form defined on $V_{h} \times V_{h}$. Let $f_{0}=f\left(\eta_{0}\right)$ and $g_{0}=g\left(\eta_{0}\right)$ for some fixed $\eta_{0} \in \Omega$ and write $D=D\left(\eta_{0}\right), S_{h}=S_{h}\left(\eta_{0}\right)$ and $V_{h}=V_{h}\left(\eta_{0}\right)$ for convenience. In addition, we assume that $g(\eta) \in H^{1}(D(\eta))^{3}$ and

$$
\left\|f(\eta)-f_{0}\right\|_{1, \infty} \leq M_{0}, \quad \forall \eta \in \Omega
$$

with some constant $M_{0}>0$. Moreover, the truncated height $h$ is chosen such that

$$
\begin{equation*}
(M-m) / \gamma<1, \tag{5.3}
\end{equation*}
$$

where $\gamma=h-\sup f_{0}\left(x^{\prime}\right)$. This condition ensures the invertiblity of the variable transform $\mathcal{H}$ which will be introduced later. Since $\Gamma_{h}$ is artificial, choosing sufficiently large $h$ will be enough.

Denote by $\operatorname{Lip}\left(\mathbb{R}^{2}\right)$ the set including all Lipschitz continuous functions on $\mathbb{R}^{2}$. Then define a product topology space

$$
\mathcal{C}=\mathcal{C}_{1} \times \mathcal{C}_{2},
$$

where

$$
\mathcal{C}_{1}:=\left\{v \in \operatorname{Lip}\left(\mathbb{R}^{2}\right): m<v<M,\left\|v-f_{0}\right\|_{1, \infty} \leq M_{0}\right\}
$$

with constant $M_{0}>0$ and

$$
\mathcal{C}_{2}:=H_{0}^{1}\left(S_{h}\right)^{3} .
$$

The topology of $\mathcal{C}_{1}$ and $\mathcal{C}_{2}$ are respectively given by the norms $\|\cdot\|_{1, \infty}$ and $\|\cdot\|_{H^{1}\left(S_{h}\right)^{3}}$.
Consider the transform $\mathcal{H}: S_{h} \rightarrow S_{h}(\eta)$ defined by

$$
\mathcal{H}(y)=y+\alpha\left(y_{3}-f_{0}\left(y^{\prime}\right)\right)\left(f\left(\eta ; y^{\prime}\right)-f_{0}\left(y^{\prime}\right)\right) e_{3}, \quad y \in D_{h},
$$

where $e_{3}$ is the unit vector in $x_{3}$ direction and $\alpha(x)$ is a cutoff function which satisfies

$$
\alpha(x)= \begin{cases}0, & x<\delta, \\ 1, & x>\gamma,\end{cases}
$$

with sufficiently small $\delta$. It is also required to satisfy

$$
\begin{equation*}
\left|\alpha^{\prime}\right|<1 /(\gamma-2 \delta) \tag{5.4}
\end{equation*}
$$

The Jacobi matrix of $\mathcal{H}$ is

$$
\mathcal{J}_{\mathcal{H}}=I_{3}+\left(\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 0 \\
J_{1} & J_{2} & J_{3}
\end{array}\right)
$$

where
$J_{i}=\alpha\left(y_{3}-f_{0}\left(y^{\prime}\right)\right)\left(\partial_{i} f\left(\eta ; y^{\prime}\right)-\partial_{i} f_{0}\left(y^{\prime}\right)\right)-\alpha^{\prime}\left(y_{3}-f_{0}\left(y^{\prime}\right)\right) \partial_{i} f_{0}\left(y_{1}\right)\left(f\left(\eta ; y^{\prime}\right)-f_{0}\left(y^{\prime}\right)\right), \quad i=1,2$ and

$$
J_{3}=\alpha^{\prime}\left(y_{3}-f_{0}\left(y^{\prime}\right)\right)\left(f\left(\eta ; y^{\prime}\right)-f_{0}\left(y^{\prime}\right)\right)
$$

Since matrix $\mathcal{J}_{\mathcal{H}}$ is required to be non-singular so that $\mathcal{H}$ is invertible, according to (5.4), we obtain

$$
\left|J_{3}\right|<\frac{M-m}{\gamma-2 \delta}
$$

Hence, by (5.3), we can choose $\delta$ sufficiently small such that

$$
\left|J_{3}\right|<\frac{M-m}{\gamma-2 \delta}<1
$$

which implies that $\mathcal{H}$ is invertible. It is easy to verify $\mathcal{H}\left(\Gamma_{h}\right)=\Gamma_{h}$. Set

$$
A=\left(\alpha_{1}, \alpha_{2}, \alpha_{3}\right), B^{\top}=\left(\beta_{1}, \beta_{2}, \beta_{3}\right) \in \mathbb{C}^{3 \times 3}
$$

then denote

$$
A: B=\operatorname{tr}\left(B^{\top} A\right)
$$

and

$$
A \otimes B=\left(\begin{array}{c}
\alpha_{2} \cdot \beta_{3}-\alpha_{3} \cdot \beta_{2} \\
\alpha_{3} \cdot \beta_{1}-\alpha_{1} \cdot \beta_{3} \\
\alpha_{1} \cdot \beta_{2}-\alpha_{2} \cdot \beta_{1}
\end{array}\right)
$$

For $u, v \in V_{h}(\eta)$, taking $x=\mathcal{H}(y)$ in (5.1) yields

$$
\begin{aligned}
\tilde{B}_{\eta}(u, v)= & 2 \mu \int_{S_{h}} \sum_{j=1}^{3} \nabla \tilde{u}_{j} \mathcal{J}_{\mathcal{H}^{-1}} \mathcal{J}_{\mathcal{H}^{-1}}^{\top} \nabla \overline{\tilde{v}}_{j} \operatorname{det} \mathcal{J}_{\mathcal{H}} \mathrm{d} y \\
& +\lambda \int_{S_{h}}\left(\nabla \tilde{u}: \mathcal{J}_{\mathcal{H}^{-1}}^{\top}\right)\left(\nabla \overline{\tilde{v}}: \mathcal{J}_{\mathcal{H}^{-1}}^{\top}\right) \operatorname{det} \mathcal{J}_{\mathcal{H}} \mathrm{d} y \\
& -\mu \int_{S_{h}}\left(J_{\mathcal{H}^{-1}} \otimes \nabla \tilde{u}\right)\left(J_{\mathcal{H}^{-1}} \otimes \nabla \overline{\tilde{v}}\right) \operatorname{det} \mathcal{J}_{\mathcal{H}} \mathrm{d} y \\
& -\omega^{2} \int_{S_{h}} \tilde{u} \cdot \overline{\tilde{v}} \operatorname{det} \mathcal{J}_{\mathcal{H}} \mathrm{d} y-\int_{\Gamma_{h}} \mathcal{T} \tilde{u} \cdot \overline{\tilde{v}} \mathrm{~d} s(y),
\end{aligned}
$$

where $\tilde{u}=u \circ \mathcal{H}, \tilde{v}=v \circ \mathcal{H}$. Similarly, for $v \in V_{h}(\eta)$, let $x=\mathcal{H}(y)$ in (5.2),

$$
\tilde{G}_{\eta}(v)=-\int_{D_{h}} \tilde{g}(\eta) \cdot \overline{\tilde{v}} \operatorname{det} \mathcal{J}_{\mathcal{H}} \mathrm{d} x
$$

Recall that we require $g(\eta) \in H^{1}(D(\eta))^{3}$ and the support of $g(\eta)$ is in $S_{h}(\eta)$, we have $\tilde{g}(\eta) \in H_{0}^{1}\left(S_{h}\right)^{3}$ for all $\eta$. So we can define the input map $c: \Omega \rightarrow \mathcal{C}$ by

$$
c(\eta):=(f(\eta), \tilde{g}(\eta))
$$

Note that $\tilde{u}, \tilde{v} \in V_{h}$. Thus we can define a continuous sesquilinear form $B_{c(\eta)}(u, v)$ on $V_{h} \times V_{h}$ by

$$
\begin{align*}
B_{c(\eta)}(u, v):= & 2 \mu \int_{S_{h}} \sum_{j=1}^{3} \nabla u_{j} \mathcal{J}_{\mathcal{H}^{-1}} \mathcal{J}_{\mathcal{H}^{-1}}^{\top} \nabla \bar{v}_{j} \operatorname{det} \mathcal{J}_{\mathcal{H}} \mathrm{d} y \\
& +\lambda \int_{S_{h}}\left(\nabla u: \mathcal{J}_{\mathcal{H}^{-1}}^{\top}\right)\left(\nabla \bar{v}: \mathcal{J}_{\mathcal{H}^{-1}}^{\top}\right) \operatorname{det} \mathcal{J}_{\mathcal{H}} \mathrm{d} y \\
& -\mu \int_{S_{h}}\left(J_{\mathcal{H}^{-1}} \otimes \nabla u\right)\left(J_{\mathcal{H}^{-1}} \otimes \nabla \bar{v}\right) \operatorname{det} \mathcal{J}_{\mathcal{H}} \mathrm{d} y \\
& -\omega^{2} \int_{S_{h}} u \cdot \bar{v} \operatorname{det} \mathcal{J}_{\mathcal{H}} \mathrm{d} y-\int_{\Gamma_{h}} \mathcal{T} u \cdot \bar{v} \mathrm{~d} s(y) . \tag{5.5}
\end{align*}
$$

It is easy to see

$$
\tilde{B}_{\eta}(u, v)=B_{c(\eta)}(\tilde{u}, \tilde{v})
$$

Similarly we can define an antilinear functional $G_{c(\eta)}$ on $V_{h}$ by

$$
\begin{equation*}
G_{c(\eta)}(v):=-\int_{S_{h}} \tilde{g}(\eta) \cdot \bar{v} \operatorname{det} \mathcal{J}_{\mathcal{H}} \mathrm{d} x \tag{5.6}
\end{equation*}
$$

Obviously, there holds the identity

$$
G_{c(\eta)}(\tilde{v})=\tilde{G}_{\eta}(v)
$$

Then the sesquilinear form $\tilde{\mathcal{B}}$ on $L^{2}\left(\Omega ; V_{h}\right) \times L^{2}\left(\Omega ; V_{h}\right)$ can be defined by

$$
\mathcal{B}(u, v):=\int_{\Omega} B_{c(\eta)}(u, v) \mathrm{d} \mathbb{P}(\eta)
$$

and the antilinear functional $\mathcal{G}$ is defined on $L^{2}\left(\Omega ; V_{h}\right)$ by

$$
\mathcal{G}(v):=\int_{\Omega} G_{c(\eta)}(v) \mathrm{d} \mathbb{P}(\eta)
$$

For convenience, we regard the sesquilinear form $B_{c(\eta)}: V_{h} \times V_{h} \rightarrow \mathbb{C}$ as the same operator in $B\left(V_{h}, V_{h}^{*}\right)$ generated by it. Here $V_{h}^{*}$ is the dual space of $V_{h}$ and $B(X, Y)$ denote the space including all bounded linear operators $X \rightarrow Y$. Similarly to (5.5)-(5.6), we can define the sesquilinear form $B_{(\phi, \psi)}$ and the antilinear functional $G_{(\phi, \psi)}$ for all $(\phi, \psi) \in \mathcal{C}$. Then we can define the map $\mathscr{B}: \mathcal{C} \rightarrow B\left(V_{h}, V_{h}^{*}\right)$ by

$$
\mathscr{B}((\phi, \psi)):=B_{(\phi, \psi)}
$$

and the $\operatorname{map} \mathscr{G}: \mathcal{C} \rightarrow V_{h}^{*}$ by

$$
\mathscr{G}((\phi, \psi)):=G_{(\phi, \psi)} .
$$

Now we can define the stochastic variation problem as follows.
Variational Problem II: find $u \in L^{2}\left(\Omega ; V_{h}\right)$ such that

$$
\begin{equation*}
\mathcal{B}(u, v)=\mathcal{G}(v), \quad \forall v \in L^{2}\left(\Omega ; V_{h}\right) \tag{5.7}
\end{equation*}
$$

We will consider the well-posedness of the stochastic variation problem (5.7). Firstly we show both the sesquilinear form $\mathcal{B}$ and the antilinear functional $\mathcal{G}$ are well-defined which is based on measurability and $\mathbb{P}$-essentially separability of $c$. For measurability and $\mathbb{P}$ essentially separability of $c$, the following condition is necessary.

Condition 5.1. The map $c_{1}: \Omega \rightarrow \mathcal{C}_{1}$ defined by

$$
c_{1}(\eta)=f(\eta)
$$

satisfies $c_{1} \in L^{2}\left(\Omega ; \mathcal{C}_{1}\right)$ and the map $c_{2}: \Omega \rightarrow \mathcal{C}_{2}$ defined by

$$
c_{2}(\eta)=\tilde{g}(\eta)
$$

satisfies $c_{2} \in L^{2}\left(\Omega ; \mathcal{C}_{2}\right)$.
It implies the following lemma (see Lemma 4.1 in [14]).
Lemma 5.1. Under Condition 5.1, the map $c$ is measurable and $\mathbb{P}$-essentially separable.
Then we can prove that the sesquilinear form $\mathcal{B}$ is well-defined by the continuity of $\mathscr{B}$ and the regularity of $\mathscr{B} \circ c$.

Lemma 5.2. (i) The map $\mathscr{B}: \mathcal{C} \rightarrow B\left(V_{h}, V_{h}^{*}\right)$ is continuous.
(ii) The map $\mathscr{B} \circ c \in L^{\infty}\left(\Omega ; B\left(V_{h}, V_{h}^{*}\right)\right)$.
(iii) The sesquilinear form $\mathcal{B}$ is well-defined on $L^{2}\left(\Omega ; V_{h}\right) \times L^{2}\left(\Omega ; V_{h}\right)$.

Proof. We only prove (i), since (ii),(iii) can be verified similarly as the two-dimensions case in [14]. For convenience, we only prove the continuity at the point $\left(f_{0}, g_{0}\right) \in \mathcal{C}$ since for other points the proof is similar. Consider the sequence $\left\{\left(f_{m}, g_{m}\right)\right\} \subset \mathcal{C}$ such that $\left(f_{m}, g_{m}\right) \rightarrow$ $\left(f_{0}, g_{0}\right)$ in $\mathcal{C}$ when $m \rightarrow \infty$. Denote the transform by

$$
\mathcal{H}_{m}(y)=y+\alpha\left(y_{3}-f_{0}\left(y^{\prime}\right)\right)\left(f_{m}\left(y^{\prime}\right)-f_{0}\left(y^{\prime}\right)\right) e_{3}, \quad y \in D_{h} .
$$

For any $u, v \in V_{h}$,

$$
\begin{aligned}
B_{\left(f_{m}, g_{m}\right)}(u, v) & -B(u, v)=2 \mu \int_{S_{h}} \sum_{j=1}^{2} \nabla u_{j}\left(I_{3}-\mathcal{J}_{\mathcal{H}_{m}^{-1}} \mathcal{J}_{\mathcal{H}_{m}^{-1}}^{\top} \operatorname{det} \mathcal{J}_{\mathcal{H}_{m}}\right) \nabla \bar{v}_{j} \mathrm{~d} x \\
& +\lambda \int_{S_{h}}(\nabla \cdot u)(\nabla \cdot \bar{v})-\left(\nabla \tilde{u}: \mathcal{J}_{\mathcal{H}_{m}^{-1}}\right)\left(\nabla \overline{\tilde{v}}: \mathcal{J}_{\mathcal{H}_{m}^{-1}}^{\top}\right) \operatorname{det} \mathcal{J}_{\mathcal{H}_{m}} \mathrm{~d} x \\
& -\mu \int_{S_{h}}\left(J_{\mathcal{H}_{m}^{-1}} \otimes \nabla \tilde{u}\right)\left(J_{\mathcal{H}_{m}^{-1}} \otimes \nabla \overline{\tilde{v}}\right) \operatorname{det} J_{\mathcal{H}_{m}}-(\nabla \times u) \cdot(\nabla \times \bar{v}) \mathrm{d} x \\
& -\omega^{2} \int_{S_{h}} u \cdot \bar{v}\left(\operatorname{det} \mathcal{J}_{\mathcal{H}_{m}}-1\right) \mathrm{d} x .
\end{aligned}
$$

By direct calculations, we have

$$
\operatorname{det} \mathcal{J}_{\mathcal{H}_{m}}=1+O\left(\left\|f_{m}-f_{0}\right\|_{1, \infty}\right), \quad \mathcal{J}_{\mathcal{H}_{m}^{-1}}=I_{3}+O\left(\left\|f_{m}-f_{0}\right\|_{1, \infty}\right),
$$

which imply that

$$
\left|B_{\left(f_{m}, g_{m}\right)}(u, v)-B(u, v)\right| \leq C\|u\|_{H^{1}\left(D_{h}\right)^{2}}\|v\|_{H^{1}\left(S_{h}\right)^{3}}\left\|f_{m}-f_{0}\right\|_{1, \infty} .
$$

It turns out when $m \rightarrow \infty$,

$$
\left\|B_{\left(f_{m}, g_{m}\right)}-B\right\|_{B\left(V_{h}, V_{h}^{*}\right)} \leq C\left\|f_{m}-f_{0}\right\|_{1, \infty} \rightarrow 0
$$

This completes the proof.
Next we give a similar lemma for the antilinear functional $\mathcal{G}$.
Lemma 5.3. (i) The $\operatorname{map} \mathscr{G}: \mathcal{C} \rightarrow V_{h}^{*}$ is continuous.
(ii) The $\operatorname{map} \mathscr{G} \circ c \in L^{2}\left(\Omega ; V_{h}^{*}\right)$.
(iii) The antilinear functional $\mathcal{G}$ is well-defined on $L^{2}\left(\Omega ; V_{h}\right)$.

The proof is similar to Lemma 4.3 in [14]. For any given sampling $\eta$, we consider the following deterministic Variational Problem III.

Find $u(\eta) \in V_{h}$ such that

$$
\begin{equation*}
B_{c(\eta)}(u(\eta), v)=G_{c(\eta)}(v), \quad \forall v \in V_{h} . \tag{5.8}
\end{equation*}
$$

The existence and uniqueness of solutions of the problem (5.8) has been given in Theorem 2.1. The a priori bound in Lemma 4.2 can also be used for (5.8). Notice that for any $\eta$ we have the upper bound

$$
L(\eta) \leq L+M_{0}
$$

Lemma 5.4. For any given $\eta$, the variational problem (5.8) admits a unique solution $u(\eta) \in$ $V_{h}$. Moreover, the a priori bound

$$
\left\|u^{*}(\eta)\right\|_{H^{1}\left(S_{h}(\eta)\right)^{3}} \leq(h-m+2)\left(C_{4}(\omega, h)+C_{5}(\omega, h)+C_{6}\left(\omega, h, L_{0}\right)\right)\|g(\eta)\|_{H^{1}\left(S_{h}(\eta)\right)^{3}}
$$

holds for $u^{*}(\eta)=u(\eta) \circ \mathcal{H}^{-1}$ with $L_{0}=M_{0}+L$.
Proof. If $u(\eta)$ is a solution to Variational Problem III (5.8), then $u^{*}(\eta)=u(\eta) \circ \mathcal{H}^{-1}$ is solution to Variational Problem $I(2.13)$ corresponding to $f(\eta)$ and $g(\eta)$. Conversely, if $u(\eta)$ is solution to Variational Problem $I(2.13)$ corresponding to $f(\eta)$ and $g(\eta)$, then $\tilde{u}(\eta)=u(\eta) \circ \mathcal{H}$ is solution to Variational Problem III (5.8). So Theorem 2.1)implies existence and uniqueness of solutions to the variation problem (5.7), and Theorem 4.3 implies the a priori bound.

Lemma 5.4 shows the existence of a solution $u(\eta)$ to (5.8) for given $\eta$. In fact, the following lemma shows $u(\eta) \in L^{2}\left(\Omega ; V_{h}\right)$.
Lemma 5.5. For the solution $u(\eta)$ to Variational Problem III (5.8), we have $u(\eta) \in L^{2}\left(\Omega ; V_{h}\right)$.
The proof is omitted here since it is similar to the two-dimensions case in Lemma 4.4 in [14]. Based on Lemmas 5.2 - 5.5, we can conclude the well-posedness of (5.7) in the framework of [12, 13, 14] and extend the a priori bound to random case as follows.

Theorem 5.6. (i) The Variational Problem II (5.7) admits a unique solution $u \in L^{2}\left(\Omega, V_{h}\right)$.
(ii) Let $u \in V_{h}(\eta)$ be a solution to the Variation Problem I (2.13) corresponding to $f(\eta)$ and $g(\eta)$ with $\eta \in \Omega$, and let $\tilde{u}(\eta) \in L^{2}\left(\Omega ; V_{h}\right)$ be the solution to the Variational Problem II (5.7). Then $u$ and $\tilde{u}$ satisfy respectively the bound

$$
\begin{aligned}
& \int_{\Omega}\|u\|_{H^{1}\left(S_{h}(\eta)\right)^{3}}^{2} \mathrm{~d} \mathbb{P} \\
& \leq(h-m+2)^{2}\left(C_{4}(\omega, h)+C_{5}(\omega, h)+C_{6}\left(\omega, h, L_{0}\right)\right)^{2} \int_{\Omega}\|g\|_{H^{1}\left(S_{h}(\eta)\right)^{3}}^{2} \mathrm{~d} \mathbb{P},
\end{aligned}
$$

and

$$
\begin{aligned}
& \int_{\Omega}\|\tilde{u}\|_{H^{1}\left(S_{h}\right)^{3}}^{2} \mathrm{~d} \mathbb{P} \\
& \leq(h-m+2)^{2}\left(C_{4}(\omega, h)+C_{5}(\omega, h)+C_{6}\left(\omega, h, L_{0}\right)\right)^{2} \int_{\Omega}\|\tilde{g}\|_{H^{1}\left(S_{h}\right)^{3}}^{2} \mathrm{~d} \mathbb{P}
\end{aligned}
$$

## 6. Conclusion

We establishes the well-posedness of the time-harmonic elastic scattering from general unbounded rough surfaces in three dimensions at an arbitrary frequency. A priori bounds which are explicit dependent on the frequency and on the geometry of the rough surface are derived both for deterministic and random cases. A possible continuation of this work is to study the elastic scattering by incident plane waves, spherical or cylindrical waves. We hope to report the progress on these results in subsequent publications.

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