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TIME-HARMONIC ACOUSTIC SCATTERING FROM LOCALLY PERTURBED PERIODIC CURVES*

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Abstract. For the Dirichlet rough-surface scattering problem in two dimensions, we prove that the Green's function defined with the angular spectrum representation radiation condition satisfies the Sommerfeld radiation condition over the half-plane. To the best of our knowledge, such an outgoing property has not been rigorously justified in the literature. We prove well-posedness for the time-harmonic acoustic scattering of plane waves from locally perturbed periodic surfaces. It will be shown that the scattered wave of an incoming plane wave is the sum of the scattered wave for the unperturbed periodic surface plus an additional scattered wave satisfying Sommerfeld's condition on the half-plane. Whereas the scattered wave for the unperturbed periodic surface has a far field consisting of a finite number of propagating plane waves, the additional field contributes to the far field by a far-field pattern defined in the half-plane directions similarly to the pattern known for bounded obstacles.

Key words. scattering problem, two-dimensional Helmholtz equation, locally perturbed periodic boundary curve, half-plane Sommerfeld radiation condition, sound-soft boundary condition

AMS subject classifications. 74J20, 76B15, 35J50, 35J08

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1. Introduction.

Scattering theory for periodic structures has many applications in near-field optics, microelectronics, nondestructive testing, and the design of photonic crystals. We refer to [42] for an introduction and historical remarks on the electromagnetic theory of gratings. Over the last twenty years, significant progress has been made concerning the mathematical analysis and the numerical approximation of grating diffraction problems for the case of incident acoustic or electromagnetic waves, using boundary integral equation (BIE) methods (e.g., [37, 39, 41, 43, 46]) and variational methods (e.g., [6, 19, 20, 29, 44]). This paper is concerned with the analysis and computation of time-harmonic scattering by a one-dimensional perfectly conducting grating with local perturbation. Physically, the local perturbation of a perfectly periodic surface can be used to model optical devices with localized defects, for instance, unmade or distorted grooves on the surface of diffraction gratings.

The diffracted field for a plane wave incident onto a perfect grating is well-known to be quasi-periodic due to the periodicity of the scattering surface and the quasi-periodicity of the incoming wave. The presence of defects will break down the quasi-periodicity property, leading to essential difficulties in the reduction of the analysis and simulation to problems over bounded domains. A limited number of approaches

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have been proposed so far for treating grating problems with local perturbations. To solve transmission problems for periodic interfaces perturbed by compact aperiodic inclusions below the interface, Ammari and Bao [1] proposed an integral equation approach. This integral equation is defined over $\mathbb{R}^2$ or $\mathbb{R}^3$ and includes a Fourier transform as well as a kernel function, which is defined by the solution of a family of variational equations. The approach relies on strong presumptions (for instance, absence of surface waves and the unique solvability of a periodic equation; see equation (3.1) in [1]), and the mathematical analysis of the unique solvability and the decay behavior of the unperturbed fields seem still to be unclear in general. Bonnet-Bendhia and Ramdani [5] treated such compact inclusions if the space beneath a planar interface is filled with media periodic in the interface direction. They employed Floquet–Bloch transforms, variational formulations, and BIE techniques. Joly et al. established an exact boundary condition with a map of Dirichlet-to-Neumann type for numerically solving an inhomogeneous source problem in a closed periodic waveguide with a local junction [27] and then extended the approach to an open waveguide, where the unperturbed medium was periodic in two directions [21]. Here and in other publications, the Floquet–Bloch transform was employed to handle scattering problems in a locally perturbed periodic medium; see [16] for a line defect, Haddar and Nguyen [24] in periodic layered medium, Lechleiter and Zhang [34] for locally perturbed sound-soft surfaces, and Hu and Lu [25] for a biperiodic photonic crystal with a bended tunnel. The resulting numerical schemes of [24, 34] required the calculation of inverse and forward Floquet–Bloch transforms or variational equations for Floquet–Bloch transformed solutions.

Motivated by recent studies on wave scattering from flat surfaces with local perturbations [2, 4, 45], in this paper we consider plane-wave scattering by a periodic grating with local perturbation and prove that the total field can be uniquely decomposed into three parts (see Theorem 3.1): the incoming wave v^{in}, the reflected field v^{sc} corresponding to the unperturbed periodic scattering interface, and the perturbed wave u_0 caused by the presence of local perturbations. We verify that u_0 satisfies the half-space Sommerfeld radiation condition (see Definition 2.1). This, in particular, implies that a local perturbation cannot give rise to any surface wave (see Remark 3.1). The characterization of the asymptotic behavior of u_0 in a periodic background medium seems to be missing in the literature and turns out to be nontrivial. In the case of flat surfaces with local perturbations, it is easy to prove that u_0 fulfills the strong Sommerfeld radiation condition uniformly for all outgoing directions in the upper half-space. Note that the splitting $u = v^{in} + v^{sc} + u_0$ is a special case of the representation $u - v^{in} = u_{pr} + u_{ev} + u_{con}$ suggested by DeSanto and Martin for general rough surfaces in [18, eq. (12)] but rigorously justified for special locally perturbed flat surfaces only. In that representation $v^{sc} = u_{pr} + u_{ev}$ is a finite sum of propagating and generalized (evanescent) plane-wave modes, and $u_0 = u_{con}$ is an integral of plane-wave modes satisfying Sommerfeld's radiation condition.

The decomposition of the scattered fields into reflected fields and Sommerfeld-type outgoing fields also applies to other cases of local perturbations, e.g., a bounded obstacle embedded in periodic background media, including inhomogeneous periodic layered media. Hence, the proposed approach can be used to handle general grating diffraction problems with defects. This requires the determination of the solution for the unperturbed periodic surface and an efficient forward solver for computing the Green's function G to the unperturbed grating diffraction problems, i.e., the computation of the total fields excited by incoming point-source waves. Since such incident waves are not quasi-periodic, special methods of computation are required. One can apply the Floquet–Bloch transform to the calculation of G (see, e.g., [33]).

The splitting $u = v^{in} + v^{sc} + u_0$ follows straightforwardly from the properties of the corresponding Green's function in the half-space. We shall prove that this Green's function, and even the non-quasi-periodic Green's function G to domains above aperiodic rough surfaces of perfectly conducting materials, fulfill this half-space Sommerfeld radiation condition (see Theorem 2.2). This includes periodic surfaces with local perturbation. For compactly supported source radiating problems, a similar property has been discussed in [12, Thm. 5.1] where the sound-soft scattering surface was supposed to be the graph of a $C^{1,1}$-smooth function. However, to the best of the authors' knowledge, a rigorous proof of the Sommerfeld outgoing property for half-space scattering problems is still open. From our proof of the Sommerfeld condition for the Green's function, we obtain Corollary 2.1, where we show that the solution to a boundary value problem for the Helmholtz equation satisfies the half-plane Sommerfeld radiation condition, provided the boundary data on a rough surface fulfills properly decaying conditions. Of course this decay excludes plane-wave incidence.

Now, for the scattering by a locally perturbed periodic grating, the above radiation condition satisfied by the Green's function enables us to establish an equivalent variational formulation over a bounded domain containing the defect. The formulation is based on a boundary integral representation of u_0 in terms of the quasi-periodic Green's function G. Thanks to solvability results for general rough surfaces [10], we show that $u = v^{in} + v^{sc} + u_0$ is the unique solution in certain weighted Sobolev spaces over a strip above the scattering surface. By the classical grating theory, the reflected field v^{sc} fulfills the upward Rayleigh expansion radiation condition. Together with the half-space Sommerfeld radiation condition for u_0, we obtain that the scattered field $v^{sc} + u_0$ still satisfies the upward angular spectral representation ([10, 11]) or, equivalently, the upward propagating radiation condition of [14]. Finally, the estimates of the present paper leading to Sommerfeld's radiation condition can be used as well to derive a far-field pattern of u_0. Note that the notion of far-field patterns can be used to model the inverse problems of finding the defect surface from measured far-field data (compare the different notion of far-field measurement in, e.g., [35]).

The remaining part of this paper is organized as follows. In the subsequent section 2 we recall solvability results for the scattering of plane and point-source waves from perfectly conducting gratings. The half-plane Sommerfeld radiation condition will be given in Definition 2.1. Section 3 is devoted to the analysis of a variational formulation over a bounded truncated domain, which is equivalent to the scattering problem. Uniqueness and existence of weak solutions will be reported in Theorem 3.1. The proof for the Sommerfeld radiation condition of the Green's function and of part u_0 of the scattering solution will be postponed to Appendix A.

2. Scattering from gratings.

2.1. Plane-wave incidence. Suppose that a perfectly conducting grating is illuminated by an incident monochromatic plane wave from above and that the grating is periodic in one surface direction and independent of the other. We consider the transverse-electric mode of polarization and let the profile of the diffraction grating be given by

$$\Gamma = \left\{ x = (x_1, x_2) \in \mathbb{R}^2 : x_2 = f(x_1) \right\}$$

with a 2π-periodic Lipschitz function $f \in C^{0,1}_{per}$. The unbounded domain $\Omega = \Omega_\Gamma$ above the grating is occupied by an isotropic homogeneous background medium. For technical reasons, we assume that Γ contains at least one line segment in each period. Note that this condition will only be used in section 2.3 below. In two dimensions, the incident wave is supposed to be a time-harmonic plane wave of the form $v^{in}(x)\exp(-i\omega t)$ with angular frequency $\omega>0$. The spatially dependent function v^{in} takes the form

$$v^{in}(x)=\exp(ik\,(\sin\theta,-\cos\theta)\cdot x), \tag{2.1}$$

where $\theta\in(-\pi/2,\pi/2)$ denotes the angle of incidence, $k:=\omega/c_0$ is the wave number, and $c_0>0$ is the speed of sound. The wave propagation is then governed by a boundary value problem for the Helmholtz equation

$$\Delta v+k^2v=0 \quad \text{in } \Omega_\Gamma, \qquad v=0 \quad \text{on } \Gamma \tag{2.2}$$

where the total field $v=v^{in}+v^{sc}$ is the sum of the incident field v^{in} and a scattered field v^{sc}, which satisfies a radiation condition.

Let $\alpha:=k\sin\theta$. Obviously, the incident field is α-quasi-periodic in the sense that $v^{in}(x)\exp(-i\alpha x_1)$ is 2π-periodic with respect to x_1 in Ω_Γ. The periodicity of the structure together with the form of the incident wave implies that the total field v must also be α-quasi-periodic. This is equivalent to

$$v(x_1+2\pi\,n,x_2)=\exp(i2\pi\alpha n)\,v(x_1,x_2) \quad \text{for all } n\in\mathbb{Z}.$$

Since the domain Ω_Γ is unbounded, a radiation condition must be imposed at infinity to ensure well-posedness of the scattering problem. For any $h>\max\{x_2: x\in\Gamma\}$, we require the scattered acoustic field v^{sc} to admit the upward Rayleigh expansion condition: There exist coefficients $v_n\in\mathbb{C}$ depending on k, θ, and Γ such that

$$v^{sc}(x)=\sum_{n\in\mathbb{Z}}v_n\,\exp(i\alpha_nx_1+i\beta_nx_2), \quad x\in U_h:=\left\{x\in\mathbb{R}^2:\ x_2>h\right\} \tag{2.3}$$

with the parameters $\alpha_n:=n+\alpha\in\mathbb{R}$ and $\beta_n\in\mathbb{C}$ defined by

$$\beta_n=\beta_n(k):=\begin{cases}(k^2-|\alpha_n|^2)^{\frac{1}{2}} & \text{if} \quad |\alpha_n|\le k,\\ i(|\alpha_n|^2-k^2)^{\frac{1}{2}} & \text{if} \quad |\alpha_n|>k.\end{cases}$$

Before stating uniqueness and existence of our scattering problem (2.1)–(2.3), we define L^2-based Sobolev spaces for a weak solution as follows. Denote by $L^2(\Omega_\Gamma)$ the Hilbert space consisting of all square-integrable functions over Ω_Γ and by $H^1(\Omega_\Gamma)$ the set of those $v\in L^2(\Omega_\Gamma)$ that have a gradient $\nabla v\in L^2(\Omega_\Gamma)$ in the weak sense. Let $H^1_0(\Omega_\Gamma)$ be the closure of $C_0^\infty(\Omega_\Gamma)$ in $H^1(\Omega_\Gamma)$. The space $H_0^{1,loc}(\Omega_\Gamma)$ denotes the set of all locally square-integrable functions $v:\Omega_\Gamma\to\mathbb{C}$ such that $\chi v\in H^1_0(\Omega_\Gamma)$ for all $\chi\in C_0^\infty(\mathbb{R}^2)$.

LEMMA 2.1. *Suppose the plane wave* v^{in} *is defined in* (2.1) *with* $k>0$ *and with* $\theta\in(-\pi/2,\pi/2)$. *For* $\alpha=k\sin\theta$, *there exists a unique* α*-quasi-periodic variational solution* $v=v^{in}+v^{sc}$ *to the scattering problem* (2.2)–(2.3) *in* $H_0^{1,loc}(\Omega_\Gamma)\cap C^2(\Omega_\Gamma)$.

Note that variational solution v means v^{sc} is a solution of a variational equation with a sesquilinear form defined over a finite domain contained in a single period of the periodic geometry (cf. [20]). As usual, this implies that the solution is locally smooth and the Helmholtz equation is fulfilled in the classical sense in the open set Ω_Γ. Furthermore, for variational solutions v, the Dirichlet boundary condition is fulfilled in the sense of traces of functions from $H^{1,loc}(\Omega_\Gamma)$ or, equivalently, as $v\in H_0^{1,loc}(\Omega_\Gamma)$. The above well-posedness result was proved by Elschner and Yamamoto in [20]. In an earlier paper by Kirsch [29], Lemma 2.1 was proved for the case where the periodic surface is given by the graph of a C^2-smooth function. Chandler-Wilde and Monk [11] proved uniqueness and existence for rough-surface scattering problems if the incident wave is generated by a compact source term and if the domain Ω_Γ fulfills the following weak assumption:

$$(x_1, x_2) \in \Omega_\Gamma \quad \Rightarrow \quad (x_1, x_2 + s) \in \Omega_\Gamma \quad \text{for all } s > 0. \tag{2.4}$$

Note that our assumption on Γ (that is, Γ is given by the graph of some Lipschitz function) excludes vertical straight-line segments in Γ and, thus, is stronger than (2.4). It deserves mention that Lemma 2.1 remains true even for periodic Lipschitz curves Γ satisfying the above weak assumption (2.4), since the classical variational theory implies Fredholm property and index zero for all Lipschitz curves and since the rough-surface result provides uniqueness for curves with (2.4) (see [10, Cor. 5.2]). In particular, the periodic Γ could be the curve of a binary grating. Indeed, the case of plane-wave incidence and sound-soft boundary conditions even for rough surfaces in two dimensions was treated in [10].

The uniqueness proofs in the above mentioned papers depend heavily on the use of Rellich's identity for scattering surfaces given by the graph of a uniformly Lipschitz function. Uniqueness to scattering problems in periodic structures cannot hold in the general case. We refer to [6] for nonuniqueness examples in inhomogeneous periodic media and to [23, 26, 28] for uniqueness examples for scattering from perfectly conducting gratings. In closed waveguides and in stratified media we refer to [22, 30, 31, 32, 47] and references therein for discussions on uniqueness, existence, and the construction of radiation conditions in an open periodic waveguide.

2.2. Point-source incidence. We now fix a $y \in \Omega_\Gamma$ and consider the case where the incident wave G^{in} is a non-quasi-periodic cylindrical wave of the form

$$G^{in}(x) = G^{in}(x; y) := \Phi(x; y) := \frac{i}{4} H_0^{(1)}(k|x-y|), \quad x \neq y, \quad x \in \Omega_\Gamma. \tag{2.5}$$

Here $H_0^{(1)}(\cdot)$ stands for the Hankel function of the first kind and of order zero. The function $\Phi(x; y)$ is the free-space fundamental solution of the Helmholtz equation $(\Delta + k^2 I)u = 0$. Since the incoming wave G^{in} is no longer quasi-periodic, the Rayleigh expansion condition (2.3) is not applicable to point-source incidence of the form (2.5). Instead we suppose that the scattered field $G^{sc}(x; y) := u(x)$ satisfies the upward angular spectrum representation (ASR) proposed in [11]:

$$u(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp\Big(i[(x_2 - h)\sqrt{k^2 - \xi^2} + x_1 \xi]\Big)\, \hat{u}_h(\xi)\, \mathrm{d}\xi, \quad x \in U_h, \tag{2.6}$$

for all $h > \max\{x_2 : x \in \Gamma\}$. Here, $\sqrt{k^2 - \xi^2} = i\sqrt{\xi^2 - k^2}$ for $\xi^2 > k^2$, and $\hat{u}_h(\xi)$ denotes the Fourier transform of $u_h(x_1) := u(x_1, h)$ with respect to x_1, i.e.,

$$\hat{u}_h(\xi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp(-i x_1 \xi)\, u(x_1, h)\, \mathrm{d}x_1, \quad \xi \in \mathbb{R}.$$

If $u_h \in L^2(\mathbb{R})$, the radiation condition (2.6) is equivalent to the representation (see, e.g., [11] and [9, p. 821])

$$u(x) = 2 \int_{\Gamma_h} \frac{\partial \Phi_h^*(x; y)}{\partial y_2} u(y_1, h)\, \mathrm{d}s(y), \quad x \in U_h,$$

which is known as the upward propagating radiation condition (UPRC) proposed in [13]. Here, we use $\Gamma_h := \{x \in \mathbb{R}^2 : x_2 = h\}$ and $\Phi_h^*(x;y) := \Phi(x;y) - \Phi(x;y_h^*)$ with $y_h^* := 2h - y_2$. If $u_h \in BC(\Gamma_h)$, it was shown in [3, Prop. 5] that the integral in the ASR can be interpreted as the bilinear duality between $\hat{u}_h(\xi) \in H^{-\sigma}(\mathbb{R})$ and $\xi \to \exp(i[(x_2 - h)\sqrt{k^2 - \xi^2} + x_1\xi]) \in H^\sigma(\mathbb{R})$ for $\sigma \in (1/2, 1)$, which was also proved to be equivalent to a "pole condition" for rough surface scattering problems. If u is α-quasi-periodic in U_h, it is known from [7, 9] that the above UPRC (and thus ASR) is equivalent to the upward Rayleigh expansion condition (2.3).

In the case that $u = G^{sc}(\cdot; y)$ with a fixed $y \in \Omega_\Gamma$, the ASR (2.6) can be understood as the duality between weighted Sobolev spaces over Γ_h following the arguments presented in [10]. We shall explain this in more detail as follows. First we state the well-posedness of the scattering problem for point-source incidence in weighted Sobolev spaces. Denote the infinite strip between Γ and Γ_h by $\Omega_{\Gamma,h} := \{x \in \Omega_\Gamma : x_2 < h\}$ and recall $U_h := \{x \in \mathbb{R}^2 : x_2 > h\}$ (cf. Figure 2.1, upper part). Define the weighted Sobolev space $V_{h,\varrho}$, for $\rho \in \mathbb{R}$, as the closure of all $u|_{\Omega_{\Gamma,h}}$ with $u \in C_0^\infty(\Omega_\Gamma)$ w.r.t. the norm

$$\|u\|_{V_{h,\varrho}} := \left[\int_{\Omega_{\Gamma,h}} \left\{ \left|(1+|x_1|^2)^{\varrho/2} u(x)\right|^2 + \left|\nabla\left[(1+|x_1|^2)^{\varrho/2} u(x)\right]\right|^2 \right\} \mathrm{d}x \right]^{1/2}.$$

Setting $H_\varrho^s(\cdot) := (1 + x_1^2)^{-\varrho/2} H^s(\cdot)$ for $\varrho, s \in \mathbb{R}$, we have the identity $V_{h,\varrho} = H_\varrho^1(\Omega_{\Gamma,h}) \cap \{u : u|_\Gamma = 0\}$ and, if $\varrho = 0$, the equality $H_\varrho^s(\mathbb{R}) = H^s(\mathbb{R})$, where the $H^s(\mathbb{R})$ are the usual nonweighted Sobolev spaces. By definition, the relation $V_{h,\varrho_1} \subset V_{h,\varrho_2}$ holds if $\varrho_1 > \varrho_2$. Let $y \in \Omega_\Gamma$ be the position of the source of the incident wave. For the scattering problem with incident wave $G^{in}(\cdot; y)$, we look for the total field $G(\cdot\,; y) = G^{in}(\cdot\,; y) + G^{sc}(\cdot\,; y)$ with $G^{sc}(\cdot\,; y) \in H_\varrho^1(\Omega_{\Gamma,h})$ such that

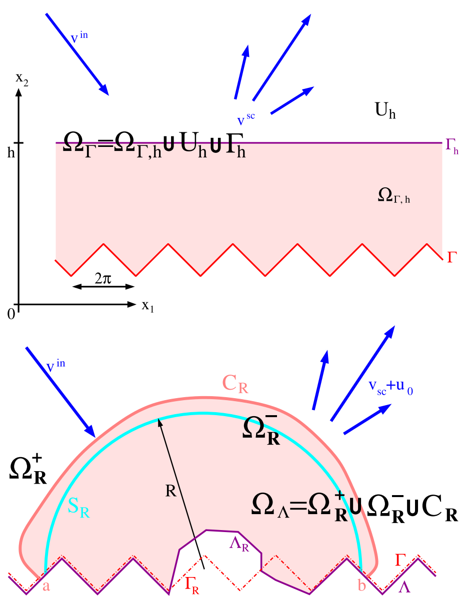

FIG. 2.1. *Geometry of unperturbed grating (upper part) and locally perturbed grating (lower part). Ω_Γ and Ω_Λ denote the domains above the unperturbed curve Γ and the perturbed curve Λ, respectively.*

$$\begin{aligned} &\Delta G^{sc}(\cdot\,;y)+k^2G^{sc}(\cdot\,;y)=0 \text{ on } \Omega_\Gamma,\\ &G^{sc}(\cdot\,;y)=-G^{in}(\cdot\,;y) \text{ on } \Gamma, \quad G^{sc}(\cdot\,;y) \text{ satisfies ASR .}\end{aligned} \tag{2.7}$$

Note that the Dirichlet condition $G^{sc}(\cdot\,;y)=-G^{in}(\cdot\,;y)$ on Γ is equivalent to $G(x;y)=0$ for any $x\in\Gamma$. As usual, the inhomogeneous Dirichlet problem for the homogeneous Helmholtz equation is reformulated into a homogeneous Dirichlet problem for the inhomogeneous Helmholtz equation. In other words, the variational problem corresponding to (2.7) is formulated with respect to the unknown solution $G^{sc}_v(\cdot\,;y):=G^{sc}(\cdot\,;y)+G^{in}_c(\cdot\,;y)\in V_{h,\varrho}$, where $G^{in}_c(\cdot\,;y)\in H^1_\varrho(\Omega_{\Gamma,h})$ is a fixed continuation of the Dirichlet data $G^{in}(\cdot\,;y)|_\Gamma$ from Γ to $\Omega_{\Gamma,h}$. Such a continuation can be chosen as the product of $G^{in}(\cdot\,;y)$ times a cutoff function, which cuts off the singularity at the critical source point y, which is identically one over Γ, and which is zero at x with large x_2. For the details and a proof of the following theorem, we refer to the arguments of [10, Thm. 4.1] (also cf. [35, sect. 2.3]).

THEOREM 2.1. *The scattering problem* (2.7) *for the incident point-source wave $G^{in}(x;y)$ with fixed $y\in\Omega_\Gamma$ has exactly one variational solution, $G^{sc}(x;y)=-G^{in}_c(x;y)+G^{sc}_v(x;y)$ with $G^{in}_c(\cdot\,;y)$ in $H^1_\varrho(\Omega_{\Gamma,h})$, with the variational solution $G^{in}_c(\cdot\,;y)|_\Gamma=G^{in}(\cdot\,;y)|_\Gamma$, and with $G^{sc}_v(\cdot\,;y)\in V_{h,\varrho}$ for all heights $h>\max\{x_2: x\in\Gamma\}$ and $-1<\varrho<0$ (cf. the variational problem in* [10, Thm. 4.1]*). In particular, we get $G^{sc}(\cdot\,;y)\in H^1_\varrho(\Omega_{\Gamma,h})\cap C^2(\Omega_{\Gamma,h})$, $\rho<0$.*

Clearly, the function $G=G^{in}+G^{sc}$ is the Green's function of the boundary value problem (2.2) with the radiation condition ASR.

The proof of Theorem 2.1 relies essentially on the decay property of G^{in} on Γ. For three dimensions, it was proved in [10] that $G^{sc}(\cdot\,;y)\in H^1_\varrho(\Omega_{\Gamma,h})$ with $\varrho\in(-1,-1/2)$. The two-dimensional case can be treated analogously with the index $\rho\in(-1,0)$ (see also the arguments presented in Appendix A). We remark that, in two dimensions, the previous well-posedness results imply that $G^{sc}_h:=G^{sc}(\cdot\,;h)\in H^{1/2}_\varrho(\Gamma_h)$ and thus, by Fourier transform, $\hat G^{sc}_h\in H^\varrho_{1/2}(\Gamma_h)$. On the other hand, it was proved in [10] for the case $h=0$ that the function $y_1\to\partial\Phi^*_h(x;y_1,h)/\partial y_2$ belongs to $H^{-1/2}_{-\rho}(\mathbb{R})$ if and only if $\rho>-1$. Hence, the integral on the right-hand side of of (2.6) can be understood as the duality between $\hat u_h=\hat G^{sc}_h\in H^\varrho_{1/2}(\Gamma_h)$ and the function $\xi\to\exp(i[(x_2-h)\sqrt{k^2-\xi^2}+x_1\xi])$ in the dual space $H^{-\varrho}_{-1/2}(\Gamma_h)$ for $\rho\in(-1,0)$ (see [10]).

For any $r>0$, write $S^\Gamma_r:=S_r:=\{x\in\Omega_\Gamma\colon |x|=r\}$. In other words, S_r is a circular arc centered at the origin and of radius r in Ω_Γ with endpoints located at Γ. Below we shall prove that, for point-source incidence, the upward ASR (2.6) is equivalent to the Sommerfeld outgoing radiation condition in a half-plane, which is defined as follows.

DEFINITION 2.1. *Let $v\in C^\infty(\Omega_\Gamma\cap\{x\in\mathbb{R}^2\colon |x|>R\})$ for a sufficiently large $R>0$. Then we say that v satisfies the half-plane Sommerfeld radiation condition (HPSRC) if, for any positive number $h>\max\{x_2\colon x\in\Gamma\}$, the function v is in $H^1_\varrho(\Omega_{\Gamma,h}\cap\{x\in\mathbb{R}^2\colon |x_1|>R\})$ for all $\varrho<0$ and if*

$$\sup_{x\in S_r\cap U_h} r^{1/2}\,|\partial_\nu v(x)-ikv(x)|\to 0,\ r\to\infty, \qquad \sup_{x\in\Omega_\Gamma\cap U_h:|x|\geq R}|x|^{1/2}\,|v(x)|<\infty. \tag{2.8}$$

If v satisfies the HPSRC with (2.8) *replaced by*

$$\int_{S_r\cap U_h}|\partial_\nu v-ikv|^2\,\mathrm{d}s\to 0,\ r\to\infty, \qquad \sup_{0<r}\int_{S_r\cap U_h}|v|^2\,\mathrm{d}s<\infty, \tag{2.9}$$

then we shall say that v fulfills the weak half-plane Sommerfeld radiation condition (wHPSRC).

Remark 2.1. A plane wave of the form (2.1) belongs to $H^1_\varrho(\Omega_{\Gamma,h}\cap\{x\in\mathbb{R}^2 : |x_1|>R\})$ with $\varrho < -1/2$; for the cylindrical wave $\Phi(x,y)$ this holds for $\varrho < 0$ for $R>|y_1|$. Hence, the upper bound $\rho=0$ in the Definition 2.1 includes cylindrical waves but excludes plane and surface waves.

The integrals in (2.9) are defined over $S_r\cap U_h$ rather than S_r, because the normal derivative $\partial_\nu v$ on $S_r\cap\Omega_{\Gamma,h}$ might not exist in the L^2-sense. The connections between the different radiation conditions are summarized as follows: Obviously, HPSRC implies wHPSRC. On the other hand, any Helmholtz solution $v=u_0$ over the domain Ω_Γ (or the perturbed domain Ω_Λ in section 3) satisfying the wHPSRC and $v|_\Gamma=0$ (or $v|_\Lambda=0$) can be represented as (3.7) (cf. the arguments leading to (3.7)), and the subsequent Lemma 3.1 implies the HPSRC. Furthermore, note that the wHPSRC for Helmholtz solutions is stronger than the ASR (cf. (2.6)). Indeed, by [15, Thm. 2.9] it holds that such a v satisfies the UPRC and equivalently the ASR (cf. [10]). Vice versa, the ASR together with the decay condition $v|_{\Gamma_h\cap\{x\in\mathbb{R}^2:\,|x|>R\}}\in L^2_\varrho$, with a ϱ s.t. $1/2<\varrho<1$, implies the HPSRC (cf. the proof of Lemma A.2). Hence in many cases, HPSRC, wHPSRC, and ASR are equivalent.

The function $x\to\Phi(x;y)$ with $y\in\mathbb{R}^2$ satisfies (2.8). For functions satisfying the HPSRC, we define the far-field pattern over the set of directions $\hat{x}\in\mathbb{S}_+$ with $\mathbb{S}_+:=\{x\in\mathbb{R}^2: x_2>0, |x|=1\}$.

Definition 2.2. *Let $v\in C^\infty(\Omega_\Gamma\cap\{x\in\mathbb{R}^2: |x|>R\})$ for a sufficiently large $R>0$. We shall call the continuous function $v_\infty\in C(\overline{\mathbb{S}_+})$ the far-field pattern of v if there is an $h>\max_{x\in\Gamma}\{x_2\}$ s.t.*

$$\sup_{x=r\hat{x}\in S_r\cap U_h}\left|v(x)-\frac{\exp(ikr)}{r^{1/2}}v_\infty(\hat{x})\right|r^{1/2}\longrightarrow 0,\quad r\to\infty. \tag{2.10}$$

In other words, by Definition 2.2, v_∞ is the far-field pattern of v in Ω_Γ if the asymptotic behavior

$$v(x)=\frac{e^{ik|x|}}{\sqrt{|x|}}v_\infty(\hat{x})+o(|x|^{-1/2})\qquad\text{as}\quad |x|\to\infty$$

holds uniformly in all $x\in U_h$ for some $h>\max_{x\in\Gamma}\{x_2\}$. We note that the above definition of far-field pattern is independent of the choice of h. The lemma below shows that the scattered field caused by G^{in} also fulfills the stronger condition of Definition 2.1 and admits an asymptotic behavior like in (2.10). The same is true for the derivatives of G.

Lemma 2.2. *For any fixed $y\in\Omega_\Gamma$ and the Green's function $G(\cdot\,;y)=G^{in}(\cdot\,;y)+G^{sc}(\cdot\,;y)$ with $G^{sc}(\cdot\,;y)$ of Theorem 2.1, the scattered field $G^{sc}(\cdot\,;y)$ satisfies the HPSRC and has a far-field pattern in $C(\overline{\mathbb{S}_+})$. Moreover, $G(\cdot\,;y)\in H^1_\varrho(\Omega_{\Gamma,h}\cap\{x\in\mathbb{R}^2: |x_1|>R\})$ for any $|\varrho|<1$ and $R>|y_1|$.*

Lemma 2.3. *Suppose l_j, $j=1,2$, are nonnegative integers. The assertion of Lemma 2.2 holds for $G(\cdot\,;y)$ replaced by the derivative $\partial^{l_1}_{y_1}\partial^{l_2}_{y_2}G(\cdot\,;y)$.*

For Γ the graph of a $C^{1,1}$-smooth function and for an incident wave with compactly supported source in Ω_Γ at a positive distance from Γ, the assertion of Lemma 2.2 is discussed already in [12, Thm. 5.1] but without proofs. If Γ is the graph of a $C^{1,1}$-function, the second condition in (2.8) and the relation $G(\cdot\,;y)\in H^1_\varrho(\Omega_{\Gamma,h}\cap\{x\in\mathbb{R}^2:$

$|x_1|>R\}$) for any $|\varrho|<1$ and $R>|y_1|$ are implicitly contained in [40, Cors. 4.2 and 4.4]. In the special case $\Gamma=\Gamma_0:=\{x\in\mathbb{R}^2: x_2=0\}$, Lemma 2.2 follows straightforwardly from the explicit formula

$$G(x;y)=\Phi(x;y)-\Phi(x;y^*),\quad y^*:=(y_1,-y_2).$$

In Appendix A we shall present a proof valid for a Lipschitz (nonperiodic) rough surface satisfying (2.4), which means an infinite boundary surface Γ of a simply connected domain Ω_Γ such that $U_0\subset\Omega_\Gamma\subset U_{h_\Gamma}$ for a real $h_\Gamma>0$ and that, for fixed numbers $\varepsilon_\Gamma>0$ and $C_\Gamma>0$ and for each $x_0\in\Gamma$, the set $\{x\in\Gamma: |x-x_0|<\varepsilon_\Gamma\}$ is a rotated graph of a Lipschitz function with Lipschitz constant C_Γ. Collecting the assertions of Theorem 2.1 and Lemma 2.2 together we get the following.

THEOREM 2.2. *For a sound-soft rough surface satisfying the condition* (2.4)*, equations* (2.7) *have a unique solution* $G^{sc}(\cdot\,;y)\in H^1_\varrho(\Omega_{\Gamma,h})$ *for all* $h>\max\{x_2: x\in\Gamma\}$ *and* $-1<\varrho<0$ *so that the Green's function* $G(\cdot\,;y)=G^{in}(\cdot\,;y)+G^{sc}(\cdot\,;y)$ *(cf.* (2.5)*) is well defined. Moreover,* $G^{sc}(\cdot\,;y)$ *satisfies the HPSRC and has a far-field pattern in* $C(\overline{\mathbb{S}_+})$*, and* $G(\cdot\,;y)\in H^1_\varrho(\Omega_{\Gamma,h}\cap\{x\in\mathbb{R}^2: |x_1|>R\})$ *for any* $|\varrho|<1$ *and* $R>|y_1|$.

We note that, using the approach of approximating the boundary curve of [10], even a larger class of nonsmooth surfaces, namely, graphs of arbitrary bounded continuous functions, can be treated.

Remark 2.2. The assertion of Lemma 2.2 does not hold for the scattered field generated by plane-wave incidence, due to the appearance of propagating wave modes, which do not decay at infinity.

As a consequence of the proof of Theorem 2.2 in Appendix A, we obtain the following well-posedness result on rough surface scattering problems.

COROLLARY 2.1. *Suppose that the boundary curve* Γ *is uniformly Lipschitz continuous, the domain* Ω_Γ *fulfills the condition* (2.4)*, and that* $f_\Gamma\in H^{1/2}_\varrho(\Gamma)$ *for some* $\varrho>1/2$*. Moreover, suppose there exists an extension* $w\in H^1_\varrho(\Omega_\Gamma)$ *of* f_Γ *(i.e.,* $w|_\Gamma=f_\Gamma$*) s.t., additionally,* $\Delta w\in L^2_\varrho(\Omega_\Gamma)$*. Then the boundary value problem* $v=f_\Gamma$ *on* Γ *for* $\Delta v+k^2v=0$ *in* Ω_Γ *under the condition ASR admits a unique solution* $v\in H^1_\rho(\Omega_{\Gamma,h})\cap C^2(\Omega_{\Gamma,h})$ *for all* $h>\max\{x_2: x\in\Gamma\}$*, which satisfies the HPSRC and has a far-field pattern in* $C(\overline{\mathbb{S}_+})$.

Remark 2.3. (i) If Γ is a bounded closed surface, the existence of w in Corollary 2.1 follows directly from the extension theorem of [38]. However, we do not know the corresponding extension theory for unbounded surfaces.

(ii) The condition on the index of decay $\varrho>1/2$ seems not to be sharp. For instance, the function $\Phi(\cdot\,;y)|_\Gamma$ for $y\in\mathbb{R}^2\backslash\overline{\Omega}_\Gamma$ belongs to $H^{1/2}_\varrho(\Gamma)$ with $\varrho<0$, and $\Phi(\cdot\,;y)$ still fulfills the HPSRC.

2.3. Local behavior of the Green's function and boundary integral operators based on the Green's function. Recalling that the unperturbed grating surface Γ contains at least one line segment in each period, for any $R>0$, we can choose two line segments Γ_a and Γ_b contained in $\Gamma\cap\{x\in\mathbb{R}^2: |x|>R\}$, Γ_a on the left of $\Gamma\cap\{x\in\mathbb{R}^2: |x|<R\}$ and Γ_b on the right. Let the curve C_R with $C_R\subset\Omega_\Gamma\cap\{x: |x|>R\}$ be an open curve with endpoints located at Γ_a and Γ_b, respectively. We emphasize that the restriction to interfaces containing straight-line segments is a technical condition needed to control the behavior of the Green's function close to the intersection of the interface and of special arcs for potential operators. This condition is needed in subsection 2.3 and in section 3 only. We suppose that, using perturbation techniques and defining the subsequent lines L_a and L_b as the tangential lines at a and b, respectively, the straight-line segments can be replaced by arcs of finite degree of smoothness. To derive the results for general Lipschitz interfaces by the arguments of the present paper might be difficult.

In section 3 we shall use the single and double layer operators defined over C_R, where the fundamental solution in the kernel function is replaced by the Green's function G. To give the correct definition of these operators and to obtain the usual strong ellipticity and compactness, respectively, we first have to look at the local behavior of the Green's kernel. Since the endpoints of C_R are chosen to be located at Γ, the behavior close to the grating surface Γ is important.

As usual, for any bounded domain of positive distance to the boundary Γ, the Green's function $G(\cdot\,;y)$ is the sum of the free-space fundamental solution $\Phi(\cdot\,,y)$ plus a smooth function. However, the behavior of the Green's function close to the boundary is hard to predict. Additional assumptions on the curve are needed. We consider bounded subdomains Ω_a, Ω_b, and Ω_c of Ω_Γ (cf. Figure 2.2) such that, for a fixed positive ε,

$$\begin{aligned}
\overline{\Omega}_a \cap \Gamma &\subset \{x \in \Gamma_a : \operatorname{dist}(x, \Gamma \setminus \Gamma_a) \geq \varepsilon\},\\
\overline{\Omega}_b \cap \Gamma &\subset \{x \in \Gamma_b : \operatorname{dist}(x, \Gamma \setminus \Gamma_b) \geq \varepsilon\},\\
\overline{\Omega}_c \quad &\subset \{x \in \Omega_\Gamma : \operatorname{dist}(x, \Gamma) \geq \varepsilon\}.
\end{aligned}$$

For $y\in\Omega_a$, we compare $G(x;y)$ with $G_a(x;y)$ the Green's function of the half-space Ω_a^{hs} bounded by the straight line L_a containing Γ_a such that Ω_Γ^{hs} and Ω_a are on the same side of Γ_a. Clearly, $G_a(x;y)=\Phi(x;y)-\Phi(x;y_a^*)$, where we denote the mirror image of $y \in \mathbb{R}^2$ w.r.t. line L_a by y_a^*. Finally, we assume that the reflected closed domain $\overline{\Omega}_a^* := \{y_a^* : y\in\overline{\Omega}_a\}$ does not intersect $\overline{\Omega}_b$ and $\overline{\Omega}_c$. In Appendix A we shall prove the following lemma.

LEMMA 2.4. *For $x\in\Omega_a\cup\Omega_b\cup\Omega_c$ and $y\in\Omega_a$, the Green's function G over Ω_Γ takes the form $G(x;y)=G_a(x;y)+R(x;y)$, where the remainder function R is smooth in the sense that R and all its derivatives are bounded and continuous over $(\overline{\Omega}_a\cup\overline{\Omega}_b\cup\overline{\Omega}_c)\times\overline{\Omega}_a$.*

Now we consider an open curve C_R with the following properties: The curve should be a twice continuously differentiable curved arc connecting the midpoints a of Γ_a and b of Γ_b. We assume that C_R has no self-intersections and that $C_R\subset\Omega_\Gamma\cup\{a,b\}$.

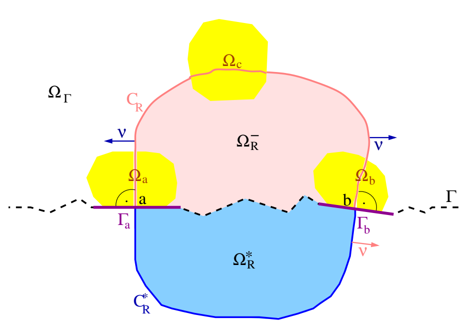

FIG. 2.2. *Straight-line segments Γ_a and Γ_b, examples of domains Ω_a, Ω_b, and Ω_c as well as curves C_R and C_R^*.*

Moreover, we assume that C_R intersects Γ_a and Γ_b at a and b under a right angle (cf. Figure 2.2). If all this is satisfied, then there exists a second arc C_R^* connecting a and b such that C_R^* has no self-intersections, that $C_R^* \subset (\mathbb{R}^2 \setminus \overline{\Omega}_\Gamma) \cup \{a, b\}$, and that C_R^* intersects Γ_a and Γ_b at a and b under a right angle. We may suppose that, for an $r_R > 0$, the arc $\{x \in C_R^* \colon |x-a| < r_R\}$ coincides with the reflected arc $\{x_a^* \colon x \in C_R,\, |x-a| < r_R\}$ and that an analogous condition holds close to b (cf. Figure 2.2). In other words, C_R is a subarc of the close curve $\widetilde{C}_R := C_R \cup C_R^*$, which is twice continuously differentiable. The single and double layer operator over C_R based on the Green's function are defined by

$$(2.11)\qquad (\mathcal{S}p)(x) \;:=\; (\mathcal{S}_{C_R}p)(x) := \int_{C_R} G(x;y)p(y)\,\mathrm{d}s(y), \qquad x \in C_R,$$

$$(2.12)\qquad (\mathcal{D}q)(x) \;:=\; (\mathcal{D}_{C_R}q)(x) := \int_{C_R} \partial_{\nu(y)} G(x;y)q(y)\,\mathrm{d}s(y), \qquad x \in C_R,$$

where $\nu(y)$ is the unit vector normal to $\widetilde{C}_R$ at $y \in \widetilde{C}_R$ pointing into the exterior of the domain $\widetilde{\Omega}_R$ enclosed by $\widetilde{C}_R$.

To get the mapping properties of $\mathcal{S}$ and $\mathcal{D}$, we need special Sobolev spaces. We choose cutoff functions $\chi_a, \chi_b \in C_0^\infty(\mathbb{R}^2)$ such that, for the r_R used before (2.11),

$$\{x \in \mathbb{R}^2 \colon |x-a| < r_R/2\} \subset \{x \in \mathbb{R}^2 \colon \chi_a(x) = 1\} \subset \operatorname{supp}\chi_a \subset \{x \in \mathbb{R}^2 \colon |x-a| < r_R\},$$
$$\{x \in \mathbb{R}^2 \colon |x-b| < r_R/2\} \subset \{x \in \mathbb{R}^2 \colon \chi_b(x) = 1\} \subset \operatorname{supp}\chi_b \subset \{x \in \mathbb{R}^2 \colon |x-b| < r_R\}.$$

With these and with x_b^* defined for b analogously as x_a^* for a, we introduce the Sobolev spaces

(2.13)
$$\widetilde{H}^{1/2}(C_R) := \left\{ u|_{C_R} \colon u \in H^{1/2}(\widetilde{C}_R) \text{ s.t. } [\chi_a u](x) = -[\chi_a u](x_a^*) \text{ and } [\chi_b u](x) = -[\chi_b u](x_b^*) \right\},$$

(2.14)
$$H^{-1/2}(C_R) := \left\{ v|_{C_R} \colon v \in H^{-1/2}(\widetilde{C}_R) \text{ s.t. } [\chi_a v](x) = -[\chi_a v](x_a^*) \text{ and } [\chi_b v](x) = -[\chi_b v](x_b^*) \right\},$$

where $\chi_a v$, $\chi_b v$, $[\chi_a v](x_a^*)$, and $[\chi_b v](x_b^*)$ are defined in the distributional sense. It is not hard to see that $\widetilde{H}^{1/2}(C_R)$ and $H^{-1/2}(C_R)$ are dual spaces, where the duality extends the scalar product of $L^2(C_R)$ such that

(2.15)
$$\int_{C_R} [\chi_a u]|_{C_R} \overline{[\chi_a v]}|_{C_R} \;=\; \frac{1}{2}\int_{\widetilde{C}_R} [\chi_a u]\overline{[\chi_a v]}, \quad \int_{C_R} [\chi_b u]|_{C_R} \overline{[\chi_b v]}|_{C_R} \;=\; \frac{1}{2}\int_{\widetilde{C}_R} [\chi_b u]\overline{[\chi_b v]},$$

which serves as the definition for $(\chi_a v)|_{C_R}$ and $(\chi_b v)|_{C_R}$. Now introduce the domains Ω_R^- and Ω_R^* as those enclosed by Γ together with C_R and C_R^*, respectively. The trace $U|_{C_R}$ of a function $U \in H^1(\Omega_R^-)$ with $U|_\Gamma \equiv 0$ is in $\widetilde{H}^{1/2}(C_R)$ since $\chi_a U$ can be extended to Ω_R^* by $[\chi_a U](x) := -[\chi_a U](x_a^*)$ such that $[\chi_a U] \in H^1(\widetilde{\Omega}_R)$ and $[\chi_a u] := [\chi_a U]|_{\widetilde{C}_R} \in H^{1/2}(\widetilde{C}_R)$ satisfies $[\chi_a U|_{C_R}](x) = -[\chi_a U|_{C_R}](x_a^*)$. Similarly, the trace $\partial_\nu V|_{C_R}$ of a function $V \in H^1(\Omega_R^-)$ with $(\Delta + k^2 I)V \equiv 0$ over Ω_R^- and with $V|_\Gamma \equiv 0$ is in $H^{-1/2}(S_R)$. Indeed, defining $[\chi_a V](x) := -[\chi_a V](x_a^*)$, we get $[\chi_a V] \in H^1(\widetilde{\Omega})$ and

$$
\begin{aligned}
&\int_{\widetilde{C}_R} [\chi_a U]|_{\widetilde{C}_R} \partial_\nu [\chi_a V]|_{\widetilde{C}_R} \\
&\quad = -\int_{\Gamma_a \cap \widetilde{\Omega}_R} [\chi_a U](x) \Big[\partial_{x_1} [\chi_a V](x_1, x_2+0) - \partial_{x_1}[\chi_a V](x_1, x_2-0) \Big] \mathrm{d}x_1 \\
&\qquad + \int_{\widetilde{\Omega}_R} [\chi_a U] \{ (\Delta + k^2 I)[\chi_a V] \} + \int_{\widetilde{\Omega}_R} \left\{ \nabla [\chi_a U] \cdot \nabla [\chi_a V] - k^2 [\chi_a U][\chi_a V] \right\} \\
&\quad = \int_{\widetilde{\Omega}_R^-} \left\{ \nabla [\chi_a U] \cdot \nabla [\chi_a V] - k^2 [\chi_a U][\chi_a V] \right\}, \\
&\int_{C_R} [\chi_a U]|_{C_R} \partial_\nu [\chi_a V]|_{C_R} = \int_{\Omega_R^-} \left\{ \nabla [\chi_a U] \cdot \nabla [\chi_a V] - k^2 [\chi_a U][\chi_a V] \right\},
\end{aligned}
$$

which defines a continuous functional.

Recalling the local behavior $G(x;y)=G_a(x;y)+R(x;y)$ of the Green's function, (2.12) turns to the representation

$$
\begin{aligned}
(\mathcal{D}[\chi_a U])(x) &= \int_{C_R} \partial_{\nu(y)} R(x;y) [\chi_a U](y)\, \mathrm{d}s(y) \\
&\quad + \int_{C_R} \partial_{\nu(y)} \Phi(x;y) [\chi_a U](y)\, \mathrm{d}s(y) - \int_{C_R} \partial_{\nu(y)} \Phi(x; y_a^*) [\chi_a U](y)\, \mathrm{d}s(y) \\
&= \int_{C_R} \partial_{\nu(y)} R(x;y) [\chi_a U](y)\, \mathrm{d}s(y) + \int_{\widetilde{C}_R} \partial_{\nu(y)} \Phi(x;y) [\chi_a U](y)\, \mathrm{d}s(y).
\end{aligned}
$$

Hence, $\mathcal{D}$ is well defined over $\widetilde{H}^{1/2}(C_R)$. The boundedness and compactness of $\mathcal{D}_{\widetilde{C}_R}$ over $H^{1/2}(\widetilde{C}_R)$ implies the boundedness and compactness of $\mathcal{D}:=\mathcal{D}_{C_R}$ in the space $\widetilde{H}^{1/2}(C_R)$. Similarly,

$$
(\mathcal{S}[\chi_a U])(x) = \int_{C_R} R(x;y)[\chi_a U](y)\, \mathrm{d}s(y) + \int_{\widetilde{C}_R} \Phi(x;y) [\chi_a U](y)\, \mathrm{d}s(y).
$$

In view of (2.15), we conclude

$$
\begin{aligned}
\int_{C_R} \Big[\mathcal{S}[\chi_a U] \Big] \overline{[\chi_a U]} &= \int_{C_R} \left[\int_{C_R} R(\cdot\,; y)[\chi_a U](y)\, \mathrm{d}s(y) \right] \overline{[\chi_a U]} \\
&\quad + \frac{1}{2} \int_{\widetilde{C}_R} \left[\int_{\widetilde{C}_R} \Phi(\cdot\,; y)[\chi_a U](y)\, \mathrm{d}s(y) \right] \overline{[\chi_a U]}.
\end{aligned}
$$

Hence, $\mathcal{S}$ is well defined as an operator, mapping $H^{-1/2}(C_R)$ into $\widetilde{H}^{1/2}(C_R)$. The boundedness and strong ellipticity of $\mathcal{S}_{\widetilde{C}_R}$, mapping $H^{-1/2}(\widetilde{C}_R)$ into $H^{1/2}(\widetilde{C}_R)$, imply the boundedness and strong ellipticity of $\mathcal{S}:=\mathcal{S}_{C_R}$.

3. Scattering from locally perturbed periodic surfaces. Now consider a one-dimensional Lipschitz curve $\Lambda \subset \mathbb{R}^2$ different from Γ, and suppose (2.4) for Λ instead of Γ. The curve Λ is said to be a local perturbation of the periodic interface Γ if Λ coincides with Γ in $\{x \in \mathbb{R}^2 : |x_1| > R\}$ for some fixed $R>0$. In other words, Λ differs from Γ in a compact set which may stand for a defect of Γ. The presence of the defect causes a perturbation u of the total wave field $v=v^{in}+v^{sc}$ that corresponds to the perfectly periodic interface Γ. In this section we show the relation between the perturbed and unperturbed scattering problems. We keep the notation used in section 2 and choose a curve C_R in accordance with the assumptions in subsection 2.3

following Lemma 2.4 such that the perturbation $\Lambda\backslash\Gamma$ of Γ is located beneath C_R, and we set (cf. Figure 2.1, lower part)

$$\Lambda_R := \{x \in \Lambda \colon x \text{ between } a \text{ and } b\}, \quad \Gamma_R := \{x \in \Gamma \colon x \text{ between } a \text{ and } b\}, \tag{3.1}$$
$$\Omega_R^- := \{x \in \Omega_\Lambda \colon x \text{ between } C_R \text{ and } \Lambda_R\}, \quad \Omega_R^+ := \Omega_\Lambda \setminus \overline{\Omega_R^-}.$$

Here, Ω_Λ denotes the unbounded Lipschitz domain above Λ, which is supposed to fulfill the geometrical condition (2.4). This admits the perturbed part Λ_R to contain vertical line segments. Assume a plane wave $v^{in}(x;\theta)$ of the form (2.1) is incident onto Λ from Ω_Λ. We seek the total field $u \in H^1_{loc}(\Omega_\Lambda)$ such that

$$\begin{cases} \Delta u + k^2 u = 0 \quad \text{in} \quad \Omega_\Lambda, \\ u = 0 \quad \text{on} \quad \Lambda, \\ u_0 := u - v \text{ satisfies the HPSRC in } \Omega_\Lambda \cap \Omega_\Gamma, \end{cases} \tag{3.2}$$

where $v = v^{in} + v^{sc}$ is the total field generated by the unperturbed surface Γ. Whereas any weak solution of the Helmholtz equation over Ω_Λ is twice continuously differentiable at any point x of Ω_Λ with $\Delta u(x) + k^2 u(x) = 0$, the Dirichlet condition $u = 0$ over Λ is defined in the sense of traces $H^{1/2,loc}(\Lambda) \ni u|_\Lambda = 0$, i.e., $u \in H_0^{1,loc}(\Omega_\Lambda)$. We shall call the u with these properties a solution of (3.2) in $H_0^{1,loc}(\Omega_\Lambda) \cap C^2(\Omega_\Lambda)$. Since both u and v vanish on $\Lambda\backslash\Lambda_R$, the function u_0 should also vanish on $\Lambda\backslash\Lambda_R$.

Next we derive a variational formulation for the problem (3.2) (cf. the subsequent (3.10)). Define the energy space X_R over the truncated domain Ω_R^- as $X_R := \{u \in H^1(\Omega_R^-) \colon u = 0 \text{ on } \Lambda_R\}$, which is equipped with the usual H^1-norm

$$\|u\|_{X_R}^2 := \int_{\Omega_R^-} \left\{ |\nabla u|^2 + |u|^2 \right\} \mathrm{d}x.$$

We introduce the Sobolev spaces on the open arc C_R by (2.13) and (2.14). It is easy to derive the following variational formulation for u:

$$\int_{\Omega_R^-} \left\{ \nabla u \cdot \nabla \overline{\phi} - k^2 u \overline{\phi} \right\} \mathrm{d}x - \int_{C_R} \partial_\nu u \overline{\phi} \, \mathrm{d}s = 0 \quad \text{for all } \phi \in X_R,$$

where ν is the unit normal on S_R pointing into Ω_R^+. Equivalently, we have

$$\int_{\Omega_R^-} \left\{ \nabla u \cdot \nabla \overline{\phi} - k^2 u \overline{\phi} \right\} \mathrm{d}x - \int_{C_R} \partial_\nu u_0 \, \overline{\phi} \, \mathrm{d}s = \int_{C_R} \partial_\nu v \, \overline{\phi} \, \mathrm{d}s \quad \text{for all } \phi \in X_R. \tag{3.3}$$

Choosing $R_1 > R$ sufficiently large and applying Green's formula to u_0, we see

$$u_0(x) = \left(-\int_{S_{R_1}} + \int_{C_R} \right) \left[u_0(y) \partial_{\nu(y)} G(x;y) - \partial_{\nu(y)} u_0(y) G(x;y) \right] \mathrm{d}s(y) \tag{3.4}$$

for $x \in \Omega_R^{\mathrm{an}} := \{x \in \Omega_\Gamma \colon |x| > R_1\} \cap \Omega_R^+$. Note that the annular domain Ω_R^{an} is a Lipschitz domain by our assumption on Γ. Moreover, both u_0 and $G = G^\Gamma$ vanish on $\Lambda\backslash\Lambda_R$. Recall the definition of U_h from (2.3) and $\Omega_{\Gamma,h} := \Omega_\Gamma \backslash \overline{U_h}$. Taking $h > \max\{x_2 \colon x \in \Gamma\}$ and making use of the wHPSRC of u_0 and G yield

$$
\begin{aligned}
(3.5)\quad &\int_{S_{R_1}\cap U_h}\left[u_0(y)\partial_{\nu(y)}G(x;y)-\partial_{\nu(y)}u_0(y)G(x;y)\right]\mathrm{d}s(y)\\
&=\int_{S_{R_1}\cap U_h}\left\{u_0(y)[\partial_{\nu(y)}G(y;x)-ikG(y;x)]-[\partial_{\nu(y)}u_0(y)-iku_0(y)]G(y;x)\right\}\mathrm{d}s(y)\\
&\le ||u_0||_{L^2(S_{R_1}\cap U_h)}\,||\partial_\nu G(\cdot;x)-ikG(\cdot;x)||_{L^2(S_{R_1}\cap U_h)}\\
&\quad +||G(\cdot;x)||_{L^2(S_{R_1}\cap U_h)}\,||\partial_\nu u_0-iku_0||_{L^2(S_{R_1}\cap U_h)}\\
&\to 0
\end{aligned}
$$

as $R_1\to\infty$. Here we have used the symmetry $G(x;y)=G(y;x)$ (cf., e.g., [35, Thm. 7]), which can be proved following the lines in the proof of Theorem 3.1.4 of [36] and using Lemma 2.2 (apply arguments like in (3.5) and (3.6) to derive the formula before equation (3.12) of [36] to prove Theorem 3.1.4 in [36]). Further, the integral over the remaining part $S_{R_1,h}:=S_{R_1}\cap\Omega_{\Gamma,h}$ of S_{R_1} can be estimated by

$$
\begin{aligned}
(3.6)\quad &\int_{S_{R_1,h}}\left[u_0(y)\partial_{\nu(y)}G(x;y)-\partial_{\nu(y)}u_0(y)G(x;y)\right]\mathrm{d}s(y)\\
&\le ||u_0(1+|y_1|^{\varrho})||_{\widetilde{H}^{1/2}(S_{R_1,h})}\,||\partial_{\nu(y)}G(x;\cdot)(1+|y_1|^{-\varrho})||_{H^{-1/2}(S_{R_1,h})}\\
&\quad +||\partial_\nu u_0(1+|y_1|^{\varrho})||_{H^{-1/2}(S_{R_1,h})}\,||G(x;\cdot)(1+|y_1|^{-\varrho})||_{\widetilde{H}^{1/2}(S_{R_1,h})}\\
&\le ||u_0||_{\widetilde{H}^{1/2}_{\varrho}(S_{R_1,h})}||\partial_{\nu(y)}G(x;\cdot)||_{H^{-1/2}_{-\varrho}(S_{R_1,h})}+||\partial_\nu u_0||_{H^{-1/2}_{\varrho}(S_{R_1,h})}||G(x;\cdot)||_{\widetilde{H}^{1/2}_{-\varrho}(S_{R_1,h})}\\
&\le C\,||u_0||_{H^1_{\varrho}(\Sigma_{R_1,h})}\,||G(x;\cdot)||_{H^1_{-\varrho}(\Sigma_{R_1,h})}.
\end{aligned}
$$

Here, we choose $\varrho\in(-1,0)$ from the wHPSRC for u_0 and take $\Sigma_{R_1,h}\subset\Omega_{\Gamma,h}$ as a small region with fixed area that contains $S_{R_1,h}$ inside. In view of the wHPSRC relation $u_0\in H^1_{\varrho}(\{x\in\Omega_{\Gamma,h}\colon |x_1|>R\})$ and of the fact that $G(x;\cdot)\in H^1_{-\varrho}(\{x\in\Omega_{\Gamma,h}\colon |x_1|>R\})$, the right-hand side of the previous inequality tends to zero as $R_1\to\infty$. This together with (3.5) implies that

$$\int_{S_{R_1}}\left[u_0(y)\partial_{\nu(y)}G(x;y)-\partial_{\nu(y)}u_0(y)G(x;y)\right]\mathrm{d}s(y)\to 0\quad\text{as } R_1\to\infty.$$

Hence, letting $R_1\to\infty$ in (3.4), we can represent the function u_0 as

$$(3.7)\qquad u_0(x)=\int_{C_R}\left[u_0(y)\partial_{\nu(y)}G(x;y)-\partial_{\nu(y)}u_0(y)G(x;y)\right]\mathrm{d}s(y),\quad x\in\Omega_R^+.$$

Take the limit of (3.7) for x tending to a point $x\in C_R$, and set $p:=\partial_\nu u_0|_{C_R}\in H^{-1/2}(C_R)$ $q:=u_0|_{C_R}\in\widetilde{H}^{1/2}(C_R)$; we then arrive at the integral equation

$$(3.8)\qquad \Big(\frac{1}{2}I-\mathcal{D}\Big)q+\mathcal{S}p=0\quad\text{on}\quad C_R.$$

Here $\mathcal{D}$ and $\mathcal{S}$ are the double and single layer potentials over C_R defined by (2.11) and (2.12), respectively. Note that the classical jump relations apply for the special Green's function. Indeed, on Ω_Γ the function G is locally the sum of the classical full-space Green's function Φ plus a smooth function since the solution of the Helmholtz equation is analytic in any domain away from the boundary. Recalling $q=(u-v)|_{C_R}$, we can rewrite (3.8) as

$$(3.9)\qquad \Big(\frac{1}{2}I-\mathcal{D}\Big)(u|_{C_R})+\mathcal{S}p=\Big(\frac{1}{2}I-\mathcal{D}\Big)(v|_{C_R})\quad\text{on}\quad C_R.$$

The equations (3.3) and (3.9) give the variational formulation for the unknown solution $u\in X_R$ and $p\in H^{-1/2}(C_R)$:

$$\begin{aligned} A\big((u,p),(\varphi,\chi)\big) &:= a_1\big((u,p),(\varphi,\chi)\big) + 2\,a_2\big((u,p),(\varphi,\chi)\big)\\ &= \int_{C_R} \partial_\nu v\,\overline{\varphi}\,\mathrm{d}s \qquad + 2\int_{C_R} (\frac{1}{2}I-\mathcal{D})(v|_{C_R})\overline{\chi}\,\mathrm{d}s \end{aligned} \tag{3.10}$$

for all $(\varphi,\chi)\in X_R\times H^{-1/2}(C_R)$, where

$$\begin{aligned} a_1\big((u,p),(\varphi,\chi)\big) &:= \int_{\Omega_R^-} \left\{\nabla u\cdot\nabla\,\overline{\varphi} - k^2 u\,\overline{\varphi}\right\}\mathrm{d}x - \int_{C_R} p\,\overline{\varphi|_{C_R}}\,\mathrm{d}s,\\ a_2\big((u,p),(\varphi,\chi)\big) &:= \int_{C_R}\left[(\frac{1}{2}I-\mathcal{D})(u|_{C_R}) + \mathcal{S}p\right]\overline{\chi}\,\mathrm{d}s. \end{aligned}$$

The variational formulation (3.10) couples the variational approach for the Helmholtz equation over Ω_R^- and the variational approach for the nonlocal boundary condition on C_R. Altogether, we have shown that (u,p) is a solution of (3.10). Recall that $u\in X_R$ is the restriction to Ω_R^- of the solution u to the Helmholtz problem

$$\Delta(u-v^{in}) + k^2(u-v^{in}) = 0 \text{ in } \Omega_\Lambda, \quad u=0 \text{ on } \Lambda, \quad u-v^{in} \text{ satisfies ASR}$$

in the variational sense of [10, Thm. 4.1]. The difference $u_0:=u-v^{in}-v^{sc}$ satisfies the HPSRC by assumption, and the solution function p is the trace of the normal derivative of u_0 on C_R. On the other hand, a solution $u\in X_R$, obtained from (3.10), can be extended from Ω_R^- to Ω_Λ via $u=v^{in}+v^{sc}+u_0$, where u_0 is expressed over $\{x\in\mathbb{R}^2\colon |x|>R\}$ by (3.7) with the traces $u_0|_{C_R}$ and $\partial_\nu u_0|_{C_R}$ replaced by $(u-v)|_{C_R}$ and the solution p of (3.10), respectively. Moreover, the extension is a solution of the Helmholtz equation and thus analytic at the interior points of C_R (observe that the second variational equation in (3.10) yields the continuity of the extension over C_R and the first equation that of the normal derivatives), and the difference of the extension and the solution v satisfies the HPSRC due to the next lemma, which will be proved in Appendix A.

LEMMA 3.1. *The function u_0 defined in* (3.7) *fulfills the HPSRC in Ω_R^+ and has a far-field pattern in the space $C(\overline{\mathbb{S}_+})$.*

Denoting the domain enclosed between C_R and Γ_R (cf. (3.1)) by $\widetilde{\Omega}_R^-$, we state the uniqueness and existence of solutions to (3.10) as follows.

LEMMA 3.2. *Suppose the squared wavenumber k^2 is not an eigenvalue for the negative Laplacian over the domain $\widetilde{\Omega}_R^-$. Then there exists a unique solution $(u,p)\in X_R\times H^{-1/2}(C_R)$ of the variational equation* (3.10).

Proof. For the quadratic form $A((u,p),(u,p))$ of the sesquilinear form A in (3.10) we conclude

$$\begin{aligned} \operatorname{Re} A\big((u,p),(u,p)\big) &= \int_{\Omega_R^-}\left\{|\nabla u|^2 - \operatorname{Re} k^2|u|^2\right\}\mathrm{d}x - \operatorname{Re}\int_{C_R} p\,\overline{u|_{C_R}}\,\mathrm{d}s\\ &\quad + \operatorname{Re}\int_{C_R}[(I-2\mathcal{D})(u|_{C_R}) + 2\mathcal{S}p]\,\overline{p}\,\mathrm{d}s\\ &= \int_{\Omega_R^-}\left\{|\nabla u|^2 - \operatorname{Re} k^2|u|^2\right\}\mathrm{d}x + \operatorname{Re} 2\int_{C_R}\mathcal{S}p\overline{p}\,\mathrm{d}s - \operatorname{Re} 2\int_{C_R}\mathcal{D}(u|_{C_R})\overline{p}\,\mathrm{d}s. \end{aligned}$$

Using the compactness of $\mathcal{D}$ and the strong ellipticity of $\mathcal{S}$ (cf. subsection 2.3) and arguing the same way as in [4], one can prove that the sesquilinear form satisfies a Gårding inequality. Such a sesquilinear form is called strongly elliptic over the space $X_R \times H^{-1/2}(C_R)$, and the corresponding variational equation (3.10) satisfies Fredholm's alternative. Let us prove that the solution for a zero right-hand side is trivial. The condition $A((u,p),(\varphi,0))=a_1((u,p),(\varphi,0))=0$ yields that u satisfies the Helmholtz equation in Ω_R^- and that $\partial_\nu u = p$ over C_R. Introducing the function $\tilde{u} := \int_{C_R} \{\partial_\nu G(\cdot\,;y)u(y) - G(\cdot\,;y)p(y)\}$ over $\Omega_\Gamma \setminus C_R$, the condition $A((u,p),(0,\chi)) = 2a_2((u,p),(0,\chi))=0$ yields that the trace on C_R of u from Ω_R^- coincides with the trace of $\tilde{u}$ from Ω_R^+. Consequently, the jump relation for the integrals in the definition of $\tilde{u}$ implies that the trace on C_R of $\tilde{u}$ from $\widetilde{\Omega}_R^-$ is zero. In other words, the restriction $\tilde{u}|_{\widetilde{\Omega}_R^-}$ is a solution of the homogeneous Dirichlet problem for the Helmholtz equation over $\widetilde{\Omega}_R^-$. If there is no nontrivial solution of the Dirichlet problem, then $\tilde{u}|_{\widetilde{\Omega}_R^-}=0$ and the trace on C_R of $\partial_\nu \tilde{u}$ from $\widetilde{\Omega}_R^-$ vanishes. The jump relation for the integrals in the definition of $\tilde{u}$ implies that the trace of $\partial_\nu \tilde{u}$ from Ω_R^+ is equal to p. If we define the function w by $w(x) := u(x)$ for $x \in \Omega_R^-$ and $w(x) := \tilde{u}(x)$ for $x \in \Omega_R^+$, then w and $\partial_\nu w$ are continuous over C_R. In other words, w is a solution of the homogeneous Dirichlet problem for the Helmholtz equation over $\Omega_\Lambda = \Omega_R^+ \cup C_R \cup \Omega_R^-$, which satisfies the radiation condition. The uniqueness of the solution to this boundary value problem (cf. [10, Thm. 4.1]) implies $w=0$ s.t. the solutions u and $p=\partial_\nu u$ vanish. Hence, the null space of the operator defined by the left-hand side of (3.10) is trivial. Applying Fredholm's alternative, we obtain existence and uniqueness of weak solutions to (3.10). ∎

For $h > \max\{x_2 : x \in \Lambda \cup \Gamma\}$, denote the strip between Λ and the straight line $\Gamma_h := \{x \in \mathbb{R}^2 : x_2 = h\}$ by $\Omega_{\Lambda,h}$. The space defined as $V_{h,\varrho}$ but with $\Omega_{\Gamma,h}$ replaced by $\Omega_{\Lambda,h}$ is denoted by $V'_{h,\varrho}$. Well-posedness of the perturbed scattering problem is stated below.

THEOREM 3.1. *Let v^{in} be a plane wave, and let Λ be the local perturbation of Γ described above. Suppose further that the perturbed domain Ω_Λ fulfills the condition* (2.4). *Then the wave scattering problem* (3.2) *over Ω_Λ admits a unique solution $u \in H_0^{1,loc}(\Omega_\Lambda) \cap C^2(\Omega_\Lambda)$ such that the difference $u - v^{in} - v^{sc}$ fulfills the HPSRC in $\Omega_\Gamma \cap \Omega_\Lambda$ and has a far-field pattern in $C(\overline{\mathbb{S}_+})$. Moreover, for any index $-1 < \varrho < -1/2$, the restriction $u|_{\Omega_{\Lambda,h}}$ is the unique variational solution of* (3.2) *over $\Omega_{\Lambda,h}$ in the weighted Sobolev space $V'_{h,\varrho}$ (cf.* [10, Thm. 4.1]*). Clearly, $u|_{\Omega_{\Lambda,h}} \in C^2(\Omega_{\Lambda,h})$.*

Proof. If the diameter of $\widetilde{\Omega}_R^-$ in the x_2-direction is sufficiently small, then the variational form of the Helmholtz operator is positive definite and k^2 is not an eigenvalue for the negative Laplacian over $\widetilde{\Omega}_R^-$. Now it is not hard to construct an analytic family of domains $\widetilde{\Omega}_R^-(\lambda)$, $0<\lambda<1$ all having the same lower boundary as $\widetilde{\Omega}_R^-$ and with an analytic family of upper boundaries $C_R(\lambda)$ such that $\widetilde{\Omega}_R^-(\lambda)$ has a small diameter in x_2-direction for $0<\lambda<\varepsilon$ and $\Lambda \setminus \Gamma \subset \widetilde{\Omega}_R^-(\lambda)$ for $1-\varepsilon<\lambda<1$. Hence, k^2 is an eigenvalue for the negative Laplacian over $\widetilde{\Omega}_R^-(\lambda)$ for λ in at most a countable set of $(1-\varepsilon, 1)$. In other words, it is possible to choose C_R such that k^2 is not an eigenvalue for the negative Laplacian over $\widetilde{\Omega}_R^-$. Furthermore, it is easy to check that a plane wave belongs to $H_\varrho^1(\Omega_{\Lambda,h})$ for any $h > \max\{x_2 : x \in \Lambda\}$ and $\varrho \in (-1, -1/2)$. Under the condition (2.4), the locally perturbed scattering problem admits a unique solution u such that $u - v^{in}$ satisfies the ASR (2.6) and belongs to the same space as the incoming wave (cf. [10, Thm. 4.1]). On the other hand, for the unique solution u to the variational problem (3.10) the difference $u-v^{in}=v^{sc}+u_0$ can be extended to a solution over Ω_Λ. In particular, the extension of u_0 for $|x|>R$ is given by (3.7). In view of Lemma 3.1, u_0 fulfills the HPSRC and has a far-field pattern. Moreover, $v^{sc}+u_0$ is in $H^1_\varrho(\Omega_{\Lambda,h})$ and satisfies the ASR (2.6), since both v^{sc} and u_0 are in $H^1_\varrho(\Omega_{\Lambda,h})$ and fulfill the ASR. Theorem 3.1 then follows from the uniqueness result of [10, Thm. 4.1]. ∎

In one of the authors' previous works [4], results similar to Theorem 3.1 were obtained in the case that Λ is a local perturbation of the ground floor $\Gamma=\{x\in\mathbb{R}^2:\ x_2=0\}$ on which an impedance boundary condition of the total field is imposed. In that case, the Green's function $G(x;y)$ to the unperturbed scattering problem is given in an explicit form, and significantly simplified arguments can be applied.

Remark 3.1. The result of Theorem 3.1 extends naturally to other incoming waves belonging to the weighted Sobolev space $V'_{h,\varrho}$ for some $h>\{x_2 : x\in\Gamma\cup\Lambda\}$ and $\rho\in(-1,-1/2)$. By Theorem 3.1, the perturbed wave field $u_0=u-v^{in}-v^{sc}$ caused by a local defect fulfills the HPSRC and thus decays as $|x_1|\to\infty$ in the strip $\Omega_{\Lambda,h}$ for any $h>\max_{x\in\Lambda\cup\Gamma}\{x_2\}$. Since it is the unique solution in the weighted Sobolev space $V'_{h,\varrho}$, Theorem 3.1 implies that a local perturbation of a grating surface does not excite any surface wave $(x_1,x_2)\mapsto c\exp(i\alpha x_1+i\beta x_2)$ with $i\beta<0$ different from those of the unperturbed grating if the domain above it still satisfies condition (2.4). Note that surface waves propagate along the grating surface and decay exponentially in the vertical direction. They belong to $V'_{h,\varrho}$ for any $\varrho<-1/2$.

Appendix A.

In this section, we give the proofs to the Lemmata 2.2–2.4 and 3.1 and Theorem 2.2. We suppose that Γ is a two-dimensional rough surface (cf. the definition before Theorem 2.2). In particular, any periodic surface is a special rough surface. Additionally, we suppose condition (2.4). All other definitions from the previous sections are retained. We prepare our proofs with two technical lemmata.

LEMMA A.1. *Fix real numbers h and h' with $h'>h$, and let $\Gamma_h:=\{z\in\mathbb{R}^2: z_2=h\}$. Suppose that $g\in L^2_\varrho(\Gamma_h)$ with $1/2<\varrho<1$. If $n+\varrho>1/2$, then there is a constant $C>0$ s.t., for all $x\in\mathbb{R}^2$ with $x_2\ge h'$,*

$$\int_{\{z\in\Gamma_h:\,|z_1|>1\}}\frac{|g(z)|}{|x-z|^n}\,\mathrm{d}s(z)\le C\|g\|_{L^2_\varrho(\Gamma_h)}\begin{cases}|x_2|^{-n} & \text{if } |x_1|\le|x_2|,\\ \left[|x_1|^{-n}+|x_2|^{-n+1/2}|x_1|^{-\varrho}\right] & \text{else.}\end{cases}$$

Proof. It follows from $g\in L^2_\varrho(\Gamma_h)$ that

$$\Big|\int_{\{z\in\Gamma_h:\,|z_1|>1\}}\frac{|g(z)|}{|x-z|^n}\,\mathrm{d}s(z)\Big|^2\le||g||^2_{L^2_\varrho(\Gamma_h)}\int_{\{z\in\Gamma_h:\,|z_1|>1\}}\frac{1}{|x-z|^{2n}\,|z_1|^{2\varrho}}\,\mathrm{d}s(z)\,.$$

Hence, we only need to estimate the integral $I=I(x_1)$ on the right-hand side. Additionally, since $|x-(0,h)|\sim|x|$ for $|x|\to\infty$, we may suppose $h=0$.

First we show the estimate $I(x_1)\le C|x_2|^{-2n}$. Under the assumption $|x_1|\le C_0x_2$ with $C_0:=1/(2h')$ this implies the estimate of the lemma. However, for the case $|x_1|>Cx_2$, it is weaker than the estimate of the lemma. Without loss of generality, we may suppose that $x_1=0$ and $x=(0,x_2)$ lying on the positive x_2-axis, so that $|x|=x_2$. Indeed, we can argue as follows. Since $x_2\ge h'>h=0$ and $|y_1|\le 1+h'$ imply $|x-(y_1,0)-z|\sim|x-z|$, we get $I(y_1)\le CI(0)$. Therefore, it remains to show $I(x_1)\le CI(0)$ for $x_1>1+h'$. We get

$$I(x_1) := \int_{\{z\in\Gamma_0:\,|z_1|>1\}} |x-z|^{-2n}|z_1|^{-2\varrho}\,\mathrm{d}s(z) \;=\; I_1+I_2+I_3, \tag{A.1}$$

$$\begin{aligned}
I_1 &:= \int_{-\infty}^{-1}[x_2^2+(x_1+|z_1|)^2]^{-n}|z_1|^{-2\varrho}\,\mathrm{d}z_1 \le \int_{-\infty}^{-1}[x_2^2+|z_1|^2]^{-n}|z_1|^{-2\varrho}\,\mathrm{d}z_1 \;\le\; I(0),\\
I_2 &:= \int_1^{x_1}[x_2^2+(x_1-z_1)^2]^{-n}|z_1|^{-2\varrho}\,\mathrm{d}z_1\\
&= \int_1^{(1+x_1)/2}\left\{[x_2^2+(x_1-z_1)^2]^{-n}|z_1|^{-2\varrho}+[x_2^2+(z_1-1)^2]^{-n}|1+x_1-z_1|^{-2\varrho}\right\}\mathrm{d}z_1\\
&\le \int_1^{(1+x_1)/2}\left\{[x_2^2+(z_1-1)^2]^{-n}|z_1|^{-2\varrho}+[x_2^2+(z_1-1)^2]^{-n}|z_1|^{-2\varrho}\right\}\mathrm{d}z_1 \le 2I(1)\\
&\le\; CI(0),\\
I_3 &:= \int_{x_1}^{\infty}[x_2^2+(z_1-x_1)^2]^{-n}|z_1|^{-2\varrho}\,\mathrm{d}z_1 \;\le\; \int_1^{\infty}[x_2^2+(z_1-1)^2]^{-n}|z_1+x_1-1|^{-2\varrho}\,\mathrm{d}z_1\\
&\le \int_1^{\infty}[x_2^2+(z_1-1)^2]^{-n}|z_1|^{-2\varrho}\,\mathrm{d}z_1 \;\le\; I(1) \;\le\; CI(0).
\end{aligned}$$

In other words, $I(x_1)\le CI(0)$. Hence, it is really sufficient to estimate $I(0)$, and we may suppose $x_1=0$.

Now denote the angle formed by $x-z$ and the positive x_2-axis by $\varphi\in(0,\pi/2)$. Then it is easy to see that $x_2=|x-z|\cos\varphi$ and $|z_1|=x_2\tan\varphi$. Changing variables, we find

$$\begin{aligned}
\int_{\{z\in\Gamma_0:\,|z_1|>1\}}\frac{1}{|x-z|^{2n}\,|z_1|^{2\varrho}}\,\mathrm{d}s(z) \;&\le\; \frac{C}{|x_2|^{2(n+\varrho-1/2)}}\int_{\arctan(1/x_2)}^{\pi/2}\frac{(\cos\varphi)^{2(n-1)}}{(\tan\varphi)^{2\varrho}}\,\mathrm{d}\varphi\\
&\le \frac{C}{|x_2|^{2(n+\varrho-1/2)}}\left\{1+\int_{\arctan(1/x_2)}^{\pi/2}\varphi^{-2\varrho}\,\mathrm{d}\varphi\right\}\\
&\le \frac{C}{|x_2|^{2(n+\varrho-1/2)}}\left\{1+\arctan(1/x_2)^{-2\varrho+1}\right\} \;\le\; \frac{C}{|x_2|^{2n}}.
\end{aligned}$$

Next we consider the case $C_0x_2\le|x_1|$, and, without loss of generality, we suppose $C_0x_2\le x_1$. We set $C_1:=C_0/2$ and get $x_1-C_1x_2\ge C_0x_2-C_1x_2=C_0/2\;x_2\ge C_0/2\;h'=1$ such that (cf. (A.1)) $I(x_1)=I_1+I_2'+I_3'+I_4'$ with

$$\begin{aligned}
I_2' &:= \int_1^{x_1-C_1x_2} f(x,z_1)\,\mathrm{d}z_1, \quad I_3' := \int_{x_1-C_1x_2}^{x_1+C_1x_2} f(x,z_1)\,\mathrm{d}z_1,\\
I_4' &:= \int_{x_1+C_1x_2}^{\infty} f(x,z_1)\,\mathrm{d}z_1, \quad f(x,z_1) := [x_2^2+(x_1-z_1)^2]^{-n}|z_1|^{-2\varrho}.
\end{aligned}$$

Then $x_1-c_1x_2\le z_1\le x_1+c_1x_2$ implies $x_2^2+(x_1-z_1)^2\sim x_2^2$ and $z_1\sim x_1$, and we arrive at

$$I_3' \le C\int_{x_1-C_1x_2}^{x_1+C_1x_2} x_2^{-2n}|x_1|^{-2\varrho}\,\mathrm{d}z_1 \;\le\; C\,x_2^{1-2n}x_1^{-2\varrho}. \tag{A.2}$$

For $x_1+C_1x_2\leq z_1$, we get $x_2^2+(x_1-z_1)^2\sim(z_1-x_1)^2$. Again, $x_1+C_1x_2\leq z_1\leq 2x_1$ implies $z_1\sim x_1$, whereas $2x_1\leq z_1$ leads to $|z_1-x_1|\sim z_1$. We obtain

$$\begin{aligned} I_4' &\leq C\int_{x_1+C_1x_2}^{2x_1}|z_1-x_1|^{-2n}x_1^{-2\varrho}\,\mathrm{d}z_1+C\int_{2x_1}^{\infty}z_1^{-2(n+\varrho)}\,\mathrm{d}z_1 \\ &\leq C[C_1x_2]^{1-2n}x_1^{-2\varrho}+Cx_1^{-2(n+\varrho-1/2)}\ \leq\ Cx_2^{1-2n}x_1^{-2\varrho}. \end{aligned} \tag{A.3}$$

Similarly, we conclude

$$I_1\ =\ \int_1^{\infty}|z_1+x_1|^{-2n}z_1^{-2\varrho}\,\mathrm{d}z_1 \tag{A.4}$$
$$\leq\ C\int_1^{x_1}x_1^{-2n}z_1^{-2\varrho}\,\mathrm{d}z_1+C\int_{x_1}^{\infty}z_1^{-2(n+\varrho)}\,\mathrm{d}z_1$$
$$\leq\ C\,x_1^{-2n},$$
$$\begin{aligned} I_2' &\leq\ C\int_1^{x_1/2}x_1^{-2n}z_1^{-2\varrho}\,\mathrm{d}z_1+C\int_{x_1/2}^{x_1-C_1x_2}|x_1-z_1|^{-2n}x_1^{-2\varrho}\,\mathrm{d}z_1 \\ &\leq\ Cx_1^{-2n}+Cx_1^{-2\varrho}x_2^{1-2n}. \end{aligned} \tag{A.5}$$

The formulas (A.4), (A.5), (A.2), and (A.3) provide us with the estimate for the case $C_0x_2\leq|x_1|$. This finishes the proof of Lemma A.1. ☐

LEMMA A.2. *Consider fixed numbers h, h', and ϱ s.t. $h>h'>0$ and $1/2<\varrho<1$, and suppose C_R is a curve satisfying the conditions of subsection* 2.3 *following Lemma* 2.4. *Choose a function $f\in L^1(C_R)$, and suppose that $\mathcal{S}_y\in L^2_\varrho(\Omega_{\Gamma,h'})$, $y\in C_R$, is a family of functions, which depend continuously on y. Extend $\mathcal{S}_y$ to Ω_Γ by $\mathcal{S}_y(x):=0$ for $x_2>h'$. By w denote the y dependent solution of the homogeneous Dirichlet problem for $\Delta w(\cdot\,;y)+k^2w(\cdot\,;y)=\mathcal{S}_y$ over the domain Ω_Γ s.t. $w(\cdot\,;y)$ satisfies the condition ASR (cf.* [10, Thm. 4.1]*). Then the functions $w(\cdot\,;y)$, $y\in C_R$ and $w_I(\cdot):=\int_{C_R}w(\cdot\,;y)f(y)\mathrm{d}y$ defined over Ω_Γ satisfy the HPSRC and have a far-field pattern in $C(\overline{\mathbb{S}_+})$.*

Proof. We only prove the more involved case of w_I. From [10, Thm. 4.1] we infer that the family of solutions $C_R\ni y\mapsto w(\cdot\,;y)\in V_{h,\varrho}$ is continuous for the fixed ϱ. Hence, $w_I\in V_{h,\varrho}$, and, for the HPSRC, it remains to prove (2.8) with $v=w_I$. Recall $\Gamma_h:=\{x\in\mathbb{R}^2:x_2=h\}$. We observe that $w(\cdot\,;y)$ is analytic near Γ_h as a Helmholtz solution and that $w(\cdot\,;y)|_{\Gamma_h}$ belongs to the space $H^{1/2}_\varrho(\Gamma_h)$ for the ϱ with $1/2<\varrho<1$ and depends continuously on $y\in C_R$. Hence, $g_h:=w_I|_{\Gamma_h}\in L^2_\varrho(\Gamma_h)\subset H^{1/2}_\varrho(\Gamma_h)$, and g_h is analytic. Moreover, since $\operatorname{supp}\mathcal{S}_y\subseteq\overline{\Omega}_{\Gamma,h'}$, the function w_I over the set $U_h:=\{x\in\mathbb{R}^2:x_2>h\}$ can be written as

$$w_I(x)\ =2\int_{\Gamma_h}\frac{\partial\Phi(x;z)}{\partial z_2}g_h(z)\,\mathrm{d}s(z),\quad x\in U_h, \tag{A.6}$$

which is known as the UPRC (see [11]). Here, z_h^* denotes the image of z with respect to reflection by the line Γ_h, and the function $\Phi_h(x;z)$ is the Green's function to the Helmholtz equation with the Dirichlet boundary condition on Γ_h. The improper integral in the above expression of w_I can be understood as the duality between

$H_\varrho^{1/2}(\Gamma_h)$ and its dual space $H_{-\varrho}^{-1/2}(\Gamma_h)$ for our ϱ; we refer to [10] for the equivalence of the UPRC and ASR in weighted Sobolev spaces.

Using a twice differentiable cutoff function, we can represent g_h as the sum of two functions, the first with compact support and the second with support in $\{z\in\Gamma_0: |z_1|>1\}$. Correspondingly, w_I is the sum of the two integrals of the type (A.6) with g_h replaced by the two functions adding up to g_h. For both integrals, we have to prove the HPSRC. The case of w_I with compact support concerns a classical double layer potential with layer function from the trace space $H^{1/2}$. The resulting w_I fulfills the classical full-space Sommerfeld condition and has the well-known far-field pattern for all directions $\hat{x}\in\mathbb{R}^2$ with $|\hat{x}|=1$. The boundedness of the norms in $H_\varrho^1(\Omega_{\Gamma,h}\cap\{x\in\mathbb{R}^2: |x_1|>R\})$ with $-1<\varrho<1$ follows from the estimate $|\partial_{z_2}\Phi(x;z)|\le C(1+|x_2|)|x|^{-3/2}$, valid for z in a bounded set and for $|x|>R$ with sufficiently large R (cf. the subsequent formulas (A.7) and (A.9)). Consequently, without loss of generality we may suppose that the support of g_h on Γ_h is contained in the set $\{z\in\Gamma_h: |z_1|>1\}$, which allows us to apply Lemma A.1.

Straightforward calculations show that for $x\in U_h$ and $z=(z_1,z_2)\in\Gamma_h$,

$$\frac{\partial\Phi_h(x;z)}{\partial z_2}\Big|_{z_2=h} = \frac{ik(x_2-z_2)H_1^{(1)}(k|x-z|)}{2|x-z|}\Big|_{z_2=h}. \tag{A.7}$$

Write $x=r(\cos\theta,\sin\theta)$, $s(r,z):=k|x(r)-z|$, and $\hat{x}:=x/r=(\cos\theta,\sin\theta)$. Here and thereafter, $H_n^{(1)}$ denotes the Hankel function of the first kind of order $n\in\mathbb{Z}$. Then we may rewrite the previous identity as

$$\frac{\partial\Phi_h(x;z)}{\partial z_2}\Big|_{z_2=h} = \frac{ik^2(x_2-h)}{2}\,\frac{H_1^{(1)}(s(r,z))}{s(r,z)}\Big|_{z_2=h}. \tag{A.8}$$

Below we shall write $s=s(r,z)$ for notational simplicity and make use of the asymptotic behavior of the Hankel functions for large argument as follows (cf., e.g., (3.82) in [17]):

$$\begin{aligned} H_n^{(1)}(s) &= \sqrt{\frac{2}{\pi s}}\,e^{i(s-(2n+1)/4\pi)} + \mathcal{O}(|s|^{-3/2}),\\ (H_n^{(1)})'(s) &= i\sqrt{\frac{2}{\pi s}}\,e^{i(s-(2n+1)/4\pi)} + \mathcal{O}(|s|^{-3/2}). \end{aligned} \tag{A.9}$$

We choose an $h''>h$ and consider $x\in\mathbb{R}^2$ with $x_2>h''$. Thus $s>k(h''-h)>0$, and the identity (A.8) implies that there exists a constant $C>0$ such that

$$\left|\frac{\partial\Phi_h(x;z)}{\partial z_2}\right| \le \frac{C(x_2+h)}{s^{3/2}} = \frac{C(r\sin(\varphi)+h)}{s^{3/2}} \tag{A.10}$$

for all $x_2>h''$ and $z_2=h$. Hence, by Lemma A.1 we obtain

$$\begin{aligned} |w_I(x)| &\le \int_{\Gamma_h}\frac{C(x_2+h)}{s(r,z)^{3/2}}|g_h(z)|\,\mathrm{d}s(z)\\ &\le C\,\|g_h\|_{L^2_\varrho(\Gamma_h)}(x_2+h)\begin{cases} |x_2|^{-3/2} & \text{if } |x_1|\le|x_2|,\\ \left[|x_1|^{-3/2}+|x_2|^{-1}|x_1|^{-\varrho}\right] & \text{else.}\end{cases} \end{aligned}$$

Since $r \sim \max\{|x_1|, |x_2|\}$, we arrive at

$$|w_I(x)| \le C\,\|g_h\|_{L^2_\varrho(\Gamma_h)} r^{-1/2}, \quad x \in U_{h''}, \tag{A.11}$$

leading to the boundedness $\sup_{r>1} \sup_{x \in S_r \cap U_{h''}} r^{1/2}|w_I(x)| < \infty$.

Further, through direct calculations we obtain

$$\frac{\partial}{\partial r}\frac{\partial \Phi_h(x;z)}{\partial z_2} = \frac{ik^2 \sin\theta}{2}\frac{H_1^{(1)}(s)}{s} + \frac{ik^2(r\sin\theta - h)}{2}\frac{d}{ds}\left(\frac{H_1^{(1)}(s)}{s}\right)\frac{ds(r,z)}{dr}. \tag{A.12}$$

As $s \to \infty$, it holds that (cf. (A.9))

$$\frac{d}{ds}\left(\frac{H_1^{(1)}(s)}{s}\right) = \frac{H_1^{(1)'}(s)\,s - H_1^{(1)}(s)}{s^2} = i\frac{H_1^{(1)}(s)}{s} + \mathcal{O}(s^{-5/2}). \tag{A.13}$$

It is easy to check that, for $z_2 = h$,

$$\frac{ds(r)}{dr} = k\frac{|x| - \hat{x}\cdot z}{|x-z|}, \quad \left|\frac{ds(r)}{dr} - k\right| \le C\frac{h+s}{s}, \tag{A.14}$$

where the constant $C>0$ is independent of $z=(z_1,h)$ and of $x \in U_h$. In the last step we have used $||x| - \hat{x}\cdot z| \le h+s$, which can be seen as follows. Consider the triangle between the points $(0,0)$, $(z_1,0)$, and $x=(x_1,x_2)$, and suppose $u=(u_1,u_2)$ is the projection of $(z_1,0)$ onto the line through $(0,0)$ and x. Then, in the rectangular triangle between x, $(z_1,0)$, and u, the hypotenuse between x and $(z_1,0)$ is longer than the side between x and u, i.e., $||x| - \hat{x}\cdot z| \le |x-(z_1,0)|$. Consequently, we get $||x| - \hat{x}\cdot z| \le |x-(z_1,h)| + h = s+h$.

Combining the relations (A.13) and (A.14) yields that, for $s \to \infty$,

$$\frac{d}{ds}\left(\frac{H_1^{(1)}(s)}{s}\right)\frac{ds(r)}{dr} - ik\frac{H_1^{(1)}(s)}{s} = \frac{iH_1^{(1)}(s)}{s}\left[\frac{ds(r,z)}{dr} - k\right] + \mathcal{O}(s^{-5/2}) = \mathcal{O}(s^{-3/2}). \tag{A.15}$$

Now we deduce from (A.8), (A.12), and (A.15) that, for $z \in \Gamma_h$ and a suitable constant $C>0$,

$$\left|\left(\frac{\partial}{\partial r} - ik\right)\frac{\partial \Phi_h(x;z)}{\partial z_2}\right| \le C\left(\frac{\sin(\theta)}{s^{3/2}} + \frac{(x_2+h)}{s^{3/2}}\right) \le C\frac{(x_2+h)}{s^{3/2}} \tag{A.16}$$

as $s \to \infty$. Similarly to the derivation of (A.11) from (A.10) and Lemma A.1, (A.16) and Lemma A.1 imply

$$\begin{aligned} |\partial_r w_I(x) - ikw_I(x)| &\le \int_{\{z \in \mathbb{R}^2:\, z_2 = h\}} \left|(\partial_r - ik)\frac{\partial \Phi_h(x;z)}{\partial z_2} g_h(z)\right| \mathrm{d}s(z) \\ &\le C\|g_h\|_{L^2_\varrho(\Gamma_h)}|x|^{-1/2} \end{aligned} \tag{A.17}$$

for all $x \in U_{h''}$. Next we choose $\varepsilon > 0$ and prove that there is a constant C independent of ε s.t. the supremum over $x \in S_r \cap U_{h''}$ of the expression $r^{1/2}|\partial_r w_I(x) - ikw_I(x)|$ is less than $C\varepsilon$ whenever r is larger than a suitable threshold. We choose an approximation

$\tilde{g}_h$ of g_h over Γ_h with compact support s.t. $\|\tilde{g}_h - g_h\|_{L^2_\varrho(\Gamma_h)} < \varepsilon$ and define $\tilde{w}_I$ by the integral on the right-hand side of (A.6) with g_h replaced by $\tilde{g}_h$. Then the proof of (A.17) implies

$$\sup_{x\in S_r\cap U_{h''}} r^{1/2}\big|\partial_\nu[w_I(x)-\tilde{w}_I(x)] - ik[w_I(x)-\tilde{w}_I(x)]\big| \le C\varepsilon. \tag{A.18}$$

On the other hand, we get $|\,|x|-\hat{x}\cdot z-|x-z|\,|<C$ if z is bounded, and the upper estimate $C(h+s)/s$ in (A.14) can be improved to a simple C/s. Hence, for the compactly supported $z\mapsto\tilde{g}_h(z)$, derivation of (A.17) implies

$$|\partial_\nu\tilde{w}_I - ik\tilde{w}_I| \le C\int_{\Gamma_h}\Big[r^{-1}x_2 s^{-3/2} + (x_2+h)s^{-5/2}\Big]\,|\tilde{g}_h|\,\mathrm{d}z,$$

$$\sup_{x\in S_r\cap U_{h''}} r^{1/2}\,|\partial_\nu\tilde{w}_I - ik\tilde{w}_I| \le C_{\tilde{g}_h}\,r^{1/2-\varrho} \le \varepsilon \tag{A.19}$$

if r is larger than a suitable threshold. Combining (A.18) and (A.19), we get that the supremum over $x\in S_r\cap U_{h''}$ of the expression $r^{1/2}|\partial_r w_I(x) - ikw_I(x)|$ is less than $(C+1)\varepsilon$ if r is sufficiently large. The proof of (2.8) for $v=w_I$ is completed.

Next we have to prove the existence of the far-field pattern. We prove it for the representation of w_I by the right-hand side of (A.6). The relations (A.7) and (A.9) lead to (cf. Lemma A.1)

$$\begin{aligned} w_I(x) &= \int_{\Gamma_h}\left\{c\frac{e^{ik|x-z|}(x_2-h)}{|x-z|^{3/2}} + \mathcal{O}\Big((x_2-h)|x-z|^{-5/2}\Big)\right\} g_h(z)\mathrm{d}s(z) \\ &= c\int_{\Gamma_h}\frac{e^{ik|x-z|}x_2}{|x-z|^{3/2}}g_{L,h}(z)\mathrm{d}s(z) + \mathcal{O}\Big(\|g_h - g_{L,h}\|\,|x|^{-1/2}\Big) + \mathcal{O}\big(|x|^{-\varrho}\big)\,, \end{aligned}$$

where

$$g_{L,h}(z) := \begin{cases} g_h(z) & \text{if } -L<z_1<L, \\ 0 & \text{else,}\end{cases} \qquad c = \frac{ik}{2}\sqrt{\frac{2}{\pi}}e^{-3i/(4\pi)}.$$

Using that, for fixed L and $|x|\gg L$,

$$\frac{1}{|x-z|^{3/2}} = \frac{1}{|x|^{3/2}} + \mathcal{O}_L(|x|^{-5/2}), \quad e^{ik|x-z|} = e^{ik|x|}e^{-ik[x/|x|]\cdot z}\left[1+\mathcal{O}_L(|x|^{-1})\right],$$

and setting $x=r\hat{x}$ with $r:=|x|$ and $\hat{x}\in C_R$, we arrive at

$$\begin{aligned} w_I(x) = c\,\frac{e^{ikr}}{r^{1/2}}\,\hat{x}_2\,e^{-ikh\hat{x}_2}\int_{-L}^{L} e^{-ikz_1\hat{x}_1}g_h(z_1,h)\mathrm{d}z_1 &+ \mathcal{O}\Big(\|g_h - g_{L,h}\|_{L^2_\varrho(\Gamma_h)}\,|x|^{-1/2}\Big) \\ &+ \mathcal{O}_L\big(|x|^{-\varrho}\big)\,. \end{aligned}$$

Here the $\mathcal{O}_L$ terms denote usual $\mathcal{O}$ expressions defined with constants depending on L. Now we get that $g_h\in L^2_\varrho(\Gamma_h)\subset L^1(\Gamma_h)$ is valid for $\varrho\in(1/2,1)$. So we obtain

$$\begin{aligned} w_I(x) = \frac{e^{ikr}}{\sqrt{r}}\,c\,\hat{x}_2 e^{-ikh\hat{x}_2}\int_{\mathbb{R}} e^{-ikz_1\hat{x}_1}g_h(z_1,h)\mathrm{d}z_1 &+ \mathcal{O}\Big(\|g_h - g_{L,h}\|_{L^2_\varrho(\Gamma_h)}\Big)\frac{1}{\sqrt{r}} \\ &+ \mathcal{O}_L(r^{-\varrho}), \end{aligned} \tag{A.20}$$

where the second term on the right-hand side is smaller than $\varepsilon/2$ for sufficiently large L. Fixing such an L, the third term is less than $\varepsilon/2$ if r is sufficiently large. All these estimates are uniform w.r.t. $\hat{x}$ s.t. the multiplicator of $\exp(ikr)r^{-1/2}$ in the first term on the right-hand side is the far-field pattern of the function w_I. ∎

Remark A.1. The relation (A.20) implies that the far-field patten $w_{I,\infty}$ of w_I takes the form

$$w_{I,\infty}(\hat{x}) = c\,\hat{x}_2 e^{-ikh\hat{x}_2}\int_{\mathbb{R}} e^{-ikz_1\hat{x}_1} g_h(z_1,h)\mathrm{d}z_1, \qquad \hat{x} = x/|x| = (\hat{x}_1,\hat{x}_2)\in\overline{\mathbb{S}^+}.$$

In particular, we observe $w_{I,\infty}(\hat{x})\to 0$ for $\mathbb{S}_+\ni\hat{x}\to(\pm 1,0)$. Note that it is natural to have a vanishing pattern function $w_{I,\infty}(\hat{x}_0)=0$ at the horizontal directions $\hat{x}_0 := (\pm 1,0)$ due to the Dirichlet boundary condition imposed on Γ.

As a corollary of the proof of the lemma and of Theorem 2.1 valid for rough surfaces, we can prove that the Green's function to rough surface scattering problems satisfies HPSRC, i.e., the assertions of Theorem 2.2 and Lemma 2.2.

Proof of Theorem 2.2 *and Lemma* 2.2. Let $y\in\Omega_\Gamma$, and suppose without loss of generality that Γ lies above $\Gamma_0 := \{x_2=0\}$. Denote the reflection image $(y_1,-y_2)$ of $y=(y_1,y_2)$ with respect to the straight line Γ_0 by y^*. Choose a cutoff function $\chi\in C_0^\infty(\Omega_\Gamma)$ such that $\chi\equiv 0$ in $x_2>h'$ for some $h'>y_2>0$ and $\chi\equiv 1$ near y. For the unique G of Theorem 2.1 valid for rough surfaces (cf. the arguments of [10, Thm. 4.1]), set

$$w(x;y) := G(x;y) - \big(G^{in}(x;y) - G^{in}(x;y^*)\big)\chi(x) \quad x\in\Omega_\Gamma.$$

Then it is easy to see $(\Delta+k^2I)w =: \mathcal{S}_y\in L^2_\rho(\Omega_{\Gamma,h'})$ for all $\rho\in(0,1)$. However, there is a unique solution $w\in V_{h,\rho}$ of $(\Delta+k^2I)w=\mathcal{S}_y$ (cf. [10, Thm. 4.1]), which yields the existence of a unique variational solution in Theorem 2.2. Applying Lemma A.2, we get the assertions on the HPSRC, the far-field pattern, and the inclusion in the corresponding Sobolev space. □

Obviously, the cutoff function in the previous proof depends on the distance between the source position y and the rough surface Γ. The corresponding estimates blow up if this distance tends to zero. Below we present a more ingenious proof to Lemma 2.2 and Theorem 2.2. Our approach has the merit that the constructed Green's function depends continuously on the source position y and does not depend on the distance between y and Γ. This will be important to derive Lemma 2.4.

Second proof of Lemma 2.2. Without loss of generality we may fix $R>0$ and $y\in\Omega_\Gamma$ such that $|y|\le R$. For a radius $r>0$, we denote the circle $\{x\in\mathbb{R}^2 : |x|<r\}$ by B_r. We consider a simple, bounded, and closed Lipschitz curve $\Theta\subset\mathbb{R}^2\backslash\Omega_\Gamma$ s.t. $\Gamma\cap B_{2R}\subseteq\Theta\cap B_{2R}$. By $G^\Theta(x;y)$ we denote the Green's function for the Dirichlet boundary problem with classical Sommerfeld radiation condition for the Helmholtz equation over the domain Ext_Θ exterior to Θ (cf. Figure A.1). Furthermore, we fix a cutoff function as

$$\chi(x) = \begin{cases} 0 & \text{if} \quad |x|<R/3, \\ 1 & \text{if} \quad |x|>2R/3, \end{cases} \qquad |\nabla^{|\alpha|}\chi| < C, \quad |\alpha| = 0,1,2.$$

Recall that Γ is located between Γ_h and Γ_0. Then we shall prove

$$\begin{aligned} G(x;y) &= G^\Theta(x;y) - G^\Theta\big(x+(0,H);y\big) + u(x;y) + w(x;y), \\ u(x;y) &:= -\chi(x-y)G^\Theta(x;y) + G^\Theta\big(x+(0,H);y\big), \\ \mathcal{S}_y(x) &:= -\left(\Delta_x + k^2\right)u(x;y), \end{aligned} \tag{A.21}$$

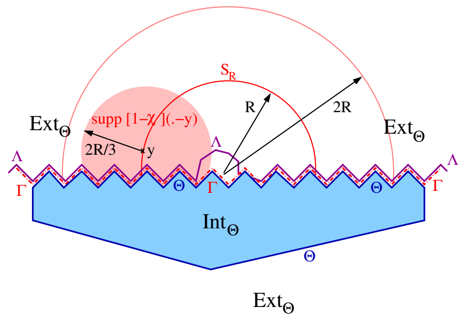

FIG. A.1. *Closed curve Θ having common part with periodic profile curve Γ.*

where H is a fixed positive constant s.t. $\max\{y,h\}\leq H/2$ and $w(\cdot\,;y)$ is the solution of the homogeneous Dirichlet problem for $\Delta w(\cdot\,;y)+k^2w(\cdot\,;y)=\mathcal{S}_y$ under the condition ASR. Concerning the term $G^\Theta(x+(0,H);y)$, we observe that, for $x_0\in\Theta$, we get $G^\Theta([x_0-(0,H)]+(0,H);y)=G^\Theta(x_0;y)$, i.e., the boundary behavior of $G^\Theta(x_0;y)$, $x_0\in\Theta$, is shifted by H into the negative x_2-direction. Moreover, the weak singularity of the Green's function at the source point appears for $x+(0,H)=y$, i.e., at $x=y-(0,H)$. In other words, the function $(x,y)\mapsto G^\Theta(x+(0,H);y)$ is the Green's function $G^{\Theta-(0,H)}(x,y\text{–}(0,H))$ of the domain $Ext_\Theta\text{–}(0,H)$ at the source point $y\text{–}(0,H)$. In particular, $G^\Theta(x+(0,H);y)$ is an analytic function on Ω_Γ. Clearly, the support of the right-hand side $\mathcal{S}_y$ over Ω_Γ is contained in the compact set $\operatorname{supp}[1-\chi](\cdot-y)\subseteq\{x\in\mathbb{R}^2: |x-y|\leq 2R/3\}$ (cf. Figure A.1) and

$$\text{(A.22)}\quad \mathcal{S}_y=-\sum_{j=1}^{2}\Big\{2\partial_{x_j}\chi(\cdot-y)\,\partial_{x_j}G^\Theta(\cdot\,;y)\;+\;\partial^2_{x_j}\chi(\cdot-y)\,G^\Theta(\cdot\,;y)\Big\}\in L^2_\varrho(\Omega_\Gamma).$$

Therefore, the solution function $w(\cdot\,;y)$ is the solution of the variational equation of [10, Thm. 4.1].

Let us define G by the right-hand side of (A.21). Then the equation $\Delta G(\cdot\,;y)+k^2G(\cdot\,;y)=\delta_y$ follows from the Green's function property of $G^\Theta(\cdot\,;y)$ and $G^\Theta(\cdot+(0,H);y)$ and from the definition of w using $\mathcal{S}_y$. The boundary condition is fulfilled since $w(\cdot\,;y)|_\Gamma=0$ holds for the solution of a homogeneous Dirichlet problem, since $G^\Theta(x;y)-G^\Theta(x+(0,H);y)+u(x;y)$ vanishes for x with $\chi(x-y)=1$, and since, for $|y|\leq R$ and for any $x\in\Gamma$ with $\chi(x-y)\neq 1$, we get $\chi(x-y)G^\Theta(x;y)=G^\Theta(x;y)=0$ by the Dirichlet condition for the Green's function G^Θ. The condition ASR is satisfied as we shall prove the stronger HPSRC below. In other words, the right-hand side (A.21) is really the Green's function $G(x;y)$ for the domain Ω_Γ.

Let us prove the radiation condition and the existence of the far-field pattern for the terms on the right-hand side of (A.21). Lemma A.2 implies the HPSRC and the existence of the far-field pattern for $w(\cdot\,;y)$. The Green's functions $G^\Theta(\cdot\,;y)$ and $G^\Theta(\cdot+(0,H);y)$ satisfy the classical full-space Sommerfeld condition implying (2.8) and have a far-field pattern even uniformly in all directions θ with $|\theta|=1$. The boundedness of the $H^1_\varrho(\Omega_{\Gamma,h}\cap\{x\in\mathbb{R}^2: |x_1|>R\})$-norms of $G^\Theta(\cdot\,;y)-G^\Theta(\cdot+(0,H);y)$ for $-1<\varrho<1$ follows from $\Phi(x+(0,H);y)=\Phi(x;y-(0,H))$ (cf. (2.5) for the definition of Φ) and from the estimate

$$\text{(A.23)}\qquad \left|\partial^{l_1}_{y_1}\partial^{l_2}_{y_2}\Phi(x;y)-\partial^{l_1}_{y_1}\partial^{l_2}_{y_2}\Phi\big(x;y-(0,H)\big)\right|\leq C\,\frac{1+|x_2|}{|x|^{3/2}}$$

valid for fixed integers $l_1,l_2\geq 0$, for any y from a bounded set, and for any $x>R$ with sufficiently large R (see below and also [8, 9]). Indeed, we can represent $G^\Theta(\cdot\,;y)$ by the representation formula as the sum of a single and double layer operator over a bounded smooth curve Θ' enclosing Θ. The weight functions in these potentials are smooth. Consequently, $G^\Theta(\cdot\,;y)-G^\Theta(\cdot\,+(0,H);y)$ is equal to the difference of the representation formula minus the same formula with the same weights but on the curve Θ' shifted by H in the direction of the negative x_2-axis. Applying (A.23), we get the estimate $|G^\Theta(x;y)-G^\Theta(x+(0,H);y)|\le C|x|^{-3/2}$ for large values of $|x|$ with $x_2<h$. Similarly, we can prove the estimate for the difference of the gradients $|\nabla_x G^\Theta(x;y)-\nabla_x G^\Theta(x+(0,H);y)|\le C|x|^{-3/2}$ for large values of $|x|$ with $x_2<h$. It is easy to see that these estimates imply the boundedness of the $H^1_\varrho(\Omega_{\Gamma,h}\cap\{x\in\mathbb{R}^2: |x_1|>R\})$-norms of the functions $G^\Theta(\cdot\,;y)-G^\Theta(\cdot\,+(0,H);y)$.

For the proof of (A.23), we observe that $\partial^{l_1}_{y_1}\partial^{l_2}_{y_2}\Phi(x;y)$ is a derivative of the function $H^{(1)}_0(k|x-y|)$ multiplied by a rational function depending on the arguments $|x-y|^{1/2}$, x_1, x_2, y_1, and y_2. The derivatives of order higher than one can be reduced to the zero and first order derivative using Bessel's differential equation. In view of (A.9), we can replace the derivatives of the Hankel functions by the expression $\exp(i(k|x-y|-(2n+1)/4\pi))$. Simple estimates of the difference for the expressions with y and with y replaced by $y-(0,H)$ give the estimate on the right-hand side of (A.23). Indeed, estimates like

$$\Big|\,|x-y|^{1/2}-\big|x-\big(y-(0,H)\big)\big|^{1/2}\Big|\le C|x|^{-1/2},$$
$$\Big|\exp\big(ik|x-y|\big)-\exp\Big(ik\big|x-\big(y-(0,H)\big)\big|\Big)\Big|\le C\frac{1+|x_2|}{|x|}$$

lead us to the additional factor $(1+|x_2|)|x|^{-1}$ in $C(1+|x_2|)|x|^{-3/2}$ in comparison to an estimate by $C|x|^{-1/2}$ following directly from (A.9) applied to a single derivative of Φ. $\square$

It follows from (A.23) that the function $v=G^\Theta(\cdot\,;y)-G^\Theta(\cdot\,;y^*)$ decays faster than $G^\Theta(\cdot\,;y)$ in U_h. As a consequence of the proof of Lemma 2.2, we obtain the well-posedness result on rough surface scattering problems formulated in Corollary 2.1.

Proof of Lemma 2.3. Replacing $G(x;y)$ by $\partial^{l_1}_{y_1}\partial^{l_1}_{y_1}G(x;y)$ in the proof of Lemma 2.2, we conclude that the modified right-hand side of (A.21) satisfies the properties of a differentiated Green's function together with the HPSRC. Applying the inverse operator $[\partial^{l_1}_{y_1}\partial^{l_1}_{y_1}]^{-1}$, i.e., integrations w.r.t. the variables y_1 and y_2, we define a new Green's function satisfying the HPSRC. From the uniqueness of the Green's function, we obtain that the modified right-hand side of (A.21) is indeed the differentiated Green's function $G(x;y)$. Hence, $\partial^{l_1}_{y_1}\partial^{l_1}_{y_1}G(x;y)$ is equal to the modified right-hand side of (A.21), and the HPSRC is satisfied. The far-field pattern exists as well. $\square$

Proof of Lemma 2.4. For definiteness, we consider the case $x\in\Omega_a$. We simply repeat the proof of Lemma 2.2 but with Θ and $G^\Theta(x\,;y)$ replaced by the line L_a containing Γ_a and by the Green's function $G_a(x\,;y)=\Phi(x\,;y)-\Phi(x\,;y^*)$, respectively. Then, due to the differentiability of $\mathcal{S}_y$ in (A.22) with G^Θ replaced by G_a, the remainder term $R(x\,;y)=w(x\,;y)$ is a solution of the boundary value problem in [10, Thm. 4.1] and, therefore, a function $\Omega_a\ni y\mapsto R(\cdot;y)\in H^1(\Omega_a)$, which together with all derivatives is continuous. By the regularity of solutions to the homogeneous Dirichlet problem for the Helmholtz equation, the function $y\mapsto R(\cdot;y)$ maps even to the Sobolev spaces of higher order. $\square$

Proof of Lemma 3.1. We first note that the integrals on the right-hand side of (3.7) are understood as the duality between the spaces $\widetilde{H}^{1/2}(C_R)$ and $H^{-1/2}(C_R)$ (cf. subsection 2.3). Applying integration by parts along the boundary C_R, we get a new integral representation with higher order derivatives w.r.t. y on G but with smoother weight function f. Without loss of generality we may suppose $f \in L^1(\Gamma_0)$. The proof of Lemma 2.2 implies Lemma 3.1 if we follow the proof of Lemma 2.2 with the Green's function replaced by its derivatives and if we apply Lemma A.2 twice with w_I equal to one of the integrals on the right-hand side of (3.7). $\square$

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