

Potential theory and geometric analysis.

① Linear potential theory, Δ

1) Sobolev space $W_0^{1,p}(\Omega)$, characterization.

2) Riesz representation thm

3) Wiener Criterion for the solvability of $\begin{cases} \Delta u = 0 \\ u|_{\partial\Omega} = \phi \end{cases}$

4) Application in Geometry.

② Nonlinear potential theory

1) Kilpeläinen & Malý thm

2) Wiener Criterion

3) Application to Geometry, $|W|\text{-Laplacian equation}$

1. Linear potential theory

1. D.H. Armitage & S.J. Gardiner, classical potential theory

Springer. 2001.

2. J. Heinonen, T. Kilpeläinen & O. Martio.

Nonlinear potential theory of degenerate elliptic equations.

1.1 Introduction

\mathbb{R}^N , $N \geq 2$. $x = (x_1, \dots, x_n)$, $\Omega \subseteq \mathbb{R}^N$.

$u \in C(\bar{\Omega})$, define $\nabla u = (\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_N})$, $\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \dots + \frac{\partial^2 u}{\partial x_N^2}$

$\alpha = (\alpha_1, \dots, \alpha_N)$ $0 \leq \alpha_i \in \mathbb{N}$, multi-index

$$|\alpha| = \alpha_1 + \dots + \alpha_N . \quad \nabla^\alpha u(x_1, \dots, x_N) = \frac{\partial^\alpha u(x_1, \dots, x_N)}{\partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N}}$$

$L^1_{loc}(\Omega) = \{ u(x) : \text{integrable on compact } K \subset \Omega \}$

Def 1. $u \in L^1_{loc}(\Omega)$, $\alpha = (\alpha_1, \dots, \alpha_N)$, $v \in L^1_{loc}(\Omega)$ is called α th weak derivative of u if

$$\int_{\Omega} \phi v \, dx = (-1)^{|\alpha|} \int_{\Omega} u \cdot \nabla^{\alpha} \phi \, dx \quad \forall \phi \in C_c^\infty(\Omega)$$

Denote $\nabla^{\alpha} u$.

Def 2. $k \geq 0$, $\alpha \in \mathbb{Z}$, $p > 1$

$W^{k,p}(\Omega) = \{ u \in L^1_{loc} : \nabla^{\alpha} u \in L^p(\Omega) \text{ for all } 0 \leq |\alpha| \leq k \}$

$$\|u\|_{W^{k,p}(\Omega)} = \sum_{|\alpha| \leq k} \left(\int_{\Omega} |\nabla^{\alpha} u|^p \, dx \right)^{1/p}$$

$$C_0^\infty(\Omega) \subseteq W^{k,p}(\Omega) \quad W_0^{k,p}(\Omega) = \overline{C_0^\infty(\Omega)}$$

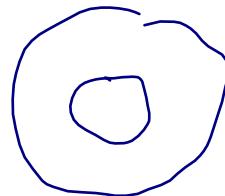
Characterize $W_0^{1,p}(\Omega)$

Problems.

1. $B_1 \subset \mathbb{R}^N$. $f(x) = 1 - |x|^2 = 1 - (x_1^2 + \dots + x_N^2)$

① $f(x) \in W_0^{1,2}(B_1 \setminus \{0\})$? \vee

② $f(x) \in W_0^{1,2}(B_1 \setminus (B_1 \setminus B_2))$ \times



③ $f(x) \in W_0^{1,2}(B_1 \setminus \{0\})$ \vee

④ If for a closed $S \subset B_1 \setminus \{0\}$ $f(x) \in W_0^{1,2}(B_1 \setminus S)$

Connect to some capacity of S .

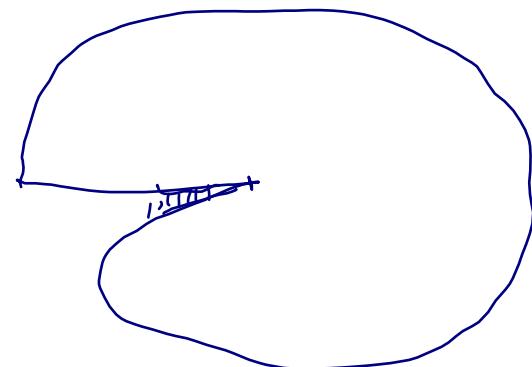
2. Consider Dirichlet problem

$$\begin{cases} \Delta u = 0 & x \in \Omega \\ u = \phi & x \in \partial\Omega \end{cases}$$

$$\phi(x) \in C(\bar{\Omega})$$

Does it admit a solution in the sense

$$\lim_{x \rightarrow x_0 \in \partial\Omega} u(x) = \phi(x_0)$$

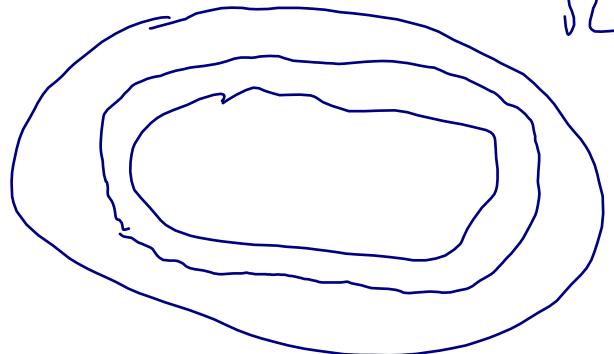


1.2 Problem 1, Assume $\Omega \subseteq \mathbb{R}^N$. Ω bounded.

Lemma 1 Suppose $u \in W_c^{1,p}(\Omega)$ (c, compact support)

$$\Rightarrow u \in W_c^{1,p}(\Omega)$$

Proof.



For $u \in W_c^{1,p}(\Omega)$,

choose $v_n \in C_0^\infty(\Omega)$ s.t. $v_n, v_n \rightarrow u$
in $W_0^{1,p}(\Omega)$ $n \rightarrow \infty$.

For example, we may choose

$$v_n = \int_{\Omega} \frac{1}{\Sigma_n^N} \phi\left(\frac{x-y}{\Sigma_n}\right) \underline{u(y)} dy.$$

$$\phi \in C_0^\infty(B_1)^0, \quad \int_{\Omega} \phi = 1, \quad \Sigma_n \rightarrow 0$$

$$\text{supp } v_n \subset \subset \Omega, \quad v_n \in C_0^\infty(\Omega) \Rightarrow u \in W_0^{1,p}(\Omega)$$

Lemma 1.2 If Ω is a bounded domain $\underline{u} \in W^{1,p}(\Omega)$

if for $\forall y \in \partial\Omega$ $\lim_{x \rightarrow y} u(x) = 0 \Rightarrow u \in W_0^{1,p}(\Omega)$

Proof. $u = u^+ + u^-$. We may assume $u \geq 0$.

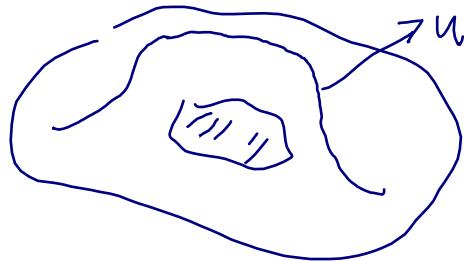
Define $u_\varepsilon = \max\{u - \varepsilon, 0\} \in W^{1,p}(\Omega)$ for any $\varepsilon > 0$.

and u_ε has compact support in Ω , $\Rightarrow u_\varepsilon \in \underline{W_0^{1,p}}$

$u_\varepsilon \xrightarrow{W^{1,p}(\Omega)} u \Rightarrow u \in W_0^{1,p}$

Definition 3. Fix $p > 1$. For domain $\Omega \subseteq \mathbb{R}^N$, and any compact subset $F \subset \Omega$. we define

$$\begin{aligned}\text{Cap}_p(F, \Omega) &= \inf \left\{ \int_{\Omega} |\nabla u|^p dx : u \in C_0^\infty(\Omega), u \geq 1 \text{ on } F \right\} \\ &= \inf \left\{ \int_{\Omega} |\nabla u|^p dx : u \in W_0^{1,p}(\Omega), u \geq 1 \text{ on } F \right\}\end{aligned}$$



For open subset U of Ω . define

$$\text{Cap}_p(U, \Omega) = \sup \left\{ \text{Cap}_p(F, \Omega) ; F \subset U, F \text{ compact} \right\}$$

For arbitrary subset E . define

$$\text{Cap}_p(E, \Omega) = \inf \left\{ \text{Cap}_p(U, \Omega) ; U \supset E, U \text{ open} \right\}$$

电容.



$$\Omega \subseteq \mathbb{R}^3$$

$$\text{Cap}_2(E, \Omega)$$

Newtonian capacity.

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$$\inf \left\{ \int_{\Omega} u^2, u \geq 1, u \in W_0^{1,2}(\Omega) \right\}$$

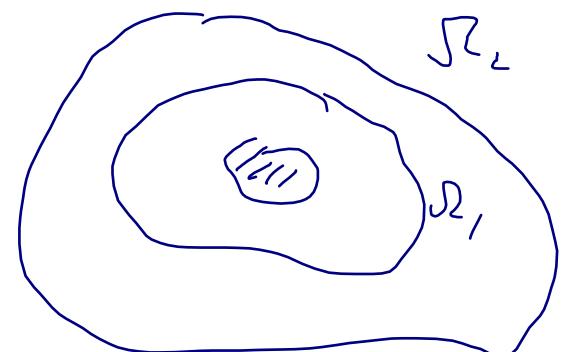
Theorem 1-3 The set function $E \mapsto \text{Cap}_p(E, \Omega)$, $E \subset \Omega$

has the following properties.

①. $E_1 \subset E_2 \quad \text{Cap}_p(E_1, \Omega) \leq \text{Cap}_p(E_2, \Omega)$

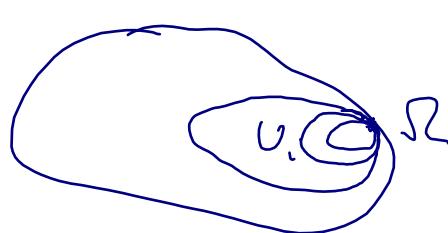
②. $\Omega_1 \subset \Omega_2$ are open $E \subset \Omega_1$

$$\text{Cap}_p(E, \Omega_2) \leq \text{Cap}_p(E, \Omega_1)$$



③ If k_i is a decreasing sequence of "compact" subset of Ω ,

$$k = \bigcap_i k_i \Rightarrow \text{Cap}_p(k, \Omega) = \lim_{i \rightarrow \infty} \text{Cap}_p(k_i, \Omega)$$



$$U_i \text{ open } \quad U_i \subset U. \quad X$$

④ If $E_1 \subset E_2 \subset \dots \subset \cup_i E_i = E \subset \Omega$

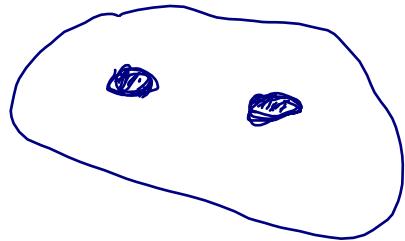
$$\text{Cap}_p(E, \Omega) = \lim_{i \rightarrow \infty} \text{Cap}_p(E_i, \Omega)$$

⑤ If $E = \cup_i E_i \subset \Omega$, then

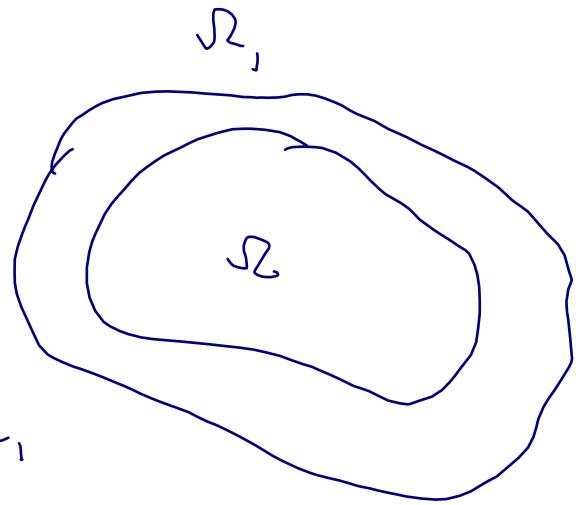
$$\text{Cap}_{p, \mu}(E, \Omega) \leq \sum_{i=1}^{\infty} \text{Cap}_p(E_i, \Omega) \Rightarrow \text{In particular}$$

$$\text{Refer to [HKM]} \quad \neg \exists i \text{ s.t. } \text{Cap}_p(E_i, \Omega) = 0 \Rightarrow \text{Cap}_p(E, \Omega) = 0.$$

Capacity is not a measure.



$$\sum_i \mu(E_i) = \mu(\bigcup E_i) \quad E_i \text{ not } \Omega.$$



Definition 4. Let Ω be a bounded domain, $\Omega \subset \Omega_1$,

for another bounded Ω ,

① A property is called to be hold ^p quasi everywhere (q.e) in Ω ,

if it holds except for a set E , with $\text{Cap}_p^*(E, \Omega_1) = 0$.

② A function f on Ω is said to be p-quasicontinuous if

for any $\varepsilon > 0$, \exists open set $U \subset \Omega$, with $\text{Cap}_p(U, \Omega_1) < \varepsilon$,

s.t. outside U , f is finite valued and continuous.

③ A sequence $\psi_j: \Omega \rightarrow \mathbb{R}$ converges p -quasimuniformly in Ω to ψ if for $\forall \varepsilon > 0$, \exists open set G s.t. $\text{Cap}_p(G, \mathcal{S}_1) < \varepsilon$ and $\psi_j \rightarrow \psi$ uniformly on $\Omega \setminus G$.

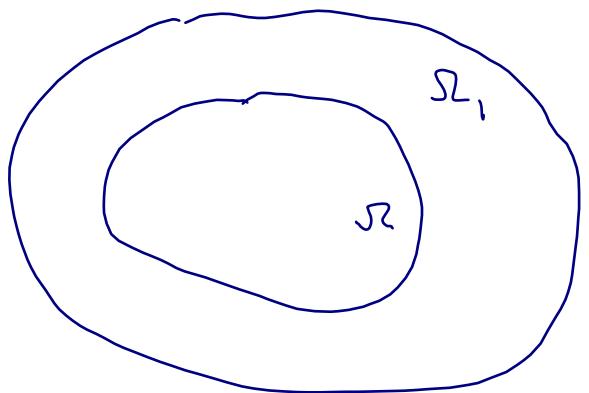
④. ψ_j converges locally p -quasimuniformly if it converges p -quasimuniformly on each $D \subseteq \Omega$.]

Lemma 1.4. A sequence ψ_j converges locally p -quasimuniformly in Ω iff for $\forall \varepsilon > 0$ \exists open $\underbrace{G \subset \Omega}$ with $\text{Cap}_p(G, \mathcal{S}_1) \leq \varepsilon$ s.t. ψ_j converges uniformly on every compact subset $\Omega \setminus G$.

Lemma 1.5. Let Ω be as before. Let $\phi_j \in ((\Omega) \cap W^{1,p}(\Omega))$

be a Cauchy sequence in $W^{1,p}(\Omega)$.

$\Rightarrow \exists$ a subsequence ϕ_k which converges
locally p -quasimuniformly in Ω to $u \in W^{1,p}(\Omega)$



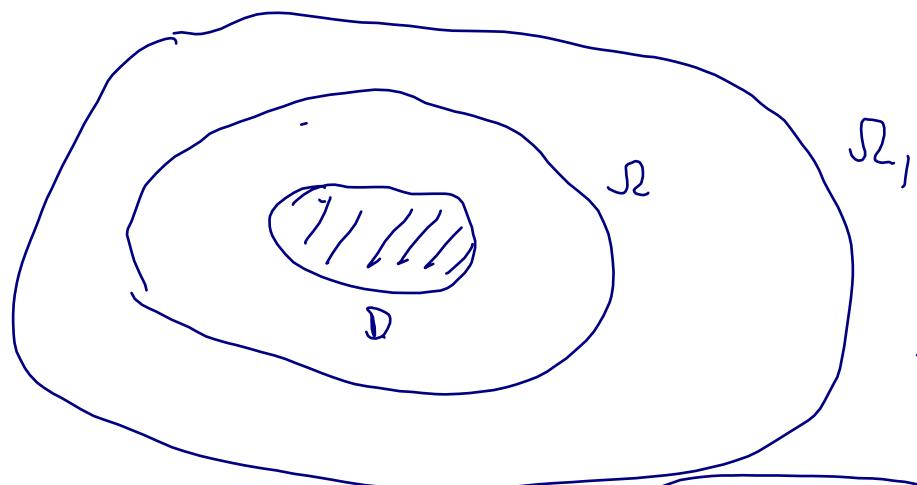
In particular, u is p -quasicontinuous and $\phi_k \rightarrow u$
pointwise p -q.e. in Ω .]

Proof. Suppose, a locally quasimuniformly convergent subsequence can be selected. It's not hard to prove the rest of the Lemma.
Since ϕ_j is a Cauchy sequence in $W^{1,p}(\Omega)$.

We may find a subsequence, ϕ_j s.t.

$$\sum_{j=1}^{\infty} \int_{\Omega} 2^{jp} \left(|\phi_j - \phi_{j+1}|^p + |\nabla(\phi_j - \phi_{j+1})|^p \right) dx < +\infty$$

For $D \subset \Omega$, let $\psi \in C_0^\infty(\Omega)$ $\psi \equiv 1$ on D . \Rightarrow



$$\sum_{j=1}^{\infty} \int_{\Omega} 2^{jp} \left(|\psi(\phi_j - \phi_{j+1})|^p + |\nabla \psi(\phi_j - \phi_{j+1})|^p \right) dx < +\infty$$

$\Rightarrow \forall \varepsilon > 0. \quad \exists j_\varepsilon$ large

s.t. $\sum_{j=j_\varepsilon}^{\infty} \int_{\Omega} 2^{jp} \left(|\psi(\phi_j - \phi_{j+1})|^p + |\nabla \psi(\phi_j - \phi_{j+1})|^p \right) dx < \varepsilon$.

Let $E_j = \{x \in D; |\phi_j(x) - \phi_{j+1}(x)| > 2^{-j}\}$

$$\text{Cap}_p(E_j, \Omega_1) \leq \int_{\Omega_1} 2^{jp} |\nabla(\psi(\phi_j - \phi_{j+1}))|^p d\mu$$

Put $E_\varepsilon = \bigcup_{j=j_\varepsilon}^{\infty} E_j$

$$\Rightarrow \text{Cap}_p(E_\varepsilon, \Omega_1) \leq \sum_{j=j_\varepsilon}^{\infty} \underbrace{\text{Cap}_p(E_j, \Omega_1)}_{< \varepsilon} < \varepsilon.$$

Moreover $j_\varepsilon \leq j \leq k$ $\underbrace{|\phi_j - \phi_k|}_{\leq \sum_{l=j}^{k-1} 2^{-l}} \leq \underbrace{2^{1-j}}_{\in D \setminus E_\varepsilon}$

$\Rightarrow \phi_j$ converges uniformly in $D \setminus E_\varepsilon$.

Hence it is not hard to prove this Lemma.

Theorem 1.6. Suppose that $u \in W^{1,p}(\Omega)$. Then there exists

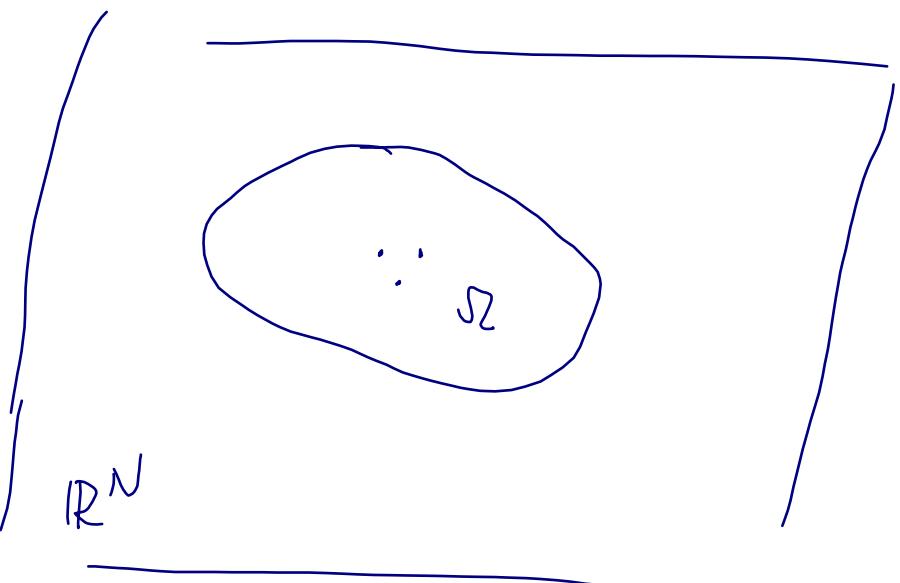
a p -quasicontinuous function $v \in W^{1,p}(\Omega)$ s.t. $\underline{u=v}$ a.e.

Theorem 1.7. Suppose $u \in W^{1,p}(\Omega)$. Then $u \in W_0^{1,p}(\Omega) \iff$

\exists ϕ -quasicontinuous function v in \mathbb{R}^N s.t.

$v=u$ a.e. in Ω and

$v=0$ g.e. in Ω^c .



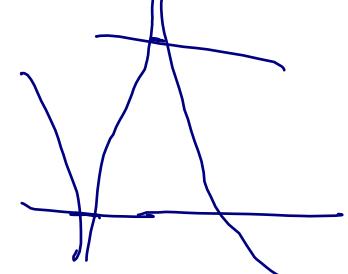
Proof

Fix $u \in W_0^{1,p}(\Omega)$, let $\phi_j \in C_c^\infty(\Omega)$ be a sequence converging to u in $W^{1,p}(\Omega)$. By thm 1.5, \exists subsequence ϕ_j - which converges p -quasieverywhere in \mathbb{R}^N to a p -quasicontinuous function v s.t. $v = u$ a.e. in Ω . & $v = 0$ g.e. in Ω^c

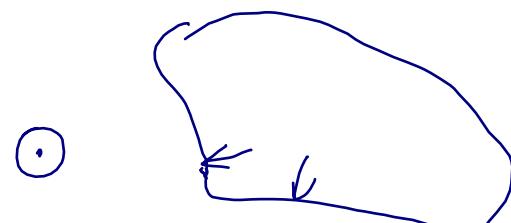
To prove the converse, first we may assume v is bounded because if we take

$$v_\sigma = \begin{cases} -\sigma & v \leq -\sigma \\ 0 & v \geq \sigma \\ v & \text{otherwise} \end{cases}$$

$$\begin{aligned} v &\leq -\sigma \\ v &\geq \sigma \\ \text{otherwise} & \end{aligned}$$



then $v_\sigma \xrightarrow{W^{1,p}} v$.



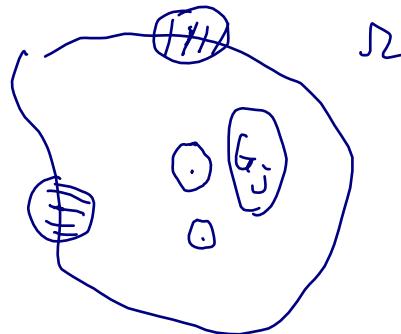
$$E = \{x \in \Omega : v(x) \neq 0\}$$

$$\begin{aligned} &1 - |x|^2 \\ &B_1(0) \setminus \{x_0\} \end{aligned}$$

Since $\text{Cap}_p(E, \Omega_1) = 0$ (key). We may choose open set G_j

such that $E \subset G_j$ s.t. $\text{Cap}_p(G_j, \Omega_1) \rightarrow 0$

and $v|_{G_j}$ is continuous.



Pick $\phi_j \in C_0^\infty(\mathbb{R}^N)$ $0 \leq \phi_j \leq 1$

$\phi_j = 1$ everywhere on G_j &

$$\int_{\mathbb{R}^n} (|\phi_j|^p + |\nabla \phi_j|^p) dx \rightarrow 0$$

Let $w_j = (1 - \phi_j)v \in W^{1,p}(\Omega)$

Moreover. $\lim_{x \rightarrow y \in \partial \Omega} w_j(x) = 0$. By lemma 1.2 $\Rightarrow w_j \in W_0^{1,p}(\Omega)$.

Clearly $w_j \rightarrow v$ in $L^p(\Omega)$.

$$\begin{aligned} \left(\int_{\Omega} |\nabla w_j - \nabla v|^p dx \right)^{\frac{1}{p}} &= \left(\int_{\Omega} |\nabla (\phi_j v)|^p dx \right)^{\frac{1}{p}} \\ &\leq \left(\int_{\Omega} |v \cdot \nabla \phi_j|^p dx \right)^{\frac{1}{p}} + \left(\int_{\Omega} |\phi_j \cdot \nabla v|^p dx \right)^{\frac{1}{p}} \end{aligned}$$

Dominated convergence thm

$$\Rightarrow w_j \rightarrow v \quad \text{in } W^{1,p}(\Omega) \quad \Rightarrow \quad v \in W_0^{1,p}(\Omega). \quad \#$$

Can be generalized to $W_0^{k,p}(\Omega)$ case.

1.3 Superharmonic function.

Let $\Omega \subseteq \mathbb{R}^N$ be a domain.

Aim: Define superharmonic functions on Ω .

$u \in C^2(\Omega)$. u superharmonic $\Leftrightarrow -\Delta u \geq 0$.

$$\liminf_{x \rightarrow x_0 \in \Omega} u(x) \geq u(x_0)$$

Definition $u: \Omega \rightarrow (-\infty, +\infty]$ is a lower semi-continuous (l.s.c.)

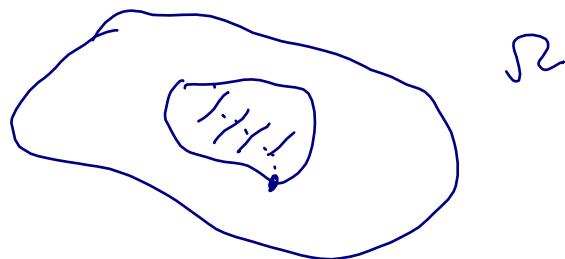
u is called supermonic if $u \not\equiv +\infty$ and

for any $D \subset \Omega$ & harmonic function v on D

if $v \leq u$ on $\partial D \Rightarrow v \leq u$ in D ,

If $-u$ is superharmonic, u is called subharmonic.

- Property: Superharmonic function is locally bounded from below.



Example. By maximum principle, $u \in C^2(\bar{\Omega})$ & $-\Delta u \geq 0$
 $\Rightarrow u$ is superharmonic.

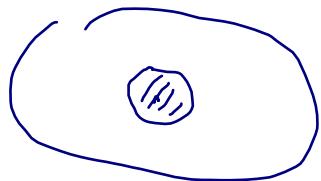
We are interested in superharmonic functions, which are
not smooth.

$$U_y(x) = \begin{cases} -\frac{1}{2\pi} \log |x-y| & N=2 \\ \frac{c(n)}{|x-y|^{n-2}} & N \geq 3 \end{cases}$$

is superharmonic in \mathbb{R}^N .

Lemma 1.8 A superharmonic function $u \in U(\Omega)$ is locally integrable.

(Moreover, $u \in L_{loc}^P(\Omega) \quad 1 \leq P < \frac{n}{n-2}$)



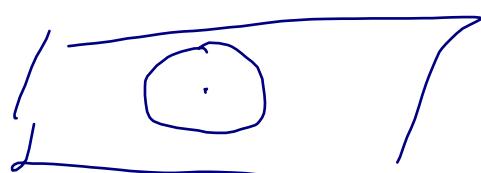
Step 1, u l.s.c. \Rightarrow u is locally bounded from below.

Step 2, From Baire's theorem, \exists a sequence $w_1 \leq w_2 \leq \dots$ of continuous functions, satisfying $\lim_{n \rightarrow \infty} w_n(x) = v(x)$

For any $\overline{B_r(x_0)} \subset \Omega$, we can solve

$$\begin{cases} \Delta v_n = 0 & x \in B_r(x_0) \\ v_n(x) = w_n(x) & x \in \partial B_r(x_0) \end{cases} \quad \text{using Poisson's integral formula.}$$

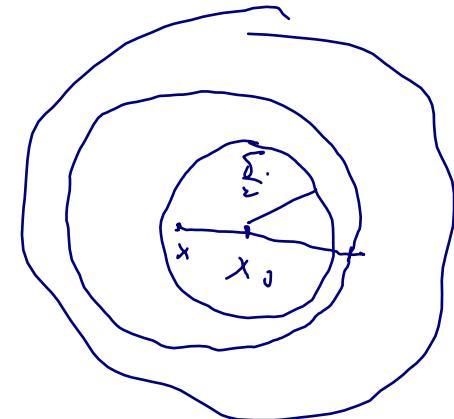
$$\int_{\partial B_r(x_0)} v_n(x) d\sigma_{\partial B_r} = v_n(x_0) \leq v(x_0)$$



Fatou's lemma

$$\int_{\partial B_r(x_0)} v(x) d\mu_{\partial B_r} \leq v(x_0)$$

Integrating w.r.t. $r \Rightarrow \int_{B_r(x_0)} v(x) dx \leq v(x_0)$



Step 3. Suppose $v(x)$ is not locally integrable.

We may assume $v(x) > 0$ in a neighborhood of x_0 .

But $\int_{B_r(x_0)} v(x) dx = +\infty$, $B_r(x_0)$ is a neighborhood of x_0 .

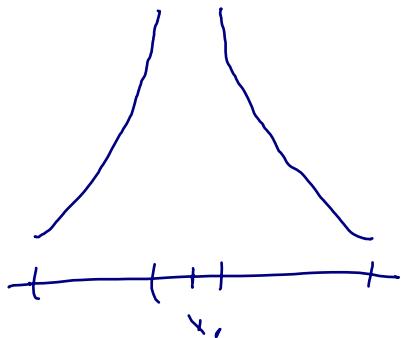
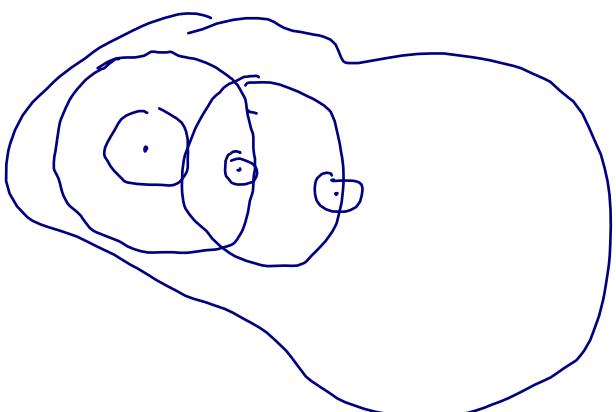
$B_\delta(x)$.

$x \in B_{\frac{\delta}{2}}(x)$. Step 2 $\Rightarrow v(x) = +\infty \quad \forall x \in B_{\frac{\delta}{2}}(x_0)$

STEP 4

$v(x) \equiv +\infty$ in $B_{\frac{R}{2}}(x_0)$ \Rightarrow If $R \geq \frac{\delta}{2}$ $B_R(x_0) \subset \Omega$

$v(x) \equiv +\infty$ on $B_R(x_0)$ $\Rightarrow v \equiv +\infty$ in Ω contradiction.

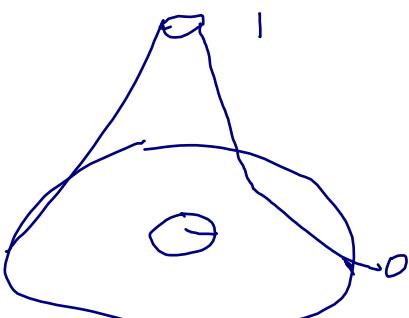


$$h(x) = \frac{\int_{|x-x_0|}^R t^{-(n-1)} dt}{\int_{\frac{R}{2}}^R t^{-(n-1)} dt}$$

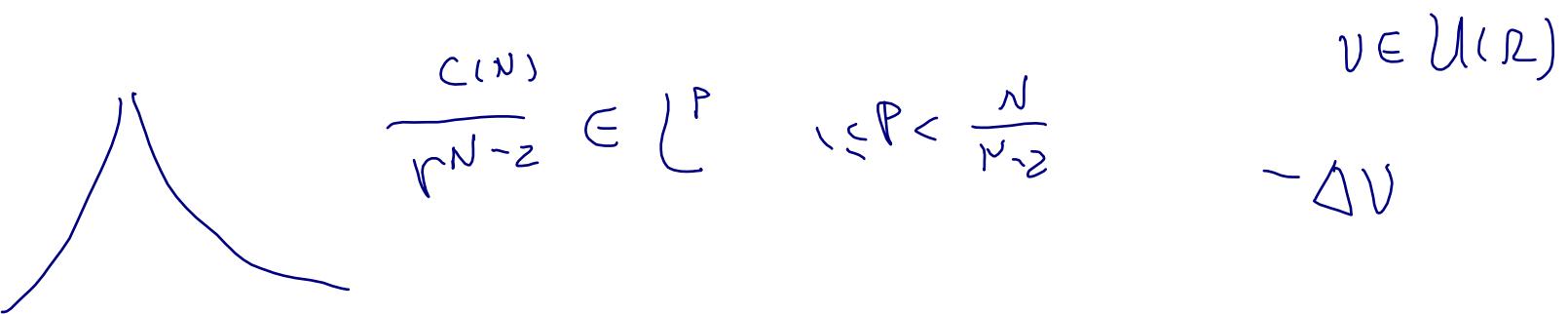
h harmonic in $\frac{\delta}{2} \leq |x-x_0| \leq R$

$\Rightarrow v - \inf_{B_R(x_0)} v \geq k h(x)$ in $B_R(x_0) \setminus B_{\frac{R}{2}}(x_0)$

$\nexists k > 0$.



Let $k \rightarrow \infty \Rightarrow v \equiv +\infty$ in $B_R(x_0)$



Definition 6. Suppose Σ is a σ -algebra of a set X . A set function.

$\mu : \Sigma \rightarrow [-\infty, +\infty]$ is called a signed measure

on X if it satisfies.

1. μ takes on at most one of values $\pm \infty$;
2. $\mu(\emptyset) = 0$;
3. For countable collection $\{E_k\}_{k=1}^{\infty}$ of pairwise disjoint sets in Σ

$$\mu(\bigcup_{k=1}^{\infty} E_k) = \sum_{k=1}^{\infty} \mu(E_k)$$

In addition, if $\mu(U) \geq 0$, $\forall U \in \Sigma$ μ is called a measure.

Definition 7. X Hausdorff topological space. Σ is all Borel sets
 μ is a measure defined on Σ .

- μ is called inner regular. If $\forall U \in \Sigma$.

$$\mu(U) = \sup \{ \mu(K) ; K \subset U, K \text{ compact} \}$$

- μ is called locally finite, if $\forall y \in X$, y has a neighborhood U s.t. $\mu(U) < \infty$.

- $\mu(U) \geq 0$ for $\forall U \in \Sigma$

Then μ is called a Radon measure.

We use $M^+(X)$ to denote all

Radon measures.

Def 8. If μ is a Radon measure on \underline{X} , and U is an open set.

If $\mu(D) = 0$, \forall open $D \subset U$, we say μ vanishes on D .

We denote $\mu|_D = 0$

Support of μ is the complement of the union of all open D such that $\mu|_D = 0$.

Let $X = \mathbb{R}^N$. Let F be a compact subset of \mathbb{R}^N .

$M^+(F) = \{ \text{Radon measure on } \mathbb{R}^N \text{ supported on } F \}$

For domain Ω , $M^+(\Omega) = \{ \text{Radon measure on } \Omega \}$.

$\Omega \subseteq \mathbb{R}^N$, domain. $u : \Omega \rightarrow [-\infty, +\infty]$. Locally integrable

$$L_u : C_0^\infty(\Omega) \rightarrow \mathbb{R}$$

$$\psi \mapsto L_u(\psi) = \underbrace{\int_{\Omega} u \circ \psi \, dx}_{\text{underbrace}}$$

L_u is called the distributional Laplacian of u .

Let v be another l. i. function on Ω $k \neq 0, \lambda \in \mathbb{R}$

$$L_{k \cdot u + \lambda v} = k L_u + \lambda L_v$$

Thm 19. If $u \in U(\Omega)$, then L_u is a Radon measure.

Pruf. L_u is well defined because u is locally integrable.

We only need to prove. for any compact $K \subset \Omega$, and any

$\psi \in C_0^\infty(\Omega)$ $\text{supp } \psi \subseteq K$, we have

$$\left| L_u(\psi) \right| \leq c(K) \|\psi\|_{C_0^\infty}$$

$$L_u \psi \geq 0 \quad \forall \psi \in C_0^\infty(K)$$

From Thm 3.3.3 [A.G. Classical potential theory].

\exists an increasing sequence $u_n \in U(\Omega) \cap C_0^\infty(\Omega)$ s.t. $u_n \rightarrow u$

pointwise to u , & $-\Delta u_n \geq 0$. $\forall \psi \in C_0^\infty(K) \quad \psi \geq 0$

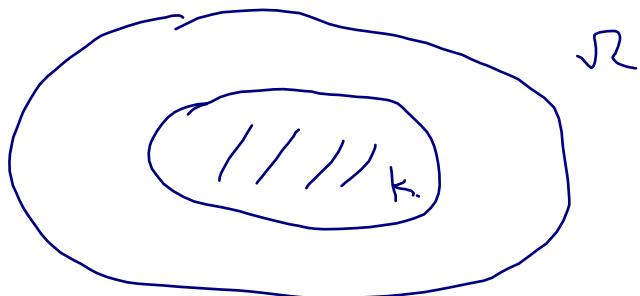
$$\Rightarrow - \int_{\Omega} (\Delta u_n) \cdot \psi \geq 0 \Rightarrow \int_{\Omega} u_n (-\Delta \psi) \geq 0 \quad ((-\Delta \psi)^+ - (-\Delta \psi)^-)$$

$$\Rightarrow \int_{\Omega} u (-\Delta \psi) \geq 0 \Rightarrow L_u(\psi) \geq 0$$

Then for contradiction, we assume $\exists \psi_n \in C_0^\infty(k) \quad \|\psi_n\|_{C^0} \leq 1$

$$\text{s.t. } L_u(\psi_n) \geq \frac{1}{n}$$

choose $0 \leq \phi \in C_0^\infty(\mathbb{R}) \quad \phi > 1 \text{ on } k$



$$L_u(\phi - \psi_n) \underset{n \text{ large}}{\leftarrow} 0 \quad \text{for } n \text{ large.}$$

which contradicts with $\phi - \psi_n \geq 0$.

L_u defines a Radon measure $\mu_u \in M^+(\mathbb{R})$

$$U(x-y) = \frac{c(N)}{|x-y|^{N-2}}$$

We call it "Riesz measure" of $u \in U(\mathbb{R})$

$$\Delta u = f \underset{\rightarrow}{\rightarrow}$$

$$u = h + \boxed{\int_U U(x-y) f(y) dy}$$

1.4 Potentials.

Def 9. If f, g functions defined on Ω .

If $f \leq g$, f is called a minorant of g
 g is called a majorant of f .

(majorant)

If f is a minorant of g , satisfying property A.

f is called a A -minorant (majorant) of g .

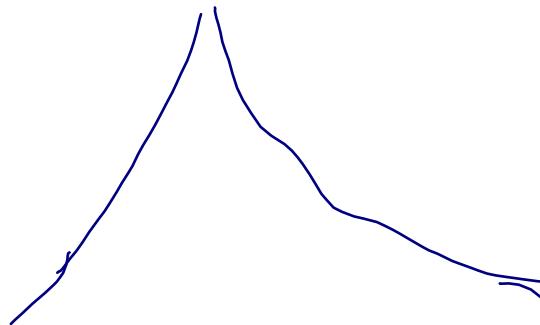
Define $U_y(x) = U_{N,y}(x) = \begin{cases} -\frac{1}{2\pi} \log|x-y| & N=2 \\ \frac{C(N)}{|x-y|^{N-2}} & N \geq 3 \end{cases}$

$$-\Delta U_y(x) = \delta(x-y)$$

Def¹⁰. An open set $\Omega \subseteq \mathbb{R}^N$ is called Greenian if

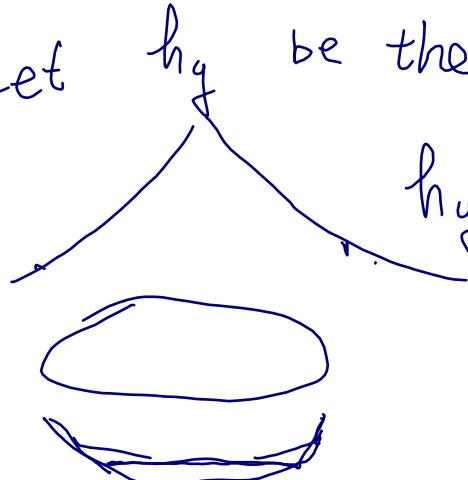
$\forall y \in \Omega$, U_y has a subharmonic minorant in Ω .

(. \mathbb{R}^2 is not Greenian)



We assume Ω is Greenian always in this subsection.

Let h_y be the greatest harmonic minorant of U_y on Ω .



$$h_y = \sup \{ f \text{ is a subharmonic minorant of } U_y \}$$

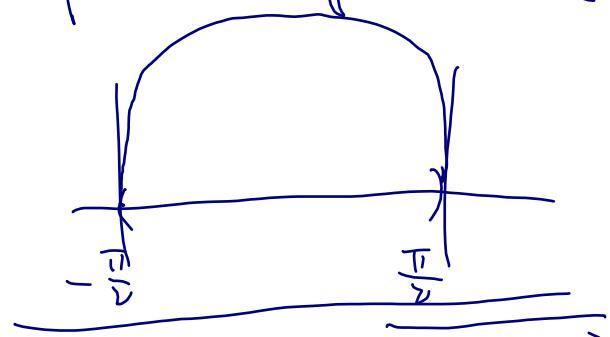
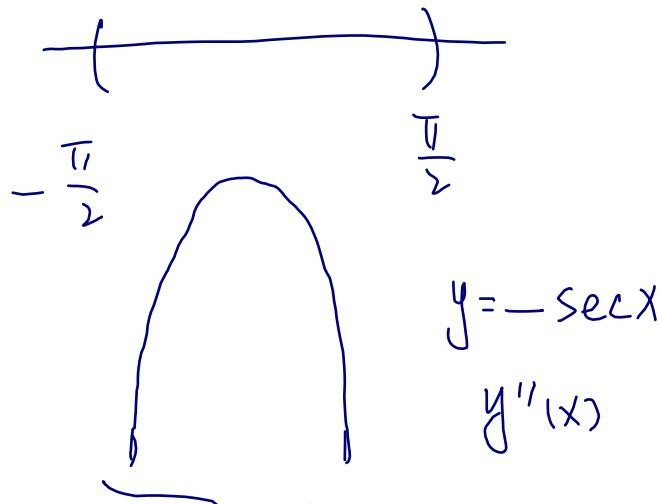
Prove that h_y is harmonic. Petrov method

Define $\underline{G}_{\Omega}(x,y) = U_y - h_y$

Definition!! Let μ be a Radon measure on Ω . define

$$\underline{G}_{\Omega} \mu(x) = \int_{\Omega} \underline{G}_{\Omega}(x,y) d\mu(y).$$

If $\underline{G}_{\Omega} \mu(x) \neq +\infty$, we call it potential generated by $\mu(x)$



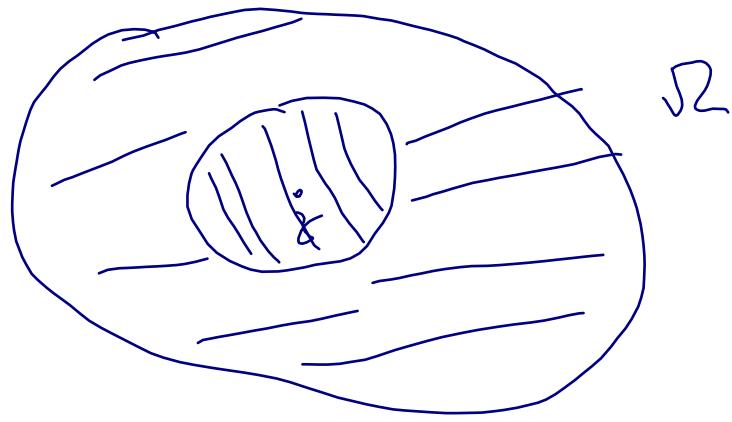
$$z = \sqrt{\frac{\pi}{2}} - x^2$$

$G_{\Omega} z''$ is a potential

$$G_{\Omega} y'' \equiv +\infty$$

Thm 10 Let μ be a measure on a connected Greenian open set Ω and let $\overline{B(b,r)} \subset \Omega$. $G_{\Omega} \mu$ is a potential \Leftrightarrow

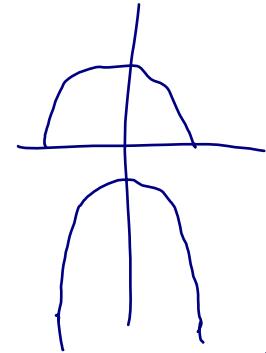
$$\int_{\Omega \setminus \overline{B(b,r)}} G_{\Omega}(z,y) d\mu(y) < \infty$$



In particular, if $\mu(\Omega) < +\infty$

or μ has compact support

$\Rightarrow G_{\Omega} \mu$ is a potential.



Thm 11 (Riesz decomposition theorem) Let u be superharmonic in Ω .

let \underline{u}_u Riesz measure of u . Suppose \underline{u} has a subharmonic minorant on Ω . $\Rightarrow G_{\Omega} \underline{u}_u$ is a potential on Ω

$\underline{u} = \underline{G}_{\Omega} \underline{u}_u + h$, where h is the greatest harmonic minorant on Ω .

Lemma

(Weyl's lemma)

Let Ω be a domain \mathbb{R}^N , $u \in L^1_{loc}(\Omega)$. If

$$\int_{\Omega} u(x) \Delta \phi(x) dx = 0 \quad \forall \phi \in C_0^\infty(\Omega)$$

$\Rightarrow u$ is harmonic in Ω . $\Rightarrow u \in C^\infty(\Omega)$.

Lemma Let Ω be a Greenian domain and let μ be a Radon measure on Ω .

If $G_\Omega \mu$ is a potential, $\Rightarrow G_\Omega \mu$ is locally integrable, superharmonic

Moreover, the Piesz measure of $G_\Omega \mu$ is μ .

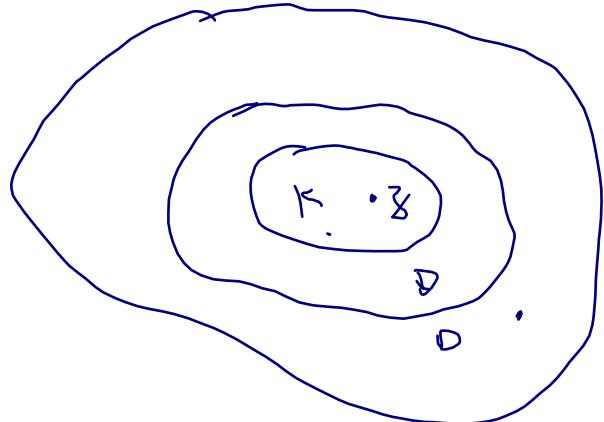
Proof. First we prove that $G_\Omega \mu$ is locally integrable.

For simplicity we assume $N \geq 3$. "N=2" left to you

$$G_{\Omega}(x, y) \leq U_y(x).$$

Let $K \subset \subset D \subset \subset \Omega$.

For $z \in K$



$$\Omega \quad G_{\Omega} \mu(z) = \int_D \int_{\Omega} G_{\Omega}(z, x) d\mu(x),$$

$\int_{\Omega} G_{\Omega}(z, x) d\mu(x)$ is harmonic in $z \in D$

$$\int_K \int_D G_{\Omega}(z, x) d\mu(x) dz = \int_D \underbrace{d\mu(x)}_{\Omega} \int_K G_{\Omega}(z, x) dz \leq \mu(D) \cdot C(\text{Vol}(K), N)$$

$$\left(\int_K G_{\Omega}(z, x) dz \leq \int_K U_x(z) dz \leq \int_B U_x(z) dz \right. \quad \begin{matrix} \text{B ball centered at } x \\ \text{Volume}(B) = \text{Vol}(K) \end{matrix}$$

$$\leq C(\text{Vol}(K), N)$$

$L_{G_n u}$ will defined

$$\begin{aligned} L_u(\phi) &= \int_{\Omega} (-\Delta \phi) G_n u \, dx = \int_{\Omega} (-\Delta \phi) \left(\int_{\Omega} G_n(x,y) d u(y) \right) dx \\ &\stackrel{\frac{U_y(x) + h}{\downarrow}}{=} \int_{\Omega} d u(y) \int_{\Omega} \underbrace{G_n(x,y)}_{-\Delta \phi(x)} (-\Delta \phi(x)) \, dx = \int_{\Omega} \phi(y) d u(y) \end{aligned}$$

Proof. Let $\{K_n\}$ be a sequence of compact subsets of Ω

$$\text{s.t. } K_n \subset \text{int}(K_{n+1}), \quad \bigcup_n K_n = \Omega$$

$$\text{Let } \underline{\underline{M_u^{(n)}}} = M_u|_{K_n}.$$

The distributional Laplacian $L G_{\Omega} \underline{\underline{M_u^{(n)}}}, L u$

acting on $C_0^\infty(\text{int } K_n)$.

Let $\underline{\underline{v_n}} = u - G_{\Omega} \underline{\underline{M_u^{(n)}}} \Rightarrow v_n$ is a harmonic function in K_n

$G_{\Omega} \underline{\underline{M_u^{(n)}}}$ is a superharmonic function

Suppose h is the GHM of u . Now we prove

$$G_{\Omega} \underline{\underline{M_u^{(n)}}} \leq u - h \quad \text{Then let } n \rightarrow \infty \quad G_{\Omega} \underline{\underline{M_u}} \leq \underline{\underline{u - h}}$$

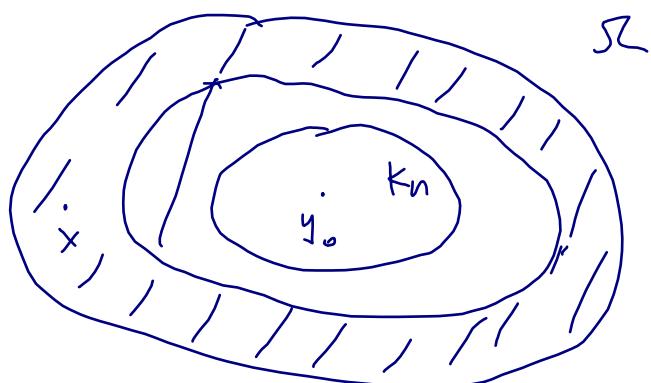
It suffices to prove $\underset{\wedge}{G} H M_i \cdot f \quad u - h - G_{\Omega} \mu_u^{(n)} \geq 0$

$$\boxed{u - h - G_{\Omega} \mu_u^{(n)}} \geq \boxed{- G_{\Omega} \mu_u^{(n)}} \Rightarrow \boxed{w_n} \geq - G_{\Omega} \mu_u^{(n)}$$

$$\boxed{G_{\Omega} \mu_u^{(n)}} \geq \boxed{- w_n}$$

It suffice to prove $G_{\Omega} \mu_u^{(n)}$'s $G H M_i = 0$

It's easy to see 0 is a harmonic minorant of $G_{\Omega} \mu_u^{(n)}$



$$S \subset \text{Harnack} \quad \underbrace{G_{\Omega}(x, y)}_{y \in K_n} \leq C \underbrace{G_{\Omega}(x, y_0)}$$

Integrate w.r.t. y

$$\underbrace{G_{\Omega} \mu_u^{(n)}(x)}_{=} \leq C \mu_u(K_n) \underbrace{G_{\Omega}(x, y_0)}_{=} \quad \text{in } S \setminus K_{n+1}$$

Let ξ_n be GHM: of $G_{\Omega} \mu_u^{(n)}$, ∞ .

$$\text{In } \Omega \setminus \{k_n\} \quad \xi_n \leq C \mu_u(k_n) G_{\Omega}(x, y_0) \Rightarrow \text{In } \Omega \quad \xi_n \leq C \mu_u(k_n) G_{\Omega}(x, y_0)$$

$$\Rightarrow \xi_n \leq 0$$

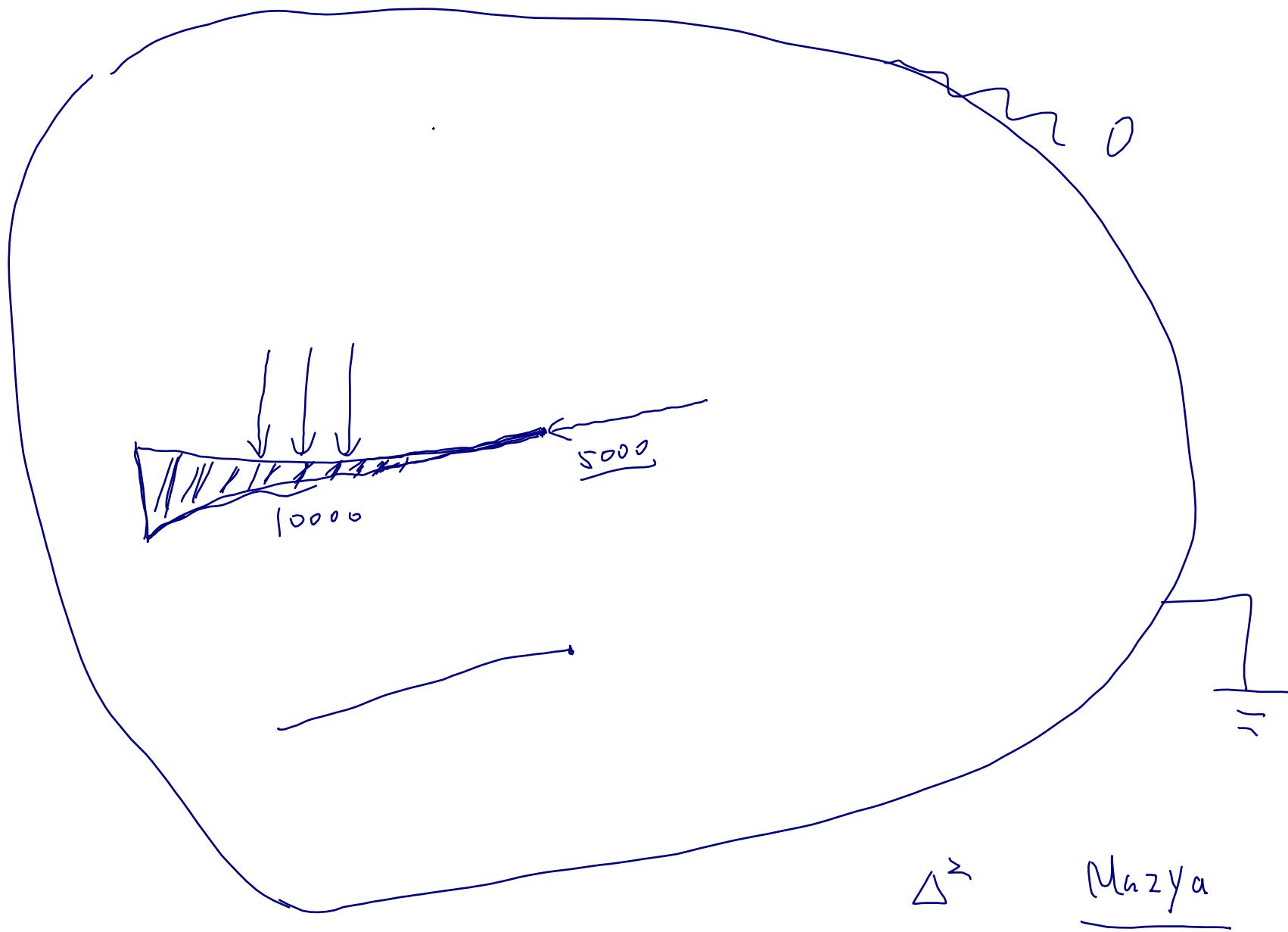
$\Rightarrow G_{\Omega} \mu_u \leq u - h$ hence is a potential.

The GHM: of $\underline{u - h} - \underline{G_{\Omega} \mu_u}$ is $\underline{0}$

Since GHM: of $u - h$ is already 0

$$\Rightarrow u = h + G_{\Omega} \mu_u . \quad \#.$$

$G_{\Omega} \mu_u$ potential



1.5. polar set

Sim Wiener's criterion on the solvability of Dirichlet problem.

Definition 12. A set E in \mathbb{R}^N is called polar if \exists superharmonic

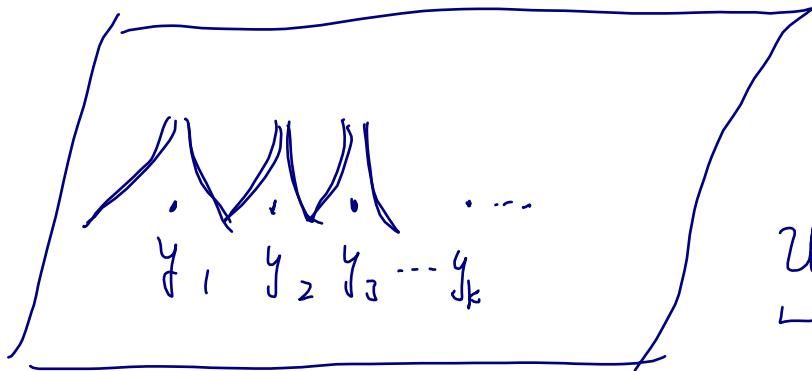
function u on some open set Ω such that

$$E \subset \{x \in \Omega; u(x) = +\infty\}.$$

Polar set has 0 - Lebesgue measure.

Example. 1. $\{y\}$ is polar set, since U_y is superharmonic
on \mathbb{R}^n & $U_y = +\infty$ on $\{y\}$.

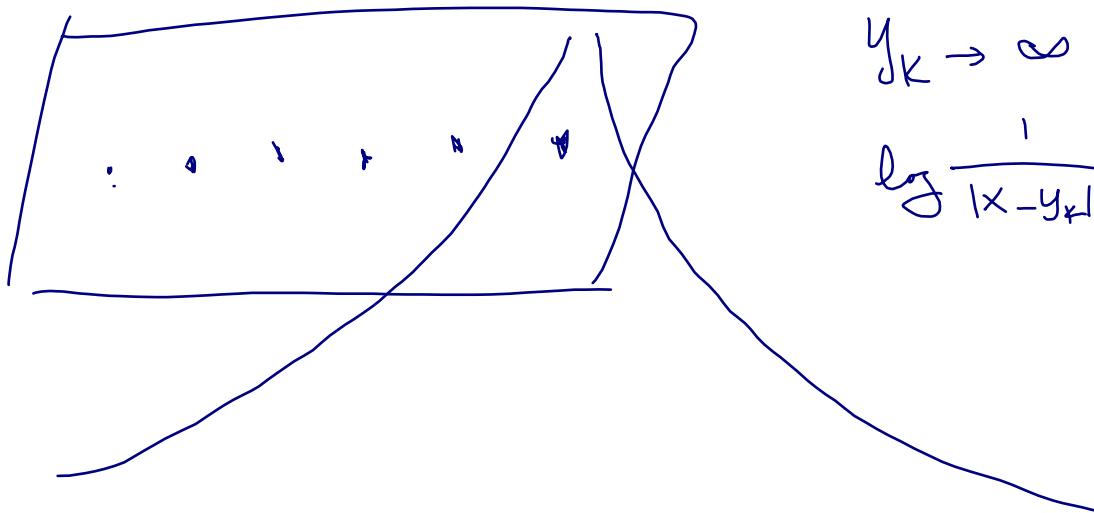
$N \geq 3$



$\{y_k ; y \in N\}$ is polar

$$u(x) = \sum_k 2^{-k} u_{y_k}(x) \rightarrow$$

$$u_{y_k}(x) = \frac{c(y_k)}{|x-y_k|^{n-2}}$$



$y_k \rightarrow \infty$

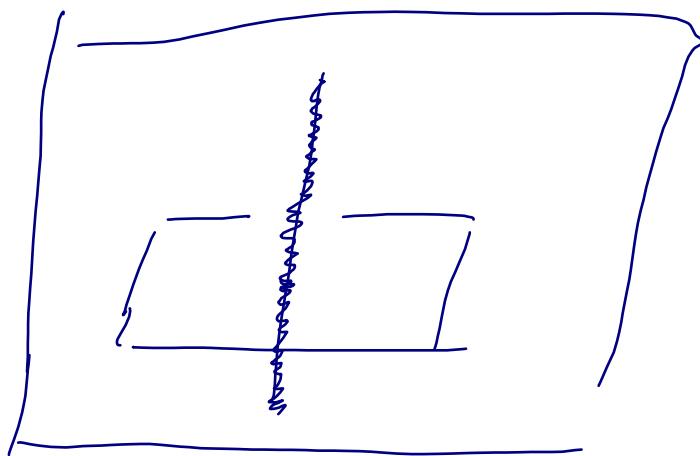
$$\log \frac{1}{|x-y_k|}$$

$N=2$

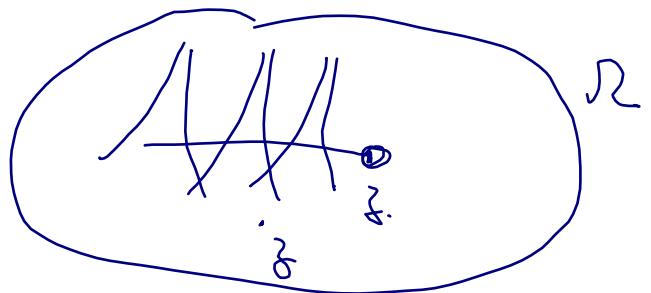
$$u(x) = \sum_k 2^{-k} \left(1 + \log^+ \|y_k\| \right)^{-1} u_{y_k}(x)$$

$$x \in \mathbb{R}^2$$

2. If $N \geq 3$, the set $E = (0,0) \times \mathbb{R}^{N-2}$ is a polar set,
 since $u(x_1, \dots, x_N) = \begin{cases} -\log(x_1^2 + x_2^2) & \\ \end{cases}$

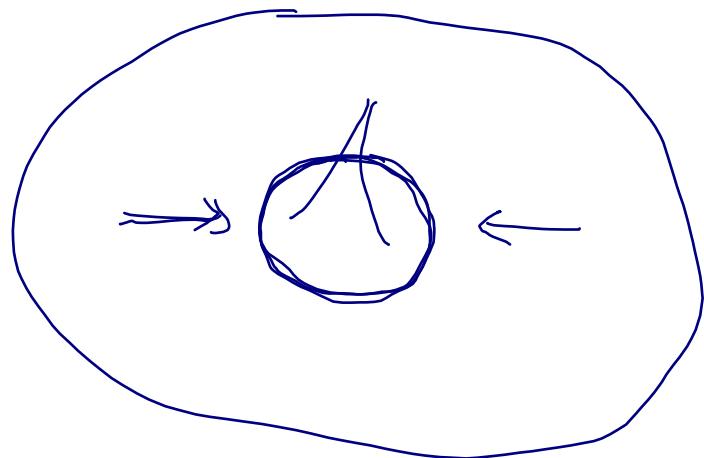
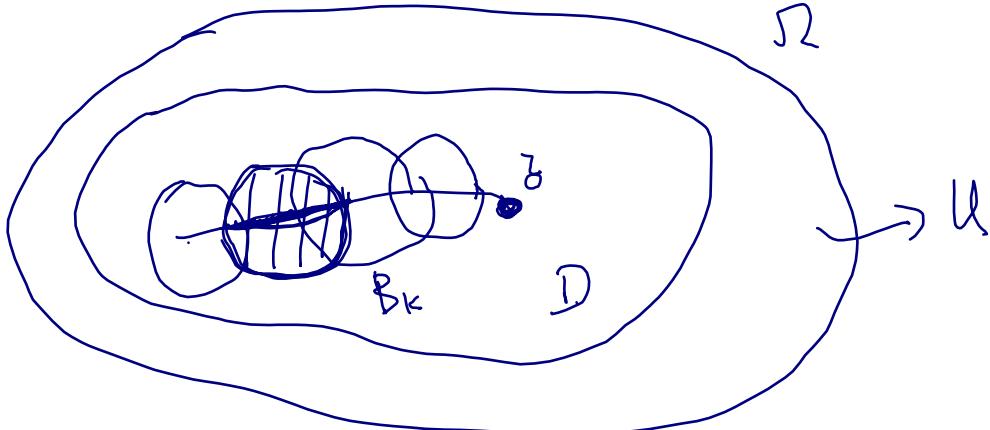


$$\frac{1}{|x-y|^{N-2}}$$



Theorem 1.12. Let E be a polar set. $E \subset \Omega$.

and let $z \in \Omega \setminus E$. If Ω is a Greenian, then there is
 a potential $G_{\Omega}\mu$ valued $+\infty$ on E such that
 $\overline{G_{\Omega}\mu(z)} < +\infty$, $\mu(\Omega) < +\infty$



Proof. E is polar. $E \subset \Omega \Rightarrow \exists u \in U(D)$ s.t.

$$u(E) = +\infty, \text{ Suppose } \delta = 0$$

Let $E \subset D \subset \Omega$ but $z \notin D$

Find $\bar{B}_k \subset D$ & $\bigcup_k B_k = D$, For $\forall k$ define

$$v_k(A) = \frac{\mu_u(A \cap B_k)}{\mu_u(B_k) + 1}$$

$U_k = \int \underbrace{U_y}_{dV_k(y)}$ is valued $+\infty$ on $E \cap B_k$.

& $U_{k(0)} < +\infty$ Since $0 \notin \overline{B_k}$.

When $N \geq 3$. let

$$\begin{aligned} \mu &= \sum_k 2^{-k} \\ &= \text{(Diagram showing a large circle } E \text{ containing several smaller circles } V_k, \text{ one of which contains the value } U_{k(0)}.) \end{aligned}$$

$\Rightarrow \mu(E) \leq 1$ & $G_E \mu(0) < +\infty$

$$G_E \mu \Big|_E = +\infty$$

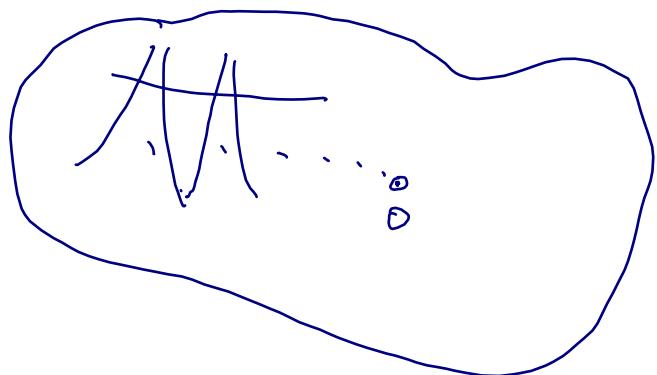
If $N=2$ $\mu = \sum_k 2^{-k} \left\{ 1 + \int_{B_k} (\log |y|) dV_k(y) \right\}^{-1} V_k$

We can prove similarly.

1.6 Fine topology.

习題.

$U \in U(\Omega)$ u is not continuous.



$\{x_n\} \subset \Omega, \lim_{n \rightarrow \infty} x_n = o \in \Omega$

$$u(x) = \sum_{n=1}^{\infty} \sum_{z^n} \frac{U_{x_n}(x)}{U_{x_n}(z)}$$

$u(o) = 1, u(x_n) = \underline{+\infty}, u$ is not continuous.

$\underline{u} = \min \{u, 2\}$ \underline{u} superharmonic. bounded.

but not continuous.

Def. T_1, T_2 topologies. $T_2 \subset T_1$

T_1 finer than T_2 , T_2 coarser than T_1

"fine topology" of "classical" potential theory

is the coarsest topology of \mathbb{R}^N , which makes every $u \in U(\mathbb{R}^N)$ continuous. In extended sense of u may take values in $[-\infty, +\infty]$.

$$U_y(x) = \frac{C(N)}{|x-y|^{N-2}}$$

$$\underbrace{\{x; U_y(x) > a\}}_{B} \subseteq \mathbb{R}^N$$

\Rightarrow Euclidean topology \subseteq Fine topology.

Lemma 1.15. (Lemma 7.1.2 of [A.G. CPT])

1. A subbase for fine topology is $\{x; u(x) < a\} \quad u \in U(\mathbb{R}^N)$

2. If $u \in U(\Omega)$, u is finely continuous on Ω .

Def 15. A set E is said to be thin at a point y if y is not a fine limit point of E . i.e. \exists a fine neighborhood of y which does not intersect with $E \setminus \{y\}$. Otherwise E is not thin at y .

Thm 1.16

A polar set is thin everywhere

Proof.

$E \subseteq \mathbb{R}$ polar set , $y \in \mathbb{R}$. $E \setminus \{y\}$ is polar set

$\exists u \in U(\mathbb{R})$ s.t. $u \begin{cases} > +\infty & E \setminus \{y\} \\ < +\infty & y \end{cases}$

y is not a fine limit of $E \setminus \{y\}$.

otherwise u is not continuous.

Thm 1.17. Let y be a limit point of $E \subseteq \Omega$.

The following are equivalent.

1. E is thin at y ;

2. $\exists u \in U(B)$ B is a neighborhood of y

$$\liminf_{\substack{x \rightarrow y \\ x \in E}} u(x) > u(y)$$

3. For \forall Green set Ω , $y \in \Omega$. $\exists u \in U_+(\Omega)$

s.t. $u(y) < +\infty$, & $\lim_{\substack{x \rightarrow y \\ x \in E}} u(x) = +\infty$

Pwf. (3) \Rightarrow (2) is clear.

(2) \Rightarrow (1) by definition

(1) \Rightarrow (3) Suppose E is thin at y

$\exists u_1, \dots, u_m \in U(\mathbb{R}^N)$ & constants a_1, a_2, \dots, a_m .

s.t. $U = \bigcap_{i=1}^m \{x : u_i(x) < a_i\}$ is a fine neighbourhood of y .

& $\bigcup_{i=1}^m (E \setminus \{y\}) = \emptyset$.

$$u'_n(x) = \frac{u_n(x) - u_n(y)}{a_n - u_n(y)}$$

$$u'_n(y) = 0$$

Let $v = \sum_{n=1}^m u'_n$ $w = \min \{u'_1, u'_2, \dots, u'_m\}$.

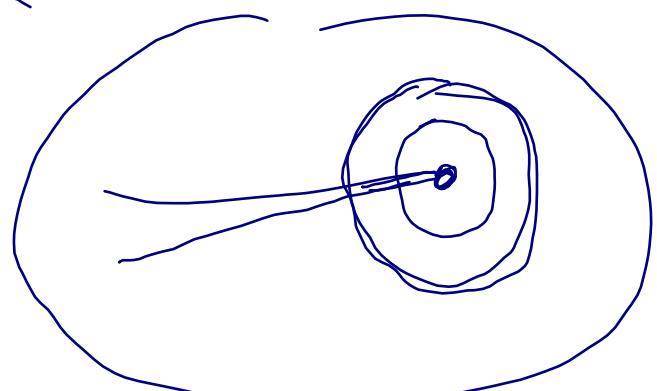
If $x \in E \setminus \{y\}$, $x \notin U \Rightarrow u_n(x) \geq 1$ for some n .

$$v \geq 1 + (m-1)w \text{ on } E \setminus \{y\}.$$

$$\Rightarrow \liminf_{x \rightarrow y, x \in E} v(x) \geq \liminf_{x \rightarrow y, x \in E} [1 + (m-1)w(x)] \geq 1 + (m-1) \underline{w(y)} = 1 = \underline{v(y)}$$

μ be the Riesz measure of v

$$\mu(\{y\}) = 0 \quad \exists r_n \rightarrow 0 \text{ s.t. } \mu(B_{r_n}(y)) = \underline{\mu_n}.$$



$$\Rightarrow \lim_{\substack{x \rightarrow y \\ x \in E}} v(x) = +\infty$$

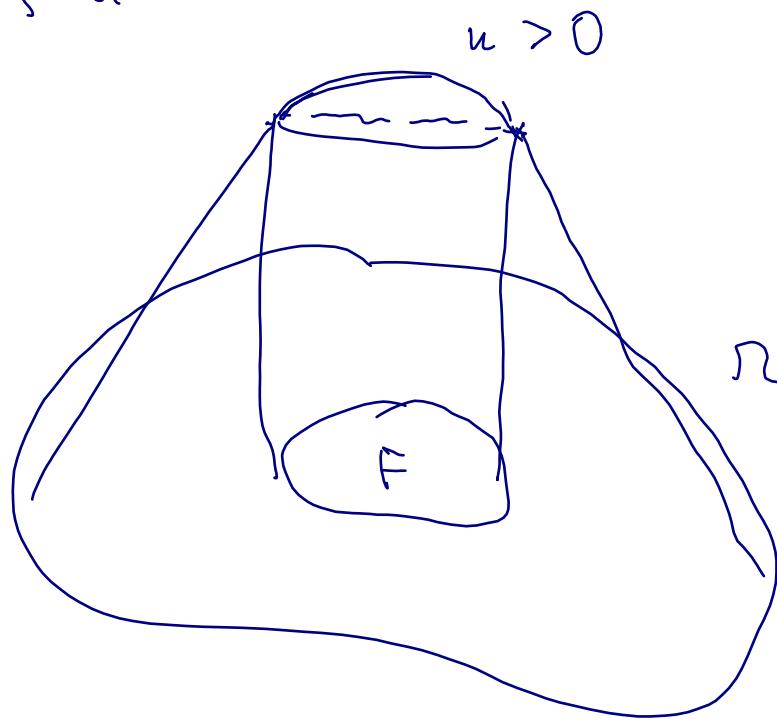
$$v = \sum_n G_{r_n} \underline{\mu_n} \quad \text{s.t. } G_{r_n} \underline{\mu_n}(y) < 2^{-n}.$$

$v(y) \leq 1 \quad \#$

$G_{r_n} \underline{\mu_n}(x) \quad \underline{v(x)}$

$\Rightarrow G_{r_n} \underline{\mu_n}(x) \quad \Big|_{B_{r_n}(y) \cap E} \geq 1$

1.7. Reduced function.



for $u \in U(\Omega)$ $u \geq 0$. put

$$\phi_{F, \Omega}^u = \{v \in U(\Omega) : v \geq 0 \text{ on } \Omega, v \geq u \text{ on } F\}$$

$$R_{F, \Omega}^u(x) = \inf_{v \in \phi_F^u} v(x) \quad \text{Reduced function of } u \text{ w.r.t. } (F, \Omega).$$

In general R_F^u may not be l.s.c.

Example.

$$\underline{N \geq 3}, \quad \Omega = \underline{\mathbb{R}^N}, \quad F = \{0\}, \quad U(x) = |x|^{2-n}$$

$$\underline{\underline{R_{F,\Omega}^u}} = \begin{cases} 0 & x \neq 0 \\ +\infty & x = 0 \end{cases}$$

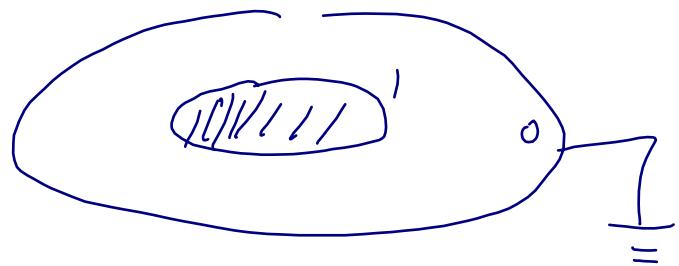
Def int

$$\hat{R}_{F,\Omega}^u(x) = \liminf_{y \rightarrow x} R_{F,\Omega}^u(y), \quad x \in \mathbb{R}^N$$

We call it regularized reduced function. or balayage.

Important

$$\underline{\underline{R_{F,\Omega}^1}}, \quad \boxed{\hat{R}_{F,\Omega}^1}$$



Lemma 1.18

1. $0 \leq \boxed{\hat{R}_{F,\Omega}^1} \leq R_{F,\Omega}^1 \leq 1$

2. $R_{F,\Omega}^1 = 1 \text{ on } F$

3. $\hat{R}_{F,\Omega}^1 = R_{F,\Omega}^1 \text{ on } \text{int}(F \cup F^c)$

$\hat{R}_{F,\Omega}^1 \neq R_{F,\Omega}^1$ may happen only on a polar set ∂F

4. $\hat{R}_{F,\Omega}^1$ is superharmonic, harmonic in F^c

It's called the capacity potential of F w.r.t. Ω

Its Riesz measure is called capacity distribution of F w.r.t. Ω .

5. $\hat{R}_{F,\Omega}^I = \hat{R}_{\text{FUE},\Omega}^I$, if E is a polar set.

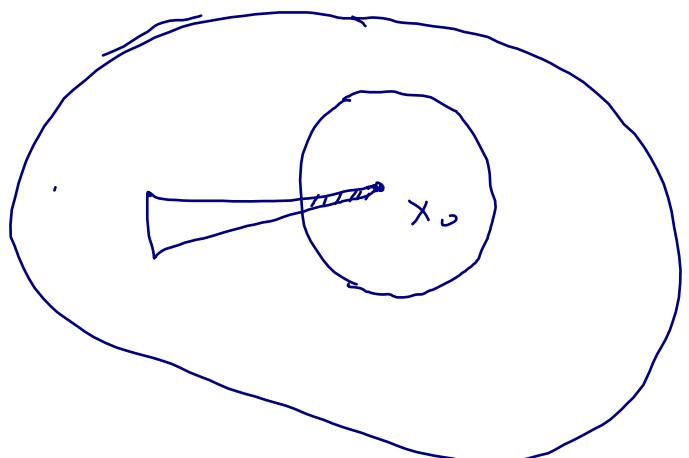
Lemma 1.19 $E \subset \Omega$. E is thin at x_0 .

$$\Rightarrow \hat{R}_{E \cap B(x_0, r)}^I \rightarrow 0, \text{ for } r \rightarrow 0^+$$

Prof. By Thm 1.17, we know.

\exists superharmonic $U \geq 0$ s.t.

$$0 < U(x_0) < +\infty = \liminf_{\substack{x \rightarrow x_0 \\ x \in E \\ x \neq x_0}} \underline{U}(x)$$



$$\sum [U(x) > U(x_0), x \in E \setminus x_0]$$

Choose $\varepsilon > 0$ small, choose $r > 0$ such that $u(x) > \frac{u(x_0)}{\varepsilon}$

$$\forall x \in (E \cap B(x_0, r)) \setminus \{x_0\} . \quad \frac{\varepsilon u(x)}{u(x_0)}$$

The set $\{x_0\}$ has 0 capacity. $\Rightarrow \hat{R}_{E \cap B(x_0, r)}^1 = \hat{R}_{(E \cap B(x_0, r)) \setminus \{x_0\}}^1$.

$$\Rightarrow \hat{R}_{E \cap B(x_0, r)}^1 \leq \varepsilon \frac{u(x)}{u(x_0)} \quad \forall x \in \mathbb{N}.$$

$$\Rightarrow \hat{R}_{E \cap B(x_0, r)}^1 (x_0) \leq \varepsilon.$$

It's easy to check for $0 < r' < r$

$$\hat{R}_{E \cap B(x_0, r')}^1 (x_0) \leq \hat{R}_{E \cap B(x_0, r)}^1 (x_0)$$

$$\Rightarrow \hat{R}_{E \cap B(x_0, r)}^1 (x_0) \rightarrow 0 \text{ as } r \rightarrow 0^+$$

Lemma 1.20. If E is not thin at x_0 , then $\hat{R}_{E \cap B(x_0, r)}^1(x_0) = 1$ for all $r > 0$.

Proof. $R_{E \cap B(x_0, r)}^1|_E = 1 \Rightarrow \hat{R}_{E \cap B(x_0, r)}^1 = 1$ outside a polar set.
on $\partial(E \cap B(x_0, r))$.

Since E is not thin at x_0 , \Rightarrow at least for a sequence
 $x_0 \neq x_n \rightarrow x_0$, $\Rightarrow \underbrace{\hat{R}_{E \cap B(x_0, r)}^1(x_n)}_n = 1$,
Since E is not thin at x_0 , $\Rightarrow x_0$ is the fine limit point of E

$$\Rightarrow \hat{R}_{E \cap B(x_0, r)}^1(x_0) = 1.$$

1.8. Regular boundary points (w.r.t. $\Delta u = 0$)

Let $\Omega \subseteq \mathbb{R}^N$, $N \geq 2$ be a bounded domain & $\phi(x) \in C(\bar{\Omega})$.

Perron method.

Define $\overline{H}_\phi = \inf \left\{ v \in U(\Omega); \liminf_{\Omega \ni y \rightarrow x} v(y) \geq \phi(x) \text{ for any } x \in \partial\Omega \right\}$

From standard argument, for example, from Chapter of [Gilbarg & Trudinger].

$\Rightarrow \overline{H}_\phi$ is harmonic in Ω . (\overline{H}_ϕ Perron solution)

Definition 17 $x_0 \in \partial\Omega$ is called a regular boundary point,

if for $\forall \phi(x) \in C(\bar{\Omega})$ $\lim_{\Omega \ni y \rightarrow x} \overline{H}_\phi(y) = \phi(x)$

Definition 18 Let $\Omega \subseteq \mathbb{R}^N$ be bounded open domain. If $x_0 \in \partial\Omega$

A function $w = w_{x_0}(x)$ is called a (local) barrier function at $x_0 \in \partial\Omega$ if it is defined in $W \cap \Omega$, W is a neighborhood of x_0 & has the following properties:

① w is superharmonic on $W \cap \Omega$

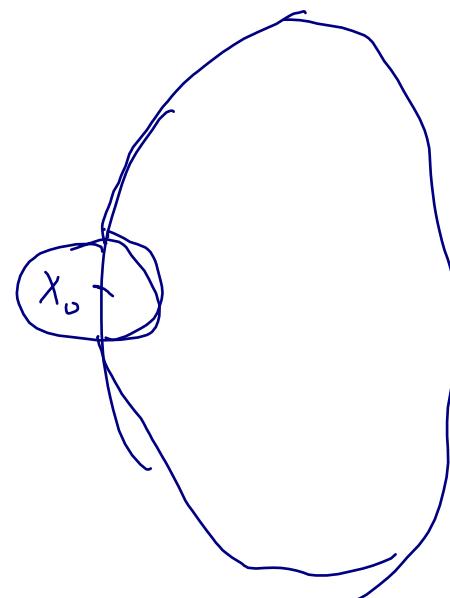
② $w > 0$ in $W \cap \Omega$

③ $\lim_{W \cap \Omega \ni x \rightarrow x_0} w(x) = 0$

From Lemma 6.6.3 & [classical Potential theory].

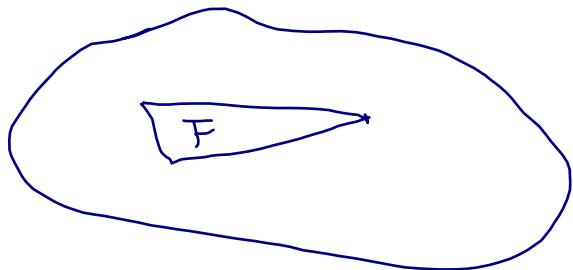
Lemma 7.21 If \exists a local barrier at $x_0 \in \partial\Omega$, \Rightarrow

\exists a global barrier at y such that $v \in U_y(\Omega)$ & $\inf_{\Omega \setminus \{y\}} v > 0$ for \forall open neighborhood of x_0 .



Lemma 1-22. Lemma 7.3.4 of [CPT]

Let $F \subset \Omega$ & $u \geq 0$. $u \in U(\Omega)$. Assume u peaks at $x_0 \in \Omega$
 $\& u(x_0) < \infty$.



Then F is thin at x_0 if and only if

$$R_{F, \Omega}^u(x_0) < u(x_0)$$

Thm 1-23. Let Ω be a bounded domain. $x_0 \in \partial\Omega$.

The following statements are equivalent.

1. x_0 is regular;

2. \exists a barrier function at x_0

3. Ω^c is not thin at x_0 .

Prof. (1) \iff (2) Suppose x_0 is regular bdy point.

Consider $w(x) = |x_0 - x|$, $x \in \partial\Omega$. \bar{H}_w is harmonic in Ω

$H_w > 0$ in Ω by strong maximum principle.

$\Rightarrow \lim_{r \rightarrow x \rightarrow x_0} \bar{H}_w(x) = 0$. $\Rightarrow \bar{H}_w(x)$ is a barrier function at x_0 .

Now we assume \exists barrier function w in $B_r(x_0) \cap \Omega$.

By lemma 1.2, we may assume w is a global one

with $\inf_{\Omega \setminus \{x_0\}} w > 0$ if open neighborhood of x_0

Let $M = \sup_{\Omega} |\phi|$. $\exists \delta > 0$, $k > 0$ s.t.

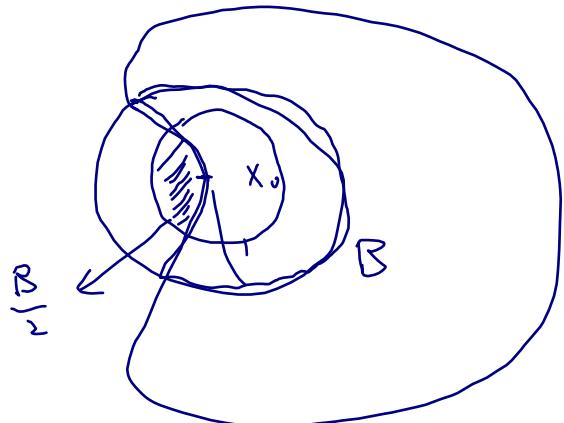
$$|\phi(x) - \phi(x_0)| < \varepsilon \quad \text{if } |x - x_0| \leq \delta \quad \& \quad k w(x) \geq 2M \quad \text{if } |x - x_0| \geq \delta$$

$$\phi(x_0) - \varepsilon - k w(x) \leq u(x) \leq \phi(x_0) + \varepsilon + k w(x)$$

$$\text{or } |u(x) - \phi(x_0)| \leq \varepsilon + k w(x) \quad \text{Since } w(x) \rightarrow 0 \text{ as } x \rightarrow x_0 \\ \Rightarrow u(x) \rightarrow \phi(x_0) \quad \text{as } x \rightarrow x_0$$

(1) \Leftrightarrow (3)

Assume Ω^c is not thin at x_0 . Consider $B(x_0, 1) = B$



Define $U(x) = 1 - |x - x_0|^2$.

$u \in U(B)$, $u = 0$ on ∂B .

$$(-\Delta u = 2N > 0)$$

Let $w := u - \hat{R}_{\Omega^c \cap \frac{B}{2}, B}^u$

$w > 0$ in $B \cap \Omega$ & w is superharmonic in $B \cap \Omega$ since

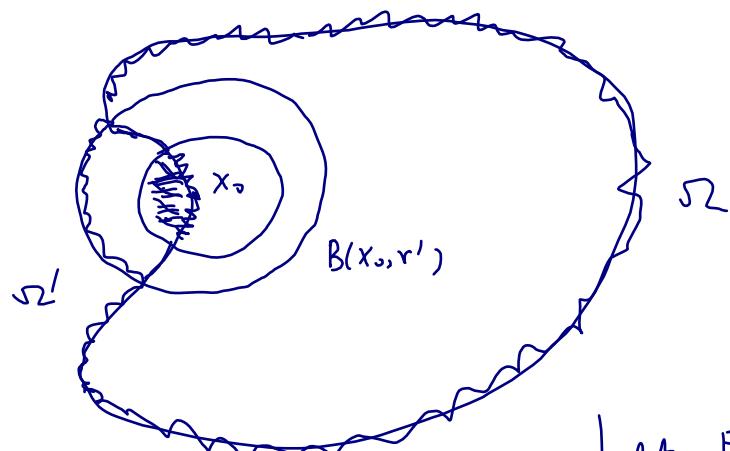
$\hat{R}_{\Omega^c \cap \frac{B}{2}, B}^u$ is harmonic in $B \cap \Omega$

Lemma 1.22

Since u peaks at $x_0 \Rightarrow w(x_0) = 0$

$\Rightarrow w$ is a barrier at $x_0 \Rightarrow x_0$ is a regular point.

Now suppose x_0 is regular. Choose r' . Let $\Omega' = \Omega \cup B(x_0, r')$.



choose $0 < r < r'$ define

$$f_r(x) = \begin{cases} 1 & x \in \partial\Omega \cap B(x_0, r) \\ 0 & x \in \partial\Omega \cap B(x_0, r) \end{cases}$$

Let $E(r) = \overline{B(x_0, r)} \setminus \Omega$.

Consider $\hat{R}_{E(r), \Omega'}^{-1}(x_0) \leq 1$

$\forall 0 \leq u \in U(\Omega')$, $u \geq 1$ on $E(r)$

$$\Rightarrow \liminf_{\Omega \ni x \rightarrow x_0} \underline{w(x)} \geq \liminf_{\Omega \ni x \rightarrow x_0} \underline{\hat{H}_{dr}(x)} = f_r(x_0) = 1 \quad \text{since } x_0 \text{ is regular.}$$

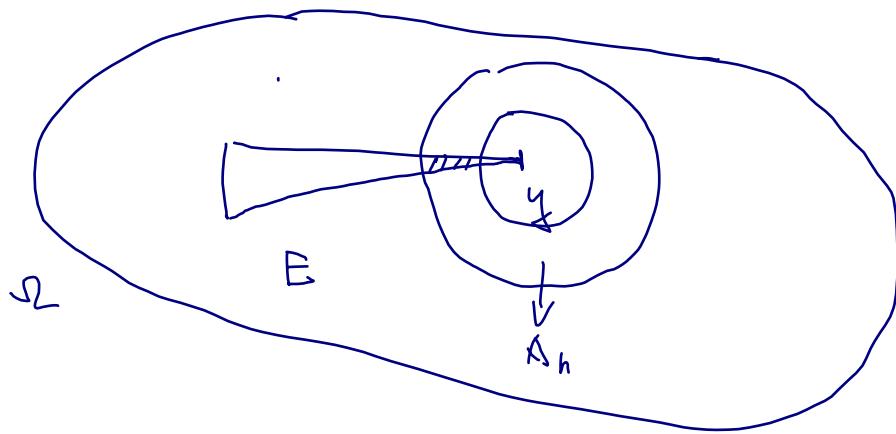
$$\Rightarrow \limsup_{x \rightarrow x_0} \hat{R}_{E(r), \Omega'}^{-1}(x) = \hat{R}_{E(r), \Omega'}^{-1}(x_0)$$

$$\Rightarrow \forall 0 < r < r' \Rightarrow \hat{R}_{E(r), \Omega'}^{-1}(x_0) = 1 \xrightarrow{\text{Lemma 1.20}} \Omega^c \text{ is not thin at } x_0$$

1.9 Wiener Criterion

Let Ω be Greenian domain & $E \subset \Omega$, $y \in \Omega$.

Fix $\alpha > 1$. Let $A_n = \{x \in \mathbb{R}^N : \alpha^n \leq U_y(x) \leq \alpha^{n+1}\}$ $n \in \mathbb{N}$.



Theorem 1.24 (Wiener Criterion for $N \geq 3$). Let $n' \in \mathbb{N}$

be such that $U_{n'} = \{x : U_y(x) \geq \alpha^{n'}\} \subseteq \Omega$. The following are equivalent.

1. E is thin at y ;

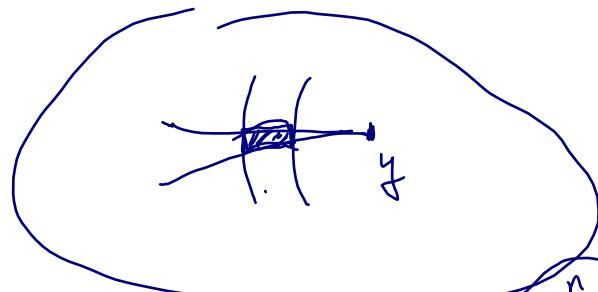
2. $\sum_{n=n'}^{+\infty} \alpha^n \cdot \text{Cap}_2(E \cap A_n, \Omega) < +\infty$;

3. $\sum_{n=n'}^{\infty} \hat{R}_{E \cap A_n}^{-1}(y) < +\infty$, $\hat{R}_{E \cap A_n}^{-1}(y) = \hat{R}_{E \cap A_n, \Omega}^{-1}(y)$.

PwF. (2) \Leftrightarrow (3)

For $n \geq n'$, $\hat{R}_{E \cap A_n}^1(x)$ is the capacity potential of $E \cap A_n$.

w.r.t. Ω . The Riesz measure μ_n of $\hat{R}_{E \cap A_n}^1(x)$.



is capacity distribution of $E \cap A_n$

$$\mu_n(\Omega) = \left[\text{Cap}_2(E \cap A_n) \right]$$

$$\hat{R}_{E \cap A_n}^1(y)$$

$$\text{Cap}_2(E \cap A_n)$$

$$\hat{R}_{E \cap A_n}^1(y) \neq$$

$$\int_{\Omega} (y|z|)^n d\mu_n(z) + h_n$$

h_n harmonic

$$U_\delta(y) = \frac{C(N)}{|y-\delta|^{N-2}}$$

$$U_\delta(y) \sim 2^n$$

$$\mu_n(\Omega) = \left[\text{Cap}_2(E \cap A_n) \right]$$

$$\Rightarrow (2) \Leftrightarrow (3) \quad \text{i.e. } \sum_{n=n'}^{\infty} \alpha^n \text{Cap}_2(E \cap A_n, \Omega) < +\infty \Leftrightarrow \sum_{n=n'}^{\infty} \hat{R}_{E \cap A_n}^1(y) < +\infty$$

(1) \Leftrightarrow (3)

Suppose (3) holds. i.e. $\sum_{n=n'}^{\infty} \hat{R}_{E \cap A_n}^{-1}(y) < +\infty$

$\Rightarrow \exists$ sequence $b_n \rightarrow +\infty$, $v(y) = \sum_{n=n'}^{\infty} b_n \underbrace{\hat{R}_{E \cap A_n}^{-1}(y)}_{=}$ $< +\infty$

Since $\hat{R}_{E \cap A_n}^{-1} = 1$ on $E \cap A_n$, except for a polar set $\underline{\bigcup F_n}$

$\Rightarrow v(x) \rightarrow +\infty$ as $x \rightarrow y$ along $E \setminus \underline{\bigcup F_n}$

$\Rightarrow E \setminus \underline{\bigcup F_n}$ is thin at y . $\Rightarrow E$ is thin at y .

Suppose (1) holds. \exists superharmonic function $0 \leq w(x) \in U(\Omega)$

s.t. $w(y) < +\infty \Rightarrow \lim_{E \ni x \rightarrow y} w(x) = +\infty$

Let μ be the Riesz measure of $w(x)$. $\mu|_{A_n} = \mu_n$

$$G_\Omega \mu = G_\Omega \left(\mu \Big|_{\Omega \setminus \{V_n\}} \right) + \sum_{n=1}^{\infty} G_\Omega \mu_n$$

$\Rightarrow G_\Omega \mu - w(x)$ is harmonic function in Ω .

$$\Rightarrow \lim_{E \ni x \rightarrow y} G_\Omega \mu = +\infty, \quad \& \quad \boxed{G_\Omega \mu(y) < +\infty}$$

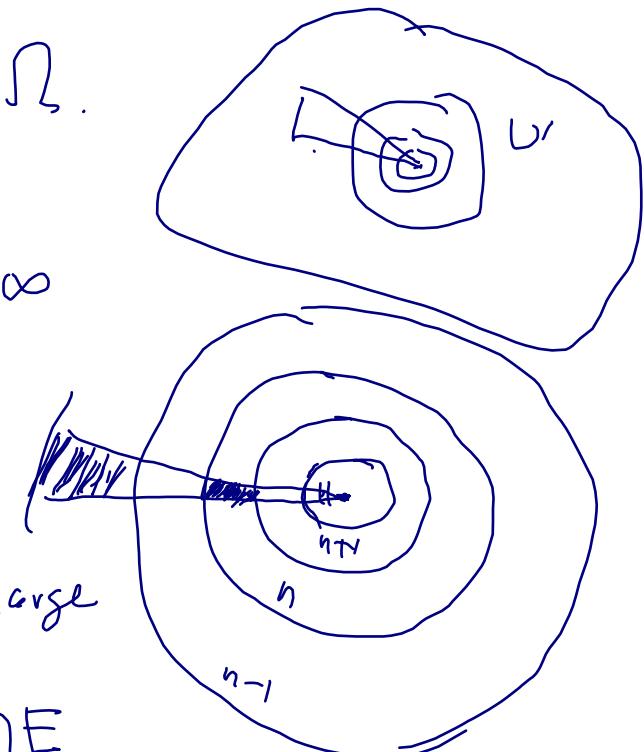
$$\Rightarrow \sum_{n=1}^{\infty} G_\Omega \mu_n(y) < +\infty.$$

As long as we prove that for $n \geq n''$ large

$$\boxed{G_\Omega(\mu_{n-1} + \underline{\mu_n} + \mu_{n+1})(x) \geq 1} \quad \forall x \in A_n \cap E$$

Then we have

$$\sum_{n''=1}^{\infty} \hat{R}_{E \cap A_n}^1(y) \leq 3 \sum_{n''=1}^{\infty} G_\Omega \mu_n(y) < +\infty$$



Für $i = n', \dots, n-2, n+2, n+3, \dots$

$$\text{dist}(A_i, A_n) \geq c \text{ dist}(A_i, y)$$

$$\Rightarrow \sup_{A_n} G_{\Omega}(\mu_{n'} + \dots + \mu_{n-2} + \mu_{n+2} + \dots)(x) \\ \leq c G_{\Omega}(\mu_{n'} + \dots + \mu_{n-2} + \mu_{n+2} + \dots)(y)$$

$$\leq c G_{\Omega} \mu(y)$$

but $G_{\Omega} \mu \Big|_{A_n} \rightarrow +\infty$

$$\Rightarrow G_{\Omega}(\mu_{n-1} + \mu_n + \mu_{n+1}) \Big|_{A_n} \rightarrow +\infty \quad \vee.$$

Ihm 1.25 Let $E \subset \mathbb{R}^2$, let $y \in \mathbb{R}^2$ & let \mathcal{S} be a Greenian set
 $y \in \mathcal{S}$. Let $\beta \in (0, 1)$ let

$$E_n = \{x \in E : \beta^{n+1} \leq |x-y| \leq \beta^n\}$$

For n' large s.t. $B(y, \beta^{n'-1}) \subset \mathcal{S}$

The two terms are equivalent

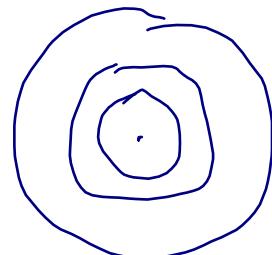
1. E is thin at y

2. $\sum_{n=1}^{\infty} n \cdot \text{Cap}_2(E, \mathcal{S}) < +\infty$

Application to Geometry.

$$u \in U(\Omega)$$

$$x_0 \in \Omega$$



$$\text{Ave } u(r) = \frac{1}{2\pi} \int_{\partial B_r(x_0)} u$$

$$\lim_{r \rightarrow 0} \frac{\text{Ave } u(r)}{\frac{1}{2\pi} \log(1/r)} = M_u(\{x_0\})$$

Theorem 1.26. (Arsene & Huber 1973. CPT Thm 7.4.3)

$\exists E$, thin at x_0 , s.t.

$$\lim_{\substack{x \rightarrow x_0 \\ x \in E}} \frac{u(x)}{\frac{1}{2\pi} \log \frac{1}{|x|}} = M_u(\{x_0\}).$$

$\mathbb{R}^2, (x_1, x_2)$

$$g_{\text{stdn}} = dx_1^2 + dx_2^2$$

$$g = e^{2u} g_{\text{stdn}}$$

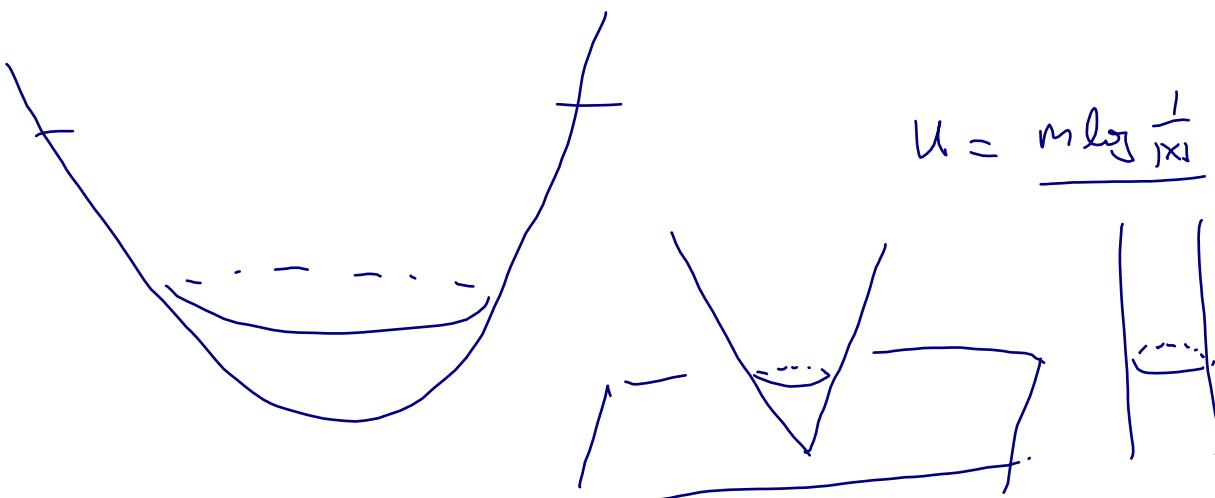
Let g be complete, $K_g \geq 0$ Then

Thm 1.27 (Bonini - M - Qing)

$\exists E$ thin at infinity. such that.

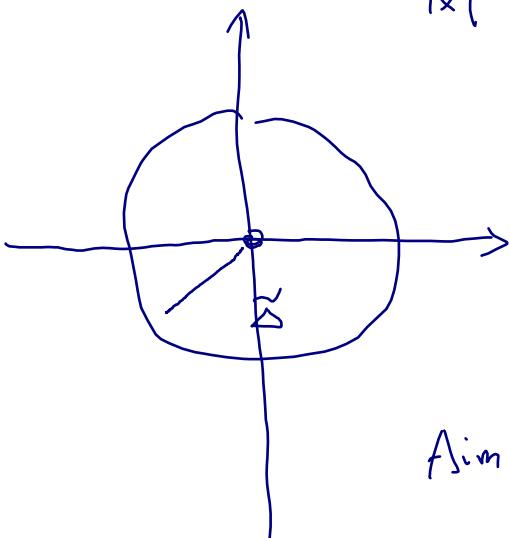
$$\lim_{x \rightarrow \infty, x \notin E} \frac{u(x)}{\log \frac{1}{|x|}} = m, \quad 0 \leq m \leq 1$$

thin at infinity $\Leftrightarrow \tilde{E} = \left\{ \frac{x}{|x|^2} : x \in E \right\}$ is thin at 0.



$$u = \frac{m \log \frac{1}{|x|}}{|x|}, \quad 0 \leq m \leq 1$$

Proof. $x = \frac{\tilde{x}}{|\tilde{x}|^2}$ $\mathbb{R}^2 \setminus B_{\mathbb{R}}(x)$ is conformal $\rightarrow B_1(0) \setminus \{0\} \subseteq \widetilde{\mathbb{R}^2}$



$$g = e^{2\tilde{u}} (dx_1^2 + dx_2^2)$$

$$\Delta \tilde{u} + k \cdot e^{2\tilde{u}} = 0$$

$$\Rightarrow -\Delta \tilde{u} = k e^{2\tilde{u}} \quad \text{in } B_1(0) \setminus \{0\}$$

$$\text{Assume: } \lim_{\tilde{x} \rightarrow 0} \tilde{u}(\tilde{x}) = +\infty \Rightarrow \tilde{u} \in U(B_1(0))$$

$$\text{From } -\Delta \tilde{u} = k e^{2\tilde{u}} \Rightarrow -e^{-2\tilde{u}} \Delta \tilde{u} = k \geq 0$$

$$\Rightarrow -\Delta_g \tilde{u} = k \geq 0$$

$$\Delta_g u = \frac{1}{\sqrt{\det g}} \partial_i \left(g^{ij} \sqrt{\det g} \partial_j u \right)$$

$$\Rightarrow \Delta_g e^{-\tilde{u}} = e^{-\tilde{u}} \left(-\Delta_g \tilde{u} + |\nabla_g \tilde{u}|^2 \right) \geq 0$$

$e^{-\tilde{u}}$ subharmonic in
 g metric.

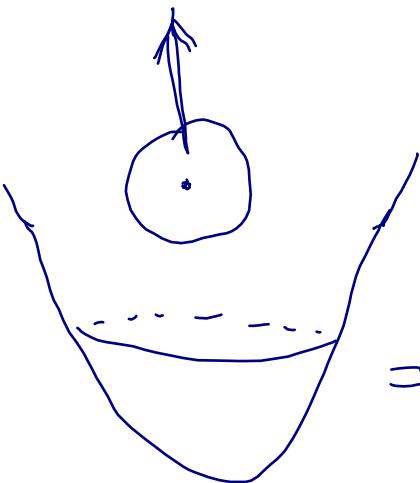
Theorem (P. Li & R. Schoen)

M^n . complete mfd, $\text{Ric} \geq -(n-1)k$ $k > 0$

$x_0 \in M$ on $v \geq 0$ is subharmonic in $B_{2r}(x_0)$.

\Rightarrow for $\forall \tau \in (0, \frac{1}{2})$

$$\sup_{B_{(1-\tau)r}^M(x_0)} v^2 \leq \tau^{-c(1+\sqrt{k}r)} \frac{1}{\text{Vol}(B_r^M(x_0))} \int_{B_r^M(x_0)} v^2 d\text{vol}_M$$



$$\Rightarrow \sup_{B_{(1-\tau)r}(x_0)} e^{-2\tilde{u}} \leq \tau^{-c}$$

$$g_{\mathbb{R}^2} = e^{-2\tilde{u}} g$$

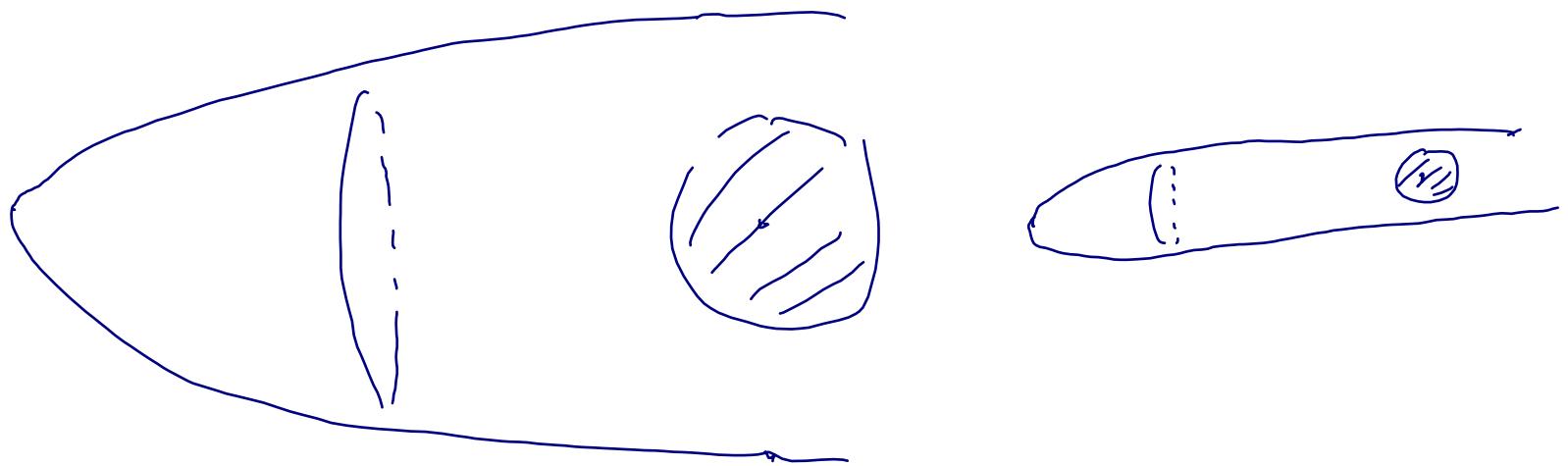


Thm (Croke & Karcher)

If (M^2, g) is complete non compact. $Kg \geq 0$

$\Rightarrow \exists$ const $C(M)$ s.t. for $\forall r \leq 1$

$$\text{vol}_g(B_r(x)) \geq C(M) r^2$$



$$\Rightarrow \tilde{u} \in U(B_1(0))$$

From Arsove & Huber's theorem, $\exists \tilde{E}$ thin at 0

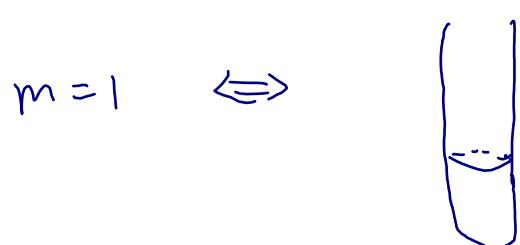
$$\lim_{\substack{x \rightarrow 0 \\ x \notin E}} \frac{\tilde{u}(\tilde{x})}{\log \frac{1}{|\tilde{x}|}} = m \geq 0$$

$$\tilde{u} \xrightarrow{\text{inversion}} u$$

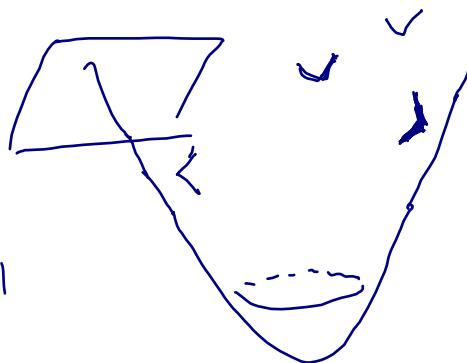
$$\Rightarrow u(x) = m \log \frac{1}{|x|} + o(\log \frac{1}{|x|})$$

$m \in [0, 1]$, outside a thin set

$$m=0 \Leftrightarrow (M, g) \cong (\mathbb{R}, g_\infty)$$



$$0 < m < 1$$



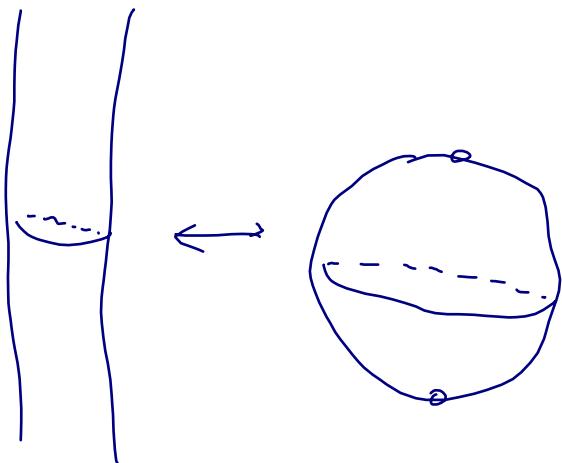
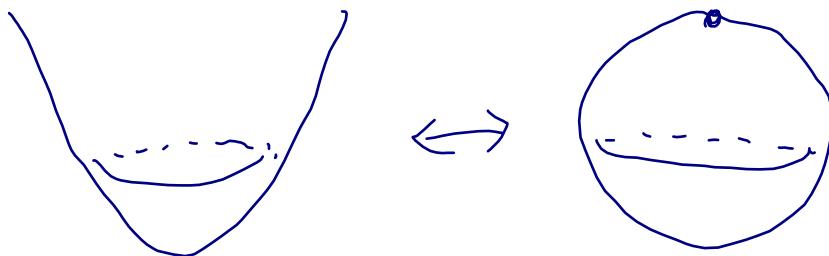
Thm 1-30 (Huber)

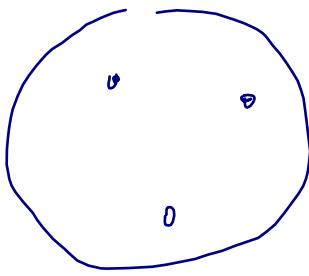
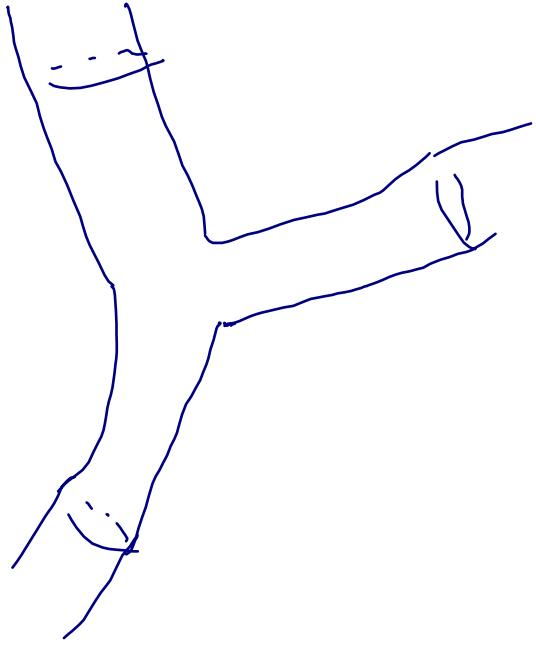
Suppose (M, g) is complete non compact R.S. with

$$\int_M k^- d\mu_g < +\infty$$

$\Rightarrow \exists$ compact R.S. Σ s.t. M is conformally equivalent to

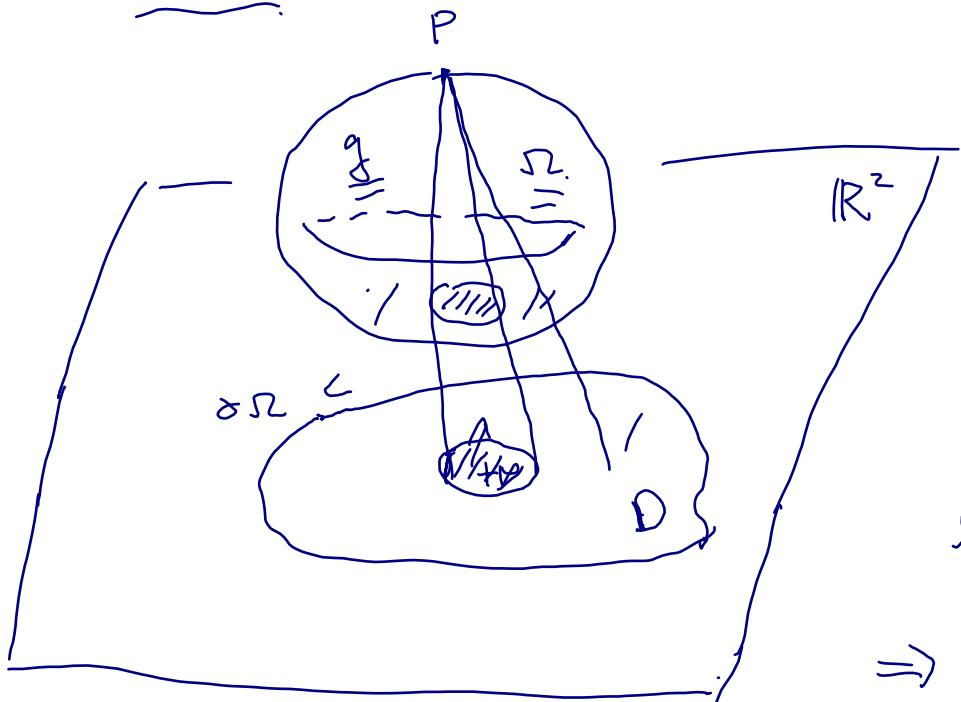
$$\sum_{-} \left(\{ p_1, \dots, p_k \} \right)$$





Thm 1.31. Suppose $\Sigma \subset (S^2, g_{S^2})$. Who has a complete conformal metric $\tilde{g} = e^{2u} g_{S^2}$. If $k_g \geq 0$
 $\Rightarrow \partial\Sigma = \underbrace{\{p\}}_{\{p, q\}} \cup \underbrace{\{p, q\}}_{\{p, q\}}$. If $\partial\Sigma = \{p, q\} \Rightarrow (\Sigma, \tilde{g})$ is a cylinder.

Proof.



In $D \setminus S$

$$g = e^{2v} \cdot g_{\mathbb{C}}$$

$$\Delta_{\mathbb{C}} v + k_g e^{2v} = 0$$

$$\lim_{x \rightarrow S} v(x) = +\infty$$

$\Rightarrow v \in U(D)$ S is polar set

$\text{Cap}_2(S, D) = 0 \Rightarrow \dim_{\mathbb{H}^2} S = 0 \Rightarrow S$ totally disconnected

Splitting thm $\Rightarrow S = \{P\}, \{P, \bar{z}\}$

2. Nonlinear Potential theory.
1. Peter Lindqvist. Notes on the p-Laplace equation
2. HKM, ✓]
3. Kilpeläinen & Mály, The Wiener test and potential estimates for quasilinear elliptic equations.
(Acta. Math. 1994).]
4. M. Rüng CUPDE
5. M. Rüng Huber-thm. Adv. Math.

$$1 < p \leq N$$

$$\text{Consider } I_p(u) = \underbrace{\int_{\Omega} |\nabla u|^p dx}.$$

$$\text{Given } \phi \in W^{1,p}(\Omega) \quad \inf \left\{ \int_{\Omega} |\nabla u|^p dx ; u - \phi \in W_0^{1,p}(\Omega) \right\}$$

$$\text{Suppose minimizer } u_0 \Rightarrow \int_{\Omega} \langle |\nabla u_0|^{p-2} \nabla u_0, \underline{\nabla \phi} \rangle dx = 0 \quad (*)$$

$\Leftarrow \forall \phi \in C_c^\infty$

$$\Leftrightarrow \operatorname{div} \left(|\nabla u_0|^{p-2} \nabla u_0 \right) = 0, \quad x \in \Omega \quad (\Delta)$$

$$\text{i.e. } \Delta_p(u_0) = 0$$

Def 19. $\Omega \subseteq \mathbb{R}^N$, $u \in W_{loc}^{1,p}(\Omega)$, u satisfies $(*)$

We call u a weak solution to p -harmonic equation (Δ)

In addition if u is continuous (always does) u p -harmonic

Def 20. $v: \Omega \rightarrow (-\infty, +\infty]$ is called p -superharmonic in Ω

i.e. D. l.s.s.

② $v \not\equiv +\infty$

③ comparison principle (with p -harmonic function)

$$\Omega \rightarrow U_p(\Omega)$$

Example. 1. $|x|^{\frac{p-N}{p-1}}$ $1 < p < N$.

$$-\log|x| \quad p = N$$

$\nearrow p$ superharmonic

2. For $1 < p < N$.

$$v(x) = \int_{\Omega} \frac{d\mu(y)}{|x-y|^{\frac{N-p}{p-1}}}$$

$$v \in U_p(\Omega)$$

$\exists, u, v \in U_p(\Omega) \quad \min\{u, v\} \in U(\Omega)$

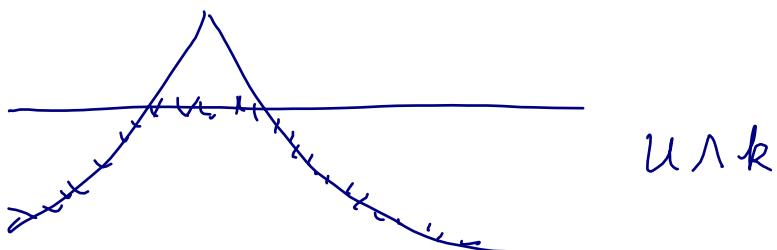
Theorem 2.1. (Summability, Lindquist.)

For $u \in U(\Omega) \Rightarrow \forall D \subset \Omega,$

$$\int_D |v|^p < \infty, \quad 0 \leq p < \frac{N(p-1)}{N-p}, \quad 1 \leq p \leq N.$$

$$\& \int_D |\nabla v|^p < \infty \quad 0 < r < \frac{N(p-1)}{N-1} \quad 1 < p \leq N$$
$$= \frac{(p-1)}{N-1} \cdot \frac{N}{N-1}$$

We use $\underline{u \wedge k}$ to represent $\min\{u, k\}.$



Thm 2.2 For $u \in U_p(\Omega)$, $\underline{\Delta_p u}$ is a well defined Radon measure

Proof

Define $\underline{L_u^p(\phi)} = \int |\nabla u|^{p-2} \nabla u \cdot \nabla \phi \, dx$ $\forall \phi \in C_0^\infty$

We prove $\forall \phi \in C_0^\infty$ $\phi \geq 0$. $\underline{L_u^p(\phi)} \geq 0$

$$\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla \phi \, dx = \lim_{k \rightarrow \infty} \int_{\Omega} |\nabla u \wedge k|^{p-2} \nabla(u \wedge k) \cdot \nabla \phi \, dx$$

$$u \wedge k \in W_{loc}^{1,p}(\Omega) \cap U(\Omega) \Rightarrow \int_{\Omega} |\nabla u \wedge k|^{p-2} (\underline{u \wedge k}) \, d\phi \, dx \geq 0$$

(Lindquist, 71)

Thm 2.3 Ω bdd. $\mu \in \mathcal{M}_{\text{finite}}^+(\Omega)$.

$\Rightarrow \exists$ P superharmonic function u s.t.

$$-\Delta_p u = \underline{\mu}$$

& $u \wedge k \in W_0^{1,p}(\Omega)$ for all $k > 0$.

2.2. Wolff potential & generalization of Riesz representation

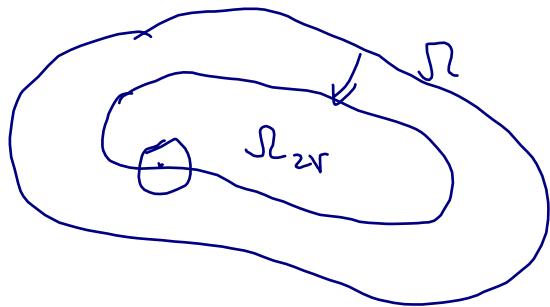
Def $1 < p \leq N$.

Ω domain $\mu \in \mathcal{M}^+(\Omega)$

$$\Omega_{2r} = \{x \in \Omega \mid \text{dist.}(x, \partial\Omega) < 2r\}$$

$$W_{1,p}^{\mu}(x_0, r) = \int_0^r \left(\frac{\mu(B(x_0, t))}{t^{n-p}} \right)^{\frac{1}{p-1}} \frac{dt}{t}$$

$$\# x_0 \in \Omega_{2r}$$



$P=2$

$$\int_0^r \frac{\mu(B(x_0, t))}{t^{n-2}} dt$$

$$\sim \int_0^r \mu(B(x_0, t)) dt \frac{1}{t^{n-2}}$$

Newtonian potential

↙ ↗

$$\sim \int \frac{1}{t^{n-2}} d\mu(B(x_0, t)) \leftrightarrow \int \frac{1}{|x-y|^{n-2}} d\mu(y)$$

↙ ↘

$$u \in U(\mathbb{R}) \quad u = \underbrace{\sum_n \mu_n}_{\sim} + \underbrace{h}_{\sim}$$

Thm 2.4. (kilpeläinen & Maly, Acta Math.)

Suppose $u \in U(B(x_0, 3r))$, $u \geq 0$, $\mu = -\Delta_p u$

$$c_1 W_{1,p}^{\mu}(x_0, r) \leq U(x_0) \leq \underbrace{c_2 \inf_{B(x_0, r)} u}_{\sim} + c_3 W_{1,p}^{\mu}(x_0, 2r)$$

2.3 Wiener Criterion

Consider $\left\{ \begin{array}{l} \Delta_p u = 0 \\ u - f \in W_0^{1,p}(\Omega) \end{array} \right.$ $x \in \Omega$, $f \in W^{1,p}(\Omega) \cap C(\bar{\Omega})$

Def 22 $x_0 \in \partial\Omega$ is called p -regular, if the solution to (10)

has the limit value $f(x_0)$, whenever $f \in W^{1,p}(\Omega) \cap C(\bar{\Omega})$

Def 23. The p -fine topology is defined similarly.

$E \subset \Omega$, $x_0 \in \Omega$. E is called thin at x_0 if
 x_0 is not the fine limit of $E \setminus \{x_0\}$

Theorem 2.5 Suppose $E \subset \Omega$, $x_0 \in \Omega$. The following statements are equivalent.

1. E is p -thin at x_0 ;
2. \exists p -superharmonic function in Ω

s.t. $\liminf_{\substack{x \rightarrow x_0 \\ x \in E \setminus \{x_0\}}} u(x) > u(x_0)$

3. $\exists u \in U_p(\Omega)$ s.t.

$$\lim_{\substack{x \rightarrow x_0 \\ x \in E \setminus \{x_0\}}} u(x) = +\infty \quad u(x_0) < +\infty.$$

Thm 2.6. bdy point $x_0 \in \partial\Omega$ is regular $\Leftrightarrow \Omega^c$ is not p -thin at x_0 .

Thm 2.7 Wiener Criterion for p -Laplace equation $1 < p \leq N$

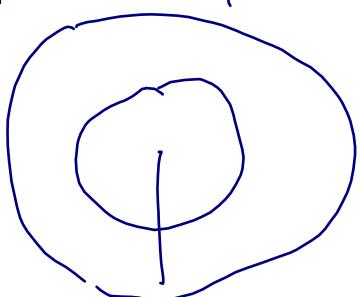
E is thin at x_0 iff

$$\int_0^1 \left(\frac{C_{\alpha,p}(B(x_0,t) \cap E, B(x_0,2t))}{C_{\alpha,p}(B(x_0,t), B(x_0,2t))} \right)^{\frac{1}{p-1}} \frac{dt}{t} < +\infty$$

Necessary Part : Maz'ya 1970

Sufficient Part : Kilpelainen & Malý 1994 Acta

$N \geq 3, p=2$ Thm 2.7 coincides with Thm 1.24



$$C_{\alpha,2}(B(x_0,t), B(x_0,2t)) \sim t^{n-2}$$

$$\begin{aligned}
& \int_0^1 \frac{\text{Cap}_2(B(x_0, t) \cap E, B(x_0, 2t))}{t^{n-2}} \frac{dt}{t} \\
& \sim \sum_n \int_{2^{-\frac{n+1}{N-2}}}^{2^{-\frac{n}{N-2}}} t^{2-n} \underbrace{\text{Cap}_2(B(x_0, t) \cap E, B(x_0, 2t))}_{\text{Cap}_2(B(x_0, 2^{-n}) \cap E, B(x_0, 2^{-n}))} \left(\frac{dt}{t} \right) \\
& \sim \sum_n 2^n \underbrace{\text{Cap}_2(E \cap A_n, \mathcal{S})}_{\text{Cap}_2(B(x_0, 2^{-n}) \cap E, B(x_0, 2^{-n}))}
\end{aligned}$$

$N \geq 3, p = N$ Wiener Criterion $\Leftrightarrow \frac{1}{p} \alpha > 1$

$$\sum_N \text{Cap}_N \left(B(x_0, 2^{-N}) \cap E, B(x_0, 2^{-N+1}) \right)^{\frac{1}{N-1}} < +\infty$$

2.4 Conformal Geometry & N-Laplacian equation

(M, g_M) , N -manifold. Ω domain, with a conformal metric

$$f = e^{2u} g_M$$

$$(Ric_g)_{ij} = R_{ij} - (\Delta u) g_{ij} + (2-N) u_{,i} u_{,j} + (N-2) u_{,i} u_{,j} + (2-N) |\nabla u|^2 (g_{ij})_{ij}$$

Take trace \Rightarrow Yamabe equation.

Take value in ∇u direction

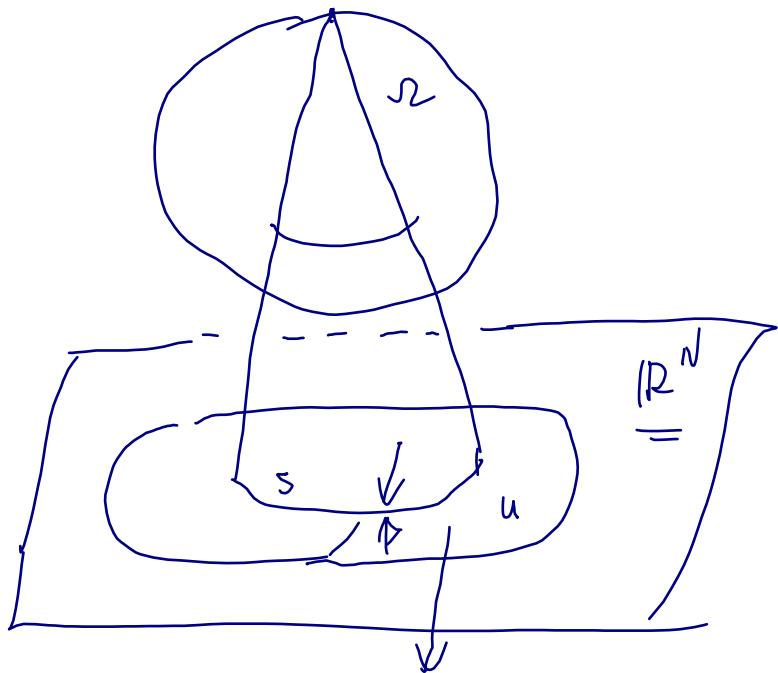
$$\Rightarrow \underbrace{\Delta_N u - |\nabla u|^{N-2} Ric\left(\frac{\nabla u}{|\nabla u|}\right)}_{+ |\nabla u|^{N-2} Ric\left(\frac{\nabla g u}{|\nabla g u|_g}\right)} e^{2u} = 0$$

$$N=2 \rightsquigarrow \Delta u - k + k_g e^{2u} = 0$$

Thm (Zhu Shunkui)

Suppose (Ω, g) is a domain of (S^N, g_S) , $N \geq 3$.

$g = e^{2u} g_S$ is complete. $\text{Ric}_g \geq 0 \Rightarrow \partial\Omega$ consists of at most two point, two point case $\Leftrightarrow S^{N-1} \times \mathbb{R}$.



$$g = e^{2v} g_{\mathbb{R}^N}$$

$$-\Delta_h v = Ric\left(\frac{\nabla v}{|\nabla v|}, |\nabla_g v|\right)^{n-2} e^{2v} \geq 0$$

Chang-Hang-Yang $\Rightarrow \lim_{x \rightarrow \infty} v(x) = +\infty$

$v \in U_N(D) \Rightarrow$

Thm (M-Qing, CPDE)

Let w be an N -superharmonic in $B^{(0,2)} \subset \mathbb{R}^N$.

$$-\Delta_N w = \mu \geq 0 \quad \mu \in \mathcal{M}^+(B^{(0,2)})$$

\exists set E , which is $\underbrace{N^{\text{-thin}}}_{\text{at } 0}$, such that

$$\lim_{x \notin E, x \rightarrow 0} \frac{w(x)}{\log \frac{1}{|x|}} = \liminf_{x \rightarrow 0} \frac{w(x)}{\log \frac{1}{|x|}} = m \geq 0$$

$$\& \quad w(x) \geq m \log \frac{1}{|x|} - C.$$

$$\omega(y, i) = \{x \in \mathbb{R}^N : 2^{-i-1} \leq |x-y| \leq 2^{-i}\}$$

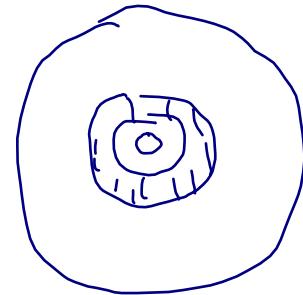
$$\Omega(y, i) = \{ \dots - 2^{-i-2} \leq |x-y| \leq 2^{-i+1} \}$$

$$\omega(\infty, i) = \{ \dots - 2^i \leq |x| \leq 2^{i+1} \}$$

$$\Omega(\infty, i) = \{ \dots - 2^{i-1} \leq |x| \leq 2^{i+2} \}$$

$$N^* - \text{thin at } y \Leftrightarrow \sum_{n=h'}^{\infty} n^{N-1} \text{cap}_N(\mathcal{E} \cap \omega(y, i), \Omega(y, i)) < +\infty$$

$$N^* - \text{thin at } \infty \Leftrightarrow y \rightarrow \infty$$



Thm 2.10 Suppose that $(\mathbb{R}^N, e^{2u}|dx|^2)$ is complete with $Ric \geq 0$.

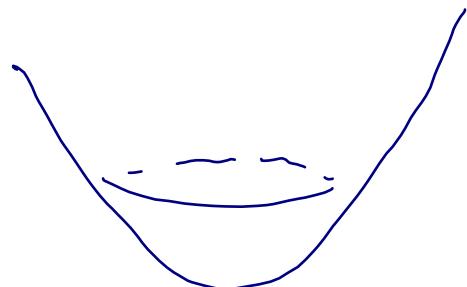
Then $\exists \epsilon \in N^*$ -thin at infinity. S.t.

$$\lim_{x \notin E, x \rightarrow \infty} \frac{u(x)}{\log \frac{1}{|x|}} = \liminf_{x \rightarrow \infty} \frac{u(x)}{\log \frac{1}{|x|}} = m \in [-\infty, 1]$$

$$u(x) \geq m \log \frac{1}{|x|} - L$$

- $m = \infty \Leftrightarrow \mathcal{S}$ flat

- $0 \leq Ric_g \leq c$ $\lim_{x \rightarrow \infty} \frac{u(x)}{\log \frac{1}{|x|}} = m$



Thm 2.11. For $N \geq 3$, let Ω be domain of (\mathbb{S}^N, g_S)

$\tilde{g} = e^{2u} g_S$ is complete on Ω .

1. $Ric_{\tilde{g}} \in L^1(\Omega, \tilde{g}) \cap L^\infty(\Omega, \tilde{g})$

2. $R_g \in L^\infty$. $|D R_g| \in L^\infty$

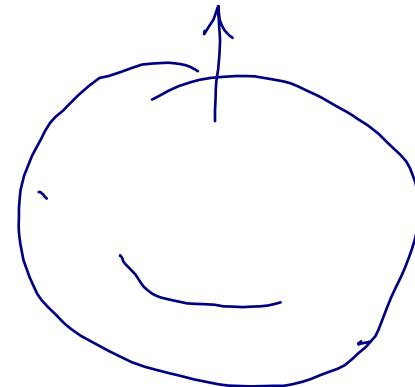
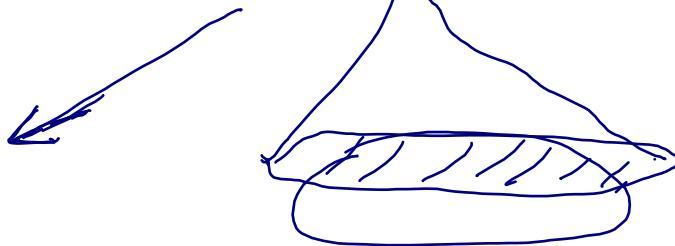
$$\left. \Rightarrow \partial \Omega = \mathbb{S}^N \setminus \{P_1, \dots, P_k\} \right\}$$

$\Delta^2, \Delta^k \leftrightarrow Q$ curvature

$\sigma_k(u_{ij}) \leftrightarrow$ Will potential



$\sigma_k(sch_{ij})$



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$$c_g(x, y) = \frac{U_g(x) - h_g(x)}{x \in \partial\Omega}$$

