Lecture notes on potential theory*

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Contents

Abstract

This note is based on the lectures given by the author during the summer school of Sun Yat-sen University in July 2022.

1 Linear Potential theory

For this part, we will study the potential theory related to Δ. And we will focus on the following topics.

1. Characterization of the Sobolev space $W_0^{1,p}(\Omega)$;

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2. Riesz decomposition theorem;
3. Wiener criterion for the solvablility of Laplacian equation with Dirichlet boundary conditon;
4. An application in geometry. The structure of noncompact surfaces with nonnegative Gaussian curvature.

We refer to [4, 6, 14] for materials of this section.

1.1 Introduction

For $N \geq 2$, let $\mathbb{R}^N$ be the N-dimensional Euclidean space and $x = (x_1, \cdots, x_N)$ be the Cartesian coordinate. Let $\Omega \subset \mathbb{R}^N$ be a bounded open domain. Suppose u is a smooth function on Ω, we denote $\nabla u = (\frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_N})$ and $\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_N^2}$. $\alpha = (\alpha_1, \cdots, \alpha_N)$, $0 \leq \alpha_i \in \mathbb{Z}$ is called a multi-index, and $|\alpha| = \alpha_1 + \cdots + \alpha_N$. We denote $\nabla^\alpha u(x_1, \cdots, x_N) = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N}} u(x_1, \cdots, x_N)$.

Let

$$L^1_{loc}(\Omega) = \{u(x) \text{ is integrable on any compact subset } K \subset \Omega\}.$$

Definition 1. *For $u \in L^1_{loc}(\Omega)$ and $\alpha = (\alpha_1, \cdots, \alpha_n)$ is a multi-index, where $\alpha_i \geq 0$ is an integer. Then a function $v \in L^1_{loc}(\Omega)$ is called αth weak derivative of u if it satisfies*

$$\int_\Omega \phi v dx = (-1)^{|\alpha|} \int_\Omega u \nabla^\alpha \phi dx, \text{ for all } \phi \in C_0^\infty(\Omega).$$

We denote $v = \nabla^\alpha u$.

Now we can define Sobolev space $W^{k,p}(\Omega)$ and $W_0^{k,p}(\Omega)$.

Definition 2. *For $k \geq 0$ and $p \geq 1$,*

$$W^{k,p}(\Omega) = \{u; \nabla^\alpha u \in L^p(\Omega) \text{ for all } 0 \leq |\alpha| \leq k\},$$

together with the norm

$$\|u\|_{W^{k,p}(\Omega)} = \sum_{|\alpha| \leq k} (\int_\Omega |\nabla^\alpha u|^p dx)^{\frac{1}{p}}. \tag{1}$$

And

$$W_0^{k,p}(\Omega) = \overline{C_0^\infty(\Omega)},$$

where the closure is taken in $W^{k,p}(\Omega)$ with respect to norm (1).

Problems

1. Let $f(x) = 1 - |x|^2$, which belongs to $W^{1,p}(B_1(0))$ obviously. Does $f(x) \in W_0^{1,p}(B_1(0))$? Does $f(x) \in W_0^{1,p}(B_1(0)\backslash B_{\frac{1}{2}}(0))$? Does $f(x) \in W_0^{1,p}(B_1(0)\backslash\{0\})$? If for a closed subset $S \subset B_1(0)$, $f(x) \in W_0^{1,p}(B_1(0))$, what can we say about S?

2. Consider Dirichlet problem

$$\begin{cases} \Delta u = 0 & x \in \Omega \\ u = \phi & x \in \partial\Omega, \end{cases}$$

$\phi(x) \in C(\bar{\Omega})$. Does it admit a solution in the sense

$$\lim_{x \to x_0 \in \partial\Omega} u(x) = \phi(x_0),$$

when $\Omega = B_1(0)$, $B_1(0)\backslash B_{\frac{1}{2}}(0)$, $B_1(0)\backslash\{0\}$ or $B_1(0)\backslash S$? What is the equivalent condition for this problem to have a solution for any continuous $\phi(x)$.

1.2 Characterization of $W_0^{1,p}(\Omega)$.

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain.

Lemma 1.1. *Suppose $u \in W_c^{1,p}(\Omega)$ (c means compact support). Then $u \in W_0^{1,p}(\Omega)$.*

Proof. Suppose $u \in W_c^{1,p}(\Omega)$. Then we can choose $v_n \in C^\infty(\Omega)$ such that

$$v_n \to u$$

in $W^{1,p}(\Omega)$ norm. For example we may choose

$$v_n = \int_\Omega \frac{1}{\varepsilon_n^N} \phi(\frac{x-y}{\varepsilon_n}) u(y) dy$$

where $\phi(x) \in C_0^\infty(\mathbb{R}^N)$ with $\int_\Omega \phi = 1$ and $0 < \varepsilon_n \to 0$. We know v_n has compact support actually for n large. $\square$

Roughly speaking, we may think of $W_0^{1,p}(\Omega)$ as the subset of $W^{1,p}(\Omega)$, with the property $u \to 0$ as $x \to \partial\Omega$, because of the following lemma.

Lemma 1.2. *If Ω is a bounded domain and $u \in W^{1,p}(\Omega)$, if for all $y \in \partial\Omega$*

$$\lim_{x \to y} u(x) = 0 \tag{2}$$

then $u \in W_0^{1,p}(\Omega)$.

Proof. Recall that $u = u^+ + u^-$ we may assume that u is nonnegative. Notice that since $\partial\Omega$ is compact, (2) holds uniformly on it. The function

$$u_\varepsilon = \max(u - \varepsilon, 0) \in W^{1,p}(\Omega)$$

for any $\varepsilon > 0$ and has compact support in Ω. Thus $u_\varepsilon \in W_0^{1,p}$ from Lemma 1.1. And it is easy to check $u_\varepsilon \to u$ in $W^{1,p}(\Omega)$.

□

The above assumption (2) is not necessary. To study further, we consider the following definition.

Definition 3. *Fix $p > 1$. For a domain $\Omega \subset \mathbb{R}^N$, any compact subsets $F \subset \Omega$, we define the p-capacity of F with respect to Ω by*

$$\begin{aligned}\mathrm{Cap}_p(F,\Omega) &= \inf\{\int_\Omega |\nabla u|^p dx; u \in C_0^\infty(\Omega), u \geq 1 \,\mathrm{on}\, F\},\\ &= \inf\{\int_\Omega |\nabla u|^p dx; u \in W_0^{1,p}(\Omega), u \geq 1 \,\mathrm{on}\, F\}.\end{aligned}$$

For any open subset U of Ω, we define the p-capacity of U w.r.t Ω by

$$\mathrm{Cap}_p(U,\Omega) = \sup\{\mathrm{Cap_p}(F,\Omega); F \subset E, F\,\mathrm{compact}\}.$$

For arbitrary subset E, we define the p-capacity of E w.r.t Ω by

$$\mathrm{Cap}_p(E,\Omega) = \inf\{\mathrm{Cap_p}(U,\Omega); U \supset E, U\,\mathrm{open}\}.$$

Theorem 1.3. *The set function $E \mapsto \mathrm{Cap}_p(E,\Omega)$, $E \subset \Omega$ enjoys the following properties.*

1. *If $E_1 \subset E_2$, then $\mathrm{Cap}_p(E_1,\Omega) \leq \mathrm{Cap}_p(E_2,\Omega)$;*
2. *If $\Omega_1 \subset \Omega_2$ are open and $E \subset \Omega_1$, then*
$$\mathrm{Cap}_p(E,\Omega_2) \leq \mathrm{Cap}_p(E,\Omega_1);$$
3. *If K_i is a decreasing sequence of compact subsets of Ω with $K = \cap_i K_i$, then*
$$\mathrm{Cap}_p(K,\Omega) = \lim_{i\to\infty} \mathrm{Cap}_p(K_i,\Omega);$$
4. *If $E_1 \subset E_2 \subset \cdots \subset \cup_i E_i = E \subset \Omega$, then*
$$\mathrm{Cap}_p(E,\Omega) = \lim_{i\to\infty} \mathrm{Cap}_p(E_i,\Omega);$$
5. *If $E = \cup_i E_i \subset \Omega$, then*
$$\mathrm{Cap}_{p,\mu}(E,\Omega) \leq \sum_i \mathrm{Cap}_p(E_i,\Omega),$$
in particular, if all E_i satisfies $\mathrm{Cap}_p(E_i,\Omega) = 0$, then $\mathrm{Cap}_{p,\mu}(E,\Omega) = 0$.

We refer to [6, Theorem 2.2] for the proof.

Definition 4. *Let Ω be a bounded domain, and $\Omega \Subset \Omega_1$, for another bounded domain Ω_1;*

A property is called to hold quasieverywhere (q.e.) in Ω, if it holds except for a set E with $\mathrm{Cap}_p(E, \Omega_1) = 0$;

A function f on Ω is called to be p-quasicontinuous if for any $\varepsilon > 0$, there is an open subset U with $\mathrm{Cap}_p(U, \Omega_1) < \varepsilon$, such that outside U, f is finite valued and continuous;

A sequence of functions $\psi_j : \Omega \to \mathbb{R}$ converges p-quasiuniformly in Ω to a function ψ if for every $\varepsilon > 0$ there is an open set G such that $\mathrm{Cap}_p(G, \Omega_1) < \varepsilon$ and $\psi_j \to \psi$ uniformly in $\Omega \backslash G$.

The sequence ψ_j converges locally p-quasiuniformly if it converges p-quasiuniformly in each open $D \Subset \Omega$.

Lemma 1.4. *A sequence ψ_j converges locally p-quasiuniformly in Ω if and only if for every $\varepsilon > 0$ there is an open set $G \subset \Omega$ with $\mathrm{Cap}_p(G, \Omega_1) < \varepsilon$ such that the sequence converges uniformly on every compact subset $\Omega \backslash G$.*

The proof is omitted. One may refer to [6, Lemma 4.2] for a proof.

Lemma 1.5. *Let Ω be as before. Let $\phi_j \in C(\Omega) \cap W^{1,p}(\Omega)$ be a Cauchy sequence in $W^{1,p}(\Omega)$. Then there is a subsequence ϕ_k which converges locally p-quasiuniformly in Ω to a function $u \in W^{1,p}(\Omega)$. In particular, u is p-quasicontinuous and $\phi_k \to u$ pointwise p-q.e. in Ω.*

Proof. Suppose that a locally quasiuniformly convergent subsequence can be selected. Then it clearly converges p-q.e. to a p-quasicontinuous function u. Moreover, $u \in H^{1,p}(\Omega)$. Then it suffices to show that a locally quasiuniformly convergent subsequence can be found.

Since ϕ_j is a Cauchy sequence in $W^{1,p}(\Omega)$, there is a subsequence, denoted again by ϕ_j, such that the series

$$\sum_{j=1}^{\infty} \int_{\Omega} 2^{jp}(|\phi_j - \phi_{j+1}|^p + |\nabla \phi_j - \nabla \phi_{j+1}|^p)dx$$

converges. For $D \Subset \Omega$, let $\psi \in C_0^{\infty}(\Omega)$ and $\psi \equiv 1$ on D. Then

$$\sum_{j=1}^{\infty} \int_{\Omega} 2^{jp}(|\psi(\phi_j - \phi_{j+1})|^p + |\nabla \psi(\phi_j - \phi_{j+1})|^p)dx$$

converges. For any $\varepsilon > 0$, there is j_ε such that

$$\sum_{j=j_\varepsilon}^{\infty} \int_{\Omega} 2^{jp}(|\psi(\phi_j - \phi_{j+1})|^p + |\nabla \psi(\phi_j - \phi_{j+1})|^p)dx < \varepsilon.$$

Let

$$E_j = \{x \in D; |\phi_j(x) - \phi_{j+1}| > 2^{-j}\}$$

we then have

$$\mathrm{Cap}_p(E_j,\Omega_1) \le \int_{\Omega_1} 2^{jp}|\nabla(\psi(\phi_j-\phi_{j+1}))|^p dx.$$

Put

$$E_\varepsilon = \cup_{j_\varepsilon}^\infty E_j.$$

Then we know

$$\mathrm{Cap}_p(E_\varepsilon,\Omega_1) \le \sum_{j=j_\varepsilon}^\infty \mathrm{Cap}_p(E_j,\Omega_1) < \varepsilon.$$

Moreover, for $j_\varepsilon \le j \le k$

$$|\phi_j-\phi_k| \le \sum_{l=j}^{k-1} 2^{-l} \le 2^{1-j}$$

in $D\backslash E_\varepsilon$, and this means that ϕ_j converges uniformly in $D\backslash E_\varepsilon$. Hence it is not hard to prove this theorem.

□

Theorem 1.6. *Suppose that $u \in W^{1,p}(\Omega)$. Then there exists a p-quasicontinuous function $v \in W^{1,p}(\Omega)$ such that $u = v$ a.e.*

Theorem 1.7. *Suppose $u \in W^{1,p}(\Omega)$. Then $u \in W_0^{1,p}(\Omega)$ if and only if there is a p-quasicontiuous function v in $\mathbb{R}^N$ such that $v = u$ a.e. in Ω and $v = 0$ q.e. in Ω^c.*

Proof. Fix $u \in W_0^{1,p}(\Omega)$ and let $\phi_j \in C_0^\infty$ be a sequence converging to u in $W^{1,p}(\Omega)$. By Theorem 1.5, there is a subsequence of ϕ_j which converges p-q.e. in $\mathbb{R}^n$ to a p-quasicontinuous function v such that $v = u$ a.e. in Ω and $v = 0$ q.e. on Ω^c. Hence v is the desired function.

To prove the converse, since the truncations of v converge to v in $W^{1,p}(\Omega)$, we may assume that v is bounded. Let

$$E = \{x \in \partial\Omega : v(x) \neq 0\}.$$

Since $\mathrm{Cap}_p(E,\Omega_1) = 0$, we may choose open sets G_j such that $E \subset G_j$,

$$\mathrm{Cap}_p(G_j,\Omega_1) \to 0,$$

and $v|_{G_j^c}$ is continuous. Pick a sequence $\phi_j \in W^{1,p}(\mathbb{R}^n)$ such that $0 \le \phi_j \le 1$, $\phi_j = 1$ everywhere in G_j, and

$$\int_{\mathbb{R}^n} (|\phi_j|^p + |\nabla\phi_j|^p) d\mu \to 0.$$

Then

$$w_j = (1-\phi_j)v \in W^{1,p}(\Omega).$$

Moreover, $\lim_{x\to y} w_j(x) = 0$ for $y \in \partial\Omega$. Thus $w_j \in W_0^{1,p}(\Omega)$ by Lemma 1.2. Clearly $w_j \to v$ in $L^p(\Omega)$. By the dominated convergence theorem

$$\begin{aligned}(\int_\Omega |\nabla w_j - \nabla v|^p d\mu)^{\frac{1}{p}} &= (\int_\Omega |\nabla(\phi_j v)|^p d\mu)^{1/p} \\ &\leq (\int_\Omega |v\nabla\phi_j|^p d\mu)^{1/p} + (\int_\Omega |\phi_j \nabla v|^p d\mu)^{1/p} \to 0.\end{aligned}$$

So $w_j \to v$ in $W^{1,p}(\Omega)$, and hence $v \in W_0^{1,p}(\Omega)$. The proof is complete. □

1.3 Superharmonic functions

Definition 5. *Suppose $u : \Omega \to (-\infty, +\infty]$ is a lower semi-continuous (l.s.c.) function. u is called superharmonic function, if $u \not\equiv +\infty$ and for any bounded domain $D \Subset \Omega$, and harmonic function v with $v \leq u$ on ∂D, there holds $v \leq u$ in D.*

If $-u$ is superharmonic, u is called a subharmonic function.

We use $\mathcal{U}(\Omega)$ to denote all superharmonic functions on Ω.

Example. *By maxmum principle, we know a function $u \in C^2(\Omega)$ with*

$$-\Delta u \geq 0$$

belongs to $\mathcal{U}(\Omega)$.

However, in this lesson, we will be interested in superharmonic functions which are not necessarily smooth.

The function

$$U_y(x) = \begin{cases} -\frac{1}{2\pi}\log|x-y|, & N = 2 \\ \frac{1}{(N-2)|\mathbb{S}_1^{N-1}||x-y|^{N-2}}, & N \geq 3 \end{cases}$$

is superharmonic on $\mathbb{R}^n$.

Lemma 1.8. *A superharmonic function $v \in \mathcal{U}(\Omega)$ is locally integrable.*

Proof. We will give a sketch of the proof. The details are left to the reader.

Step 1 From v is lower semi-continuous, we know that it is locally bounded from below.

Step 2 From Baire's theorem, we know, there is a sequence $w_1 \leq w_2 \leq \cdots$ of continuous functions, which satisfies

$$\lim_{n\to\infty} w_n(x) = v(x)$$

for any $x \in \Omega$. For any $\overline{B_r(x_0)} \subset \Omega$, we can solve

$$\begin{cases} \Delta v_n = 0 & x \in B_r(x_0) \\ v_n(x) = w_n(x) & x \in \partial B_r(x_0) \end{cases}$$

using Possion's integral formula. Then

$$\fint_{\partial B_r(x_0)} v_n(x) d\mu_{\partial B_r} = v_n(x_0) \leq v(x_0).$$

Using Fatou's lemma, we see

$$\fint_{\partial B_r(x_0)} v(x) d\mu_{\partial B_r} \leq v(x_0).$$

Integrating with respect to r, we know

$$\fint_{B_r(x_0)} v(x) dx \leq v(x_0). \tag{3}$$

Step 3 Suppose $v(x)$ is not locally integrable, then we may assume there is some $x_0 \in \Omega$ such that $v(x)$ is not integrable in any neighborhood $B_r(x)$ of x_0, where $x \in B_{\frac{\delta}{2}}(x_0)$ and $0 < r < \delta$. Then from (3), we know

$$v(x) \equiv +\infty, \forall x \in B_{\frac{\delta}{2}}(x_0).$$

Step 4 As long as $v(x) \equiv +\infty$ in $B_{\frac{\delta}{2}}(x_0)$, then $v(x) = +\infty$ in any $B_R(x_0) \Subset \Omega$ for $R > \frac{\delta}{2}$. Once this is proved, it is not hard to verify taht $v \equiv +\infty$ in Ω, which is a contradiction. To prove this, we let

$$h(x) = \frac{\int_{|x-x_0|}^{R} t^{-(n-1)} dt}{\int_{\frac{\delta}{2}}^{R} t^{-(n-1)} dt}.$$

Then h is harmonic when $x \neq x_0$, hence in $\frac{\delta}{2} \leq |x - x_0| \leq R$. It takes the boundary value 0 on ∂B_R and 1 on $\partial B_{\frac{\delta}{2}}$.

Since v is locally bounded from below, we know

$$v - \inf_{B_R(x_0)} v \geq k h(x)$$

for any $k > 0$ in $B_R \backslash B_{\frac{\delta}{2}}$. Let $k \to \infty$, we know $v \equiv +\infty$ in $B_R(x_0)$.

$\square$

Definition 6. *Suppose Σ is the σ-algebra of a set X. A set function*

$$\mu : \Sigma \to [-\infty, +\infty]$$

is called a signed measure on X if it satisfies

1. *μ takes on at most one of values $\pm\infty$;*
2. *$\mu(\emptyset) = 0$;*
3. *For countable collections $\{E_k\}_{k=1}^{\infty}$ of pairwise disjoint sets in Σ,*

$$\mu(\cup_{k=1}^{\infty} E_k) = \sum_{k=1}^{\infty} \mu(E_k).$$

If $\mu(U) \geq 0$ for any $U \in \Sigma$, then μ is called a nonnegative Radon measure.

Definition 7. *Let μ be a measure on the σ algebra Σ of Borel sets of a Hausdorff topological space X.*

- *μ is called inner regular, if for any open $U \in \Sigma$,*

$$\mu(U) = \sup\{\mu(K); K \subset U \text{ is compact}\};$$

- *μ is called locally finite if every point $y \in \mathbb{R}^n$ has a neighborhood U such that $|\mu(U)| < +\infty$.*

The measure μ is called a signed Radon measure if it is inner regular and locally finite. If $\mu(U) \geq 0$ for any $U \in \Sigma$, μ is called a Radon measure.

We use $\mathcal{M}(X)$ to deonte all signed Radon measures on X, and use $\mathcal{M}^+(X)$ to denote all (nonnegative) Radon measures on X.

Definition 8. *If μ is a Radon measure on X, and U is an open subset. If*

$$\mu(D) = 0$$

holds for any open $D \subset U$, we say μ vanishes on D. We denote $\mu|_D = 0$.

The support of μ is defined as the complement of the union of all open D such that $\mu|_D = 0$.

Let F be a compact subset of $\mathbb{R}^n$, we use $\mathcal{M}^+(F)$ to denote the set of all nonnegative Radon measures on $\mathbb{R}^n$ supported in F. For a domain Ω, we use $\mathcal{M}^+(\Omega)$ to denote the set of all nonnegative Radon measures in Ω.

If $u : \Omega \to [-\infty, +\infty]$ is locally integrable in Ω, we define a linear functional on $C_0^\infty(\Omega)$ by

$$L_u(\psi) = -\int_\Omega u\Delta\psi dx.$$

We call L_u the minus distributional Laplacian of u. If v is another locally integrable function on Ω, then clearly

$$L_{k\cdot u + l\cdot v} = kL_u + lL_v.$$

We need the following Riesz representation theorem. One may refer to [15, Theorem 2.14] for the proof.

Theorem 1.9. *Let X be a locally compact Hausdorff space, and let Λ be a positive linear functional on $C_c(X)$, which is the real linear space of continuous functions with compact support. Then there exists a σ-algebra Σ which contains all Borel sets in X, and a unique positive measure μ on Σ (), which represents Λ in the sense that*

1. $\Lambda f = \int_X f d\mu, \forall f \in C_c(X)$ *, and which has the following additional properties:*
2. μ *is locally finite;*
3. $\forall E \in \Sigma$*, we have*
$$\mu(E) = \inf\{\mu(V) : E \subset V, V \text{ open}\};$$
4. *The relation*
$$\mu(E) = \sup\{\mu(K) : K \subset E, K \text{ compact}\}$$
holds for $\forall$ *open set* E *and* $\forall F \subset \Sigma$ *with* $\mu(F) < \infty$.
5. *If* $E \subset \Sigma, A \subset E$ *and* $\mu(E) = 0$*, then* $A \in \Sigma$.

In particular, from Item 2 and 4, μ *is a Radon measure.*

Theorem 1.10. *If* $u \in \mathcal{U}(\Omega)$*, then* L_u *is a nonnegative Radon measure.*

Proof. First if $u \in \mathcal{U}(\Omega)$, u is locally integrable. Then L_u is well defined. We only need to prove
$$|L_u(\psi)| \le C(K)\|\psi\|_{L^\infty} \tag{4}$$
for any $\psi \in C_0^\infty(\Omega)$ and $supp\psi \subset K$. Then from theorem 3.3.3 of [4], we know there is an increasing sequence $u_n \in \mathcal{U}(\Omega) \cap C^\infty(\Omega)$ such that $u_n \to u$ pointwise on u. In particular, $\Delta u_n \ge 0$ for each n. Thus
$$\int_\Omega u_n \Delta\psi \ge 0$$
for any $\psi \in C_0^\infty$ and $\psi \ge 0$. So by writting $\Delta\psi = (\Delta\psi)^+ - (\Delta\psi)^-$, we have
$$\int_\Omega u\Delta\psi \ge 0$$
for $\psi \in C_0^\infty$ and $\psi \ge 0$.

Then the conclusion follows from Theorem 1.9.

□

From the above theorem, we know L_u defines a Radon measure $\mu_u \in \mathcal{M}^+(\Omega)$. We call it "Riesz measure" defined by the superharmonic function u.

1.4 Potentials

Now we study potentials.

Definition 9. *For functions f, g defined on Ω, if $f \leq g$, we call f is a minorant of g and g is a majorant of f.*

If f is a minorant (or majorant) of g, satisfying property A, we call f is A-minorant (or A-majorant), e.g. harmonic minorant, superharmonic majorant, etc.

Definition 10. *An open set Ω in $\mathbb{R}^N$ is said to be Greenian if, for each $y \in \Omega$, U_y has a subharmonic minorant on Ω. ($\mathbb{R}^2$ is not Greenian)*

If this subsection, we will always assume that Ω is Greenian.

Suppose $G_\Omega(x, y) = U_y - h_y$, where h_y is the greatest harmonic minorant of U_y on Ω (which can be found using Perron method). $G_\Omega(x, y)$ is called the Green function of Ω.

Definition 11. *Let μ be a Radon measure in Ω, we define*

$$G_\Omega\mu(x) = \int_\Omega G_\Omega(x, y) d\mu(y), x \in \Omega.$$

If $G_\Omega\mu(x) \not\equiv +\infty$, we call it the potential generated by $\mu(x)$.

Example. *Let $\Omega = B_1(0) \subset \mathbb{R}^2$ and let $G_\Omega(x, y)$ be the corresponding Green function. Let $f(x) = -\sec|x|$ and $g(x) = \sqrt{1 - |x|^2}$. It is easy to prove that $f, g \in \mathcal{U}(\Omega)$. However, one readily checks that*

$$G_\Omega\mu_f \equiv +\infty$$

while $G_\Omega\mu_g$ defines a potential. The reason for this lies in that $f(x)$ does not have a subharmonic minorant while $g(x)$ does.

We state a theorem and we refer to [4, Theorem 4.2.4] for the proof.

Theorem 1.11. *Let μ be a Radon measure on a connected Greenian open set Ω and let $\overline{B(z, r)} \subset \Omega$. Then $G_\Omega\mu$ is a potential if and only if*

$$\int_{\Omega \setminus B(z,r)} G_\Omega(z, y) d\mu(y) < \infty.$$

In particular, if $\mu(\Omega) < +\infty$ or μ has compact support, then $G_\Omega\mu$ is a potential.

Lemma 1.12. *(Weyl's lemma)*

Let Ω be an open subsut of $\mathbb{R}^N$, and $u \in L^1_{loc}(\Omega)$. If

$$\int_\Omega u(x)\Delta\phi(x)dx = 0$$

for any $\phi \in C_c^\infty(\Omega)$ then u is a harmonic function.

Lemma 1.13. *Let Ω be a Greenian domain and let μ be a Radon measure on Ω. If $G_\Omega \mu$ is a potential, then $G_\Omega \mu$ is locally integrable. Moreover, the Riesz measure of $G_\Omega \mu$ is μ.*

Proof. First we prove that $G_\Omega \mu$ is locally integrable. For simplicity, we assume $N \geq 3$, $N = 2$ case are left to the readers.

For this case $G_\Omega(x, y) \leq U_y(x)$. For a compact subset $K \subset \Omega$, we choose open set D such that $K \subset D \Subset \Omega$. For $z \in k$

$$G_\Omega \mu(z) = \int_D + \int_{\Omega \setminus D} G_\Omega(z, x) d\mu(x).$$

It is easy to see that the harmonic function $\int_{\Omega \setminus D} G_\Omega(z, x) d\mu(x)$ is bounded on K. For the first term

$$\int_K \int_D G_\Omega(z, x) d\mu(x) dz = \int_D d\mu(x) \int_K G_\Omega(z, x) dz.$$

So

$$\begin{aligned}
\int_K G_\Omega(z, x) dz &\leq \int_K U_x(z) dz \\
&\leq \int_B U_x(z) dz \leq C(vol(K), N),
\end{aligned}$$

here B is a ball centered at x with the same volume as K. So

$$\int_K \int_D G_\Omega(z, x) d\mu(x) dz \leq \mu(D) C(vol(K), N).$$

Then we have proved that $G_\Omega \mu$ is locally integrable.

Then we know $L_{G_\Omega \mu}$ is well defined and

$$\begin{aligned}
\int_\Omega (-\Delta \phi) G_\Omega \mu dx &= \int_\Omega (-\Delta \phi) (\int_\Omega G_\Omega(x, y) d\mu(y)) dx \\
&= \int_\Omega (\int_\Omega G_\Omega(x, y)(-\Delta \phi(x)) dx) d\mu(y) \\
&= \int_\Omega \phi(y) d\mu(y),
\end{aligned}$$

So we know that the distribution Laplacian $L_{G_\Omega \mu}$ equals to μ.

□

Theorem 1.14. *(Riesz decomposition theorem) Let u be superharmonic in Ω, let μ_u denote its associated Riesz measure and suppose that u has a subharmonic minorant on Ω. Then $G_\Omega \mu_u$ is a potential on Ω and $u = G_\Omega \mu_u + h$, where h is the greatest harmonic minorant of u on Ω.*

Proof. Let $\{K_n\}$ be a sequence of compact subsets of Ω such that $K_n \subset int(K_{n+1}^0)$ for each n, and such that $\cup_n K_n = \Omega$. Further, let $\mu_u^{(n)}$ denote the restriction of μ_u to K_n. Then from Lemma 1.13, we know $L_{G_\Omega \mu_u^{(n)}} = L_u$ on $C_0^\infty(intK_n^0)$. We define $v_n = u - G_\Omega \mu_u^{(n)}$. Then by Weyl's lemma, v_n is harmonic in K_n^0. So we see that $G_\Omega \mu_u^{(n)}$ is a superharmonic function in Ω.

Since u has a subharmonic minorant, we may assume that h is the greatest harmonic minorant of u. Now if we prove

$$G_\Omega \mu_u^{(n)} \leq u - h, \tag{5}$$

then from

$$\lim_{n\to\infty} G_\Omega \mu_u^{(n)} = G_\Omega \mu_u,$$

we know $G_\Omega \mu_u \leq u - h$ is a potential. Now we will prove (5). It suffices to prove the greatest harmonic minorant of $u - h - G_\Omega \mu_u^{(n)}$, $w_n \geq 0$. Since

$$u - h - G_\Omega \mu_u^{(n)} \geq -G_\Omega \mu_u^{(n)}$$

and $-G_\Omega \mu_u^{(n)}$ is subharmonic, we know $w_n \geq -G_\Omega \mu_u^{(n)}$. Then it suffice to prove that the greatest harmonic minorant of $G_\Omega \mu_u^{(n)}$ is 0. It is obvious that 0 is a harmonic minorant of $G_\Omega \mu_u^{(n)}$. On the other hand, fix $y_0 \in K_n$, for $x \in \Omega \backslash K_{n+1}$, there holds Harnack's inequality

$$G_\Omega(x, y) \leq C G_\Omega(x, y_0)$$

for any $y \in K_n$. Integration with respect to $y \in K_n$, we know

$$G_\Omega \mu_u^{(n)}(x) \leq C \mu_u(K_n) G_\Omega(x, y_0).$$

Suppose ξ_n is the greatest harmonic minorant of $G_\Omega \mu_u^{(n)}(x)$, then in $\Omega \backslash K_{n+1}$, $\xi_n \leq C\mu_u(K_n) G_\Omega(x, y_0)$. By the definition of superharmonic function, we know $\xi_n \leq C\mu_u(K_n) G_\Omega(x, y_0)$ holds in Ω. Since the greatest harmonic minorant of $G_\Omega(x, y)$ of 0, we know $\xi_n \leq 0$.

Then we know $G_\Omega \mu_u \leq u - h$ is a potential. The greatest harmonic minorant of

$$u - G_\Omega \mu_u - h$$

is 0 since the greatest harmonic minorant of $u - h$ is 0, and 0 is a harmonic minorant of $u - G_\Omega \mu_u - h$. However $u - G_\Omega \mu_u - h$ is harmonic. So it is 0. We have

$$u = G_\Omega \mu_u + h.$$

□

1.5 Polar set

Now we turn to the Wiener's Criterion for the solvablity of Dirichlet problem.

Definition 12. *A set E in $\mathbb{R}^N$ is called polar if there is a superharmonic function u on some open set Ω such that $E \subset \{x \in \Omega; u(x) = +\infty\}$.*

It is clear from the local integrability of superharmonic functions that polars sets have zero Lebesgue measure.

Example. 1. *Single point set $\{y\}$ is a polar set, since U_y is a superharmonic function in $\mathbb{R}^N$. In fact, any countable set $\{y_k : y \in \mathbb{N}\}$ is polar. When $N \geq 3$,*

$$u(x) = \sum_k 2^{-k} U_{y_k}(x), x \in \mathbb{R}^N$$

is a potential. When $N = 2$, the function

$$u(x) = \sum_k 2^{-k}(1 + \log^+ \|y_k\|)^{-1} U_{y_k}(x), x \in \mathbb{R}^2$$

is a logarithmic potential.

2. *If $N \geq 3$, the set $E = (0,0) \times \mathbb{R}^{N-2}$ is a polar set, since*

$$u(x_1, \cdots, x_N) = -\log(x_1^2 + x_2^2)$$

is superharmonic on $\mathbb{R}^N$.

Theorem 1.15. *Let E be a polar set such that $E \subset \Omega$, and let $z \in \Omega \backslash E$. If Ω is Greenian, then there is a potential $G_\Omega \mu$ valued $+\infty$ on E such that $G_\Omega \mu(z) < +\infty$ and $\mu(\Omega) < +\infty$.*

Proof. Since E is polar, there exist an open set D and a superharmonic function $u \in \mathcal{U}(D)$ such that $u(E) = +\infty$. We may assume that $D \subset \Omega$ and $z = 0 \notin D$ (or we remove it from D and D is still open). Let (B_k) be a sequence of open balls such that $\bar{B}_k \subset D$ for each k and $\cup_k B_k = D$. For each k we define a measure ν_k by

$$\nu_k(A) = \frac{\mu_u(A \cap B_k)}{\mu_u(B_k) + 1}$$

for any Borel set A, where μ_u is the Riesz measure associated with u. Define

$$u_k = \int U_y d\nu_k(y)$$

is valued $+\infty$ on $E \cap B_k$. Also $u_k(0) < +\infty$, since $0 \notin \overline{B_k}$.

When $N \geq 3$, let

$$\mu = \sum_k 2^{-k} \frac{\nu_k}{1 + u_k(0)}.$$

Then $\mu(\Omega) \leq 1$ and $G_\Omega \mu$ is a potential on Ω, which is valued $+\infty$ on E but is finite at 0.

If $N = 2$, let

$$\mu = \sum_k 2^{-k}\{1 + \int_{B_k} |\log \|y\||d\nu_k(y)\}^{-1}\nu_k.$$

We can prove similarly.

□

1.6 Fine topology

First we study a superharmonic function, which is not continuous.

Example. *Let $\{x_n\}$ be a sequence of points in $B_1\backslash\{0\} \subset \mathbb{R}^N$, satisfying $x_n \to 0$. Let*

$$u(x) = \sum_{n=1}^{\infty} 2^{-n} U_{x_n}(x)/U_{x_n}(0), x \in \mathbb{R}^N.$$

Then u is superharmonic, $u(x_n) = +\infty$ for each n and $u(0) = 1$. So u is discontinuous at 0. It can be modified to become a bounded one by simply considering $\min\{u, 2\}$.

This tell us that, we have to use another topology on $\mathbb{R}^N$, to make superharmonic functions continuous.

Definition 13. *If $\mathcal{T}_1$ and $\mathcal{T}_2$ are topologies on the same set E, if $\mathcal{T}_2 \subset \mathcal{T}_1$, then we say that $\mathcal{T}_1$ is finer that $\mathcal{T}_2$, and that $\mathcal{T}_2$ is coarser than $\mathcal{T}_1$. The "fine topology" of classical potential theory is the coarsest topology on $\mathbb{R}^N$ which makes every superharmonic function on $\mathbb{R}^N$ continuous in the extended sense of functions taking values in $[-\infty, +\infty]$.*

Since $B(y, r)$ can be written in the form $\{x : U_y(x) > a\}$ for a suitable choice of a, it is clear that the fine topology is finer than the Euclidean one.

Lemma 1.16.

1. *A subbase for the fine topology is given by the collection of all sets of the form $\{x : u(x) < a\}$, where $u \in \mathcal{U}(\mathbb{R}^N)$ and $a \in \mathbb{R}$.*
2. *If u is superharmonic on an open set Ω, then u is finely continuous on Ω.*

We omit the proof. One can refer to Lemma 7.1.2 of [4].

Definition 14. *A set E is said to be thin at a point y if y is not a fine limit point of E; that is, if there is a fine neighborhood U of y which does not intersect $E\backslash\{y\}$. Otherwise E is said to be non-thin at y.*

Theorem 1.17. *A polar set is thin everywhere.*

Proof. Let E be a polar set and let $y \in \mathbb{R}^N$. Then, by Theorem 1.15, there exists u in $\mathcal{U}(D)$, where D is a neighborhood of y such that $u = +\infty$ on $(E \cap D)\backslash\{y\}$ and $u(y) < +\infty$. Hence the set $\{x : u(x) < u(y) + 1\}$ is a fine neighborhood of y which does not intersect $E\backslash\{y\}$; that is, E is thin at y.

□

Theorem 1.18. *Let y be a limit point of a set E. The following are equivalent:*

1. *E is thin at y;*
2. *there is a superharmonic function u on a neighborhood of y such that*
$$\liminf_{x\to y, x\in E} u(x) > u(y);$$
3. *for every Greenian set Ω which contains y, there exists $u \in \mathcal{U}_+(\Omega)$, such that $u(y) < +\infty$ and $u(x) \to +\infty$ as $x \to y$ along E.*

Proof. Clearly (3) $\Rightarrow$ (2).

(2)$\Rightarrow$(1). If (2) holds, then we choose positive numbbers δ and ε such that $u \in \mathcal{U}(B(y,\delta))$ and $u(x) \geq u(y) + \varepsilon$ on $B(y,\delta) \cap (E\backslash\{y\})$. It follows that the set $\{x \in B(y,\delta) : u(x) < u(y) + \varepsilon\}$ is a fine neighborhood of y which does not intersect $E\backslash\{y\}$, so (1) holds.

It remains to show that (1)$\Rightarrow$(3). Suppose that E is thin at y and that Ω is a Greenian set which contains y. From Lemma 1.16, there exist $u_1, \cdots, u_m \in \mathcal{U}(\mathbb{R}^N)$ and constants $a_1, a_2, \cdots, a_m$ such that the set $U = \cap_1^m\{x : u_n(x) < a_n\}$ is a fine neighborhood of y which does not intersect $E\backslash\{y\}$. We define the superharmonic functions

$$\begin{aligned} u'_n(x) &= \frac{u_n(x) - u_n(y)}{a_n - u_n(y)}, (x \in \mathbb{R}^N; n = 1, 2, \cdots, m) \\ v &= \sum_{n=1}^{m} u'_n \end{aligned}$$

and $w = \min\{u'_1, u'_2, \cdots, u'_m\}$.

If $x \in E\backslash\{y\}$, then $x \notin U$, so $u'_n(x) \geq 1$ for some n. Hence $v \geq 1 + (m-1)w$ on $E\backslash\{y\}$, and it follows that

$$\liminf_{x\to y, x\in E} v(x) \geq 1 + (m-1)w(y) = 1 = v(y) + 1. \tag{6}$$

Let μ denote the Riesz measure associated with v. Then $\mu(\{y\}) = 0$, since $v(y) < +\infty$. Let $\mu_n = \mu|_{B(y,r_n)}$ for $r_n > 0$ small to be chosen. Since $G_\Omega\mu_n(y) \to 0$, we may choose r_n small enough such that $G_\Omega\mu_n(y) < 2^{-n}$. Let $u = \sum_n G_\Omega\mu_n$. Then $u \in \mathcal{U}_+(\Omega)$ and $u(y) < 1$. However, v differs from $G_\Omega\mu_n$ by a function harmonic on $B(y, r_n)$, so from (6) we have

$$\liminf_{x\to y, x\in E} G_\Omega\mu_n(x) \geq G_\Omega\mu_n(y) + 1 \geq 1.$$

It follows that $u(x) \to +\infty$ as $x \to y$ along E, so (3) holds.

□

1.7 Reduced functions

For $u \in \mathcal{U}(\Omega)$, $u \geq 0$, put

$$\Phi^u_{F,\Omega} := \{v \in \mathcal{U}(\Omega) : v \geq 0 \,\text{on}\, \Omega, v \geq u \text{ on } F\}.$$

and define the reduced function of u with respect to (F, Ω)

$$R^u_{F,\Omega}(x) := \inf_{v \in \Phi^u_F} v(x).$$

In general, R^u_F is not l.s.c. For example, for $n \geq 3$, let $\Omega = \mathbb{R}^n$, $F = \{0\}$, and $U(x) = |x|^{2-n}$. Then we have

$$R^U_{F,\Omega}(x) = \begin{cases} 0 & x \neq y, \\ +\infty & x = y. \end{cases}$$

So it is not l.s.c.

Definition 15. *We let*

$$\hat{R}^u_{F,\Omega}(x) = \liminf_{y \to x} R^u_{F,\Omega}(y), x \in \mathbb{R}^n.$$

We call it the regularized reduced function or balayage.

One reduced function and balayage we will usually use is

$$R^1_{F,\Omega}, \hat{R}^1_{F,\Omega}.$$

Lemma 1.19. *With the same notations as above:*

1. $0 \leq \hat{R}^1_{F,\Omega} \leq R^1_{F,\Omega} \leq 1$;
2. $R^1_{F,\Omega} = 1$ *on* F*;*
3. $\hat{R}^1_{F,\Omega} = R^1_{F,\Omega}$ *on* $\text{int}F \cup F^c$*, they are different only on a polar set of* ∂F*.*
4. *Assume* $\bar{F}$ *is a compact subset of* Ω*, then* $\hat{R}^1_{F,\Omega}$ *is a potential* $\int_\Omega G_\Omega(x, y) d\mu_y$*. Then* $\hat{R}^1_{F,\Omega}$ *is superharmonic in* Ω *and harmonic in* $\bar{F}^c$*. It is also called the capacitary potential of* F *with respect to* Ω*. Its Riesz measure is supported on* $\bar{F}$ *and is called the capacitary distribution of* F*.*
5. $\hat{R}^1_{F,\Omega} = \hat{R}^1_{F \cup E,\Omega}$ *if* E *is a polar set.*

One may refer to [4, Theorem 5.3.4, 5.3.5] for the proof.

Lemma 1.20. *Let* $E \subset \Omega$ *be thin at* x_0*. Then* $\hat{R}^1_{E \cap B(x_0,r)} \to 0$ *for* $r \to 0^+$*.*

Proof. By Theorem 1.18, we know there exists a superharmonic function $u \geq 0$ such that

$$0 < u(x_0) < +\infty = \liminf_{E \ni x \to x_0, x \neq x_0} u(x).$$

Choose $\varepsilon > 0$, we can pick $r > 0$ so that $u(x) > \frac{u(x_0)}{\varepsilon}$ for all $x \in E \cap B(x_0, r) \backslash \{x_0\}$. The set $\{x_0\}$ has capacity 0, hence $\hat{R}^1_{E \cap B(x_0,r)} = \hat{R}^1_{E \cap B(x_0,r) \backslash \{x_0\}}$. Then we know

$$\hat{R}^1_{E \cap B(x_0,r)} \leq \varepsilon \frac{u(x)}{u(x_0)}$$

on $E \cap B(x_0, r)$ and moreover $\hat{R}^1_{E \cap B(x_0,r')} \leq \hat{R}^1_{E \cap B(x_0,r)}$ for any $0 < r' < r$, as $F(r)$ is a decreasing sequence.

□

Lemma 1.21. *If E is not thin at $x_0 \in \partial\Omega$, then $\hat{R}^1_{E \cap B(x_0,r)}(x_0) = 1$ for all $r > 0$.*

Proof. $R^1_{E \cap B(x_0,r)}|_E = 1$, hence

$$\hat{R}^1_{E \cap B(x_0,r)} = 1$$

outside a polar set F on the boundary of $E \cap B(x_0, r)$. Still $(E \backslash F) \cap B(x_0, r)$ is not thin at x_0. Then we may assume there is a sequence $(E \backslash F) \backslash \{x_0\} \ni x_n \to x_0$ in fine topology. So $\hat{R}^1_{E \cap B(x_0,r)}(x_0) = 1$.

□

1.8 Regular boundary points

Now we consider the third problem. Let $\Omega \subset \mathbb{R}^N, N \geq 2$ be a bounded open set and $\phi(x) \in C(\bar{\Omega})$. One useful method to solve this problem is due to Perron. Consider

$$\bar{H}_\phi = \inf\{v \in \mathcal{U}(\Omega); \liminf_{\Omega \ni y \to x} v(y) \geq \phi(x) \text{ for any } x \in \partial\Omega\}.$$

From standard argument, for example, from Chapter 2 of [3], we know $\bar{H}_\phi$ is harmonic in Ω.

Definition 16. *$x_0 \in \partial\Omega$ is called a regular boundary point, if for any $\phi(x) \in C(\bar{\Omega})$,*

$$\lim_{\Omega \ni y \to x} \bar{H}_\phi(y) = \phi(x).$$

Now we characterize the regular points.

Definition 17. *Let $\Omega \subset \mathbb{R}^n$ be an open set. A function $w = w_{x_0}(x)$ a called a (local) barrier function at $x_0 \in \partial\Omega$ if it is defined on $W \cap \Omega$ for some neighborhood W of x_0 and has the following properties:*

1. *w is superharmonic on $W \cap \Omega$;*

2. $w > 0$ *on* $W \cap \Omega$*;*

3.
$$\lim_{W \cap \Omega \ni x \to x_0} w(x) = 0.$$

From Lemma 6.6.3 of [4], we know the following lemma holds.

Lemma 1.22. *If there is a (local) barrier at* $y \in \partial\Omega$*, then there exists a global barrier* v *at* y *such that* $v \in \mathcal{U}_+(\Omega)$ *and* $\inf_{\Omega \setminus \omega} v > 0$ *for every open neighborhood* ω *of* y*.*

Lemma 1.23. *(Lemma 7.3.4 of [4])*
Let $F \subset \Omega$ *and* $u \geq 0$*,* $u \in \mathcal{U}(\Omega)$*. Assume* u *peaks at* $x_0 \in \Omega$ *and* $u(x_0) < \infty$*. Then* F *is thin at* x_0 *if and only if* $\hat{R}^u_{F,\Omega}(y) < u(y)$*.*

Theorem 1.24. *Let* Ω *be a bounded open set that has a Green function and* $x_0 \in \partial\Omega$*. Then the following statements are equivalent:*

1. x_0 *is a regular boundary point;*

2. *There exists a barrier at* x_0*;*

3. Ω^c *is not thin at* x_0*.*

Proof. (1) $\Longleftrightarrow$ (2)
Suppose first that x_0 is regular and consider the function $w(x) = |x_0 - x|, x \in \partial\Omega$; Notice that $\bar{H}_w$ is harmonic in Ω, $H_w > 0$ in Ω by strong maximum principle, and
$$\lim_{\Omega \ni x \to x_0} \bar{H}_w(x) = 0.$$
Then we know $\bar{H}_w(x)$ is a barrier function at x_0.
Now we assume there is a barrier function w in $B_r(x_0) \cap \Omega$. Then from Lemma 1.22, we may assume w is a barrier function defined in Ω with $\inf_{\Omega \setminus \omega} w > 0$ for every open neighborhood ω of y.
Then let $M = \sup |\phi|$. By the continuity of ϕ, there are constants δ and k such that $|\phi(x) - \phi(x_0)| < \varepsilon$ if $|x - x_0| < \delta$ and $kw(x) \geq 2M$ if $|x - x_0| \geq \delta$. The functions
$$\phi(x_0) - \varepsilon - kw(x) \leq u(x) \leq \phi(\xi) + \varepsilon + kw(x)$$
or
$$|u(x) - \phi(\xi)| \leq \varepsilon + kw(x).$$
Since $w(x) \to 0$ as $x \to \xi$, we obtain $u(x) \to \phi(\xi)$ as $x \to \xi$.
(1) $\Longleftrightarrow$ (3)
Assume Ω^c is not thin at x_0 and consider the ball $B = B(x_0, 1)$. Define the positive superharmonic function $u(x) = 1 - |x - x_0|^2$ on B ($-\Delta u = 2n > 0$); now build the positive superharmonic function
$$w := u - \hat{R}^u_{\Omega^c \cap \frac{B}{2}}$$

where the balayage is with respect to B. w has to be strictly positive on $B \cap \Omega$; in fact, if there is $x \in B \cap \Omega$ such that $w(x) = 0$, by maximum principle $w = 0$ on all $B \cap \Omega$, hence $u = \hat{R}^u_{\Omega^c \cap B}$ there and u is harmonic on $B \cap \Omega$, but this would be a contradiction since $\Delta u < 0$ there. As u peaks in x_0, Theorem 1.23 and the assumption imply that $w(x_0) = 0$, therefore w is a barrier at x_0 and this implies x_0 is regular.

Suppose now x_0 is regular. Pick r' small enough so that $\Omega' = \Omega \cup B(x_0, r')$ still has a Green function; for each $0 < r < r'$ define

$$f_r(x) = \begin{cases} 1 & x \in \partial\Omega \cap B(x_0, r), \\ 0 & x \in \partial\Omega \backslash B(x_0, r). \end{cases}$$

Let $E(r) = \overline{B(x_0, r)} \backslash \Omega$. Taking the reduced functions with respect to Ω' and recalling Lemma (1.2.1):

$$\hat{R}^1_{E(r),\Omega'}(x_0) \leq 1.$$

However for any superharmonic function $0 \leq w \in \mathcal{U}(\Omega')$, $w \geq 1$ on $E(r)$, we have

$$\liminf_{\Omega \ni x \to x_0} w(x) \geq \liminf_{\Omega \ni x \to x_0} \bar{H}_{f_r}(x) = f_r(x_0) = 1$$

as x_0 is regular.

So we know

$$\liminf_{x \to x_0} R^1_{E(r),\Omega'}(x) = 1.$$

Then

$$\liminf_{x \to x_0} \hat{R}^1_{E(r),\Omega'}(x) = 1.$$

Hence $\hat{R}^1_{E(r),\Omega'}(x_0) = 1$ for all $0 < r < r'$; Then from Lemma 1.21, we know Ω^c is not thin at x_0. □

1.9 Wiener Criterion

Let Ω be a Greenian domain and $E \subset \mathbb{R}^N$ and $y \in \Omega$. We fix $\alpha > 1$. Let

$$A_n = \{x \in \mathbb{R}^N : \alpha^n \leq U_y(x) \leq \alpha^{n+1}\}, n \in \mathbb{N}.$$

Theorem 1.25. *Suppose $N \geq 3$. Let $n' \in \mathbb{N}$ be such that $U_{n'} = \{x : U_y(x) \geq \alpha^{n'}\}$ is contained in Ω. The following are equivalent:*

1. *E is thin at y;*
2. *$\sum_{n'}^{\infty} \alpha^n \mathrm{Cap}_2(E \cap A_n, \Omega) < +\infty$;*
3. *$\sum_{n'}^{\infty} \hat{R}^1_{E \cap A_n}(y) < +\infty$, where $\hat{R}^1_{E \cap A_n}(y) = \hat{R}^1_{E \cap A_n, \Omega}(y)$.*

Proof. (2) $\Longleftrightarrow$ (3)

For $n \geq n'$, $\hat{R}^1_{E \cap A_n}(y)$ is the capacitary potential of $E \cap A_n$ with respect to Ω. Let μ_n be the Riesz measure associated with the potential $\hat{R}^1_{E \cap A_n}(y)$ on Ω, which is the capacitary distribution ([4, Definition 5.4.1]) on $E \cap A_n$ and is supported on $\overline{E \cap A_n}$. Then $\mu_n(\Omega) = \mathrm{Cap}_2(E \cap A_n, \Omega)$.

Notice that from the Item 4 of Lemma 1.19

$$\hat{R}^1_{E\cap A_n}(y) = \int_\Omega G_\Omega(y,z) d\mu_n(z).$$

Notice that for $z \in E \cap A_n$,

$$G_\Omega(y,z) = U_z(y) - h_z(y),$$

where $h_z(y)$ is the greatest harmonic minorant of $U_z(y)$. As long as z stays in a compact subset of Ω, we see that $h_z(y)$ is bounded. Then we know $G_\Omega(y,z) \sim U_z(y) \sim \alpha^n$ for $z \in E \cap A_n$.

So we have

$$\hat{R}^1_{E\cap A_n}(y) \sim \alpha^n \|\mu_n\| = \alpha^n \mathrm{Cap}_2(E \cap A_n).$$

So we know (2) $\Longleftrightarrow$ (3).

(1) $\Longleftrightarrow$ (3)

For $n \geq n'$, suppose that (3) holds. Then there is a sequence $b_n \to +\infty$ and $v(y) < +\infty$, where superharmonic function

$$v(x) = \sum_{n'}^{\infty} b_n \hat{R}^1_{E\cap A_n}(x), x \in \Omega.$$

Since $\hat{R}^1_{E\cap A_n} = 1$ on $E \cap A_n$, except for a polar set F_n. We have $v(x) \to +\infty$ as $x \to y$ along $E \backslash \cup F_n$. Hence $E \backslash \cup F_n$ is thin at y. So E is thin at y.

Suppose (1) holds, then there is a superharmonic function $0 \leq w(x) \in \mathcal{U}(\Omega)$ such that $w(y) < +\infty$ and

$$\lim_{E \ni x \to y} w(x) = +\infty.$$

We let μ be the Riesz measure of $w(x)$ and $\mu|_{A_n} = \mu_n$.

$$G_\Omega\mu = G_\Omega\mu|_{\Omega\backslash \cup_{n'}^{\infty} A_n} + \sum_{n'}^{\infty} G_\Omega \mu_n.$$

We now $G_\Omega\mu - w(x)$ is harmonic in Ω. So

$$\lim_{E \ni x \to y} G_\Omega \mu = +\infty$$

and $G_\Omega\mu(y) < +\infty$. So $\sum_{n'}^{\infty} G_\Omega\mu_n((y)) < +\infty$. As long as we prove that for $n \geq n''$ large

$$G_\Omega(\mu_{n-1} + \mu_n + \mu_{n+1}) \geq 1, \ x \in A_n \cap E, \tag{7}$$

we can show

$$\sum_{n''}^{\infty} \hat{R}^1_{E\cap A_n}(y) \leq 3 \sum_{n''-1}^{\infty} G_\Omega \mu_n(y) < +\infty.$$

We know for $i = n', \cdots, n-2, n+2, n+3, \cdots$, the distance from the ring A_i to A_n is larger than $C \cdot d(A_i, \{0\})$.

$$
\begin{aligned}
&\sup_{A_i} G_\Omega(\mu_{n'} + \cdots + \mu_{n-2} + \mu_{n+2} + \cdots)(x) \\
\leq &C G_\Omega(\mu_{n'} + \cdots + \mu_{n-2} + \mu_{n+2} + \cdots)(0) \\
\leq &C G_\Omega \mu(0).
\end{aligned}
$$

So (7) must hold.

□

Theorem 1.26. *Let $E \subset \mathbb{R}^2$, let $y \in \mathbb{R}^2$ and let Ω be a Greenian domain containing y. Further, let $\beta \in (0,1)$, let*

$$
E_n = \{x \in E : \beta^{n+1} \leq |x - y| \leq \beta^n\}
$$

and let n' be such that $B(y, \beta^{n'-1}) \subset \Omega$. The following are equivalent:

1. *E is thin at y;*
2. *$\sum_{n'}^{\infty} n \mathrm{Cap}_2(E_n, \Omega) < +\infty$.*

1.10 Structure of noncompact surface with nonnegative Gaussian curvature.

Let u be a function, supermonic in $B_1(0) \subset \mathbb{R}^2$. What is the behavior, of u near $x = 0$.

Let $\mathrm{Ave}u(r) = \fint_{\partial B_r} u$. Then $\mathrm{Ave}u(r)$ is also a superharmonic function. By studying the ordinary different inequality, we can prove

$$
\lim_{r \to 0} \frac{\mathrm{Ave}u(r)}{\frac{1}{2\pi} \log(1/r)} = M.
$$

The question is whether there holds

$$
\lim_{r \to 0} \frac{u(x)}{\frac{1}{2\pi} \log \frac{1}{|x|}} = M.
$$

Theorem 1.27. *(Theorem 1.1-1.3 of [7], or Theorem 7.4.3. of [4])*
There is a set E, which is thin at 0, such that

$$
\lim_{x \to 0, x \notin E} \frac{u(x)}{\frac{1}{2\pi} \log \frac{1}{|x|}} = M.
$$

Let $g_{edu} = dx^2 + dy^2$ be the standard flat metric on $\mathbb{R}^2$. Let $g = e^{2u}(dx^2 + dy^2)$ be a complete comformal metric. We use K to denote the Gaussian curvature of g. The question is: if $K \geq 0$, what is the behavior of u near infinity?

Theorem 1.28. *(Theorem 4.2 of [14])*
There is a set E, which is 2-thin at infinity, such that

$$\lim_{x\to\infty, x\notin E} \frac{u(x)}{\log\frac{1}{|x|}} = m,$$

where $0 \leq m \leq 1$. Here 2-thin at infinity means that inversion

$$\tilde{E} = \{\frac{x}{|x|^2}; x \in E\}$$

is 2-thin at 0.

Proof. First we do the inversion, and we may consider the end as defined in the punctured ball $B_1(0)\backslash\{0\}, e^{2\tilde{u}}(d\tilde{x}_1^2 + d\tilde{x}_2^2)$. From the Gaussian curvature equation we have

$$\Delta\tilde{u} + Ke^{2\tilde{u}} = 0.$$

We use Δ_g to denote the Laplacian with respect to the metric g, we then have

$$-\Delta_g\tilde{u} = K \geq 0.$$

So

$$\Delta_g e^{-\tilde{u}} = e^{-\tilde{u}}(-\Delta_g\tilde{u} + |\nabla_g\tilde{u}|^2) \geq 0.$$

Consider the following theorem of Peter Li and R. Schoen in [8].

Theorem 1.29. *Suppose M^N is a complete Riemannian manifold with $Ric \geq -(N-1)k, k > 0$. Let $x_0 \in M$ for a nonnegative subharmonic function defined in $B_{2r}(x_0)$, there holds for any $\tau \in (0, \frac{1}{2})$*

$$\sup_{B^M_{(1-\tau)r}(x_0)} v^2 \leq \tau^{-C(1+\sqrt{k}r)} \frac{1}{vol(B^M_r(0))} \int_{B^M_r(x_0)} v^2 dvol_M.$$

Then let $v = e^{-\tilde{u}}$, we have

$$\begin{aligned} \sup_{B_{(1-\tau)r}(x_0)} e^{-2\tilde{u}} &\leq \tau^{-C} \frac{1}{vol(B^M_r(0))} \int_{B^M_r(x_0)} e^{-2\tilde{u}} dvol_M \\ &\leq \tau^{-C} \frac{vol_{|d\tilde{z}|^2}(B_r(x_0))}{vol_g(B_r(x_0))}. \end{aligned}$$

Now consider

Theorem 1.30. *(Theorem A of [2])*
If (M^2, g) is a complete and $K_g \geq 0$, then there is a constant $C(M)$ such that, for $r \leq 1$,

$$vol_g(B_r(x)) \geq C(M)r^2.$$

Now we have $\tilde{u} \to +\infty$ as $\tilde{x} \to 0$ in $B_1(0)\backslash\{0\}$. So $\tilde{u} \in \mathcal{U}(B_1(0))$ and $-\Delta\tilde{u}$ is a Radon measure on $B_1(0)$.

Then from Theorem 1.27, we know for a set $\tilde{E}$, 2-thin at 0

$$\lim_{\tilde{E} \not\ni (\tilde{x},\tilde{y}) \to 0,} \frac{\tilde{u}}{\log \frac{1}{|\tilde{x}|}} = \tilde{m}.$$

After do the inversion again, we can prove theorem, and $0 \leq m \leq 1$ follows from the fact that the metric is complete.

□

One result of Huber in [1] is the following.

Theorem 1.31. *Suppose (M, g) is a complete noncompact Riemann surface with*

$$\int_M K^- d\mu_g < +\infty.$$

Then there is a compact Riemann surface Σ such that M is comformally equivalent to $\Sigma\backslash\{p_1, \cdots, p_k\}$.

Here we use potential theory to prove a weak version.

Theorem 1.32. *Suppose $\Omega \subset (\mathbb{S}^2, g_{\mathbb{S}^2})$ is a domain, with a complete conformal metric $g = g_{\mathbb{S}^2}$. If $K_g \geq 0$, then $\partial\Omega = \{p\}$ or $\{p, q\}$. If $\partial\Omega = \{p, q\}$, then (Ω, g) is a cylinder.*

Proof. Choose $N \in \Omega$ as the north pole and use stereographic projection π_N to map $\mathbb{S}^2\backslash N$ to $\mathbb{R}^2$. Let S be $\pi_N(\partial\Omega) \subset D$, where D is a bounded domain of $\mathbb{R}^2$. Assume $g = e^{2v} g_{edu}$. We know

$$\lim_{x \to S} v(x) = +\infty.$$

Then we know that v is superharmonic in D and S is a polar set since $v|_S = +\infty$. The Hausdorff dimension of S is 0. Then it is totally disconnected. Each point of S corresponds to an end of Ω. From splitting theorem, there are at most two points in S, and two point case corresponds to a cylinder.

□

2 Nonlinear Potential theory

We reter to [9, 6, 5, 11, 12] for materials of this section.

2.1 Introduction

For $1 < p < \infty$, first we consider

$$I_p(u) = \int_\Omega |\nabla u|^p dx.$$

Given a function $\phi \in W^{1,p}(\Omega)$, we consider

$$\inf\{\int_\Omega |\nabla u|^p dx; u - \phi \in W_0^{1,p}(\Omega)\}.$$

Then the minimizer u_0 must satisfy

$$\frac{d}{dt}\int_\Omega |\nabla(u_0 + t\phi)|^p dx|_{t=0} = 0 \text{ for any } \phi \in \mathrm{C}_0^\infty.$$

Then we have

$$\int_\Omega < |\nabla u|^{p-2}\nabla u, \nabla\phi > dx = 0. \tag{8}$$

Then it satisfies the p-harmonic equation

$$\Delta_p u = 0, x \in \Omega. \tag{9}$$

Definition 18. *Let Ω be a domain in $\mathbb{R}^N$. We say $u \in W_{loc}^{1,p}(\Omega)$ is a weak solution of the p-harmonic equation (9) if for $\forall\phi \in C_0^\infty(\Omega)$, (8) holds.*

If in addition, u is continuous (actually it always has a continuous representative), then we say that u is p-harmonic function.

Definition 19. *A function $v : \Omega \to (-\infty, \infty]$ is called p-superharmonic in Ω, if*

1. *v is l.s.c. in Ω;*
2. *$v \not\equiv +\infty$ in Ω;*
3. *for each domain $D \Subset \Omega$ the comparison principle holds: if $h \in C(\bar{D})$ is p-harmonic in D and $h|_{\partial D} \leq v|_{\partial D}$, then $h \leq v$ in D.*

The set of all p−superharmonic functions on Ω are denoted by $\mathcal{U}_p(\Omega)$.

A function $u : \Omega \to [-\infty, \infty)$ is called p-subharmonic, if $-u$ is called superharmonic.

Example. *1. $|x|^{\frac{p-N}{p-1}}, p \neq N$ and $-\log|x|, p = N$ are p-superharmonic functions;*

2. *Although p-Laplace equation is not linear,*

$$v(x) = \int_\Omega \frac{d\mu(y)}{|x-y|^{(N-p)/(p-1)}}$$

is still superharmonic for $1 < p < N$ and μ is a nonnegative Radon measure (with compact support for instance);

3. *For $u, v \in \mathcal{U}(\Omega)$, $\min\{u, v\} \in \mathcal{U}(\Omega)$.*

Theorem 2.1. *([9, Theorem 5.11, 5.15])*
For $u \in \mathcal{U}(\Omega)$, we have for any $D \Subset \Omega$,

$$\int_D |v|^q < \infty$$

for $0 \leq q < \frac{N(p-1)}{N-p}$ in the case $1 < p \leq N$. And

$$\int_D |\nabla v|^r < \infty$$

for $0 < q < N(p-1)/(N-1)$ in the case $1 < p \leq N$ and $q = p$ in the case $p > N$.

From now on we assume $1 \leq p \leq n$. We use $u \wedge k$ to represent $\min\{u, k\}$.

Theorem 2.2. *For a superharmonic function $u \in \mathcal{U}(\Omega)$. $-\Delta_p u$ is a well defined Radon measure.*

Proof. Since $|\nabla u|^{p-2}\nabla u \in L^1$ from Theorem 2.1. We know

$$L_u^p(\phi) = \int_\Omega |\nabla u|^{p-2}\nabla u \nabla\phi d\mu$$

is well defined for $\phi \in C_0^\infty$. Then from dominated convergence theorem

$$\int_\Omega |\nabla u|^{p-2}\nabla u\nabla\phi d\mu = \lim_{k\to\infty}\int_\Omega |\nabla u \wedge k|^{p-2}(u \wedge k)\cdot\nabla\phi dx.$$

Notice that $u \wedge k \in W_{loc}^{1,p}(\Omega)$ and still superharmonic. Then we know

$$\int_\Omega |\nabla u \wedge k|^{p-2}(u \wedge k)\cdot\nabla\phi dx \geq 0$$

for $0 \leq \phi \in C_0^\infty(\Omega)$. Let $k \to \infty$, we know $L_u^p(\phi) \geq 0$ for $0 \leq \phi \in C_0^\infty(\Omega)$. Then from Riesz representation theorem, we can prove the conclusion.

□

On the other hand, we have the following theorem.

Theorem 2.3. *Suppose that Ω is bounded and $\mu \in \mathcal{M}^+(\Omega)$ is finite. Then there is an p-superharmonic function u in Ω such that*

$$-\Delta_p u = \mu$$

and $u \wedge k = W_0^{1,p}(\Omega)$ for all $k > 0$.

2.2 Wolff Potential and generalization of Riesz decomposition theorem

Definition 20. *Let $1 < p \leq n$. Let Ω be a domain, and μ is a Radon measure in Ω, we define the p-Wolff Potential by*

$$W^{\mu}_{1,p}(x_0, r) = \int_0^r \left(\frac{\mu(B(x_0, r))}{t^{n-p}}\right)^{1/(p-1)} \frac{dt}{t}.$$

One easily infers that $W^{\mu}_{1,2}(x_0, \infty)$ is the Newtonian potential of μ.

As the generalization of Riesz decomposition theorem for Δ, we have the following very important theorem.

Theorem 2.4. *([5, Theorem 1.6])*

Suppose u is a nonnegative p-superharmonic function in $B(x_0, 3r)$. If $\mu = -\Delta_p u$, then

$$c_1 W^{\mu}_{1,p}(x_0, r) \leq u(x_0) \leq c_2 \inf_{B(x_0,r)} u + c_3 W^{\mu}_{1,p}(x_0, 2r).$$

2.3 Wiener Criterion

Now we study the Wiener Criterion for p-harmonic equation. Consider

$$\begin{cases} \Delta_p u = 0 & x \in \Omega \\ u - f \in W^{1,p}_0(\Omega) \end{cases} \tag{10}$$

We assume $f \in W^{1,p}(\Omega) \cap C(\bar{\Omega})$. Then there is a unique solution u_f.

Definition 21. *$x_0 \in \partial\Omega$ is called p-regular, if the solution u_f to (10) has the limit value $f(x_0)$ whenever $f \in W^{1,p}(\Omega) \cap C(\bar{\Omega})$.*

Definition 22. *The p-fine topology is the coarsest topology that makes every p-superharmonic function continuous.*

Suppose $E \subset \Omega$ and $x_0 \in \Omega$. E is called p-thin at x_0, if x_0 is not the fine limit of $E \backslash \{x_0\}$.

Theorem 2.5. *Suppose $E \subset \Omega$ and $x_0 \in \Omega$, then the following statements are equivalent:*

1. *E is p-thin at x_0;*
2. *There is a p-superharmonic function in Ω such that*

$$\liminf_{x \to x_0, x \in E \backslash \{x_0\}} u(x) > u(x_0);$$

3. *There is a p-superharmonic function in Ω such that*

$$\liminf_{x \to x_0, x \in E \backslash \{x_0\}} u(x) = \infty$$

and $u(x_0) < +\infty$;

Theorem 2.6. *A boundary point x_0 is p-regular if and only if Ω^c is not p-thin at x_0.*

Theorem 2.7. *Wiener Criterion for p-Laplacian operator, $1 < p \leq n$*
Let $E \subset \Omega$ and $x_0 \in \Omega$, E is p-thin at x_0 if and only if

$$\int_0^1 (\frac{\mathrm{Cap}_p(B(x_0,t) \cap E, B(x_0,2t))}{\mathrm{Cap}_p(B(x_0,t), B(x_0,2t))})^{1/(p-1)} \frac{dt}{t} < +\infty$$

The necessary part is proved by Maz'ya in 1970 and the sufficient part is proved by Kilpelainen and Maly in 1994, in [5].

When $p = 2$ and $N \geq 3$, the above Wiener Criterion coincide with Theorem 1.25. For this, we note that

$$\mathrm{Cap}_p(B(x_0,t), B(x_0,2t)) \sim t^{n-2}.$$

So

$$\begin{aligned}
&\int_0^1 (\frac{\mathrm{Cap}_p(B(x_0,t) \cap E, B(x_0,2t))}{\mathrm{Cap}_p(B(x_0,t), B(x_0,2t))})^{1/(p-1)} \frac{dt}{t} \\
\sim &\int_{\alpha^{-\frac{n+1}{n-2}}}^{\alpha^{-\frac{n}{n-2}}} t^{2-n} \mathrm{Cap}_2(B(x_0,t) \cap E, B(x_0,2t)) \frac{dt}{t} \\
\sim &\sum_n \alpha^n \mathrm{Cap}_2(A_n \cap E, \Omega).
\end{aligned}$$

For $N \geq 3$ and $p = N$, the Wiener Criterion is equivalent to for some $\alpha > 1$

$$\sum_n \mathrm{Cap}_p(B(x_0, 2^{-n}) \cap E, B(x_0, 2^{-n+1}))^{\frac{1}{n-1}} < +\infty.$$

In our study, we will propose a different N-thin notion.

2.4 Conformal geometry and N-Laplacian equation.

First we have a new observation. Suppose (M, g_M) is an N-manifold and Ω is a domain, with a conformal metric $g = e^{2u} g_M$.

Then due to direct calculations we have

$$\tilde{R}_{ij} = R_{ij} - g_{ij}\Delta_M u + (2-N)u_{ij} + (N-2)u_i u_j + (2-N)|\nabla u|^2 (g_M)_{ij}.$$

Usually we take trace on both sides, we get Yamabe equation. Here we take values in ∇u direction. We get

$$\Delta_n u - |\nabla u|^{N-2} Ric(\frac{\nabla u}{|\nabla u|}) + |\nabla u|^{N-2} Ric(\frac{\nabla_g u}{|\nabla_g u|_g}) e^{2u} = 0.$$

This is another generaliztion of

$$\Delta u - K + K_g e^{2u} = 0$$

for $N = 2$.

Theorem 2.8. *([13, Theorem 1])*

Suppose (Ω, g) is a domain of $(\mathbb{S}^N, g_{\mathbb{S}})$ for $N \geq 3$, and $g_{\mathbb{S}}$ is the standard round metric. If $g = e^{2u} g_{\mathbb{S}}$ is complete and $Ric_g \geq 0$, $\partial\Omega$ consists of at most two points. The two point case corresponds to $\mathbb{S}^{N-1} \times R$.

Proof. We use N-Laplacian equation to prove this theorem.

Choose $q \in \Omega$ and use stereographic projection π to map $\partial\Omega$ to $S = \pi(\partial\Omega) \subset \mathbb{R}^N$. In a neighborhood D with metric g_{edu}, we have

$$g = e^{2u} g_{edu}.$$

And

$$\Delta_n u + Ric_g\left(\frac{\nabla_g u}{|\nabla_g u|_g}\right)|\nabla_g u|^{n-2} e^{2u} = 0.$$

From [10, Proposition 8.1], we know

$$\lim_{x \to S} u(x) = +\infty.$$

Then we know u is N-superharmonic in D, even across S and S is N-polar set, which has 0 N-capacity.

So $\partial\Omega$ is totally disconnected. From splitting theorem, we know the conclusion holds.

□

Now it is interesting to understand the case when $\partial\Omega = \{p\}$, that is Ω is conformally equivalent to $\mathbb{R}^N$.

Consider a complete conformal metric $g = e^{2u} g_{edu}$, if $Ric_g \geq 0$, does

$$\lim_{x \to \infty} \frac{u(x)}{\log \frac{1}{|x|}} = m$$

hold?

Theorem 2.9. *([11, Theorem 1.1])*

Let w be an N-superharmonic function in $B(0, 2) \subset \mathbb{R}^N$ and

$$-\Delta_N w = \mu \geq 0$$

for a Radon measure $\mu \in \mathcal{M}^+(B(0, 2))$. Then there is a set $E \subset \mathbb{R}^N$ which is N^*-thin at 0, such that

$$\lim_{x \notin E, x \to 0} \frac{w(x)}{\log \frac{1}{|x|}} = \liminf_{x \to 0} \frac{w(x)}{\log \frac{1}{|x|}} = m \geq 0$$

and

$$w(x) \geq m \log \frac{1}{|x|} - C$$

for $x \in B(0, 1) \backslash \{0\}$ and some C. Moreover if $w \in C^2(B(0, 2) \backslash \{0\})$ and $(B(0, 2) \backslash \{0\}, e^{2w}|dx|^2)$ is complete at 0, then $m \geq 1$.

Let

$$\omega(y,i) = \{x \in \mathbb{R}^N : 2^{-i-1} \le |x-y| \le 2^{-i}\}$$
$$\Omega(y,i) = \{x \in \mathbb{R}^N : 2^{-i-2} \le |x-y| \le 2^{-i+1}\}$$
$$\omega(\infty,i) = \{x \in \mathbb{R}^N : 2^{i} \le |x-y| \le 2^{i+1}\}$$
$$\Omega(\infty,i) = \{x \in \mathbb{R}^N : 2^{i-1} \le |x-y| \le 2^{i+2}\}.$$

A set E is called N^* -thin at y if

$$\sum_{n=n'}^{\infty} n^{N-1}\mathrm{Cap}_N(E \cap \omega(y,i), \Omega(y,i)) < +\infty.$$

A set E is called N^* -thin at ∞ if

$$\sum_{n=n'}^{\infty} n^{N-1}\mathrm{Cap}_N(E \cap \omega(\infty,i), \Omega(\infty,i)) < +\infty.$$

With this we proved

Theorem 2.10. *([11, Theorem 1.3])*
Suppose that $(\mathbb{R}^N, e^{2u}|dx|^2)$ is complete with nonnegative Ricci. Then there is a set E, N^*-thin at infinity, such that

$$\lim_{x\notin E, x\to\infty} \frac{\phi(x)}{\log\frac{1}{|x|}} = \liminf_{x\to\infty} \frac{\phi(x)}{\log\frac{1}{|x|}} = m$$

and

$$\phi(x) \ge m\log\frac{1}{|x|} - L$$

for some constant L, where $m \in [0,1]$. Moreover,

- $m = 0$ *if and only if g is flat;*
- *if Ric_g is bounded, then*
$$\lim_{x\to\infty} \frac{\phi(x)}{\log\frac{1}{|x|}} = m.$$

From Theorem 2.9, we can generalize the Huber's theorem to the following one.

Theorem 2.11. *([12, Corollary 1.1])*
For $N \ge 3$, let Ω be a domain in the standard unit round sphere $(\mathbb{S}^N, g_{\mathbb{S}})$. Suppose on Ω, there is a complete conformal metric $g = e^{2u}g_{\mathbb{S}}$ satisfying

1. $Ric_g^- \in L^1(\Omega, g) \cap L^\infty(\Omega, g)$;
2. $R_g \in L^\infty(\Omega, g)$ *and* $|\nabla^g R_g| \in L^\infty(\Omega, g)$;

Then $\partial\Omega = \mathbb{S}^N \backslash \Omega$ is a finite point set.

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